CH- QUADRATIC EQUATION

- Representation :- $ax^2 + bx + c$
  where $a \neq 0$ & $a, b, c \in R$
  ↳ Leading coefficient
- Roots Of Quadratic Eq. :-

$$\alpha ; \beta = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

where $b^2 - 4ac = D$.

$$a \times (x - \alpha)(x - \beta) = ax^2 + bx + c.$$

→ Sum of Roots :- $-\frac{b}{a} = (\alpha + \beta)$

→ Product Of Roots :- $c/a = (\alpha\beta)$

→ Difference Of Roots :-

$$|\alpha - \beta| = +ve \text{ quantity}.$$

$$(\alpha - \beta)^2 = \alpha^2 + \beta^2 - 2\alpha\beta$$
$$= \alpha^2 + \beta^2 + 2\alpha\beta - 4\alpha\beta$$

$$(\alpha - \beta)^2 = (\alpha + \beta)^2 - 4\alpha\beta$$
$$= \left(\frac{-b}{a}\right)^2 - 4\left(\frac{c}{a}\right)$$

$$(\alpha - \beta)^2 = \frac{b^2 - 4ac}{a^2} = \frac{\sqrt{b^2 - 4ac}}{a} = \frac{\sqrt{D}}{a}$$

Q. Given a quadratic Equation $ax^2+bx+c=0$ whose two roots are α & β; then find:-

i). $\alpha^2+\beta^2 = ??$

Sol^n: $(\alpha+\beta)^2 = \alpha^2+\beta^2+2\alpha\beta$

$\Rightarrow (\alpha+\beta)^2 - 2\alpha\beta = \alpha^2+\beta^2 \Rightarrow \left(\frac{-b}{a}\right)^2 - 2\left(\frac{c}{a}\right)$

D

$\Rightarrow \boxed{\frac{b^2-2ac}{a^2}}$

ii). $\alpha^3+\beta^3 = (\alpha+\beta)(\alpha^2+\beta^2-\alpha\beta)$

$= (\alpha+\beta)((\alpha+\beta)^2-2\alpha\beta-\alpha\beta)$

$= (\alpha+\beta)((\alpha+\beta)^2-3\alpha\beta)$

$= \left(-\frac{b}{a}\right)\left(\left(\frac{-b}{a}\right)^2-3\left(\frac{c}{a}\right)\right)$

$= \left(\frac{-b}{a}\right)\left(\frac{b^2-3ac}{a^2}\right)$

$= \boxed{\frac{3abc-b^3}{a^3}}$

iii) $\alpha^2\beta+\beta^2\alpha = (\alpha\beta)(\alpha+\beta) = \left(\frac{c}{a}\right)\left(\frac{-b}{a}\right)$

$= \boxed{\frac{-bc}{a^2}}$

$\frac{\sqrt{D}}{a}$

iv). $\frac{\alpha^2+\beta^2}{\alpha\beta} = \frac{(\alpha+\beta)^2-2\alpha\beta}{\alpha\beta} = \frac{\frac{b^2}{a^2}-\frac{2c}{a}}{\frac{c}{a}}$

$\Rightarrow \boxed{\frac{b^2-2ac}{ac}}$

vi) $\frac{\beta}{\alpha^2} + \frac{\alpha}{\beta^2} = \frac{\beta^3 + \alpha^3}{\alpha^2 \beta^2}$

$= \frac{3abc - b^3}{a^3 \times \frac{c^2}{a^2}} = \boxed{\frac{3abc - b^3}{ac^2}}$

Q. Find $\frac{\alpha}{\beta} + \frac{\beta}{\alpha} = ??$; if $2\alpha^2 = 3\alpha - 5$ and $2\beta^2 = 3\beta - 5$.

Sol^n. From the given equation, we can conclude that α & β are definitely its roots.

