T.I.P. → Imaginary roots too exist in conjugate pairs. *1

- Conditions for common Root :–

(A) When both Roots are common

$a_1x^2 + b_1x + c_1 = 0 <^{\alpha}_{\beta}$

$a_2x^2 + b_2x + c_2 = 0 <^{\alpha}_{\beta}$

$$\boxed{\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}}$$

For Example :–

⟹ $kx^2 - 2x + 3 = 0$
& $x^2 - \alpha x + 1 = 0$
} have both roots in common.

Then, find k & α.

Sol^n $\frac{k}{1} = \frac{-2}{-\alpha} = \frac{3}{1}$ < $\boxed{\alpha = \frac{2}{3}}$ $\boxed{k = 3}$

Imp

⟹ $2x^2 + x + 7 = 0$
* & $ax^2 + bx + c = 0$
] have one common root.

Then, find $a : b : c = ?$

Sol^n Here,

$D = 1 - 4(2)(7) < 0$
↳ Imaginary ↳ both roots common

$$\frac{a}{2} = \frac{b}{1} = \frac{c}{7}$$

$\therefore$ $a : b : c = 2 : 1 : 7$

(B) Condition for 1 common Root :-

$$a_1 x^2 + b_1 x + c_1 = 0 \; < {\alpha \atop \beta}$$

$$a_2 x^2 + b_2 x + c_2 = 0 \; < {\alpha \atop \beta}$$

Then,

$$\begin{matrix} a_1 & & b_1 & & c_1 \\ & \times & & \times & \\ a_2 & & b_2 & & c_2 \end{matrix}$$

$$(a_1 c_2 - a_2 c_1)^2 = (a_1 b_2 - a_2 b_1) \times (b_1 c_2 - b_2 c_1)$$

For Example :-

$3x^2 + 4kx + 2 = 0$

& $2x^2 + 3x - 2 = 0$

have One root in common.

Find $k = ??$

Soln.

$$\begin{matrix} 3 & & 4k & & 2 \\ & \times & & \times & \\ 2 & & 3 & & -2 \end{matrix}$$

$$(-6-4)^2 = (9-8k)(-8k-6)$$

$$100 = -72k - 54 + 64k^2 + 48k$$

By Applying quadratic formula ; we get:-

$$k = \frac{7}{4} \text{ or } -\frac{11}{8}.$$

- For finding common root from the given equations.
  - Make coefficient of x^2 same in both eqn.
  - Subtract both equation.
  - Solve.

Q. $x^2 + 3x + 2 = 0$ and $x^2 - 5x - 6 = 0$ find. the common roots.

Solⁿ Coefficient of both x^2 is same.

$\therefore$ On Subtracting, both equatⁿ; we get

$$8x + 8 = 0$$

$$\boxed{x = -1}$$

Q. for $f(x) = x^2 + cx + b = 0 <^{\beta}_{\alpha}$

& $f(x) = x^2 + bx + c = 0 <^{\alpha}_{\gamma}$

form an equatⁿ with uncommon root

Solⁿ On subtracting both equations; we get

$$(c - b)x + b - c = 0$$

$$x = \frac{c - b}{c - b} = 1$$

$$\therefore \boxed{\alpha = 1}$$

Also,

$\alpha + \beta = -c$ — (1) & $\alpha\beta = b$

$\alpha + \gamma = -b$ — (2) $\quad \alpha\gamma = c$

$\therefore \boxed{\beta = b}$ & $\boxed{\gamma = c}$

①

Now, the equat^n formed will be :-

$$x^2 - 5x + P = 0$$
$$x^2 - (\beta + \gamma)x + \beta\gamma = 0$$
$$x^2 - (b + c)x + bc = 0 \quad —③$$

By subtracting eqn. ① & ②; we get:-

$$2\alpha + \beta + \gamma = -b - c$$
$$2 + b + c = -b - c$$
$$2b + 2c = -2$$

$$\therefore \boxed{b + c = -1} \quad —④$$

Putting value of ④ in ③; we get:-

$$\boxed{x^2 + x + bc = 0}$$

M. Imp.

Q. $f(x) = \dfrac{x^2 - x + c}{x^2 + x + 2c}$ has $x \in R$ and $y \in$

Now, find value of c for which these conditions exist.

