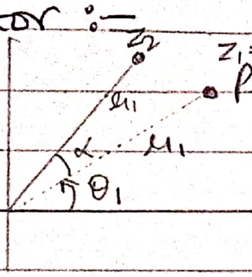
(10) Vectorial Representation and Rotation of the Complex No. Vector :-



$z_1 = x_1 + iy_1 = r_1 e^{i\theta_1}$

$Z_2 = r_1 e^{i(\theta+\alpha)}$

$Z_2 = r_1 e^{i\theta_1} . e^{i\alpha}$

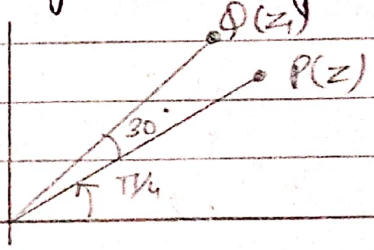
$\boxed{Z_2 = Z_1 e^{i\alpha}}$ → Rotate by an angle 'α' in A CW

Q. Find the new complex no. formed, if the original $Z = 1+i$ vector is rotated by $\pi/6$ angle, away from the +ve - x-axis.

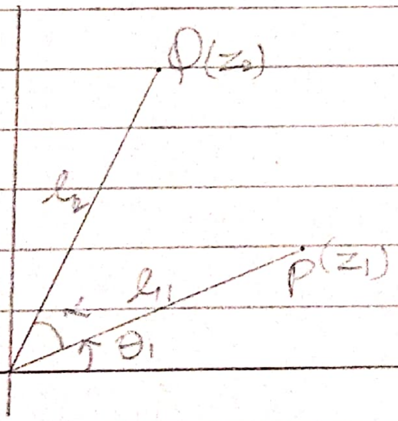
Sol^n $Z_2 = Z e^{i\pi/6} = (1+i) e^{i\pi/6}$

$Z_2 = \sqrt{2}\, e^{i\pi/4} . e^{i\pi/6}$

$Z_2 = \sqrt{2}\, e^{i5\pi/12}$



* Also, for two different complex no. vectors:-



$Z_1 = r_1 e^{i\theta_1}$

$Z_2 = r_2 e^{i(\theta_1+\alpha)}$

$Z_2 = r_2 e^{i\theta_1} . e^{i\alpha}$

$Z_2 = \frac{r_2}{r_1} r_1 e^{i\theta_1} . e^{i\alpha} = \frac{r_2}{r_1} . Z_1 e^{i\alpha}$

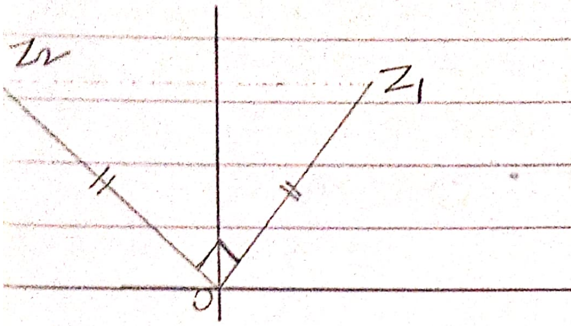
$\Rightarrow \boxed{\frac{Z_2}{Z_1} = e^{i\alpha} . \frac{r_2}{r_1} = \left|\frac{Z_2}{Z_1}\right| e^{i\alpha}}$

NOTE :- Since, $e^{i\pi/2} = i \sin\pi/2 + \cos\pi/2 = i$, thus, to rotate a complex no. vector by $\pi/2$ angle, multiply it by i.