$\therefore$ Eq. formed $= 2x^2 - 3x + 5 = 0$

Now

$\frac{\alpha^2 + \beta^2}{\alpha\beta} = \frac{(\alpha+\beta)^2 - 2\alpha\beta}{\alpha\beta} = \frac{\left(\frac{3}{2}\right)^2 - 2\left(\frac{5}{2}\right)}{\frac{5}{2}}$

$\Rightarrow \boxed{-\frac{11}{10}}$

Q. If $x^2 + px - q = 0 <^{\alpha}_{\beta}$ and $x^2 + px + r = {}^{\gamma}_{\delta}$

Then, find the value of

$\frac{(\alpha - \gamma)(\alpha - \delta)}{(\beta - \gamma)(\beta - \delta)} = ??$

Sol^n. $\dfrac{(\alpha-\gamma)(\alpha-\delta)}{(\beta-\gamma)(\beta-\delta)} =$

$$\Rightarrow \frac{\alpha^2-\gamma\delta-\gamma\alpha+\gamma\delta}{\beta^2-\beta\delta-\gamma\beta+\gamma\delta} \Rightarrow \frac{\alpha^2-\overbrace{(\delta+\gamma)}^{-p}\alpha+\gamma\delta}{\beta^2-\underbrace{(\delta+\gamma)}_{-p}\beta+\gamma\delta}$$

$$\Rightarrow \frac{\alpha^2-(-p)\alpha+q}{\beta^2-(-p)\beta+q} = \frac{\alpha^2+p\alpha+q}{\beta^2+p\beta+q} = 1$$

Imp

Q. $x^2+px+1=0 <^{\alpha}_{\beta}$ & $x^2+qx+1=0 <^{\gamma}_{\delta}$

Then, find $(\alpha-\gamma)(\beta-\gamma)(\alpha+\delta)(\beta+\delta) =$

Sol^n $(\alpha\beta-(\alpha+\beta)\gamma+\gamma^2)(\alpha\beta+(\alpha+\beta)\delta+\delta^2)$

$\Rightarrow (1-(-p)\gamma+\gamma^2)(1+(-p)\delta+\delta^2)$

$\Rightarrow (\gamma^2+p\gamma+1)(\delta^2-p\delta+1)$

Now, Putting γ & δ in $x^2+qx+1=0$

$\gamma^2+q\gamma+1=0$ & $\delta^2+q\delta+1=0$

$\therefore (-q\gamma+p\gamma)(-q\delta-p\delta)$

$\Rightarrow (-1)(p\delta+q\delta)(p\gamma-q\gamma) \Rightarrow \boxed{q^2-p^2}$

Q. $px^2 + qx + 2 = 0$; find p if roots are reciprocal of each other.

Soln: If roots are reciprocal of each other, then, let them be :-

$$\alpha \ \& \ \frac{1}{\alpha}$$

$$\therefore \alpha \times \frac{1}{\alpha} = \boxed{1} = \frac{2}{p}$$

$$\boxed{\therefore p = 2}$$

Q. If $(k-2)x^2 - (k-4)x - 2 = 0$ and Difference of roots is 3 ; then find k

given that,

$$|\alpha - \beta| = 3$$

$$\therefore (\alpha - \beta)^2 = 9$$

$$\Rightarrow \alpha^2 + \beta^2 - 2\alpha\beta = 9$$

$$\Rightarrow (\alpha + \beta)^2 - 4\alpha\beta = 9$$

$$\Rightarrow \left(\frac{k-4}{k-2}\right)^2 - 4\left(\frac{-2}{k-2}\right) = 9$$

$$\Rightarrow \left(\frac{k-4}{k-2}\right)^2 + \frac{8}{(k-2)} = 9$$

$$\Rightarrow 2k^2 - 9k + 9 = 0$$

$$\therefore k = 3 ; 3/2$$

- Important Points :-
  - $D > 0 \rightarrow$ Real & Distinct Roots
  - $D < 0 \rightarrow$ Imaginary Roots
  - $D = 0 \rightarrow$ Real & Equal Roots
  - $a, b, c \in Q$ (rational) then roots will be rational if D is perfect square.
  - Irrational roots occur in conjugate pair i.e. $a + \sqrt{b}$ & $a - \sqrt{b}$, only if $a, b, c \in Q$

