⑤ Trig. Ratio of Sum and Difference of Angles :-

(A) $\sin(A+B) = \sin A \cos B + \sin B \cos A$

(B) $\sin(A-B) = \sin A \cos B - \sin B \cos A$

(C) $\cos(A+B) = \cos A \cos B - \sin A \sin B$

(D) $\cos(A-B) = \cos A \cos B + \sin A \sin B$

Q. Find the Value of :-

(i) $\sin 75^\circ = \sin(45^\circ + 30^\circ) = \sin 45^\circ \cos 30^\circ + \cos 45^\circ \sin 30^\circ$

$\sin\left(\frac{5\pi}{12}\right) = \boxed{\frac{\sqrt{3}+1}{2\sqrt{2}}}$

(ii) $\cos 15^\circ = \cos(45^\circ - 30^\circ) = \cos 45^\circ \cos 30^\circ + \sin 45^\circ \sin 30^\circ$

$= \boxed{\frac{\sqrt{3}+1}{2\sqrt{2}}}$

(iii) $\cos 75^\circ = \cos(45^\circ + 30^\circ) = \cos 45^\circ \cos 30^\circ - \sin 45^\circ \sin 30^\circ$

$\cos \frac{5\pi}{12} = \boxed{\frac{\sqrt{3}-1}{2\sqrt{2}} = \sin \frac{\pi}{12}}$

(E) $\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$

(F) $\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$

(G) $\cot(A+B) = \cancel{\frac{\cot B + \cot A}{\cot B \cot A - 1}} = \frac{\cot A \cot B - 1}{\cot B + \cot A}$

(H) $\cot(A-B) = \cancel{\frac{\cot B - \cot A}{\cot A \cdot \cot B + 1}} = \frac{\cot A \cdot \cot B + 1}{\cot B - \cot A}$

Q. Find the value of :-

(i) $\tan 75^\circ = \tan(45^\circ + 30^\circ) = \dfrac{\tan 45^\circ + \tan 30^\circ}{1 - \tan 45^\circ \tan 30^\circ}$

$\tan 5\pi/12 = 2 + \sqrt{3} = \cot \pi/12$

(ii) $\tan 15^\circ = \tan(45^\circ - 30^\circ) = \dfrac{\tan 45^\circ - \tan 30^\circ}{1 + \tan 45^\circ \tan 30^\circ}$

$\tan \dfrac{\pi}{12} = 2 - \sqrt{3} = \cot 5\pi/12$

(I) $\sin(A+B) \cdot \sin(A-B) = \sin^2 A - \sin^2 B$
$= \cos^2 B - \cos^2 A$

(J) $\cos(A+B) \cdot \cos(A-B) = \cos^2 A - \sin^2 B$
$= \cos^2 B - \sin^2 A$

(K) $\tan A \pm \tan B = \dfrac{\sin(A \pm B)}{\cos A \cdot \cos B}$

Q. $\sin \dfrac{7\pi}{12} \cdot \cos \dfrac{\pi}{4} - \cos \dfrac{7\pi}{12} \cdot \sin \dfrac{\pi}{4} = \sin\left(\dfrac{7\pi}{12} - \dfrac{\pi}{4}\right) = \dfrac{\sqrt{3}}{2}$

Q. $\sin(n+1) \cdot A \cdot \sin(n+2) \cdot A + \cos(n+1) \cdot A \cdot \cos(n+2) A = ??$

Sol^n / Let $(n+1)A \longrightarrow P$
$(n+2)A \longrightarrow Q$

$\therefore \cos(P - Q) = \cos(-A) = \underline{\underline{\cos A}}$

Q. $\tan A \cdot \tan B = 1/3 \Rightarrow \dfrac{\cos(A-B)}{\cos(A+B)} = ??$

Sol^n $\tan A \cdot \tan B = 1/3 \Rightarrow \dfrac{\sin A}{\cos A} \cdot \dfrac{\sin B}{\cos B} = \dfrac{1}{3}$

$$\Rightarrow \frac{\cos A \cdot \cos B}{\sin A \cdot \sin B} = 3 \Rightarrow \frac{\cos A \cdot \cos B + \sin A \sin B}{\cos A \cdot \cos B - \sin A \cdot \sin B} = 2$$

(C & D. Method)

Q. $\sin^2\left(\frac{\pi}{8} + \frac{A}{2}\right) - \sin^2\left(\frac{\pi}{8} - \frac{A}{2}\right) = k \sin A$; $k = ??$