T.I.P. :- Condition for Equilateral $\triangle$:-

$$z_3^2 + z_1^2 + z_2^2 = z_1 z_2 + z_2 z_3 + z_3 z_1$$

Special Case) :- Isosceles right angled $\triangle$ at Origin

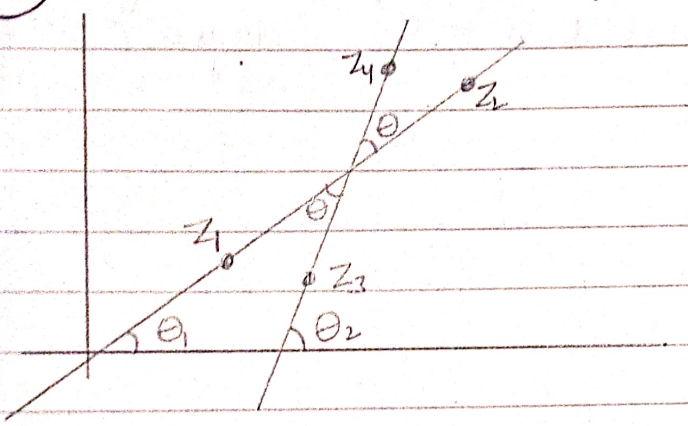


Here, $z_2 = z_1 i$

$\Rightarrow z_2^2 = z_1^2 i^2 = -z_1^2$

$$\Rightarrow \boxed{z_1^2 + z_2^2 = 0}$$

(II) Rotation Theorem / Coni's Method :-



By Ext. Angle property,

$\theta_2 = \theta_1 + \theta$

$\theta_2 - \theta_1 = \theta$

$\theta = \text{Arg}(z_4 - z_3) - \arg(z_2 - z_1)$

$$\theta = \arg\left(\frac{z_4 - z_3 \rightarrow \text{final}}{z_2 - z_1 \rightarrow \text{Initial}}\right)$$

Also, $\dfrac{z_4 - z_3}{z_2 - z_1} = \left|\dfrac{z_4 - z_3}{z_2 - z_1}\right| e^{i\theta}$

Special Case-I). $\arg.\left(\dfrac{z_4 - z_3}{z_2 - z_1}\right) = 0$ or π $\rightarrow$

Here, $z_1, z_2; z_3, z_4$ are collinear

and $\dfrac{z_4 - z_3}{z_2 - z_1}$ = purely real.

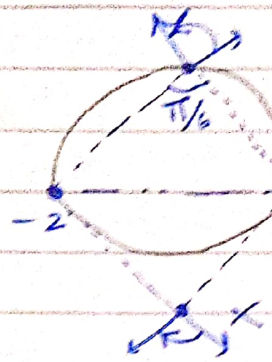
Special Case-II) $\arg\left(\dfrac{z_4 - z_3}{z_2 - z_1}\right) = \dfrac{\pi}{2}$ or $\dfrac{-\pi}{2}$

Here, $\dfrac{z_4 - z_3}{z_2 - z_1}$ = Purely imaginary

TIP :- $\arg\left(\frac{z-z_1}{z-z_2}\right) = \theta \Rightarrow$ Part of circle

Q. If $\arg\left(\frac{z-2}{z+2}\right) = \frac{\pi}{6}$, then find the locus of z.

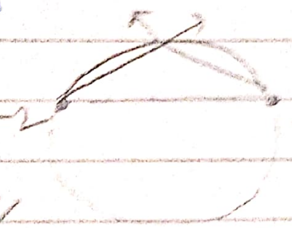
Sol^n



z will lie on the major arc of the circle passing from (2,0) & (-2,0)

Q. If $\arg\left(\frac{z-2}{z+2}\right) = \frac{2\pi}{3}$, then find the locus of z.

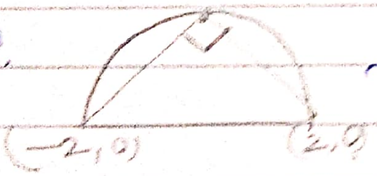
Sol^n



z will lie on the minor arc of the circle passing from (2,0) & (-2,0)

Q. If $\arg\left(\frac{z-2}{z+2}\right) = \frac{\pi}{2}$, then find the locus of z.

Sol^n



z will lie on the semicircle with diametric end points (2,0) & (-2,0)