Q. Find whether the roots of the equation
* $(a + c - b)x^2 + 2cx + (b + c - a) = 0$ are rational or Irrational?

Soln $D = (2c)^2 - 4(a + c - b)(b + c - a)$

$= 4c^2 - 4(ab + \cancel{ac} - a^2 + \cancel{bc} + c^2 - \cancel{ac} - b^2 - \cancel{bc} + ab)$

$= \cancel{4c^2} - 4(ab - a^2 + \cancel{c^2} - b^2 + ab)$

$= -8ab + 4a^2 + 4b^2$ ↳ Perfect sq.

$\boxed{\therefore \text{ Root are rational}}$

Q. $x^2 - 2x(1 + 3m) + 7(3 + 2m) = 0$ has equal roots; find 'm'.

Soln. If eq. has equal roots; then $D = 0$.

$\therefore 4(1 + 3m)^2 - 4(7)(3 + 2m) = 0$

$\Rightarrow (1 + 9m^2 + 6m) - (7)(3 + 2m) = 0$

$\Rightarrow 9m^2 - 8m - 20 = 0$

$\therefore \boxed{m = \frac{-10}{9} \text{ \& } 2}$

Q. $x^2 - 2ax + a^2 - b^2 - c^2 = 0$. find whether the roots of equation are real or imaginary.

Soln.

$$D = b^2 - 4ac$$
$$= 4a^2 - 4(1)(a^2 - b^2 - c^2)$$
$$= \cancel{4a^2} - \cancel{4a^2} + 4b^2 + 4c^2$$
$$= 4b^2 + 4c^2$$

$\therefore$ Roots are real as $D > 0$.

- Identities in Quadratic Equation :-

A quadratic equation with infinite soln is reffered to as an Identity.

for Example :-

$$(x+1)^2 = x^2 + 2x + 1$$

$\hookrightarrow$ Infinite Soln.

→ If $ax^2 + bx + c = 0$ is an identity then, $a = b = c = 0$.

Q. $(p+2)(p-1)x^2 + (p-1)(1+2p)x + p^2 - 1 =$

find the value of 'p' for which above equatn is an identity?

1

Sol^n $(p+2)(p-1)=0$; $(p-1)(2p+1)=0$; $p^2-1=0$

$\therefore$ $p=-2$; $p=1$ | $p=1$; $p=\frac{-1}{2}$ | $p=\pm 1$

Intersect^n

$$\boxed{p=1}$$

Q. $(x^2-1)a^2-(2x+3)a-(x^2+2x+2)=0$

* is an identity in 'x'. Find 'a' = ??

