RELATIONS & FUNCTIONS

*-1

- Ordered Pair :-

→ Ordered Pair (a, b) and (b, a) are different.

→ Ordered pairs (a, b) and (c, d) are equal if a = c and b = d.

- Cartesian Product Of Two Sets:-

A	B
1	a
2	b
3	c
	d

∴ $A \times B =$ {(1, a), (1, b), (1, c), (1, d), (2, a), (2, b), (2, c), (2, d), (3, a), (3, b), (3, c), (3, d)}

→ Every Possible Pair Should be Written.

∴ Total no. Of Cartesian Product Of Two Sets ⇒

$n(A) = m$

$n(B) = n$

$$\boxed{\therefore\ n(A \times B) = mn}$$

Also, $B \times A =$ {(a, 1), (b, 1), (c, 1), (d, 1), (a, 2), (b, 2), (c, 2), (d, 2), (a, 3), (b, 3), (c, 3), (d, 3)}

Q. $A = \{1, 2, 3\}$ and $B = \{2, 3, 4\}$.
find $(A \times B) \cap (B \times A)$.

Sol^n $A \times B = \{(1,2), (1,3), (1,4), (2,2), (2,3), (2,4), (3,2), (3,3), (3,4)\}$

$B \times A = \{(2,1), (2,2), (2,3), (3,1), (3,2), (3,3), (4,1), (4,2), (4,3)\}$

$\therefore A \times B \cap B \times A = \{(2,2), (3,3), (2,3), (3,2)\}$

V. Imp

$\boxed{\begin{array}{ll} \text{If} & n(A \cap B) = K \\ \therefore & n(A \times B \cap B \times A) = K^2 \end{array}}$

Q. $A = \{1, 2, 3\}$ and $B = \{a, b\}$
Then, find subsets of $A \times B$.

Sol^n $A \times B = \{(1,a), (1,b), (2,a), (2,b), (3,a), (3,b)\}$

$\therefore$ Subsets formed by $A \times B = 2^6$

- From our above example, we derive our new type of subset :-

$\boxed{R : A \longrightarrow B}$

$\boxed{R \subseteq A \times B}$

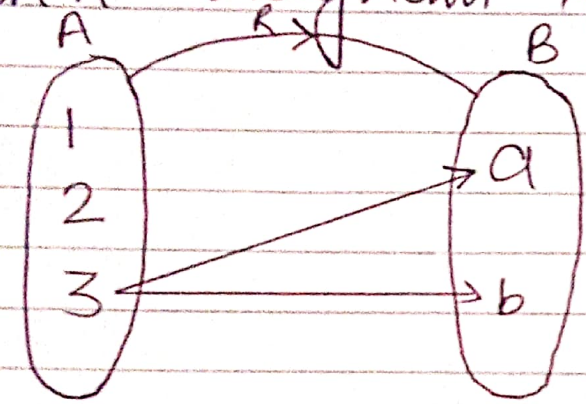
$\boxed{\begin{array}{llll} \text{If} & n(A) & = m & \text{elements} \\ \text{and} & n(B) & = n & \text{elements} \\ \therefore & n(A \times B) & = mn & \text{elements} \\ \text{Then, no. of subsets} & & = 2^{mn} & \end{array}}$

★ $\boxed{\text{No. Of Relations} = \text{No. Of Subsets of } A \times B}$

$$\downarrow$$

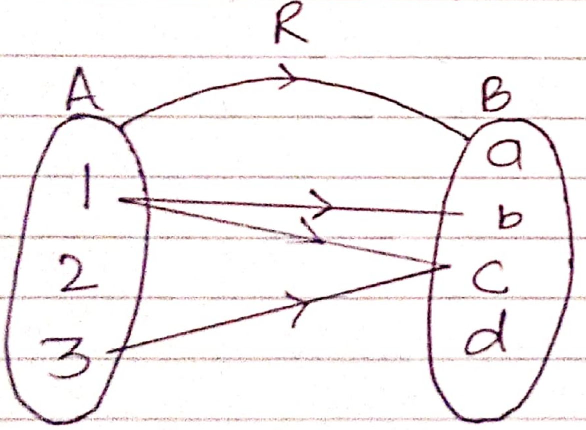
$$2^{mn}$$

- Venn Diagrams for a Relation :-



- DOMAIN, RANGE, & CODOMAIN OF A RELATION :-

Q.



$R : A \rightarrow B$

$R_1 = \{(1, b), (1, c), (3, c)\}$

∴ Range Of Relatⁿ = Associated elements Of Set B. = {b, c}

→ Domain Of Relation = Associated elements Of Set A = {1, 3}

→ Codomain Of a Relation = Comp. Set B = {a, b, c, d}

• FUNCTIONS :- Function is a relation but with specific conditions.

→ All relations are not functions but all functions are relations.