Sol^n $\sin^2 A - \sin^2 B = \sin(A+B) \cdot \sin(A-B)$

$$\sin^2\left(\frac{\pi}{8} + \frac{A}{2}\right) - \sin^2\left(\frac{\pi}{8} - \frac{A}{2}\right) = \sin A \cdot \sin\frac{\pi}{4}$$

$$\therefore \boxed{k = 1/\sqrt{2}}$$

Q. If α, β are acute such that $\tan\alpha = \dfrac{m}{m+1}$; $\tan\beta = \dfrac{1}{2m+1}$; $\alpha + \beta = ??$

Sol^n $\tan(\alpha+\beta) = \dfrac{\tan\alpha + \tan\beta}{1 - \tan\alpha \cdot \tan\beta} = \dfrac{\frac{m}{m+1} + \frac{1}{2m+1}}{1 - \frac{m}{m+1} \cdot \frac{1}{2m+1}}$

$$\Rightarrow \tan(\alpha+\beta) = 1 \Rightarrow \boxed{\alpha + \beta = \pi/4}$$

Q. $\tan(45+\theta) \cdot \tan(45-\theta) = ??$

Sol^n $\dfrac{\tan 45 + \tan\theta}{1 - \tan 45^\circ \tan\theta} \cdot \dfrac{\tan 45^\circ - \tan\theta}{1 + \tan 45^\circ \tan\theta}$

$$= \frac{1+\tan\theta}{1-\tan\theta} \cdot \frac{1-\tan\theta}{1+\tan\theta} = \underline{\underline{1}} \text{ Ans}$$

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Q. $\tan 70^\circ = \tan 20^\circ + 2\tan 50^\circ$ ← Prove that

Sol^n. $\tan 70^\circ = \tan(20^\circ + 50^\circ) = \dfrac{\tan 20^\circ + \tan 50^\circ}{1 - \tan 20^\circ \cdot \tan 50^\circ}$

$\Rightarrow \tan 70^\circ (1 - \tan 20^\circ \tan 50^\circ) = \tan 20^\circ + \tan 50^\circ$

$\Rightarrow \tan 70^\circ - \cancel{\tan 70^\circ} \cancel{\tan 20^\circ} \tan 50^\circ = \tan 20^\circ + \tan 50^\circ$

$\Rightarrow \tan 70^\circ = \tan 20^\circ + 2\tan 50^\circ$

Hence, Proved

Q. $\tan 3A \cdot \tan 2A \cdot \tan A = \tan 3A - \tan 2A - \tan A$
← Prove that

Sol^n. $\tan 3A = \tan(2A + A) = \dfrac{\tan 2A + \tan A}{1 - \tan 2A \cdot \tan A}$

$\Rightarrow \tan 3A (1 - \tan 2A \cdot \tan A) = \tan 2A + \tan A$

$\Rightarrow \tan 3A - \tan 3A \cdot \tan 2A \cdot \tan A = \tan 2A + \tan A$

$\Rightarrow \tan 3A - \tan 2A - \tan A = \tan 3A \cdot \tan 2A \cdot \tan A$

Hence, Proved.

Q. $\dfrac{\cos 9^\circ + \sin 9^\circ}{\cos 9^\circ - \sin 9^\circ} = \tan\theta$; $\theta = ??$

Sol^n. $\dfrac{\cancel{\cos 9^\circ}\left(1 + \dfrac{\sin 9^\circ}{\cos 9^\circ}\right)}{\cancel{\cos 9^\circ}\left(1 - \dfrac{\sin 9^\circ}{\cos 9^\circ}\right)}$ $\dfrac{\tan 45^\circ + \tan 9^\circ}{1 - \tan 45^\circ \cdot \tan 9^\circ} = \tan(54^\circ)$

Q.* Prove that :- $\dfrac{\tan(A-B)}{\tan A} + \dfrac{\sin^2 C}{\sin^2 A} = 1$

$\dfrac{\tan A \tan B}{\tan^2 C} = ??$

Solⁿ. Since, $\frac{\tan(A-B)}{\tan A} + \frac{\sin^2 C}{\sin^2 A} = 1$, then,

$$\Rightarrow \frac{\sin^2 C}{\sin^2 A} = 1 - \frac{\tan(A-B)}{\tan A} = \frac{\tan A - \tan(A-B)}{\tan A}$$

