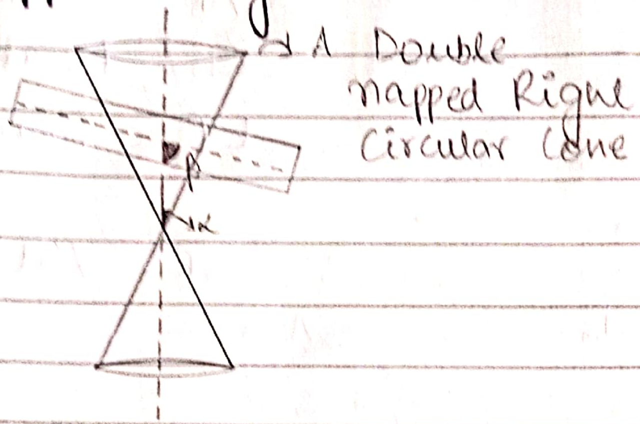
CH:- CONIC SECTION

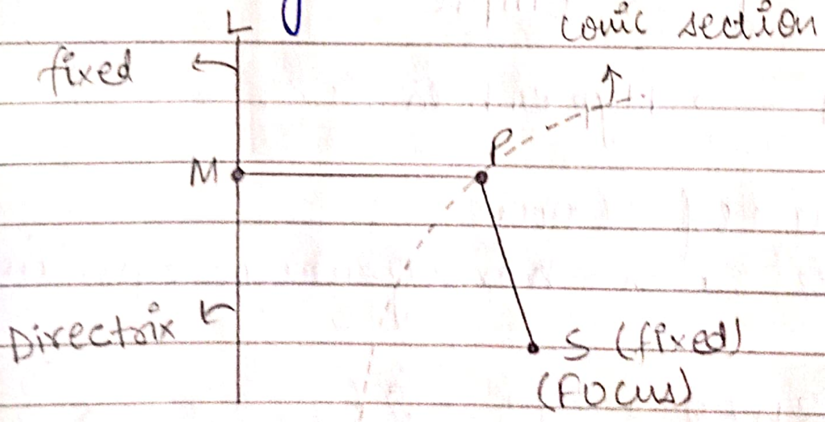
① Derivations of Conic Sections from a Double Napped Rigid Circular Cone :-



- When $\beta = 90^\circ$ → Circle is formed.
  When $\alpha < \beta < 90^\circ$ → Ellipse is formed
  When $\alpha = \beta$ → Parabola is formed
  When $0 \leq \beta < \alpha$ → Hyperbola is formed.

② Some Important Definations in conic sect

- Conic Sections:- Locus of a point which moves in a plane such that the ratio of its distance from a fixed point to its ⊥ar distance, from fixed line is always constant.