Sol^n

$$a^2x^2-a^2-2ax-3a-x^2-2x-2=0$$

$$(a^2-1)x^2-2(a+1)x-a^2-3a-2=0$$

$a^2-1=0$ | $-2(a+1)=0$ | $-a^2-3a-2=0$

$\boxed{a=\pm 1}$ | $\boxed{a=-1}$ | $a^2+3a+2=0$

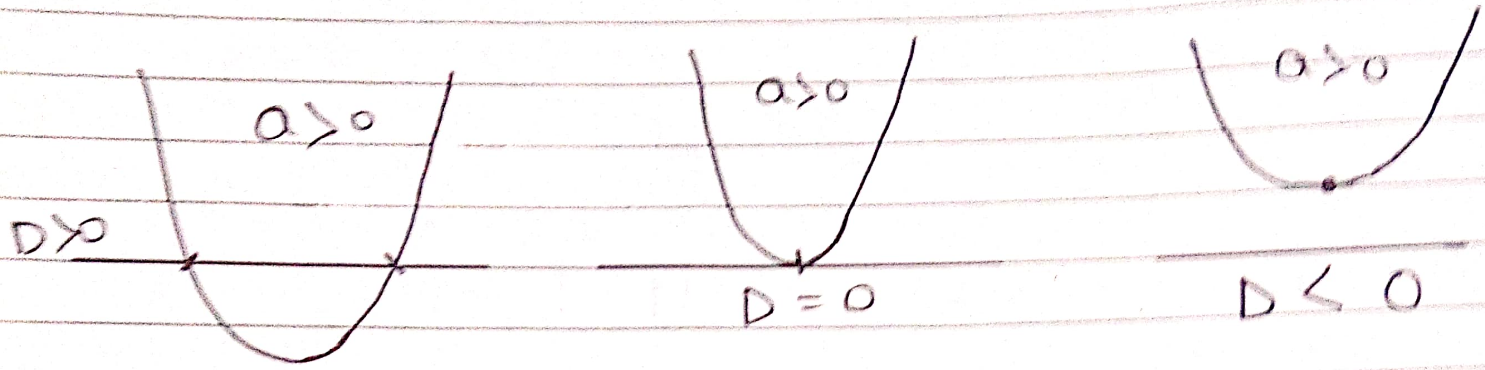
$(a+1)(a+2)$

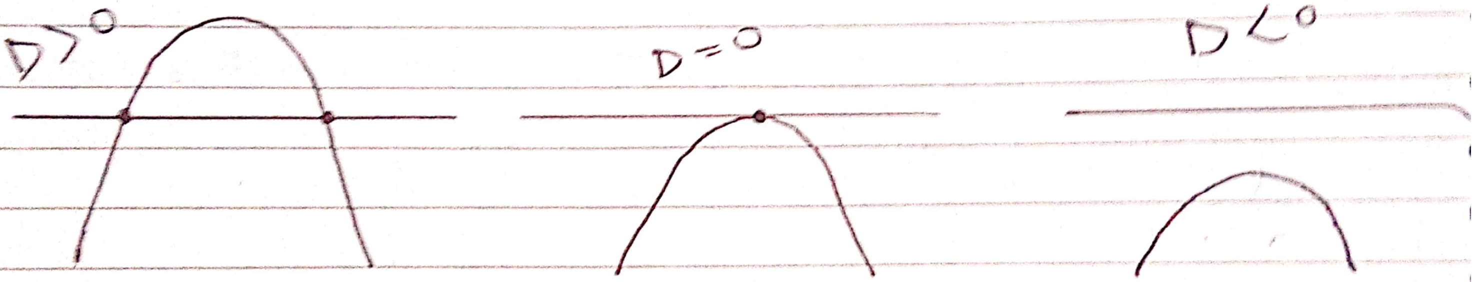
$\boxed{a=-1}$ $\boxed{a=-2}$

$\therefore$ $\boxed{a=-1}$

- Graphs Of Quadratic Expression / function

For Equation = $ax^2 + bx + c$.





→ Important Observations :-

↳ If $f(x)$ is always +ve $\longleftrightarrow$ $a>0$ & $D<0$.

↳ $f(x)$ is always -ve $\longleftrightarrow$ $a<0$ & $D<0$

↳ $f(x) \geq 0$ if $a>0$ and $D \leq 0$.

↳ $f(x) \leq 0$ if $a<0$ and $D \leq 0$

- Signs of a, b, c from graph of quadratic equation :-

(I) for a :-

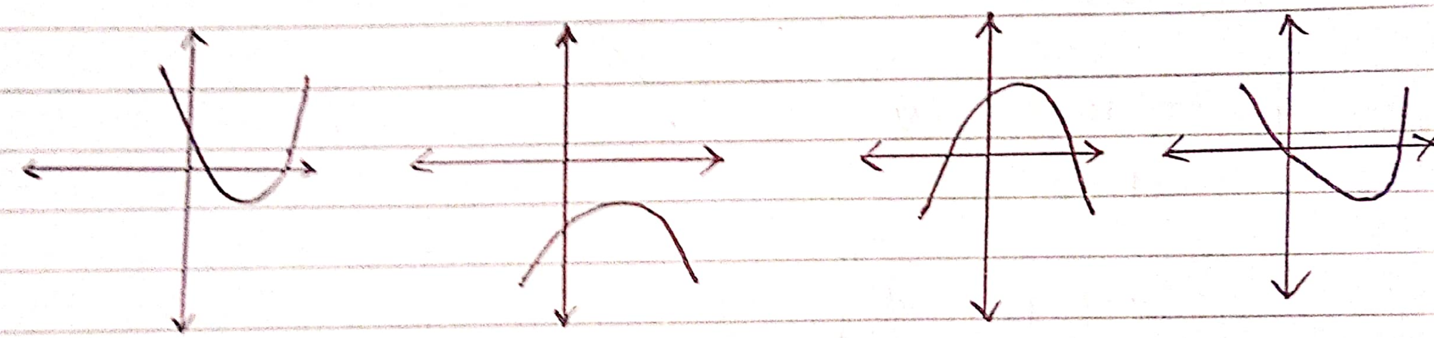
$a > 0 \longleftrightarrow$ Upward Parabola