Also, since, $\tan A - \tan B = \frac{\sin(A-B)}{\cos A \cdot \cos B}$, so,

$$\Rightarrow \frac{\sin^2 C}{\sin^2 A} = \frac{\sin(A-(A-B))}{\cancel{\cos A} \cdot \cos(A-B) \cdot \cancel{\frac{\sin A}{\cos A}}}$$

$$\Rightarrow \sin^2 C = \frac{\sin A \cdot \sin B}{\cos(A-B)}$$

$$\therefore \tan^2 C = \frac{\sin^2 C}{\cos^2 C} = \frac{\sin^2 C}{1-\sin^2 C} = \frac{\frac{\sin A \sin B}{\cos(A-B)}}{1 - \frac{\sin A \sin B}{\cos(A-B)}}$$

$$\Rightarrow \frac{\sin A \cdot \sin B}{\cos(A-B) - \sin A \sin B} = \frac{\sin A \sin B}{\cos A \cos B} = \underline{\tan A \cdot \tan B}$$

$$\therefore \frac{\tan A \cdot \tan B}{\tan^2 C} = 1 \qquad \left[\because \tan^2 C = \frac{\tan A}{\tan B}\right]$$

Q. If $A+B = \pi/4$, then, $(1+\tan A)(1+\tan B) = ??$

Solⁿ $\because A+B = \pi/4 \Rightarrow \tan(A+B) = \tan \pi/4$

$$\Rightarrow \frac{\tan A + \tan B}{1 - \tan A \cdot \tan B} = 1 \Rightarrow \tan A + \tan B + \tan A \cdot \tan B = 1$$

$$\therefore (1+\tan A)(1+\tan B) = 1 + \tan A + \tan B + \tan A \cdot \tan B = 2$$

(6) Product to Sum formulae :-

(A) $2\sin A \cos B = \sin(A+B) + \sin(A-B)$

(B) $2\cos A \cos B = \cos(A+B) + \cos(A-B)$

(C) $2\sin A \sin B = \cos(A-B) - \cos(A+B)$

Q. $\sin 75^\circ \cos 15^\circ = ??$

Solⁿ. $\frac{1}{2}[2\sin 75 \cdot \cos 15] = \frac{1}{2}[\sin(75+15) + \sin(75-15)]$

$= \frac{1}{2}[\sin 90^\circ + \sin 60^\circ] = \frac{2+\sqrt{3}}{4}$

Q. $\tan(60^\circ+\theta)\cdot\tan(60^\circ-\theta) = \frac{2\cos 2\theta + 1}{2\cos 2\theta - 1}$ ← Prove that

Solⁿ $\tan(60^\circ+\theta)\cdot\tan(60-\theta)$

$= \frac{2\sin(60^\circ+\theta)}{2\cos(60+\theta)} \cdot \frac{\sin(60-\theta)}{\cos(60-\theta)}$

$= \frac{\cos(60+\theta-(60-\theta)) - \cos(60+\theta+60^\circ-\theta)}{\cos(60+\theta+60-\theta) + \cos(60+\theta-(60^\circ-\theta))}$

$= \frac{\cos 2\theta - \cos 120^\circ}{\cos 120^\circ + \cos 2\theta} = \frac{2\cos 2\theta + 1}{2\cos 2\theta - 1}$

Hence, Proved.

Q. $\sin A \cdot \sin(B-C) + \sin B \cdot \sin(C-A) + \sin C \cdot \sin(A-B) = ??$

Sol^n. $\frac{1}{2}\left[2\sin A \sin(B-C) + 2\sin B \sin(C-A) + 2\sin C \cdot \sin(A-B)\right]$

$$= \frac{1}{2}\Big[\cancel{\cos(A-B+C)} - \cancel{\cos(A+B-C)} + \cancel{\cos(B-C+A)} - \cancel{\cos(B+C-A)} + \cancel{\cos(C-A+B)} - \cancel{\cos(C+A-B)}\Big]$$

$\Rightarrow \underline{\underline{0}}$

Q.* $\sin 10° \sin 30° \sin 50° \sin 70° = ??$

Sol^n. $\frac{1}{2}\left[\sin 10° \cdot \sin 50° \cdot \sin 70°\right]$

$= \frac{1}{2} \cdot \frac{1}{2}\left[2\sin 10° \sin 50° \sin 70°\right]$

$= \frac{1}{4}\left[\left(\cos(10-50) - \cos(10+50°)\right)\sin 70°\right]$

$= \frac{1}{4}\left[\left(\cos(-40°) - \cos 60°\right)\sin 70°\right]$

$= \frac{1}{4}\left[\left(\cos 40° - \frac{1}{2}\right)\sin 70°\right] = \frac{1}{4}\left[\sin 70° \cos 40° - \frac{1}{2}\sin 70°\right]$