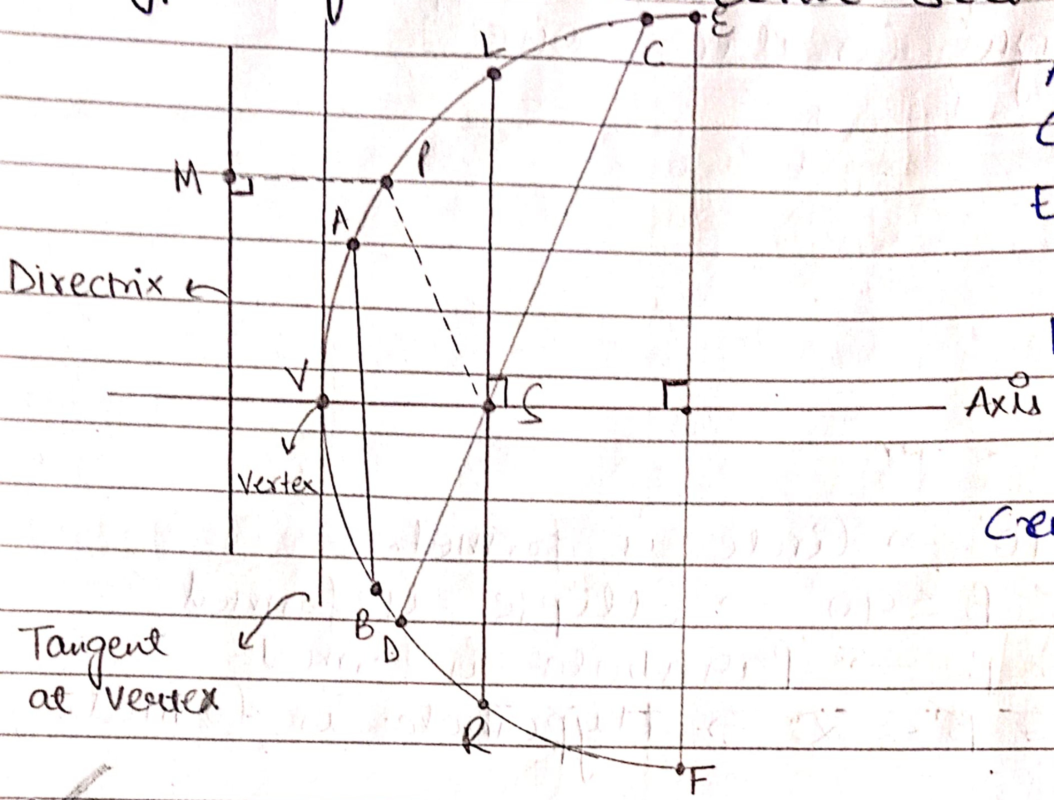


- Eccentricity :-

$$\boxed{\frac{SP}{PM} = e}$$

→ $\boxed{S.P. = e.(PM)}$

• Types Of Chord in Conic Section :-



AB = Chord

CD = Focal Chord

EF = Double ordinate

LR = Latus Rectum

Centre ⇒ Point which bisects every chord passing through it

③ Identification Of Conic Section :-

M-1) Through α and β of Double napped Right Angled Cone and plane.

M-2) Through Eccentricity :-

$$e \Rightarrow \begin{cases} 1 \longrightarrow \text{Parabola} \\ 0<e<1 \longrightarrow \text{Ellipse} \\ e>1 \longrightarrow \text{Hyperbola} \end{cases}$$

④ General Equation Of Conic :-

From 2^{nd} Degree, non-homogenous Eqn. in x and y :-

$$ax^2 + by^2 + 2hxy + 2gx + 2fy + c = 0$$

If $\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} \neq 0$, then conics are formed.

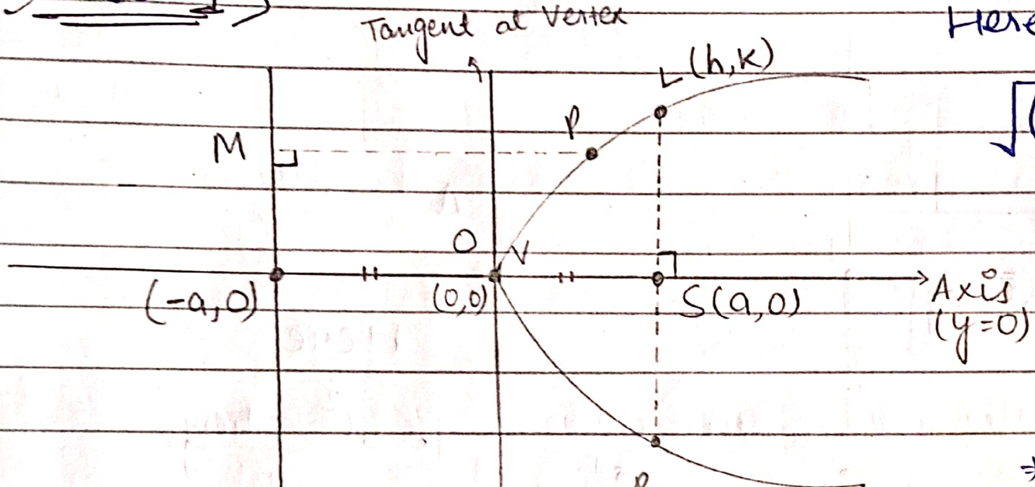
$\Delta \equiv abc + 2fgh - af^2 - bg^2 - ch^2$

Also, verify that,

$h^2 = ab$	→ Parabola
$h^2 < ab$	→ Ellipse
$h^2 > ab$	→ Hyperbola.

5) PARABOLA :-

Case - I)



Here, $\boxed{S.P. = PM}$, Thus,

$$\sqrt{(h-a)^2 + (k-0)^2} = \left|\frac{h+a}{\sqrt{1}}\right|$$

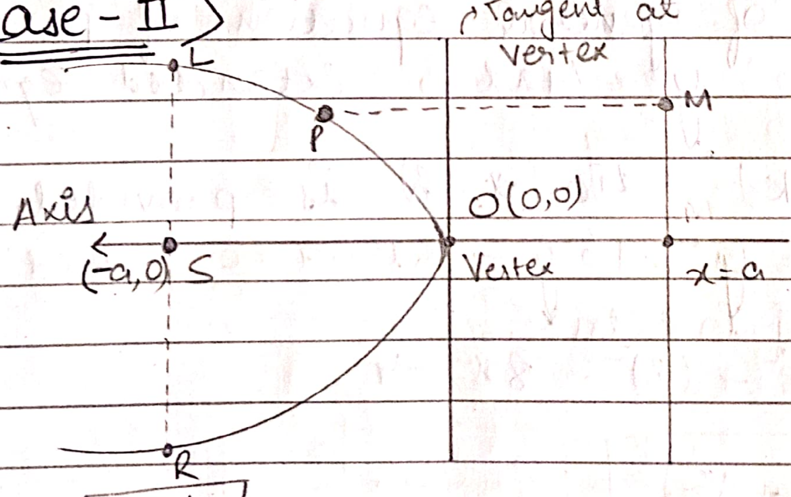
$$\Rightarrow \cancel{h^2} + \cancel{a^2} - 2ah + k^2 = \cancel{h^2} + \cancel{a^2} + 2ah$$

$$\Rightarrow k^2 = 4ah$$

$$\boxed{y^2 = 4ax}$$

Where, $\boxed{\text{Latus Rectum} = LR = 4a}$

Case - II)

