CH – COMPLEX NUMBERS

① Complex Nos :–

* $\sqrt{-1}$ = iota = i

→ $-1 = i^2$

→ $i^3 = i^2 \cdot i = -i$

→ $i^4 = +1$

* Sum of 4 consecutive integral power of iota is zero.

eg. :– $i^{318} + i^{319} + i^{320} + i^{321} = i^2 + i^3 + i^0 + i^1 = 0$

eg. :– $i^{-305} + i^{-304} + i^{-303} + i^{-302} =$

$= (i^{305})^{-1} + (i^{304})^{-1} + (i^{303})^{-1} + (i^{302})^{-1}$

$= i^{-1} + (1)^{-1} + (-i^3)^{-1} + (-1)^{-1} = 0$

* $\sqrt{ab} = \sqrt{a} \cdot \sqrt{b}$, only if atleast one of a or b $\geq$

Eg. :– $\sqrt{2} \cdot \sqrt{3} = \sqrt{6}$

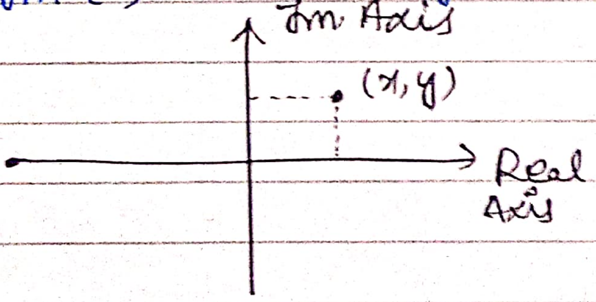
eg. :– $\sqrt{-2}\,\sqrt{3} = \sqrt{2}\, i \sqrt{3} = \sqrt{6}\, i$

But, eg. :– $\sqrt{-2} \cdot \sqrt{-3} = \sqrt{2}\, i \sqrt{3}\, i = -\sqrt{6} \neq \sqrt{6}$

* A Complex No. is of the form $Z = x + iy$, where,

$x \longrightarrow Re(Z)$
$y \longrightarrow Im(Z)$ & $x, y \in$ Real No.

is a point (x, y) on the Argand Plane



* Inequalities doesn't exist in complex No., i.e. two complex no. cannot be compared.

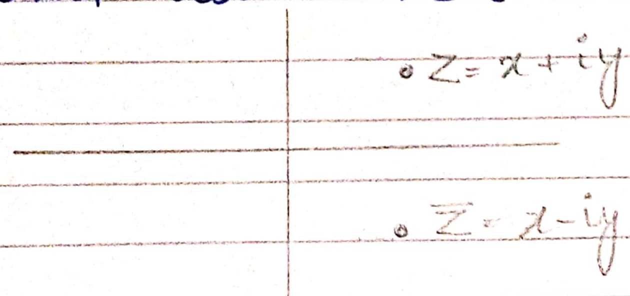
But, if for two complexes stated as $Z_1 > Z_2$ i.e., $Z_1 = x_1 + iy_1$ & $Z_2 = x_2 + iy_2$, then, $y_1 = y_2 = 0$ and $x_1 > x_2$

* For two complex no. $Z_1 = x_1 + iy_1$, and $Z_2 = x_2 + iy_2$
if $Z_1 = Z_2$, then, $x_1 = x_2$, and $y_1 = y_2$.

② Conjugate of a Complex No. :-

→ for $Z = x + iy$, $\bar{Z} = x - iy$ is its conjugate.

→ Also, Z & $\bar{Z}$ are mirror images of each other about Real Axis.



Q. If $Z_1 = 2 - 3i$ and $Z_2 = 4 + 5i$, then, $Z_1/Z_2 = ??$

Solⁿ. $\dfrac{Z_1}{Z_2} = \dfrac{2-3i}{4+5i} \times \dfrac{4-5i}{4-5i} = \dfrac{-7-22i}{41}$ Ans.