(12) De-Moivre's Theorem :-

* If $(\cos\theta + i\sin\theta)^n$ is given, then, it has 'n' solutions, its value of one of its solution will be :-

$$(e^{i\theta})^n = e^{in\theta} = \boxed{\cos n\theta + i\sin n\theta}$$

eg:- Find any one solution of $z = \left(2\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\right)^{1/3}$

Sol^n $z = 2^{1/3}\left(\cos\pi/4 + i\sin\pi/4\right)^{1/3} = 2^{1/3}\left(e^{i\pi/4}\right)^{1/3} = 2^{1/3}\left(e^{i\pi/12}\right)$

$$z = 2^{1/3}\left(\cos\pi/12 + i\sin\pi/12\right) = \boxed{2^{1/3}\left(\cos\pi/12 + i\sin\pi/12\right)}$$ Ans

$*$ for Other roots of the eqn :-

If $Z = (Z_0)^{1/n} \Rightarrow Z^n = Z_0$

$$\Rightarrow Z^n = Z_0 = re^{i\theta} = r\,e^{i(\theta + 2k\pi)}$$

$$\Rightarrow \boxed{Z = r^{1/n}\, e^{i\left(\frac{\theta + 2k\pi}{n}\right)}}$$

where, k = any 4 consecutive value

Eg :- $k = 0, 1, 2, 3$.

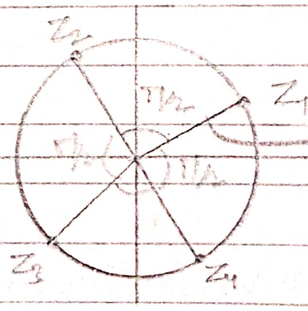
Also, the solutions of Z are symmetrically placed on the Argand plane, with each having $\boxed{\alpha = \frac{2\pi}{n}}$

Q.$*$ Find the roots of the eqn. $Z = (1+i)^{1/4}$.

Sol^n for roots, $Z^4 = (1+i)$

$$Z^4 = \sqrt{2}\, e^{i(\pi/4 + 2k\pi)}$$

$$Z = (\sqrt{2})^{1/4}\, e^{i(\pi/16 + k\pi/2)}$$



$\boxed{\alpha = \frac{2\pi}{4} = \frac{\pi}{2}}$

$\boxed{r = 2^{1/8}}$

$\therefore$
$$Z_1 = 2^{1/8}\, e^{i\pi/16}$$
$$Z_2 = 2^{1/8}\, e^{i9\pi/16}$$
$$Z_3 = 2^{1/8}\, e^{17\pi/16\, i}$$
$$Z_4 = 2^{1/8}\, e^{i\,25\pi/16}$$

(13) n^{th} roots of Unity :-

$*$ Here, $Z = (1)^{1/n}$

$$\Rightarrow Z^n = 1 = 1e^{i0} = e^{i(0 + 2k\pi)} = \boxed{e^{i\frac{2k\pi}{n}}}$$

$\boxed{\alpha = \frac{2\pi}{n}}$

Now, for diff. values of k,

$$Z_0 = 1$$
$$Z_1 = e^{i2\pi/n} = \alpha$$
$$Z_2 = e^{i4\pi/n} = \alpha^2$$
$$Z_3 = e^{i6\pi/n} = \alpha^3$$
$$\vdots$$
$$Z_{n-1} = e^{\frac{i2(n-1)\pi}{n}} = \alpha^{n-1}$$

T.I.P. :- Product of n^{th} roots of unity $\Rightarrow (-1)^{n+1} / (-1)^{n-1}$

$*$ Also, $z^n = 1 \Rightarrow z^n - 1 = 0 \Rightarrow z^n + 0z^{n-1} \dots - 1 = 0$