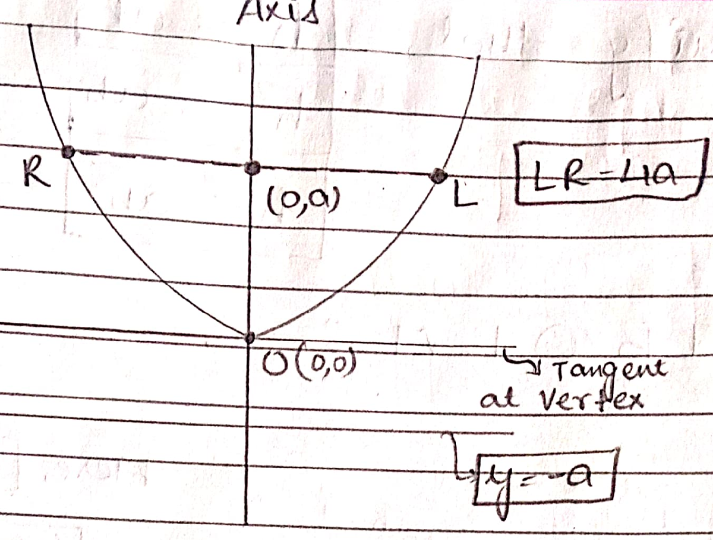


$\boxed{LR = 4a}$

Here, S.P. = PM

$$\boxed{y^2 = -4ax}$$

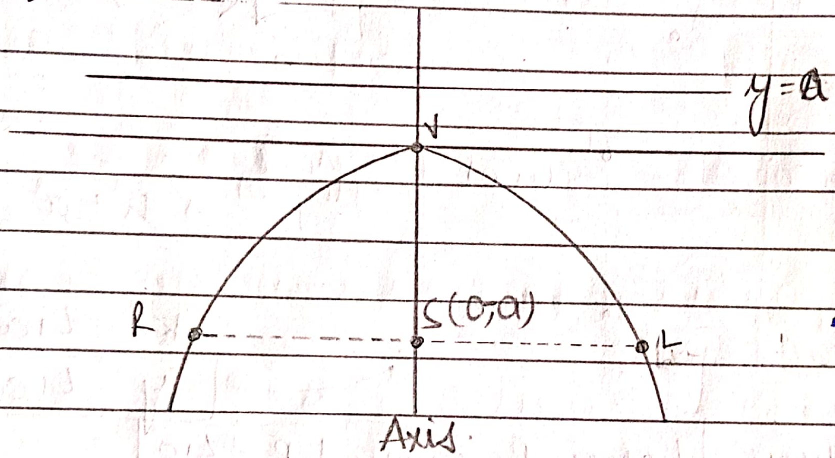
Case - III)



Here, Parabola can be represente-d through equat^n,

$$\boxed{x^2 = 4ay}$$

Case - IV)



Here,

$$\boxed{x^2 = -4ay}$$

- **Shifted Parabola :-**
  Some-times, instead of general equation of parabo-la such as '$y^2 = 4ax$', extended eqn such as,

$$(y-k)^2 = 4a(x-h) \text{ is provided.}$$

Q. For a parabola, $y^2 + 16y - 8x + 4 = 0$, find Vertex, focus, Directrix ?

Sol^n.
$$y^2 + 16y + (8)^2 - (8)^2 = 8x - 4$$

$$\boxed{(y+8)^2 = 8\left(x + \frac{15}{2}\right)}$$

Let suppose, $\boxed{X = x + \frac{15}{2}}$ & $\boxed{y + 8 = Y}$, then,

$$\boxed{Y^2 = 8X}$$

Thus, for $\boxed{y^2 = 8x}$,

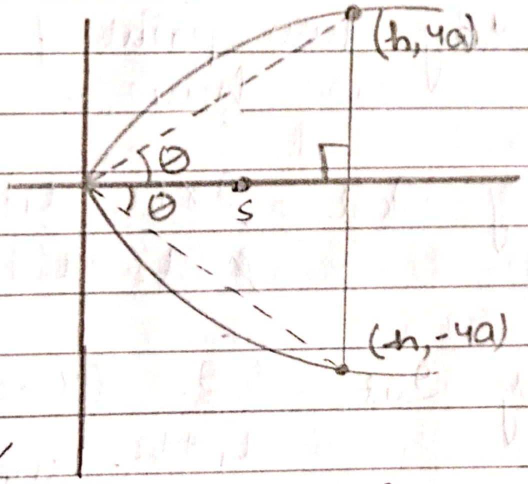
Vertex = $(0,0)$; $S = (2,0)$; $D = x = -2$

$\therefore$ Substituting these values into $(y+8)^2 = 8\left(x+\frac{15}{2}\right)$ gives us,

Vertex = $\left(-\frac{15}{2}, -8\right)$; $S = \left(-\frac{11}{2}, -8\right)$

and $D = \left(-\frac{19}{2}, -8\right)$.