Sol^n $y = \dfrac{x^2 + c - x}{x^2 + x + 2c}$

$$yx^2 + yx + 2cy = x^2 - x + c$$

$$(y-1)x^2 + (y+1)x + 2y - c = 0$$

~~Now~~ Now, $\boxed{x \in R}$

$\therefore D \geq 0$

$\Rightarrow (y+1)^2 - 4(y-1)(2cy - c) \geq 0$

$\Rightarrow y^2 + 2y + 1 - 8cy^2 + 4cy + 8cy - 4c \geq 0$

$\Rightarrow (1-8c)y^2 + (12c+2)y + (1-4c) \geq 0$

$D \leq 0$

$\Rightarrow (12c+2)^2 - 4(1-8c)(1-4c) \leq 0$

$\Rightarrow 144c^2 + 4 - 48c - 4(1 - 12c + 32c^2) \leq 0$

$\Rightarrow (144 - 128)c^2 + 96c \leq 0$

$\Rightarrow 16c^2 + 96c \leq 0$

$c(c+6) \leq 0$

$\therefore c \in [-6, 0]$

Case - I) Two common Root :-

$\hookrightarrow \frac{1}{1} = \frac{-1}{+1} = \frac{c}{2c}$ → Not Possible

Case - II). One common Root :-

$$\begin{bmatrix} 1 & -1 & c \\ 1 & 1 & 2c \end{bmatrix}$$

$(2c - c)^2 = (1+1)(-2c - c)$

$c^2 = 2(-3c)$

$(c)(c+6) = 0$

$\therefore \boxed{c = 0}$ & $\boxed{c = -6}$

$\therefore \; c \in [-6, 0] - \{-6, 0\}$

- Remainder Theorem :-

  ↳ $\frac{p(x)}{(x-a)} \Rightarrow p(a)$

- Factor Theorem :-

  ↳ If $(x-a)$ is a factor of $p(x)$ then $p(a) = 0$

- Cubic Equation :-

  $ax^3 + bx^2 + cx + d = 0$

  ↳ Roots

  ↳ $\alpha + \beta + \gamma = -\frac{b}{a}$

  ↳ $\alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a}$

  ↳ $\alpha\beta\gamma = \frac{-d}{a}$

$\Rightarrow$ Method to find roots :- cubic or any other

↳ Hit and Trial $\rightarrow \alpha$

↳ Then,

$x - \alpha \overline{)\text{Cubic}}\,(\text{Quad.})$ → factorise → 2 Roots

$\underline{\quad 0 \quad}$

Q. $2x^3 - x^2 - 22x - 24 = 0 < \alpha, \beta, \gamma$

If 2 of its roots are in the ratio 3 : 4 then find $\alpha; \beta; \gamma$.

Sol^n $\rightarrow \alpha + \beta + \gamma = +\frac{1}{2}$ — (1)

$\rightarrow \alpha\beta + \beta\gamma + \gamma\alpha = -\frac{22}{2} = -11$ — (2)

$\rightarrow \alpha\beta\gamma = \frac{24}{2} = 12$ — (3)

Now, let $\alpha = 3k$; $\beta = 4k$; $\gamma = \gamma$

In equat^n (1); we have :-

$3k + 4k + \gamma = \frac{1}{2} \Rightarrow \boxed{7k + \gamma = \frac{1}{2}}$

and in equat^n (3); we have :-

$(3k)(4k)\gamma = 12$

$12k^2\gamma = 12$

$\boxed{k^2\gamma = 1}$

In equation (2); we have :-

$12k^2 + 4k\gamma + 3k\gamma = -11$

$\therefore$ Root $= -\frac{3}{2}, -2, 4.$

- Transformation of equations :-

$$ax^2 + bx + c = 0 < {}^{\alpha}_{\beta}$$

$\big\downarrow$ Transform

Equatⁿ with roots $-\alpha$ & $-\beta$

<u>M-1</u> :- Change the signs of both roots. Then, form a new equatⁿ by sum & products of roots.