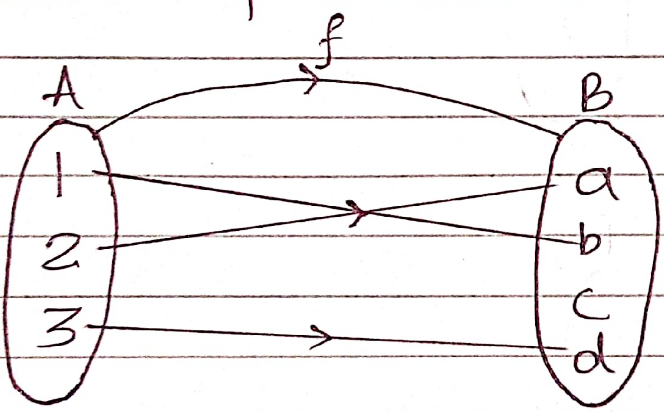
• Specific Conditions to be a Funct^{n} :-

(i) All elements of set A must be associated.

(ii) Each element of set A can/must associate with only one element of set B.

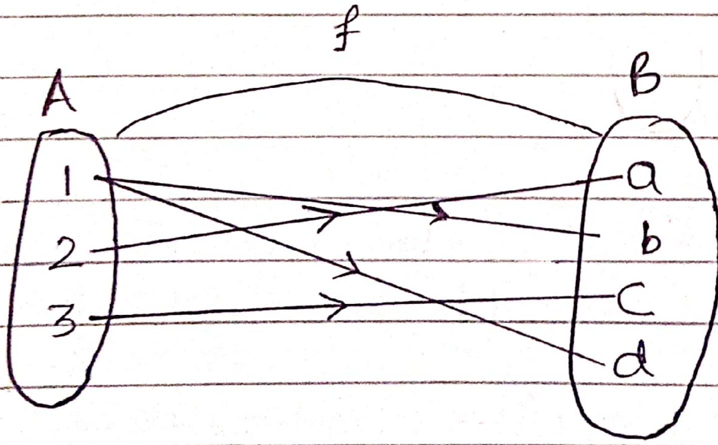
Q. Define whether the following relations are functions or not :-

(i) $f : A \to B$



→ Yess

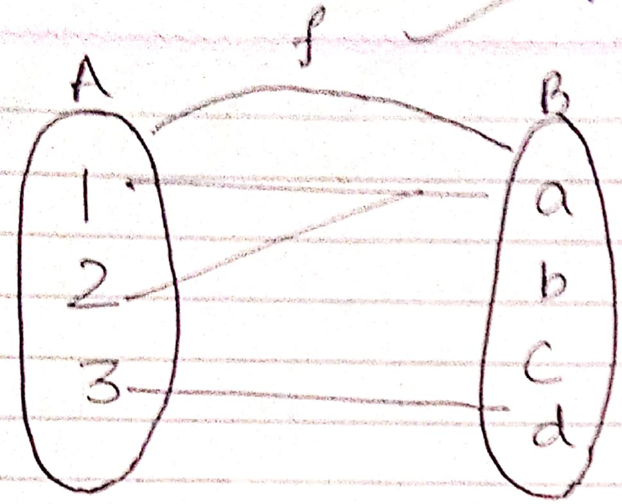
(ii)



→ NO

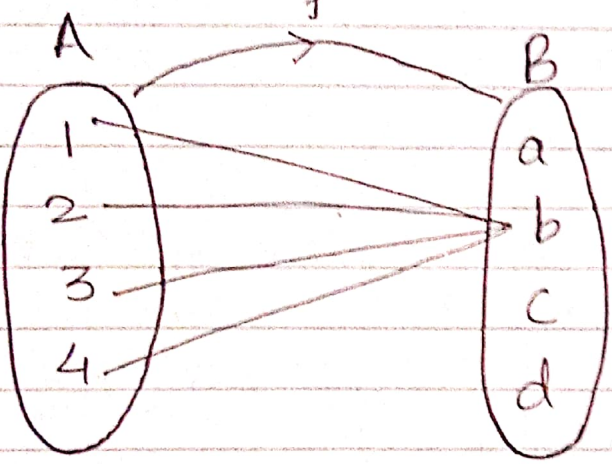
T.I.P. → Range $\subseteq$ Codomain

(iii)



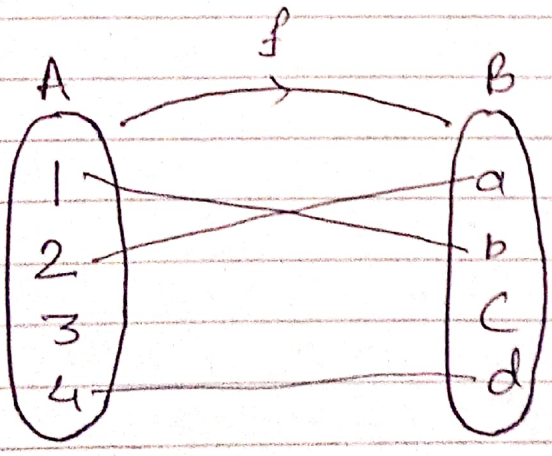
→ yes

(iv)



→ yes

(v)



→ No

DOMAIN (FUNCTION) :- Asso. elements of Set A

→ Set A (Comp.)