Q. For a parabola, $\boxed{y^2 = 4ax}$, Double Ordinate is $8a$. Find the angle b/w lines from vertex to its ends.



Also, here, 'h' $(4a)$ point on parabola

$\therefore (4a)^2 = 4a \times h$

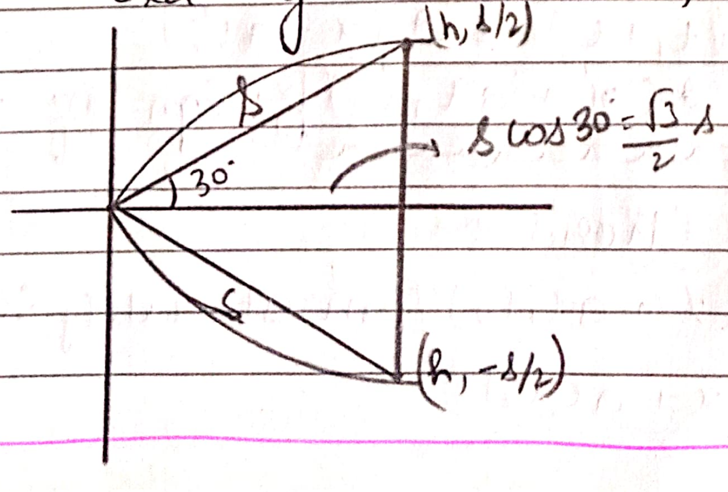
$\boxed{h = 4a}$

$\therefore \tan\theta = 1$

$\theta = \tan^{-1}(1)$

$\boxed{\theta = 45^\circ}$

Q. A Equilateral $\triangle$ is inscribed inside a parabola $y^2 = 4ax$. Then, find its side length?

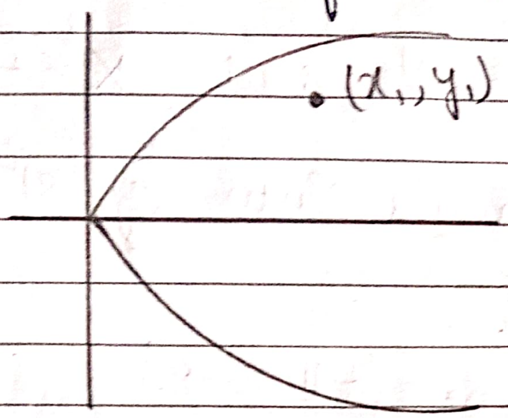


$\therefore \boxed{h = \frac{\sqrt{3}}{2}s}$

Since, $\left(\frac{\sqrt{3}}{2}s, \frac{s}{2}\right)$ is a path on parabola, thus,

$\boxed{s = 8\sqrt{3}a}$

(6) Position of a point w.r.t Parabola :-



Let the eqn. of parabola be

$\boxed{y^2 = 4ax}$

Then, find $\boxed{S_1 = y_1^2 - 4ax_1}$

If $S_1 > 0 \rightarrow$ Outside

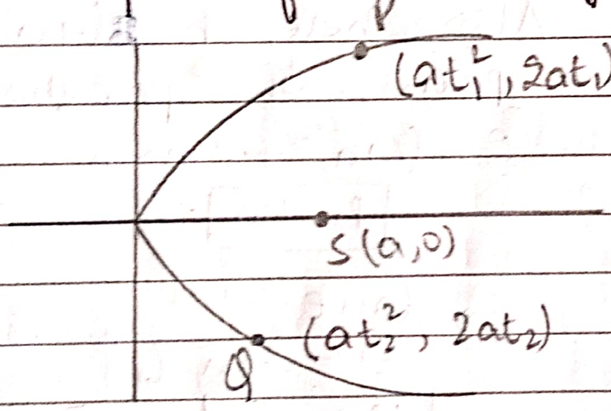
$S_1 = 0 \rightarrow$ On

$S_1 < 0 \rightarrow$ Inside.

(7) Parametric Eqns. :-

For Parabola, $y^2 = 4ax$	For Parabola, $x^2 = 4ay$
$P(at^2, 2at)$	$P(2at, at^2)$

(8) Eqn of chord joining t_1 & t_2 :-



By two point form, we have :-

$\Rightarrow y - 2at_1 = \dfrac{2\cancel{a}(t_2 - t_1)}{\cancel{a}(t_2^2 - t_1^2)}(x - at_1^2)$

$\Rightarrow y - 2at_1 = \underbrace{\dfrac{2}{t_1 + t_2}}_{\text{Slope of Chord}}(x - at_1^2)$