$\therefore$ Sum of roots of eqn. $= 0$

i.e. $1 + \alpha + \alpha^2 + \alpha^3 + \dots \alpha^{n-1} = 0$ — V. Imp

$*$ Also, $z^n - 1 = (z-1)(z-\alpha)(z-\alpha^2)(z-\alpha^3)\dots(z-\alpha^{n-1})$

$*$ Also, $(z-1)(z-\alpha)(z-\alpha^2)\dots(z-\alpha^{n-1}) = z^n - 1$

$= (z-1)\left(\dfrac{z^n - 1}{z-1}\right) \rightarrow \boxed{\text{G.P}}$

Q. Find the product of roots for $z^4 - 1 = 0$

Soln $z^4 + 0z^3 + 0z^2 + 0z - 1 = 0 \Rightarrow 1 \cdot \alpha \cdot \alpha^2 \cdot \alpha^3 = -1$

◇ Cube root of unity :-

i.e. $z = (1)^{1/3}$

$z = e^{i\frac{2k\pi}{3}}$ →
- 1
- $\omega = e^{i2\pi/3}$
- $\omega^2 = e^{i4\pi/3}$

Sum $= 0$
Product $= 1$
$(\omega^3 = 1)$

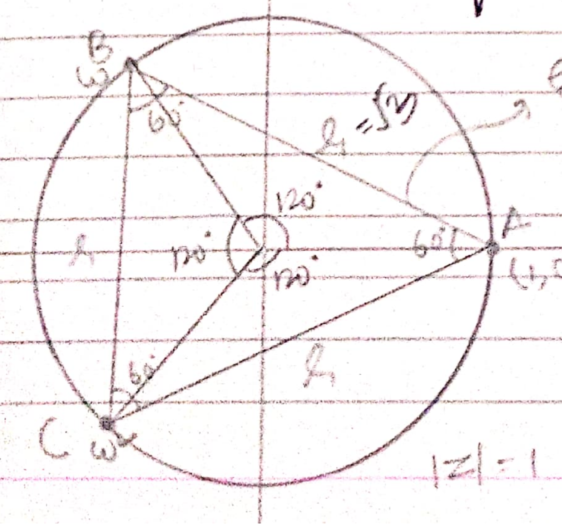
Q. Find the value of :-

(i) $\omega^{23} = \omega^2 = \cos 4\pi/3 + i \sin 4\pi/3 = -1/2 - \frac{\sqrt{3}}{2} i$

(ii) $\omega^{12378} = \omega^0 = 1$

(iii) $\omega^{19} = \omega^1 = \cos 2\pi/3 + i \sin 2\pi/3 = +\frac{\sqrt{3}}{2} i - 1/2$

$*$ Cube roots of unity on Argand Plane :-



eq. Δ → Area $= \frac{\sqrt{3}}{4} a^2$

→ O = G = C = I Coincides (origin)

$\overline{\omega^2} = \omega$

$\overline{\omega} = \omega^2$

$\rightarrow \omega^n + \omega^{n+1} + \omega^{n+2} = \omega^n(1+\omega+\omega^2) = \underline{0}$

$\rightarrow 1 + \omega^p + (\omega^2)^p = 1^p + \omega^p + \omega^{2p} = \begin{cases} 3 & p = 3n \\ 0 & p \neq 3n \end{cases}$

Q. Find the Value of $\sum_{r=1}^{56}\left(1+\frac{1}{\omega^r}+\omega^r\right)^2$

Sol^n. $\sum_{r=1}^{56}\left(1^r+\left(\frac{1}{\omega}\right)^r+\omega^r\right)^2 = \sum_{r=1}^{56}\left(1^r+\left(\frac{\omega^3}{\omega}\right)^r+\omega^r\right)^2$

$\Rightarrow \sum_{r=1}^{56}\left(1^r+\omega^{2r}+\omega^r\right)^2$ $\begin{cases} 3 & (r=3n) \rightarrow 3, 6, 9, \ldots 54 \\ 0 & (r \neq 3n) \end{cases}$