CODOMAIN (FUNCTION) :- Complete Set B

Range (FUNCTION) :- Asso. elements of Set B.

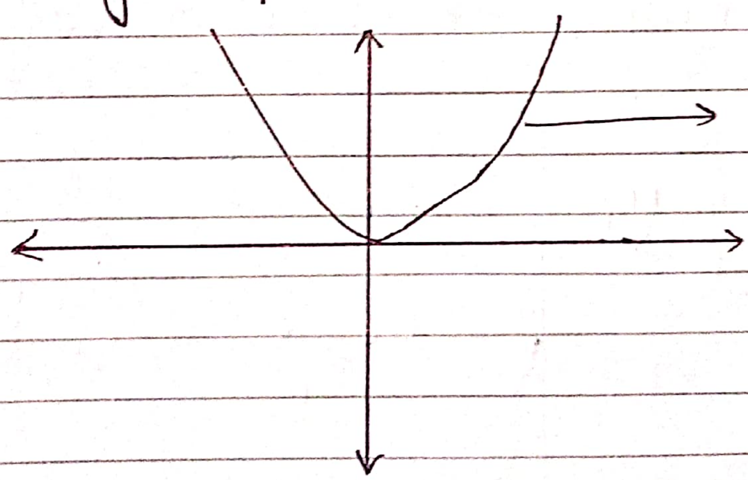
• function → Machine

Input		Output
−3		9
2	$f(x) = x^2$	4
0		0
−9		81

Domain → x

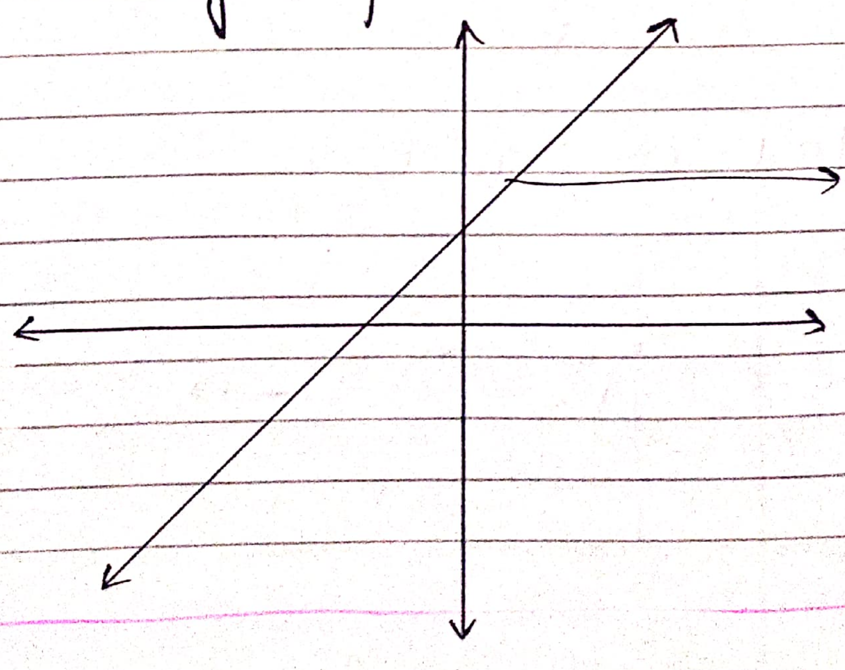
Range → y

• When $y = f(x) = x^2$



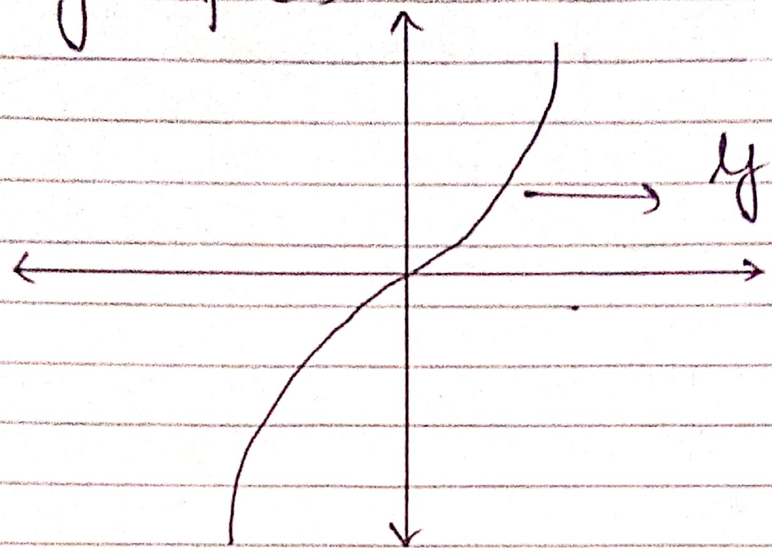
Graph of $f(x)$ (Quadratic Eq.)