$\Rightarrow y(t_1 + t_2) - 2at_1(t_1 + t_2) = 2(x - at_1^2)$

$\Rightarrow \boxed{y(t_1 + t_2) = 2(x + at_1t_2)}$ → Eqn. of chord

- In case of focal chord :-

$\Rightarrow y(t_1 + t_2) = 2(x + at_1t_2)$ must satisfy $S(a,0)$

$\Rightarrow 0(t_1 + t_2) = 2(x + at_1t_2)$

$\Rightarrow \boxed{t_1 t_2 = -1}$

T.I.P.:- Two Parabolas are said to be equal if their Latus Rectum (L.R.) are equal

Then, the coordinates of P & Q become:-

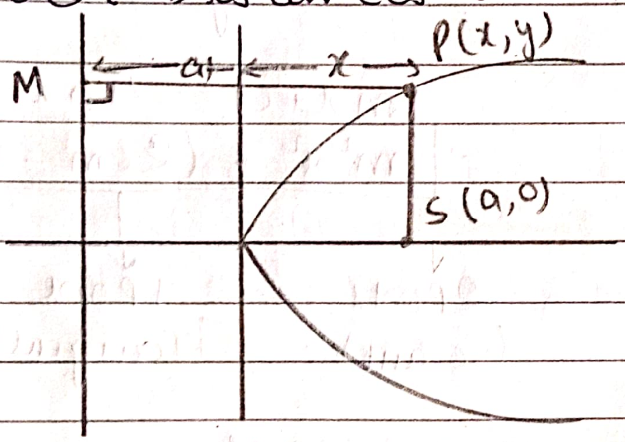
$\boxed{t_1 = t}$ $\boxed{t_2 = \frac{-1}{t}}$ $\Rightarrow$ $P(at^2, 2at)$ & $Q\left(\frac{a}{t^2}, \frac{-2a}{t}\right)$

(9) Length of focal chord :-

By applying distance formula at PQ, we get:

$$\boxed{PQ = a\left(t + \frac{1}{t}\right)^2}$$

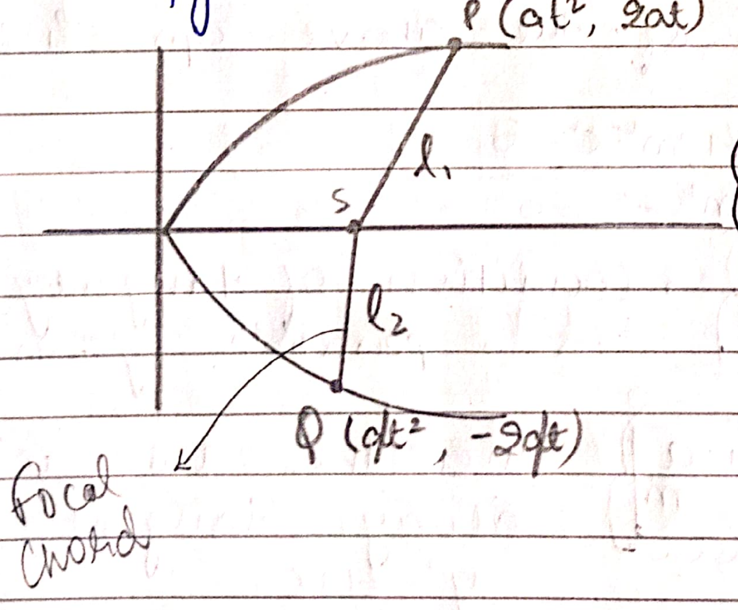
(10) Focal Distances :-



$SP = PM = |x + a|$

Or Focal Distance is the sum of distance from Tangent at Vertex and one-fourth its L.R.

If l_1 = S.P. and $SQ = l_2$ as shown in the figure below :-



In such a case,

$$\boxed{2a = \frac{2 l_1 l_2}{l_1 + l_2}}$$

2a (Semi-Latus Rectum)

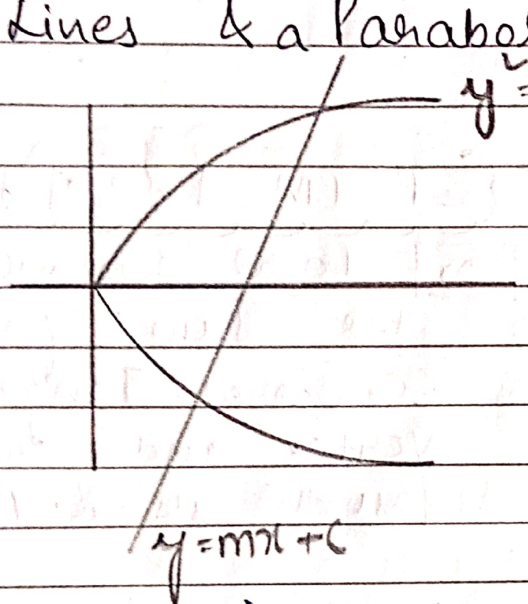
$\frac{2 l_1 l_2}{l_1 + l_2}$: Harmonic Mean of l_1 & l_2