Q. Find the values of :-

(i) $(1+i)^2 = 2i$

(ii) $(1-i)^2 = -2i$

(iii) $\left(\dfrac{1+i}{1-i}\right) = i$

(iv) $\left(\dfrac{1-i}{1+i}\right) = -i$

(v) $(1+i)^{100} = ((1+i)^2)^{50} = (2i)^{50} = 2^{50} \cdot i^{50} = 2^{50} \cdot i^2 = -2^{50}$

Q. Simplify :-

$(x^4 + 2xi) - (3x^2 + yi) = (3-5i) + (1-2yi)$

Solⁿ. $x^4 - 3x^2 + (2x - y)i = 4 - (5 + 2y)i$

$x^4 - 3x^2 = 4$

Let $x^2 = t$, then,

$t^2 - 3t - 4 = 0 < \begin{matrix} \boxed{4 = x^2} \\ -1 \cancel{=} x^2 \end{matrix}$

$2x - y = -(5 + 2y)$

$y + 2x + 5 = 0$

$\boxed{y = -9 \text{ \& } x = 2}$

& $\boxed{y = -1 \text{ \& } x = -2}$

Q. find x & y = ??

$$\frac{x-1}{3+i} + \frac{y-1}{3-i} = i.$$

Solⁿ. On simplifying, $\frac{3x-3-ix+\cancel{i}+3y+iy-\cancel{i}}{9+1} = i$

$$\Rightarrow 3x+3y-6-(x-y)i = 10i$$

So, $3x+3y-6=0$ and $x+y=2$

$\Rightarrow$ $\boxed{x=-4}$ and $\boxed{y=6}$

(3) Additive Identity :- $Z + \underset{\text{Additive Identity}}{\underset{\downarrow}{0}} = Z$

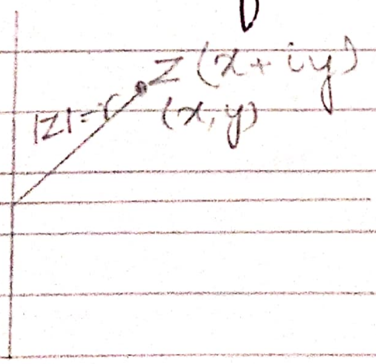
Additive Inverse :- $Z + \underset{\text{Additive Inverse}}{(-Z)} = 0$

Multiplicative Inverse :- $Z\left(\frac{1}{Z}\right) = 1$

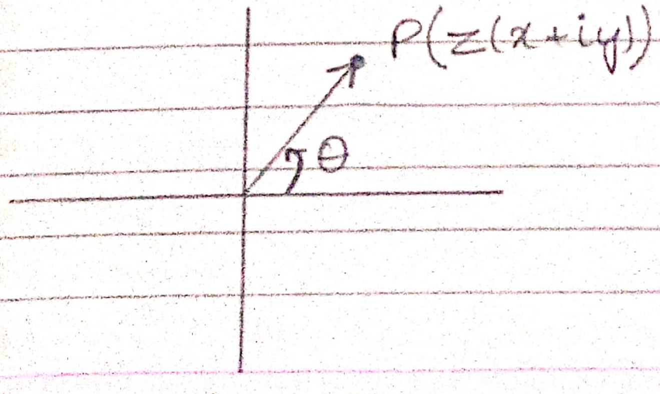
(4) Modulus of a Complex Number :-

$$|Z| = \sqrt{(x-0)^2+(y-0)^2}$$

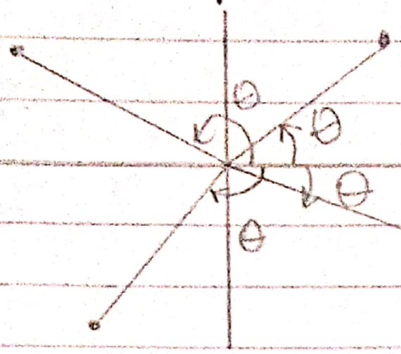
$$|Z| = \sqrt{x^2+y^2}$$



(5) Argument :- Angle made with real Axis in ACW sense.



* Principal Argument :-



$\boxed{-\pi < \theta \le \pi}$

Method to calculate Principal Argument :-

for $z = x + iy$

Calculate, $\tan\alpha = \left|\frac{y}{x}\right| \Rightarrow \alpha \to 0^\circ - 90^\circ$

② $z \to$ lies in which quad.