• When $y = f(x) = mx + c$



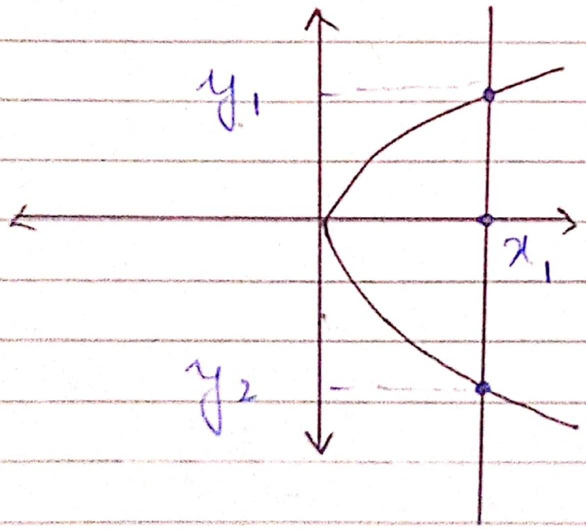
Graph of $f(x)$ (Linear Eq.)

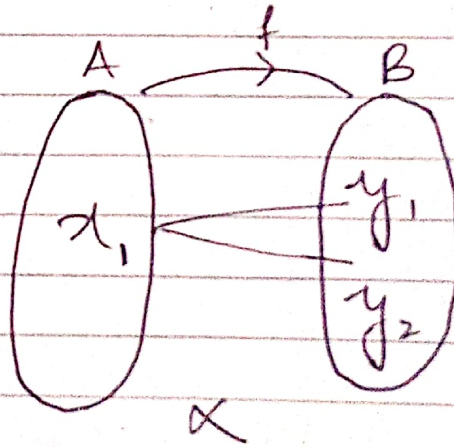
• When $y = f(x) = x^3$



→ Graph of $f(x)$ (Cubic Eq.)

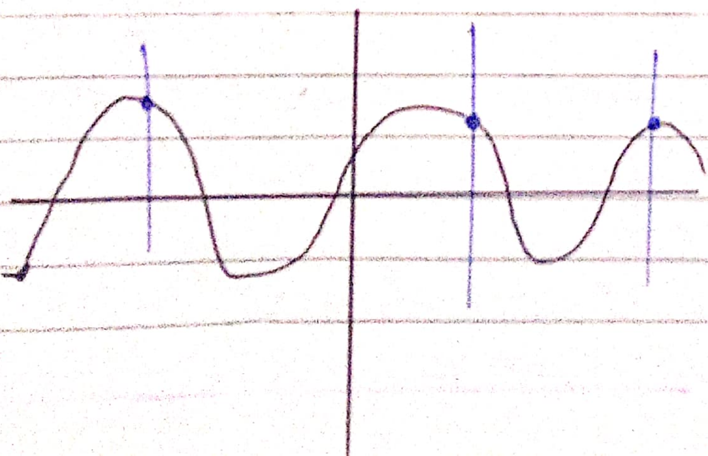
• Method to check whether a graph is a function or not :- Vertical Line Test





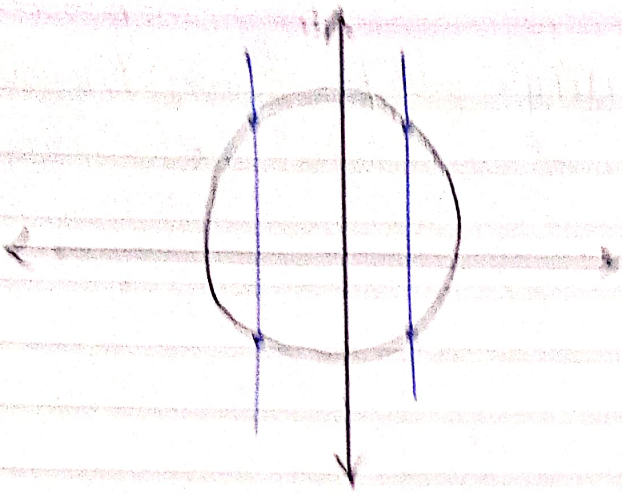
∴ Not a function.

Q.



→ Yes, it is a function

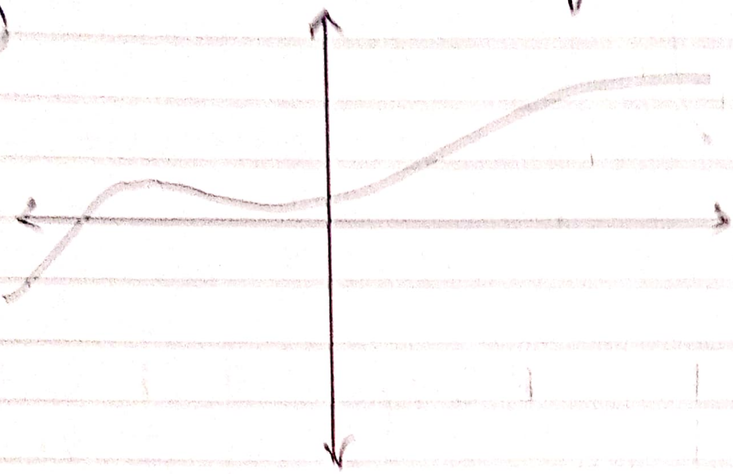
Q.