$a < 0 \longleftrightarrow$ Downward Parabola.

(II) for c :-

→ If the parabolic graph cuts the y-axis which is +ve then $c > 0$.

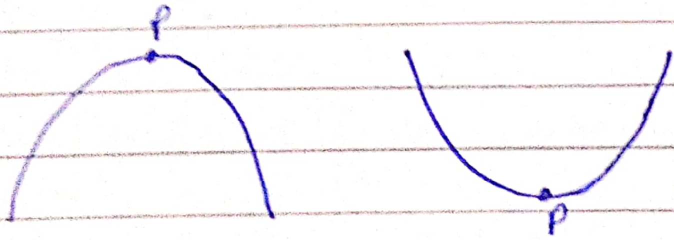
→ If the parabolic graph cuts the y-axis which is −ve then $c < 0$.



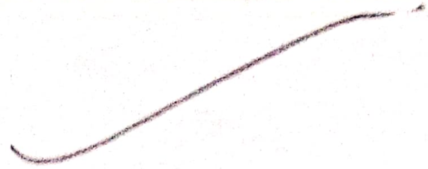
$\boxed{c > 0}$ $\boxed{c < 0}$ $\boxed{c > 0}$ $\boxed{c = 0}$

(III) for b :-

for $f(x) = ax^2 + bx + c$



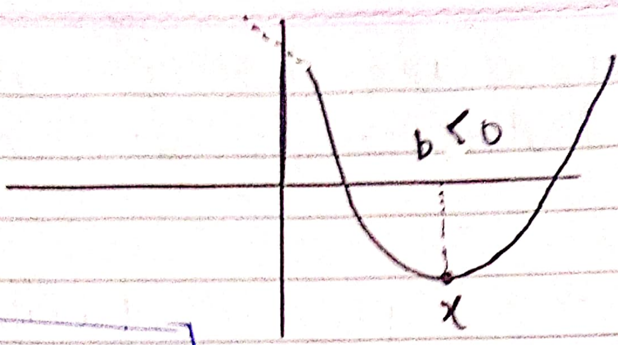
$\rightarrow P\left(\frac{-b}{2a}, \frac{-D}{4a}\right)$



Here,

$a > 0$
$c > 0$
$\boxed{x > 0}$

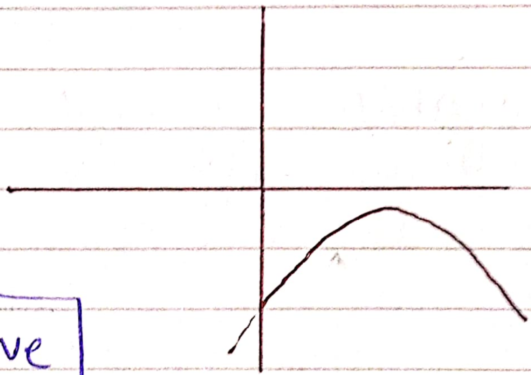
$\therefore \boxed{\dfrac{-b}{2a} = +ve}$



then, b should be $-ve$.