$= \frac{1}{4} \times \frac{1}{2}\left[2\sin 70° \cos 40° - \sin 70°\right]$

$= \frac{1}{8}\left[\sin(70°+40°) + \sin(70°-40°) - \sin 70°\right]$

$= \frac{1}{8}\left[\sin 110° + \sin 30° - \sin 70°\right] = \frac{1}{8}\left[\cancel{\sin 70°} - \cancel{\sin 70°} + \sin 30°\right]$

$= \frac{1}{8} \times \frac{1}{2} = \boxed{\frac{1}{16}\ \underline{\text{Ans}}}$

(D) $\sin(60-\theta)\cdot\sin\theta\cdot\sin(60+\theta) = \frac{1}{4}\sin 3\theta$

(E) $(60^\circ-\theta)\ \cos\theta\cdot\cos(60+\theta) = \frac{1}{4}\cos 3\theta$

(F) $\tan(60-\theta)\cdot\tan\theta\cdot\tan(60^\circ+\theta) = \tan 3\theta$

Q. $\cos 20^\circ\cdot\cos 40^\circ\cdot\cos 60^\circ\cdot\cos 80^\circ = ??$

Sol^n. $\frac{1}{2}\left[\cos(60-20^\circ)\cdot\cos 20^\circ\cdot\cos(60+20^\circ)\right]$

$= \frac{1}{2}\left[\cos 3(20^\circ)\cdot\frac{1}{4}\right] = \frac{1}{8}\cos 60^\circ = \frac{1}{16}$

Q. $\tan 20^\circ\cdot\tan 40^\circ\ \tan 60^\circ\ \tan 80^\circ = ??$

Sol^n. $\sqrt{3}\,(\tan 20^\circ\cdot\tan 40^\circ\cdot\tan 80^\circ)$

$= \sqrt{3}\,(\tan(60-20^\circ)\cdot\tan 20^\circ\cdot\tan(60+20^\circ))$

$= \sqrt{3}\,(\tan 3(20^\circ)) = \tan 60^\circ\cdot\sqrt{3} = 3$

Q. $\cos 6^\circ\cdot\cos 42^\circ\ \cos 66^\circ\ \cos 78^\circ = ??$

Sol^n. $\dfrac{\cos(60^\circ-6^\circ)\cdot\cos 6^\circ(\cos(60+6^\circ))}{\cos(60^\circ-6^\circ)}\cdot\dfrac{\cos(60-18)\cdot\cos 18\cdot\cos(60+18)}{\cos 18^\circ}$

$= \dfrac{1}{4}\dfrac{\cancel{\cos 3(6^\circ)}}{\cancel{\cos 54^\circ}}\cdot\dfrac{\cancel{\cos 3(18)}}{\cancel{4\cos 18}} = \frac{1}{16}$

(7) C – D Formulae :–

(A) $\sin C + \sin D = 2\sin\left(\frac{C+D}{2}\right)\cdot\cos\left(\frac{C-D}{2}\right)$

(B) $\sin C - \sin D = 2\sin\left(\frac{C-D}{2}\right)\cdot\cos\left(\frac{C+D}{2}\right)$

(C) $\cos C - \cos D = 2\cos\left(\frac{C+D}{2}\right)\cdot\cos\left(\frac{C-D}{2}\right)$

(D) $\cos C - \cos D = -2\sin\left(\frac{C+D}{2}\right)\cdot\sin\left(\frac{C-D}{2}\right)$

Q. Prove that:- $\frac{\cos 7A + \cos 5A}{\sin 7A - \sin 5A} = \cot A$

Solⁿ. $\frac{2\cos\left(\frac{7A+5A}{2}\right)\cdot\cos\left(\frac{7A-5A}{2}\right)}{2\cos\left(\frac{7A+5A}{2}\right)\cdot\sin\left(\frac{7A+5A}{2}\right)} = \frac{\cos A}{\sin A} = \cot A$

∴ Hence, Proved.