Not a function

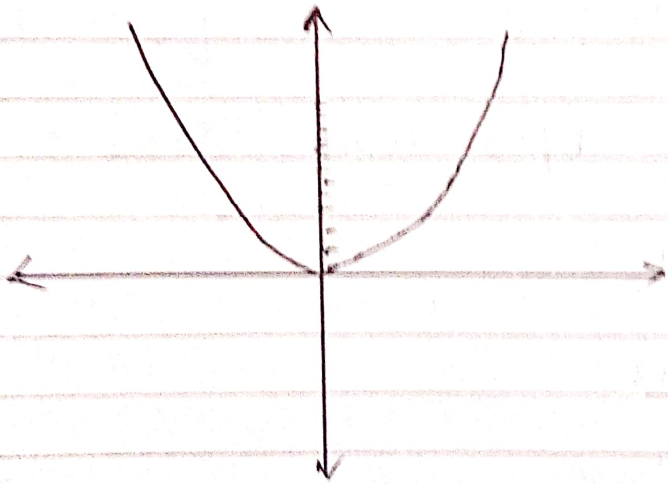
• Find Domain of the given graphs :-

(i)



$D_f \in R$
↓
Real Nos
↓
as there is every no. on the graph which can be treated as input.

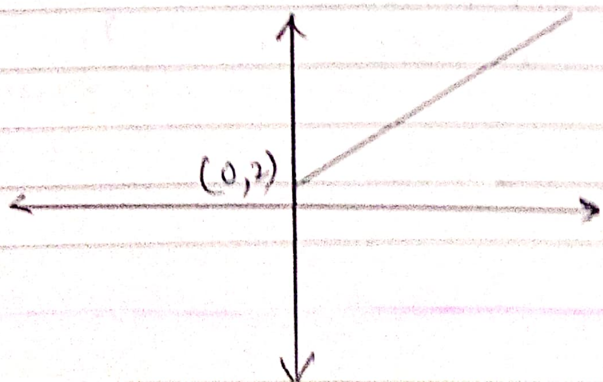
(ii)



$D_f \in R$

$R_f \in [0, \infty)$

(iii)

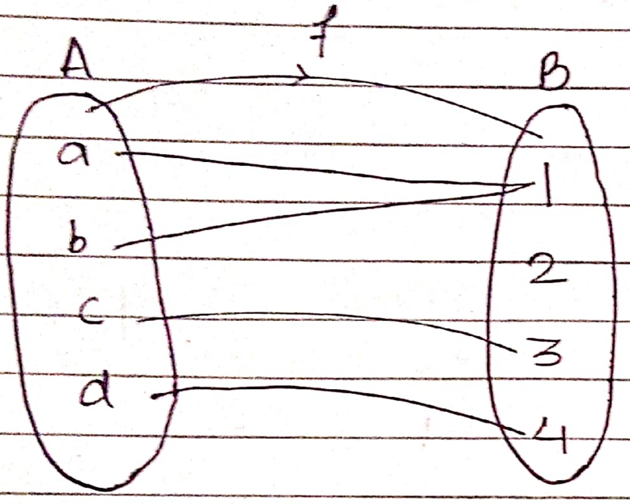


$D_f \in (0, \infty)$

$R_f \in [2, \infty)$

- Image and Pre-Image of function :-

For Example :-



∴ Here,

Image of a is 1	P. Image of 1 is a
Image of b is 1	" of 1 is b
Image of c is 3	" of 3 is c
Image of d is 4	" of 4 is d

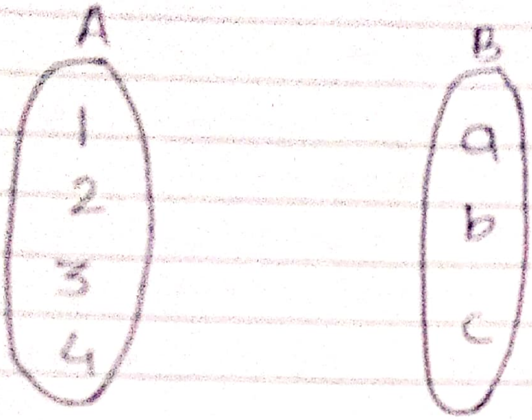
- No. of function b/w two sets :-

$n(A) \longrightarrow m$ elements

$n(B) \longrightarrow n$ elements

∴ Total no. of functions that can be made = $\boxed{n^m \text{ functions}}$

1. Find the no of functions that can be made out of two sets :-



No. of functions $= 3 \times 3 \times 3 \times 3 = \boxed{3^4}$

- Rules for finding domain :-

1. $\boxed{\text{Denominator} \neq 0}$

For Example :-

(A) $f(x) = \dfrac{1}{2x-3}$, find the domain

For Domain, $2x - 3 \neq 0$

$\Rightarrow x \neq 3/2$

$\therefore D_f \in R - \{\frac{3}{2}\}$

(B) $f(x) = \dfrac{2x-1}{2x+3}$, find the Domain

For Domain, $2x + 3 \neq 0$

$\Rightarrow \boxed{x \neq -3/2}$

$\therefore D_f \in R - \{-3/2\}$

T.I.P. → Do not cancel any component of numerator with the denominator in the case of a function.