$\therefore \sum_{r=1}^{56}\left(1+\omega^{2r}+\omega^r\right)^2 = (3)^2 \times 18 = \underline{162}$

Q. Find $(1+\sqrt{3}i)^{100} + (1-\sqrt{3}i)^{100} = ??$

Sol^n. $2^{100}\left(\left(\frac{1+\sqrt{3}i}{2}\right)^{100} + \left(\frac{1-\sqrt{3}i}{2}\right)^{100}\right)$

$= 2^{100}\left(\left(\frac{-1}{2}-\frac{\sqrt{3}}{2}i\right)^{100} + \left(\frac{-1}{2}+\frac{\sqrt{3}}{2}i\right)^{100}\right)$

$= 2^{100}\left((\omega^2)^{100} + \omega^{100}\right) = 2^{100}(\omega^2+\omega) = \underline{-2^{100}}$

Q. Find $(1+\omega-\omega^2)^{25}(3+3\omega^2-7\omega)^6 = ??$

Sol^n. $(-\omega^2-\omega^2)^{25}\left[3(1+\omega^2)-7\omega\right]^6$

$= (-2\omega^2)^{25}(3(-\omega)-7\omega)^6$

$= (-2)^{25}\omega^{50}(10)^6\omega^6 = \underline{-2^{25}\times 10^6 \times \omega^2}$ Ans.

$\rightarrow$ ω and ω^2 are sq. roots of each other.

$x = \sqrt{\omega} = \pm\omega^2$ $\quad$ $\sqrt{\omega^2} = \pm\omega$

Q. Find $\dfrac{a+b\omega+c\omega^2}{a\omega+b\omega^2+c} = ??$ $= \dfrac{a+b\omega+c\omega^2}{a\omega+b\omega^2+c(1)}$ ω^3

$\Rightarrow \dfrac{a+b\omega+c\omega^2}{a\omega+b\omega^2+c\omega^3} = \dfrac{\cancel{a+b\omega+c\omega^2}}{\omega(\cancel{a+b\omega+c\omega^2})} = \dfrac{1}{\omega} = \omega^2 = \boxed{\dfrac{-1}{2} - \dfrac{\sqrt{3}i}{2}}$

Q.* Find $z^3 = 8 \Rightarrow z = ??$

Sol^n $z = (8)^{1/3}(1)^{1/3}$

$= 2 \left\langle \begin{matrix} 1 \\ \omega \\ \omega^2 \end{matrix} \right. = 2, 2\omega, 2\omega^2$

Q.* Find z, if $z^3 = -8$.

Sol^n $z^3 = (-8) \Rightarrow z = (-8)^{1/3} = -2 \left\langle \begin{matrix} 1 \\ \omega \\ \omega^2 \end{matrix} \right. = \boxed{-2, -2\omega, -2\omega^2}$

Q.* $(z-1)^3 + 8 = 0 \Rightarrow$ Find $z = ??$

Sol^n $(z-1)^3 = -8$

$(z-1) = (-8)^{1/3} \left\langle \begin{matrix} -2 \\ -2\omega \\ -2\omega^2 \end{matrix} \right. \Rightarrow \boxed{z = -1, 1-2\omega, 1-2\omega^2}$

$\rightarrow$ Since, $z^3 = 1 \longrightarrow 1, \omega, \omega^2$

and $(z^3-1) = (z-1)(z-\omega)(z-\omega^2)$

$= \cancel{(z-1)}(z^2+z+1) = \cancel{(z-1)}(z-\omega)(z-\omega^2)$

$\Rightarrow z^2+z+1 = (z-\omega)(z-\omega^2)$

Also, $z^3 = -1 \Rightarrow -1, -\omega, -\omega^2$

$z^3+1 = 0 = (z+1)(z+\omega)(z+\omega^2)$

$(z^2-z+1) = (z+\omega)(z+\omega^2)$

(14) Square Root of a Complex No. :—

Q. What will be the value $\sqrt{3+4i}$?