Q. $\frac{\sin A + \sin B}{\cos A + \cos B} = ?? = \frac{\cancel{2}\sin\left(\frac{A+B}{2}\right)\cdot\cancel{\cos\left(\frac{A-B}{2}\right)}}{\cancel{2}\cos\left(\frac{A+B}{2}\right)\cdot\cancel{\cos\left(\frac{A-B}{2}\right)}}$

$= \tan\left(\frac{A+B}{2}\right)$

Q. $(\sin 3x + \sin x)\sin x + (\cos 3x - \cos x)\cos x = ??$

Solⁿ. $\left[2\sin\left(\frac{3x+x}{2}\right)\cdot\cos\left(\frac{3x-x}{2}\right)\right]\sin x + \left[-2\sin\left(\frac{3x+x}{2}\right)\cdot\sin\left(\frac{3x-x}{2}\right)\right]\cos x$

$= 2\sin 2x\cdot\cos x\cdot\sin x - 2\sin 2x\cdot\sin x\cos x = 0$

Q. $\sin\alpha + \sin\left(\frac{2\pi}{3}+\alpha\right) + \sin\left(\frac{4\pi}{3}+\alpha\right) = ??$

Solⁿ. $\sin\alpha + 2\sin\left(\frac{2\pi+2\alpha}{2}\right)\cdot\cos\left(-\frac{2\pi/3}{2}\right)$

$= \sin\alpha + 2\sin(\pi+\alpha)\cdot\cos(\pi/3) = \sin\alpha - \sin\alpha$
$= 0$

Q. $\frac{\sin(\theta+\alpha)}{\cos(\theta-\alpha)} = \frac{1-m}{1+m}$, then, $\tan\left(\frac{\pi}{4}-\alpha\right)\cdot\tan\left(\frac{\pi}{4}-\theta\right) = ?$

Solⁿ. By applying C 4 D, we get :-

$\frac{\sin(\theta+\alpha)+\cos(\theta-\alpha)}{\sin(\theta+\alpha)-\cos(\theta-\alpha)} = \frac{1-m+1+m}{1-m-1-m} = \frac{2}{-2m} = \frac{-1}{m}$

$$\Rightarrow \frac{\cos(\theta-\alpha)-\sin(\theta+\alpha)}{\sin(\theta+\alpha)+\cos(\theta-\alpha)} = m$$

$$\Rightarrow \frac{\cos(\theta-\alpha)-\cos\left(\frac{\pi}{2}-(\theta+\alpha)\right)}{\cos\left(\frac{\pi}{2}-(\theta+\alpha)\right)+\cos(\theta-\alpha)} = m$$

$$\Rightarrow m = \frac{-2\sin\left(\frac{\cancel{\theta}-\alpha+\frac{\pi}{2}-\cancel{\theta}-\alpha}{2}\right)\cdot\sin\left(\frac{\theta-\alpha+\theta-\frac{\pi}{2}+\alpha}{2}\right)}{2\cos\left(\frac{\frac{\pi}{2}-\cancel{\theta}-\alpha+\cancel{\theta}-\alpha}{2}\right)\cdot\cos\left(\frac{\frac{\pi}{2}-\theta-\cancel{\alpha}-\theta+\cancel{\alpha}}{2}\right)}$$

$$\Rightarrow m = +\frac{\sin(\pi/4-\alpha)\ \sin(\pi/4-\theta)}{\cos(\pi/4-\alpha)\ \cos(\pi/4-\theta)}$$

$$m = \tan(\pi/4-\theta)\cdot\tan(\pi/4+\alpha)$$

(8) Multiple Angle Formula :-

(A) $\sin 2A = 2\sin A\cos A = \frac{2\tan A}{1+\tan^2 A}$

(B) $\cos 2A = \cos^2 A - \sin^2 A$

$\Rightarrow 1+\cos 2A = 2\cos^2 A$

$\Rightarrow 1-\cos 2A = 2\sin^2 A$

$2\cos^2 A - 1$

$1-2\sin^2 A$

$\frac{1-\tan^2 A}{1+\tan^2 A}$

(C) $\tan 2A = \frac{2\tan A}{1-\tan^2 A} = \frac{\sin 2A}{\cos 2A}$

Q. $\frac{1+\sin 2\theta+\cos 2\theta}{1+\sin 2\theta-\cos 2\theta} = ??$

Sol^n. $\frac{1+\cos 2\theta+\sin 2\theta}{1-\cos 2\theta+\sin 2\theta} = \frac{2\cos^2 A+2\sin\theta\cos\theta}{2\sin^2\theta+2\sin\theta\cdot\cos\theta}$

$= \cot\theta$