(C) $f(x) = \dfrac{x-3}{x^2-4}$, find the domain.

For Domain, $x^2 - 4 \neq 0$

$x^2 \neq 4$

$\boxed{x \neq \pm 2}$

$\therefore D_f \in R - \{-2, +2\}$

(D) $f(x) = \dfrac{x-2}{x^2-4}$, find the domain.

For Domain, $x^2 \neq 4$

$\boxed{x \neq \pm 2}$

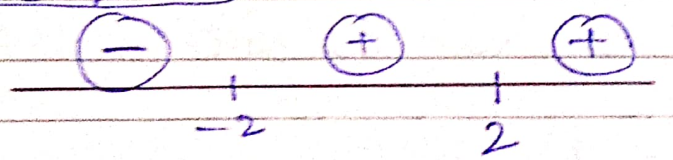
$\therefore D_f \in R - \{-2, +2\}$

(E) $\dfrac{x-2}{x^2-4} > 0$, find the domain.

For domain, $x^2 - 4 \neq 0$

$\boxed{x \neq \pm 2}$

$\left[\dfrac{1}{(x-2)^0 (x+2)}\right]$



$\therefore D_f \in (-2, \infty) - \{2\}$

(F) $f(x) = \dfrac{1}{x^2+4}$ → Always +ve

$\boxed{\therefore x \in R}$

Imp.

$$\boxed{\begin{array}{c} f(x) = ax^2 + bx + c \\ \downarrow \\ a > 0 \quad \text{and} \quad D < 0 \\ \downarrow \\ \therefore \{ f(x) > 0 \} \end{array}}$$

(2) $\boxed{\text{If } \sqrt{f(x)}}$, then Resultant of $f(x) \geq 0$

for Example :-

(A) $f(x) = \sqrt{4-x^2}$, find the domain

$\therefore \quad 4 - x^2 \geq 0$

$-(x^2 - 4) \geq 0$

$x^2 - 4 \leq 0$

$x^2 \leq 4 \longrightarrow (x-2)(x+2) \leq 0$

$\oplus \quad \ominus \quad \oplus$

$-2 \qquad 2$

$\therefore \quad x \in [-2, 2]$

(B) $f(x) = \sqrt{4+x}$

$\therefore \quad 4 + x \geq 0$

$x \geq -4$

$\therefore \quad x \in [-4, \infty)$

3

(C) $f(x) = \sqrt{\dfrac{(x-1)(x-2)}{(x-3)}}$, find domain.

A

$\therefore \dfrac{(x-1)(x-2)}{(x-3)} \geq 0$

$\ominus \quad \oplus \quad \ominus \quad \oplus$

$1 \quad 2 \quad 3$

$x \in [1,2] \cup (3,\infty)$

Imp

(D) $f(x) = \sqrt{\dfrac{4-8x^2}{3+x}}$, find domain

B

$\therefore \dfrac{4-8x^2}{x+3} \geq 0$

$\Rightarrow \dfrac{-8\left(x^2 - 1/2\right)}{x+3} \geq 0$

$\Rightarrow \dfrac{\left(x-\frac{1}{\sqrt{2}}\right)\left(x+\frac{1}{\sqrt{2}}\right)}{x+3} \leq 0$

$\ominus \quad \oplus \quad \ominus \quad \oplus$

$-3 \quad -1/\sqrt{2} \quad 1/\sqrt{2}$

Imp

C

$\therefore x \in (-\infty, -3) \cup \left[-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right]$

③ If $\frac{1}{\sqrt{f(x)}}$, then $\boxed{f(x) > 0}$

Ⓐ $f(x) = \frac{1}{\sqrt{x-5}}$, find the domain.

$\therefore \quad x - 5 > 0$

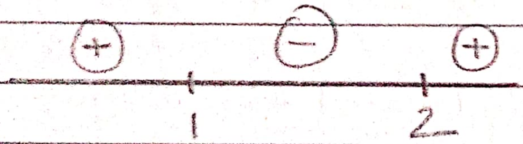
$\Rightarrow \quad x > 05$

$\therefore x \in (5, \infty)$

Imp

Ⓑ $f(x) = \frac{1}{\sqrt{\frac{x-1}{2-x}}}$, find domain.

$\therefore \quad \frac{x-1}{2-x} > 0 \quad \rightarrow \quad \frac{x-1}{x-2} < 0$

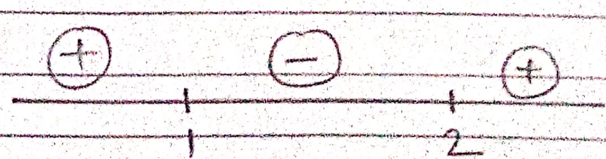


$\therefore \quad x \in (1, 2)$

Imp

Ⓒ $f(x) = \sqrt{\frac{2-x}{x-1}}$, find domain.