Q. For a parabola, $\boxed{y^2 = 16x}$ if $SP = 10$ units then, find $\boxed{SQ = l_1}$

Sol^n. ∴ $4a = 16 \Rightarrow \boxed{a = 4}$

∴ $\dfrac{2(10)(l_2)}{10 + l_2} = 2(4) \Rightarrow \boxed{l_2 = \dfrac{20}{3}} = SQ.$

(11) Lines & a Parabola :-



Since, $y = mx + c$ & $y^2 = 4ax$, thus,

$(mx+c)^2 = 4ax$

$\Rightarrow \boxed{m^2x^2 + (2cm - 4a)x + c^2 = 0}$

2 Roots (Secant) | 1 Root (Tangent) | 0 Roots (Non-Intersecting)

i.e., $D > 0 \rightarrow$ Secant

$D = 0 \rightarrow$ Tangent

$D < 0 \rightarrow$ Non-Intersecting

Now, Discriminant of the above eqn is :-

$\Rightarrow (2cm - 4a)^2 - 4m^2c^2 = 0$

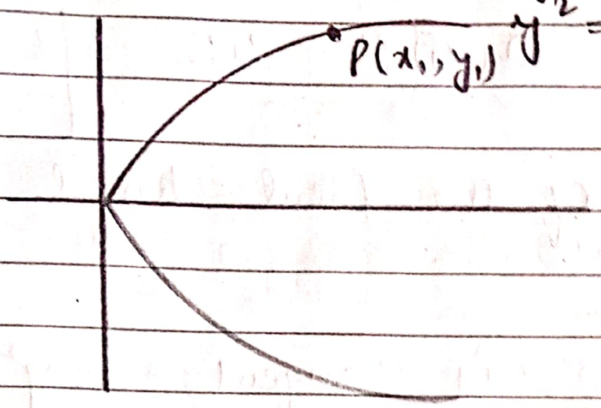
$\Rightarrow (cm - 2a)^2 - m^2c^2 = 0$

$\Rightarrow \boxed{c = \dfrac{a}{m}}$ → Condition of tangency for a parabola $y^2 = 4ax$

∴ If $\boxed{y = mx + \dfrac{a}{m}}$, then this line will always tangent to $y^2 = 4ax$.

P-2

12. Equation of tangent at a point :-



$y^2 = 4ax$, $P(x_1, y_1)$

By $\boxed{T=0}$ we get Equation of tangent a

$$\boxed{y y_1 = 4a\left(\frac{x+x_1}{2}\right)}$$

If $(x_1, y_1) = (at^2, 2at)$

then,

$$y(2at) = 2a(x + at^2) \Rightarrow \boxed{ty = x + at^2}$$

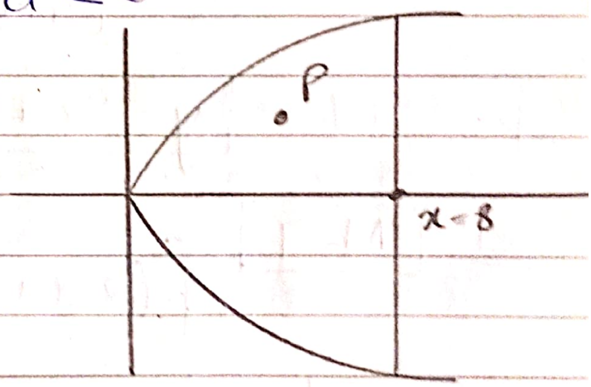
$\boxed{\text{Slope} = \frac{1}{t}}$

Q. If $(a, 2a)$ is a point which lies inside the region bounded by parabola & L.R find the range of values of a?

Sol^n. Since, the point lies inside the parabola, thus, $S_1 < 0$,

$\therefore \quad (2a)^2 - 32a < 0$

$\Rightarrow \boxed{a \in (0, 8)}$



Also, $x - 8 = 0$

$a - 8 < 0$

$\boxed{a < 8}$

$\therefore \boxed{a \in (0, 8)}$

Q. If Parabola $\boxed{y^2 = 16x}$ has one end of the focal chord at $(2, 4\sqrt{2})$, then find the other end point of the focal chord.

Sol^n. Here, $\boxed{a = 4}$

Let the first end point be $(at^2, 2at)$ and the other be $\left(\frac{a}{t^2}, \frac{-2a}{t}\right)$

Thus, $(at^2, 2at) = (2, 4\sqrt{2})$

$\therefore$ $2at = 4\sqrt{2} \Rightarrow 2(4)t = 4\sqrt{2} \Rightarrow \boxed{t = \frac{1}{\sqrt{2}}}$

So, the other end of the focal chord is

$(8, -8\sqrt{2})$

Q. If line $lx + my - n = 0$ is Tangent to $y^2 = 8x$ Then, find the relation b/w l, m, n

Sol^n. From line $lx + my + n = 0$, we get :-

$y = -\frac{l}{m}x - \frac{n}{m} \longrightarrow \boxed{M = -l/m}$ & $\boxed{C = -n/m}$

Now, let the line be $y = Mx + C$ then,

$C = \frac{a}{M} \Rightarrow \frac{-n}{m} = \frac{2}{-l/m} \Rightarrow \frac{n}{m} = +\frac{2m}{l}$

$\therefore$ $\boxed{nl = 2m^2}$ is the relation b/w l, m, n.