③ Quad. θ

Quad.	θ
I	$\theta = \alpha$
II	$\theta = \pi - \alpha$
III	$\theta = -(\pi - \alpha)$
IV	$\theta = -\alpha$

Q. Calculate principal argument for :-

(A) $z = -1 + \sqrt{3}\, i$

Sol^n Here, $\tan\alpha = \left|\frac{\sqrt{3}}{-1}\right| = \sqrt{3} \Rightarrow \alpha = \frac{\pi}{3}$

and, since, $z \to$ IInd Quad.,

$\therefore \theta = \pi - \alpha = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$

(B) $z = -\sqrt{3} - i$

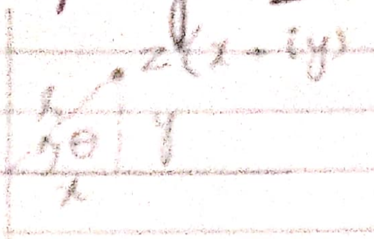
Sol^n Here, $\tan\alpha = \left|\frac{1}{\sqrt{3}}\right| = \frac{1}{\sqrt{3}} \Rightarrow \boxed{\alpha = \frac{\pi}{6}}$

and, since, $z \to$ IIIrd Quad.

$\therefore \theta = -(\pi - \alpha) = -\frac{5\pi}{6}$

{for Purely Real No. :- $z - \bar{z} = 0$
{for Purely Imaginary No. :- $z + \bar{z} = 0$}

(6) Polar / Trig. and Euler's form of Complex No.



$\because x = r\cos\theta$

$y = r\sin\theta$

$\therefore \boxed{z = x + iy = r(\cos\theta + i\sin\theta)}$ ★★ Imp.

$\rightarrow$ Polar / Trig form

θ : Principal Argument

Also, $\boxed{z = re^{i\theta}}$ ★★ Imp. V.V. $\Rightarrow$ Euler's form

NOTE :- for a complex No. $z = x + iy$:-

$\boxed{z^{-1} = \frac{1}{z} = \frac{\bar{z}}{|z|^2}}$ & $\boxed{z\bar{z} = |z|^2}$ ★★ Imp. ✓

(7) Properties of a Conjugate of a Complex No. :-

(A) If $z = \bar{z} \iff$ then, $z \rightarrow$ Purely Real No.
i.e., $z = x + iy \Rightarrow \boxed{y = 0}$ & $\boxed{z = x}$

(B) If $\bar{z} = -z \iff z \rightarrow$ Purely Imaginary No.
i.e., $z = x + iy \Rightarrow \boxed{x = 0}$ & $\boxed{z = iy}$.

(C) $\frac{z + \bar{z}}{2} = Re(z)$ and $\frac{z - \bar{z}}{2i} = Im(z)$

$\therefore \boxed{z = \frac{z + \bar{z}}{2} + i\left(\frac{z - \bar{z}}{2i}\right)}$

(D) $\overline{z_1 \pm z_2} = \bar{z}_1 \pm \bar{z}_2$

(E) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$

(F) $\left(\overline{z^n}\right) = (\bar{z})^n$

(G) $\left(\bar{\bar{z}}\right) = z$

(H) $\left(\overline{z_1 z_2}\right) = \bar{z}_1 \bar{z}_2$

(I) $z_1\bar{z}_2 + \bar{z}_1 z_2 = 2\,Re(z_1\bar{z}_2)$
$= 2\,Re(\bar{z}_1 z_2)$

Q. Find the Real part of $(1-\cos\theta + 2i\sin\theta)^{-1}$.

Sol. $z = \dfrac{1}{1-\cos\theta + 2i\sin\theta} = \dfrac{(1-\cos\theta) - 2i\sin\theta}{(1-\cos\theta)^2 + (2\sin\theta)^2}$

So, $Re(z) = \dfrac{1-\cos\theta}{1+\cos^2\theta - 2\cos\theta + 4\sin^2\theta} = \dfrac{(\cos\theta - 1)}{(\cos\theta - 1)(3\cos\theta + 5)}$