$\therefore \quad \frac{(x-2)(-1)}{x-1} \geq 0$

$\Rightarrow \quad \frac{x-2}{x-1} \leq 0 \quad \rightarrow$ 

$\therefore x \in (1, 2]$

(D) $\sqrt{x-1} + \dfrac{1}{\sqrt{2-x}}$

$x - 1 \geq 0$

$x \geq 1$

$2 - x > 0$

$x - 2 < 0$

$\boxed{x < 2}$

$\therefore \quad x \in [1, 2)$

(E) $f(x) = \dfrac{1}{x-2} + \sqrt{16-x^2} + \dfrac{1}{\sqrt{4-x}}$

$x - 2 \neq 0$ $\boxed{x \neq 2}$	$16 - x^2 \geq 0$ $x^2 - 16 \leq 0$ $(x-4)(x+4) \leq 0$ $x \in [-4, 4]$	$4 - x > 0$ $x < 4$

$\therefore \quad x \in [-4, 2) \cup (2, 4)$

(F) $f(x) = \dfrac{1}{x^2 - 5x - 6}$

$x^2 - 5x - 6 \neq 0$

$(x+1)(x-6) \neq 0$

$\therefore \quad x \neq -1 \quad \text{and} \quad x \neq 6$

$\therefore \quad x \in R - \{-1, 6\}$

(1)

(G) $f(x) = \frac{1}{\sqrt{2x^2 + x + 3}}$

Since, $D < 0$ $\xrightarrow{}$ Always +ve, $\therefore$ $\boxed{x \in R}$

Imp

(H) $f(x) = \frac{1}{\sqrt{-2x^2 + x - 3}}$

Since, $D < 0$ and $a < 0$, $\therefore$ $f(x) < 0$ $\rightarrow$ always −ve

$$\boxed{\therefore \; x \in \phi}$$

Imp

(I) $x + 4 - \frac{1}{\sqrt{x-1}}$

$x+4$ ↙ * Not meant for solving

$\sqrt{x-1}$ ↓

$x - 1 > 0$

$x > 1$

$\therefore \; x \in (1, \infty)$

(J) $f(x) = \sqrt{x+3} + \frac{1}{\sqrt{9-x^2}}$

$x + 3 \geq 0$

$\boxed{x \geq -3}$

$9 - x^2 > 0$

$x^2 - 9 < 0$

$(x+3)(x-3) < 0$

$x \in (-3, 3)$

$\therefore x \in \underline{(-3, 3)}$ Ans

(k) $f(x) = \dfrac{\sqrt{1-x^2}}{x} \longrightarrow x \neq 0$, either undefined function

$\therefore \; 1 - x^2 \geq 0$

$x^2 - 1 \leq 0$

$x \in [-1, 1]$

$\therefore x \in [-1, 1] - \{0\}$

(L) $f(x) = \dfrac{\sqrt{x+2}}{x^2 - 3x + 2}$

$x + 2 \geq 0$

$x \geq -2$

$x^2 - 3x + 2 \neq 0$

$(x-1)(x+2) \neq 0$

$x \neq 1$ and $x \neq 2$

$x \in [-2, \infty) - \{1, 2\}$

- **Rules for finding Range of a function :-**

$\boxed{P-1} \rightarrow \; f(x) = \dfrac{ax+b}{cx+d}$ Linear / Linear

Method → Cross Multiply

$\Rightarrow \; y = \dfrac{ax+b}{cx+d} \Rightarrow cyx + dy = ax + b$

$\Rightarrow \; cyx - ax = b - dy$

$\therefore x = \frac{b - dy}{cy - a}$

v. Imp.

$\boxed{\text{Short Trick :- } y \in R - \left\{\frac{a}{c}\right\}}$

Q. Find the range of the following functions :-

(A) $y = \frac{2x - 3}{3 - x}$

Soln $3y - xy = 2x - 3$

$3y + 3 = 2x + xy$

$3y + 3 = (2 + y)x$

$\therefore \boxed{x = \frac{3y + 3}{2 + y}}$

$\therefore y \in R - \{-2\}$ → Range of Domain.

(B) $y = \frac{x - 1}{x + 3}$

Soln $yx + 3y = x - 1$

$yx - x = -1 - 3y$

$x(y - 1) = -(1 + 3y)$

$\therefore x = \frac{-(1 + 3y)}{y - 1}$

$\therefore y \in R - \{1\}$

(c) $f(x) = \dfrac{x - 1/3}{3/5 - 2x}$

Soln $y = \dfrac{\frac{3x-1}{3}}{\frac{3-10x}{5}} \Rightarrow y = \dfrac{15x - 5}{9 - 30x}$

$\Rightarrow 9y - 30xy = 15x - 5$

$\Rightarrow 9y + 5 = 30xy + 15x$

$9y + 5 = (30y + 15)x$

$\therefore x = \dfrac{9y+5}{30y+15}$

$\therefore y \in R - \{-1/2\}$

$\boxed{P-2} \rightarrow$ Quadratic Equations.