M-2 :- Symmetry of operations :- (Based on same changes in both roots)

$\hookrightarrow \boxed{x = -\alpha} \Rightarrow \boxed{\alpha = -x}$

$\hookrightarrow a\alpha^2 + b\alpha + c = 0$

$\hookrightarrow a(-x)^2 + b(-x) + c = 0$

$\hookrightarrow \boxed{ax^2 - bx + c = 0}$

Q. If $ax^2 + bx + c = 0 < {}^{\alpha}_{\beta}$. Then, form an equation with roots $\frac{1}{\alpha}, \frac{1}{\beta}$.

Solⁿ. Here,

$$x = \frac{1}{\alpha} \Rightarrow \boxed{\alpha = \frac{1}{x}}$$

$$\therefore \quad a\left(\frac{1}{x}\right)^2 + b\left(\frac{1}{x}\right) + c = 0$$

$$\boxed{a + bx + cx^2 = 0}$$

Q. $ax^2 + bx + c = 0 <^{\alpha}_{\beta}$

Then, form an equation whose roots are $\alpha + 1$ and $\beta + 1$.

Soln Here,

$$x = \alpha + 1 \Rightarrow \alpha = (x - 1)$$

$$\therefore \quad a(x-1)^2 + b(x-1) + c = 0$$

$$\Rightarrow a(x^2 + 1 - 2x) + b(x-1) + c = 0$$

Q. * Let S be the set of all non-zero real numbers α such that the quadratic equation $\alpha x^2 - x + \alpha = 0$ has two distinct real roots x_1 & x_2 satisfying the inequality $|x_1 - x_2| < 1$. Which of the following interval -s is/are a subset of S?

A). $\left(-\frac{1}{2}, -\frac{1}{\sqrt{5}}\right)$ B). $\left(-\frac{1}{\sqrt{5}}, 0\right)$

C). $\left(0, \frac{1}{\sqrt{5}}\right)$ D). $\left(\frac{1}{\sqrt{5}}, \frac{1}{2}\right)$

Soln. $|x_1 - x_2| < 1$

$$\Rightarrow (x_1 - x_2)^2 < 1$$

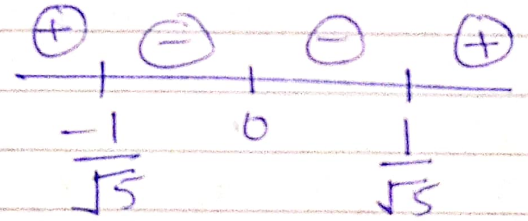
$$\Rightarrow (x_1 + x_2)^2 - 4x_1x_2 < 1$$

$\Rightarrow \left(\frac{1}{\alpha}\right)^2 - 4(1) < 1$

$\Rightarrow \frac{1}{\alpha^2} - 5 < 0 \quad \Rightarrow \frac{1 - 5\alpha^2}{\alpha^2} < 0$

$\Rightarrow \frac{5\alpha^2 - 1}{\alpha^2} > 0$

$\Rightarrow \frac{\alpha^2 - \frac{1}{5}}{(\alpha - 0)^2} > 0$



$\therefore \alpha \in \left(-\infty, \frac{-1}{\sqrt{5}}\right) \cup \left(\frac{1}{\sqrt{5}}, \infty\right)$ — ①

$\Rightarrow D > 0$

$\Rightarrow 1 - 4\alpha^2 > 0$

$\Rightarrow \alpha^2 - \frac{1}{4} < 0 \quad \Rightarrow \left(\alpha - \frac{1}{2}\right)\left(\alpha + \frac{1}{2}\right)$

$\therefore \alpha \in \left(-\frac{1}{2}, \frac{1}{2}\right)$ — ②