$\therefore \boxed{Re(z) = \dfrac{1}{5+3\cos\theta}}$

(8) Properties of Modulus :-

(A) $|z| = |-z| = |\bar{z}| = |-\bar{z}| = |zi| = |-iz| = |i\bar{z}|$

(B) $|z_1 z_2| = |z_1||z_2|$

$\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$; where, $|z_2| \neq 0$

(C) If $|z| = 0 \Rightarrow z = 0 + 0i$

(D) $|z^n| = (|z|)^n$

* (E) $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2Re(\bar{z}_1 z_2)$

$= |z_1|^2 + |z_2|^2 + 2Re(z_1\bar{z}_2)$

$= |z_1|^2 + |z_2|^2 + (\bar{z}_1 z_2 + z_1\bar{z}_2)$

$|z_1 - z_2|^2 = |z_1|^2 + |z_2|^2 - 2Re(\bar{z}_1 z_2)$

Also, $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2(|z_1|^2 + |z_2|^2)$

(F) $|z_1 \pm z_2| \neq |z_1| \pm |z_2|$ (In general)

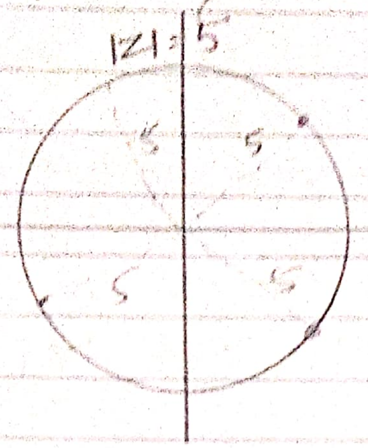
*** V. Imp. (G) $\underbrace{||z_1| - |z_2|| \leq}_{\text{II}^{nd}\ \triangle\ \text{Inequality}} \underbrace{|z_1 + z_2| \leq |z_1| + |z_2|}_{\text{I}^{st}\ \triangle\ \text{Inequality}}$

IInd $\triangle$ Inequality: Equality holds when $z_1, 0, z_2$ are collinear & origin is in between.

Ist $\triangle$ Inequality: Equality hold when the argument of z_1 & z_2 are same, and $0, z_1, z_2$ are collinear.

①

* Graphical Representation of $|z|$:-



Z can lie anywhere on the circle ($r=5$).

→ Defines a locus / Path / curve / possible locat" on the Argand Plane.

$|z|<5$ ⟶ Inside the circle

$|z|\le 5$ ⟶ Inside & boundaries

$|z|>5$ ⟶ Outside, excluding boundaries

→ $|z|$ or $|z-0| = r$ ⟹ Distance b/w z & origin.

→ $|z_1 - z_2| = r$ ⟹ Distance b/w z_1 & z_2 is 'r' units

Q. If $|z-2+i| \le 2$, then, find the greatest & least value of $|z|$.

Sol^n M-1) By using $\triangle$ Inequality :-

$$\big||z| - |-2+i|\big| \le |z + (-2+i)| \le |z| + |-2+i|$$

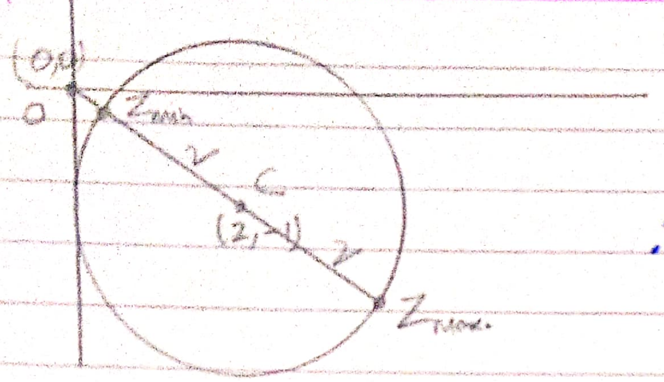
Thus, from here, $\big||z| - |-2+i|\big| \le 2$

$\Rightarrow -2 \le |z| - \sqrt{5} \le 2$

$\Rightarrow \boxed{-2+\sqrt{5} \le |z| \le 2+\sqrt{5}}$

Least Greatest

M-2)