Here,

$a < 0$
$c < 0$
$x > 0$



$\therefore \boxed{\dfrac{-b}{2a} = +ve}$

$\boxed{\therefore \; b > 0}$

Range Of Quadratic Expression :-

For $f(x) = y = ax^2 + bx + c$

→ If $a > 0 \longrightarrow y \in \left[\dfrac{-D}{4a}, \infty\right)$

→ If $a < 0 \longrightarrow y \in \left(-\infty, \dfrac{-D}{4a}\right)$

Q. Find the range of $x^2 + x - 2 = y$.

Here, $a > 0$

$\therefore D = 1 - 4(-2) = \boxed{9}$

and $\dfrac{-D}{4a} = \dfrac{-9}{4(1)} = \dfrac{-9}{4}$

$\therefore y \in \left[\dfrac{-9}{4}, \infty\right)$

Imp

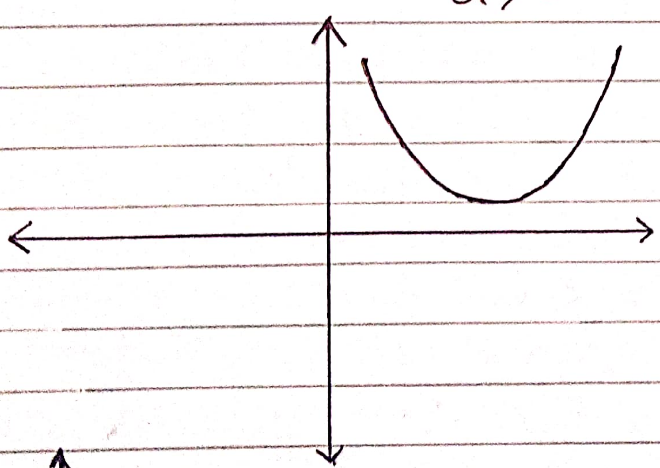
* Q. Given equation $ax^2 + bx + c = 0$ does not have real roots and $a + b + c < 0$. Then find the sign of $c = ??$

Soln

Given that, $D < 0$

Case-I)

$a > 0$

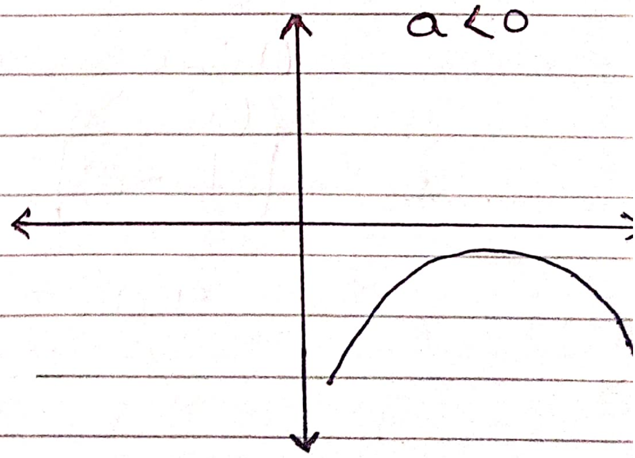


Not Possible, due to

$f(x) = f(1) = a + b + c < 0$

Case-II)

$a < 0$



$\boxed{\therefore C = -ve}$

V. Imp

Q. For $f(x) = x^2 + x - 2$ having domain in $x \in [-1, 1]$, find the range.

Soln Here,

$$-1 \leq x \leq 1$$

$$-1/2 \leq x + 1/2 \leq 3/2$$

$$0 \leq (x + 1/2)^2 \leq 9/4$$

$$-\frac{9}{4} \leq \left(x + \frac{1}{2}\right)^2 - \frac{9}{4} \leq 0$$

$$\therefore \boxed{f(x) \in \left[-\frac{9}{4}, 0\right]}$$

Q. For $f(x) = x^2 - 2ax + 5 > 0$. find 'a'.

Soln Here,

In this equation,

$a > 0$

$f(x) > 0$

$\therefore \boxed{D < 0}$

$$\therefore 4a^2 - 4(5) < 0$$

$$4a^2 - 20 < 0$$

$$(a - \sqrt{5})(a + \sqrt{5}) < 0$$

$$\therefore a \in (-\sqrt{5}, \sqrt{5})$$