Q. If $\boxed{y = 2x + k}$ is a tangent to parabola $y^2 = 16(x+4)$, then find 'k'?

Sol^n $\boxed{M-1}$

$\boxed{(2x+k)^2 = 16(x+4)} \longrightarrow \boxed{D = 0}$

$\boxed{M-2}$

Let $\boxed{y = Y}$ and $\boxed{x + 4 = X}$ thus,

eqn. becomes $\boxed{Y^2 = 16X}$

Also, $y = 2x + k$ becomes

~~$Y = 2(X - 4) + k$~~

$\boxed{Y = 2X + (k - 8)}$

Now, applying $\boxed{c = \frac{a}{m}}$, therefore,

$$k - 8 = \frac{4}{2} \Rightarrow \boxed{k = 10}$$

Q. If Parabola $\boxed{y^2 = 16x}$ has a tangent of slope $\boxed{m = 5}$ then find point of contact and equatⁿ of tangent.

Solⁿ. Here, $\boxed{a = 4}$, and the equation of tangent will be of form $\boxed{y = 5x + \frac{a}{m}}$

$$\therefore \text{ Eqn.} \Rightarrow \boxed{y = 5x + \frac{4}{5}}$$

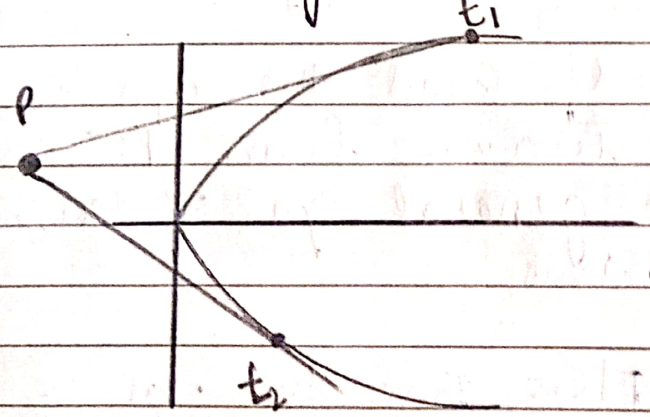
Also, from $\boxed{T = 0}$, we get,

$$\boxed{y^2 = 16x} \longrightarrow yk = 16\frac{(x+h)}{2} = 8x + 8h$$

Since, $\boxed{yk = 8x + 8h}$ and $\boxed{25x - 5y + 4 = 0}$ are same, thus,

$$\frac{8}{25} = \frac{-k}{-5} = \frac{8h}{4} \quad \boxed{h = 4/25} \quad \boxed{k = 8/5}$$

(13) Point of Intersection of tangents at t_1 & t_2 for $y^2 = 4ax$:-



Here, $\boxed{T = 0}$ w.r.t. t_1 $= (at_1^2, 2at_1)$

$$t_1 y = x + at_1^2 \quad \text{—— (1)}$$

$$\text{and } t_2 y = x + at_2^2 \quad \text{—— (2)}$$

$$\therefore \boxed{x = at_1 t_2} \;\&\; \boxed{y = a(t_1 + t_2)}$$

$\therefore$ $\boxed{P(at_1t_2,\ a(t_1+t_2))}$

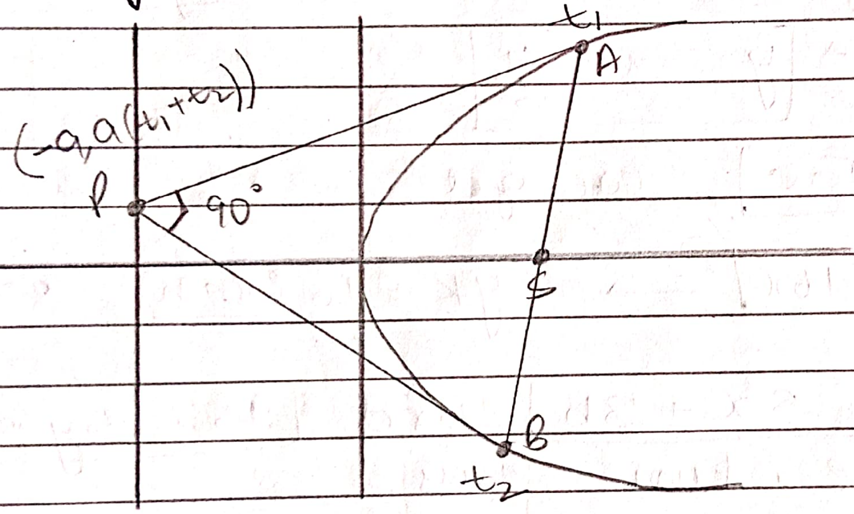
G.M. $\sqrt{at_1^2 \cdot at_2^2}$

A.M. $\frac{2at_1 + 2at_2}{2}$

- Condition for tangent to meet on x-axis :-

$\therefore \boxed{y=0} \Rightarrow a(t_1+t_2) = 0 \Rightarrow \boxed{t_2 = -t_1}$

- Tangents are drawn at extremities of focal chords of $y^2 = 4ax$:-



In case of focal chords, t_1 and t_2,

$\boxed{t_1 \cdot t_2 = -1}$

$\therefore$ Point of Intersectn.