$OC = \sqrt{5}$

$\therefore$ Min (z) $= SD = \sqrt{5} - 2$

Max (z) $= LD = \sqrt{5} + 2$

Q. If $\left|\dfrac{z_1 - 2z_2}{2 - z_1\bar{z}_2}\right| = 1$, and $|z_2| \neq 1$, then $|z_1| = ??$

Solⁿ $\because$ $|z_1 - 2z_2| = |2 - z_1\bar{z}_2|$

$\Rightarrow$ On squaring both sides, we have :-

$\Rightarrow |z_1 - 2z_2|^2 = |2 - z_1\bar{z}_2|^2$

$\Rightarrow (z_1 - 2z_2)(\bar{z}_1 - 2\bar{z}_2) = (2 - z_1\bar{z}_2)(2 - \bar{z}_1 z_2)$ $\quad [|z|^2 = z\bar{z}]$

$\Rightarrow z_1\bar{z}_1 - \cancel{2\bar{z}_1 z_2} - \cancel{2z_1\bar{z}_2} + 4z_2\bar{z}_2 = 4 - \cancel{2z_1\bar{z}_2} - \cancel{2\bar{z}_1 z_2} + |z_1|^2|z_2|^2$

$\Rightarrow |z_1|^2 + 4|z_2|^2 - |z_1|^2|z_2|^2 - 4 = 0$

$\Rightarrow |z_1|^2[1 - |z_2|^2] + 4[|z_2|^2 - 1] = 0$

$\Rightarrow [|z_1|^2 - 4][\cancel{1 - |z_2|^2}] = 0 \quad \Rightarrow \quad |z_1|^2 = 4 \quad \Rightarrow \boxed{|z_1| = 2}$

Q. If $|z| = 1$, then find out whether z_1 is a purely real / imaginary, where $z_1 = \dfrac{z-1}{z+1}$?

Solⁿ. If $z_1 \rightarrow$ Purely imaginary, then, $z_1 + \bar{z}_1 = 0$, thus

$$z_1 + \bar{z}_1 = \frac{z - 1}{z+1} + \frac{\bar{z} - 1}{\bar{z}+1}$$

$$= \frac{z\bar{z} + \cancel{z} - \cancel{\bar{z}} - 1 + z\bar{z} - \cancel{z} + \cancel{\bar{z}} - 1}{(z+1)(\bar{z}+1)}$$

$$= \frac{2|z| - 2}{(z+1)(\bar{z}+1)} = 0 \Rightarrow z_1 \text{ is a purely imaginary.}$$

Q. If $|\beta| = 1$, then $\left|\frac{\beta-\alpha}{1-\bar{\alpha}\beta}\right| = ??$

Sol^n $\because$ $|z|^2 = z\bar{z}$, thus,

$$\left|\frac{\beta-\alpha}{1-\bar{\alpha}\beta}\right|^2 = \left(\frac{\beta-\alpha}{1-\bar{\alpha}\beta}\right)\left(\frac{\bar{\beta}-\bar{\alpha}}{1-\alpha\bar{\beta}}\right)$$

$$= \frac{\beta\bar{\beta} - \alpha\bar{\beta} - \beta\bar{\alpha} + \alpha\bar{\alpha}}{1-\bar{\alpha}\beta - \alpha\bar{\beta} + |\alpha|^2|\beta|^2} = \frac{1 - \alpha\bar{\beta} - \beta\bar{\alpha} + |\alpha|^2}{1 - \bar{\alpha}\beta - \alpha\bar{\beta} + |\alpha|^2}$$

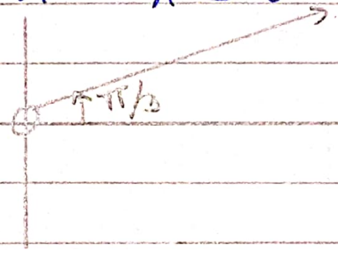
$$\therefore \left|\frac{\beta-\alpha}{1-\bar{\alpha}\beta}\right|^2 = 1 \Rightarrow \boxed{\left|\frac{\beta-\alpha}{1-\bar{\alpha}\beta}\right| = 1}$$

(9) Properties Of Argument :-

(A) Arg $(z) = \theta \Rightarrow$ A ray starting from origin at an angle θ in A.C.W. with +ve Real Axis.