$\boxed{M-1} \rightarrow$ Completing the squares.

$f(x) = ax^2 + bx + c$

$= a\left[x^2 + \dfrac{b}{a}x\right] + c$

$= a\left[x^2 + \dfrac{b}{a}x + \left(\dfrac{b}{2a}\right)^2 - \left(\dfrac{b}{2a}\right)^2\right] + c$

for $(a+b)^2 = a\left[x^2 + bx/a + (b/2a)^2\right] + c$

for $(a-b)^2 = a\left[x^2 + bx/a - (b/2a)^2\right]$

For Example :-

$$f(x) = 3x^2 - 2x - 5$$

Solⁿ $\longrightarrow$ $3\left(x^2 - \frac{2}{3}x\right) - 5$

$\Rightarrow 3\left(x^2 - \frac{2x}{3} + \left(\frac{1}{3}\right)^2 - \left(\frac{1}{3}\right)^2\right) - 5$

$\Rightarrow 3\left(\left(x - \frac{1}{3}\right)^2 - \left(\frac{1}{9}\right)\right) - 5$

$\Rightarrow 3\left(x - \frac{1}{3}\right)^2 - \frac{1}{3} - 5$

$\Rightarrow \boxed{3\left(x - \frac{1}{3}\right)^2 - 16/3 = y}$

Here,

$\Rightarrow -\infty < x < \infty$

$\Rightarrow -\infty - \frac{1}{3} < x - \frac{1}{3} < \infty - \frac{1}{3}$

$\Rightarrow -\infty < x - \frac{1}{3} < \infty$

$\Rightarrow 0 \leq \left(x - \frac{1}{3}\right)^2 < \infty$

$\Rightarrow 0 \leq 3\left(x - \frac{1}{3}\right)^2 < \infty$

$\Rightarrow -\frac{16}{3} \leq 3\left(x - \frac{1}{3}\right)^2 - \frac{16}{3} < \infty$

$\boxed{M-2}$ → Completing the Sq. Method → Logic

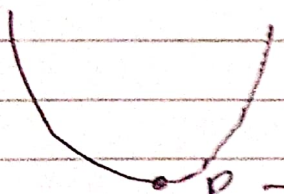
In the above example,

Min value of $f(x) = -16/3$
Max. value of $f(x) = \infty$

$\boxed{M-3}$

$$f(x) = ax^2 + bx + c$$

$\boxed{a > 0}$



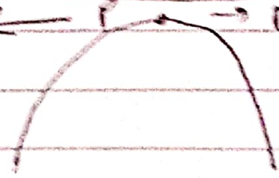
P → minima

$\left(\frac{-b}{2a}, \frac{-D}{4a}\right)$

x-coordinate → y-coordinate

$\boxed{a < 0}$

$\left(\frac{-b}{2a}, \frac{-D}{4a}\right)$ ← → maxima



★ → For Range, calculate only value of y-coordinates; i.e. $y = \frac{-D}{4a}$

★ For $a > 0$, Range $= \left[\frac{-D}{4a}, \infty\right)$

★ For $a < 0$, Range $= \left(-\infty, \frac{-D}{4a}\right]$

Q. Find Range of the following functions:-

(A) $f(x) = -3x^2 + 5x + 7$

Here, $\boxed{a < 0}$

and $\frac{-D}{4a} = \frac{-(25 - 4(-3)(7))}{4(-3)}$

$= \boxed{109/12}$

$\therefore$ Range $= (-\infty, 109/12]$

(B) $f(x) = x^2 - 7x + 28$

Here, $a > 0$

and $\frac{-D}{4a} = \frac{-(49 - 4(28))}{4(1)} = \frac{63}{4}$

$\therefore$ Range $= [\frac{63}{4}, \infty)$

P-3 When $f(x) = \sqrt{\text{Quad}}$.

Q. Find the range of the following functions:-

(A) $f(x) = \sqrt{x^2 + 4x + 11}$

Since, $a > 0$ and $\frac{-D}{4a} = 7$

$\therefore$ Output $= [7, \infty)$

By putting them in underoot, we get :-

Range $\in [\sqrt{7}, \infty)$

(1)

(B) $f(x) = \sqrt{-x^2 - 2x + 1}$

Since $a < 0$; and $\frac{-D}{4a} = 12$

$\therefore$ Output $= (-\infty, 12]$

$\therefore$ Range $(y) \in (-\cancel{\infty}^{0}, \sqrt{12}]$

(C) $f(x) = \sqrt{x^2 + 5x + 4}$

$a > 0$ and $\frac{-D}{4a} = -9/4$

$\therefore$ Output $= [\frac{-9}{4}, \infty)$

$\therefore$ Range $(y) \in [0, \infty)$

- Constant Function :-

For Ex :-

$f(x) = 0.25$; $f(x) = e$

$f(x) = \pi$; $f(x) = \sqrt{2}$

- Identity function :-

where $f(x) = \cancel{y}\, x$ $\boxed{or}$ $\boxed{y = x}$

- Linear function :-

Where $f(x) = ax + b$