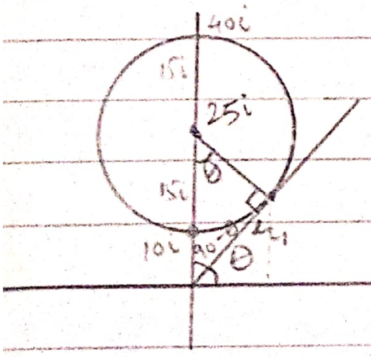
For ex :-

Arg $z = \frac{\pi}{3} \Rightarrow$



Q. If $|z - 25i| \le 15$, then find the z for which the argument is least & Max. ?

Sol^n for Min. Argument,

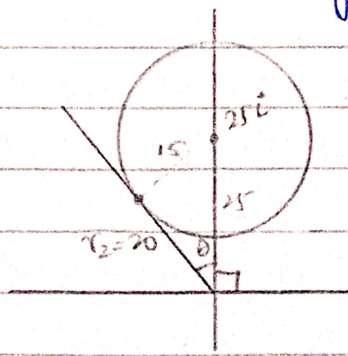


$\boxed{r_1 = 20 \text{ cm}}$

$\boxed{\tan\theta = \frac{4}{3}}$

$\boxed{z = 12 + 16i}$

For Max. Argument,



$z = r_2(\cos\theta + i\sin\theta)$

$= 20(\cos(90+\theta) + i\sin(90+\theta))$

$= 20[-\sin\theta + i\sin\theta]$

$= 20\left[+\frac{4}{5}i + \frac{3}{5}\right]$

$= -12 + 16i$

$(z-z_0)(\bar{z}-\bar{z}_0)=r^2$

$z\bar{z}-z_0\bar{z}-z\bar{z}_0+z_0\bar{z}_0-r^2=0$

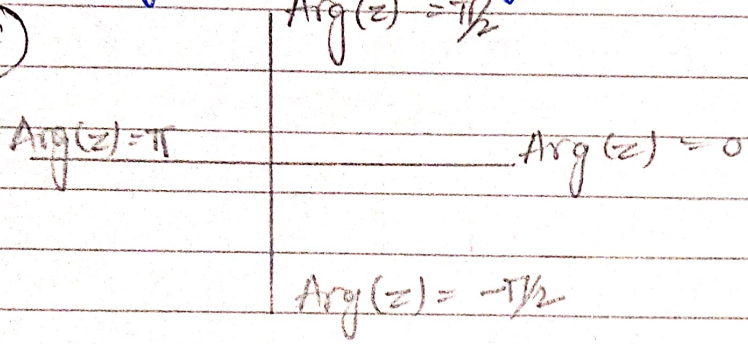
(B) $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2) + 2k\pi$

(C) $\arg(z_1/z_2) = \arg z_1 - \arg(z_2) + 2k\pi$

(D) $\arg(\bar{z}) = -\arg(z)$

(E) $\arg(z^n) = n\cdot\arg(z) + 2k\pi$

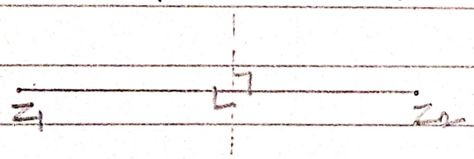
(F)



(G) $\arg(z\bar{z}) = \arg(|z|^2) = 0$

(H) If $|z-z_0| = r \Rightarrow$ Locus = Circle, Centre = z_0, radius = r

(I) $|z-z_1| = |z-z_2| \Rightarrow$ z lies on ⊥r bisector of the line segment joining z_1 & z_2.



(J) $\left|\dfrac{z-z_1}{z-z_2}\right| = k \Rightarrow$ Locus = Circle $(k \neq 1)$

(K) For $|z-z_1| + |z-z_2| = k$,

If $|z_1-z_2| > k \Rightarrow$ No Locus

If $|z_1-z_2| = k \Rightarrow$ Line segment joining z_1 & z_2

If $|z_1-z_2| < k \Rightarrow$ Ellipse is formed, z_1 & z_2 foci and k = Major Axis.

(L) For $\big||z-z_1| - |z-z_2|\big| = k$,

If $|z_1-z_2| > k \Rightarrow$ Complete Hyperbola, z_1 & z_2 = foci and Transverse Axis = k

If $|z_1-z_2| = k \Rightarrow$ 2 Outward Rays along z_1 & z_2 Line

If $|z_1-z_2| < k \Rightarrow$ NO Locus.