$(at_1t_2,\ a(t_1+t_2))$

$\Rightarrow (-a,\ a(t_1+t_2))$

$\Rightarrow$ Meets at Directrix.

Here, $m_{AP} = +\frac{1}{t_1}$

and $m_{BP} = \frac{1}{t_2}$

$\boxed{m_{AP} \cdot m_{BP} = \frac{1}{t_1 \cdot t_2}} = \frac{1}{-1}$

$\Rightarrow \boxed{m_{AP} \times m_{BP} = -1}$

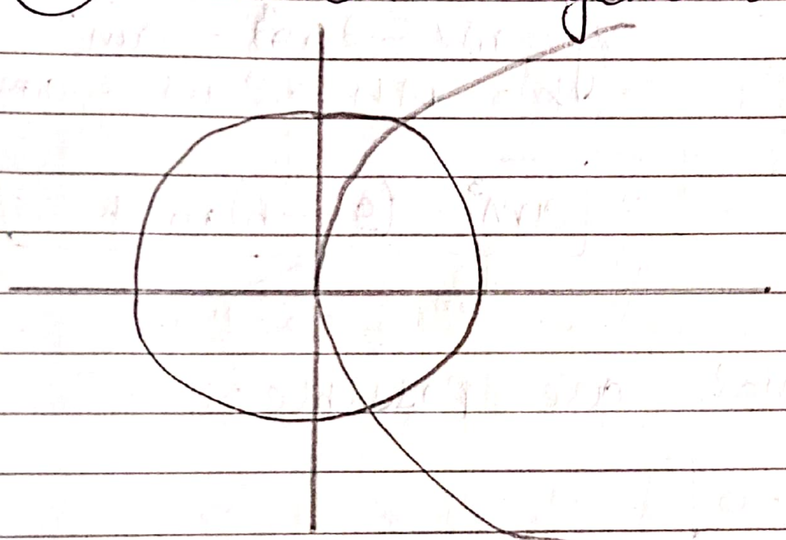
Q. For a parabola, $\boxed{y^2 = 8x}$ find the eqn of P.O. External Tangent drawn from P(5, 20)

Sol.ⁿ Let us consider a general eqn. of tangent lar to $\boxed{y^2 = 8x}$, thus,

$\boxed{y = mx + \frac{2}{m}}$. Now, since the eqn of tangent should pass from $P(5,20)$, therefore

$20 = 5m + \frac{2}{m} \Rightarrow \boxed{20m = 5m^2+2}$ — m_1, m_2 → 2 Eqn. of Tangent

(14) Common Tangent :-



for a circle, $\boxed{x^2+y^2=4}$ and a parabola, $\boxed{y^2=16x}$,

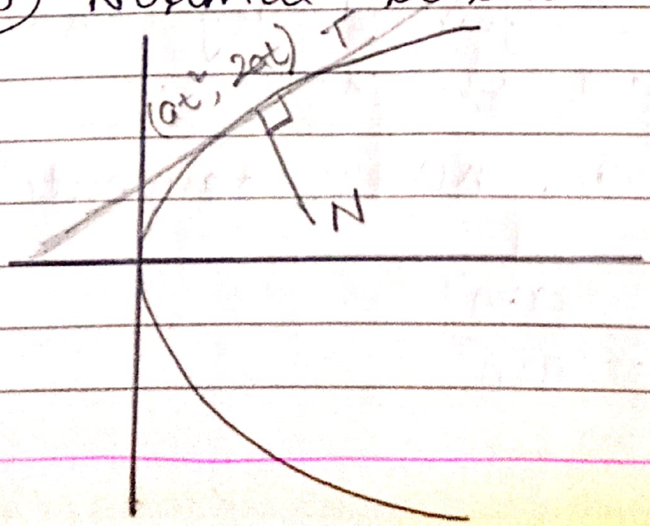
$\boxed{y = mx + \frac{4}{m}}$ is the eqn. of common tangent to circle and parabola. There fore,

By applying Distance formula, we have

$\Rightarrow \left|\frac{m^2(0) - m(0) + 4}{\sqrt{(m^2)^2 + m^2}}\right| = 2 \Rightarrow \frac{\cancel{16}^{4}}{m^4+m^2} = \cancel{4}$

$\Rightarrow 4 = m^4 + m^2 \Rightarrow \boxed{m^4+m^2-4=0}$ — m_1, m_2 → Eqn. of Common Tangent

(15) Normal to the Parabola :-



Eqn. of Tangent $\Rightarrow$

$ty = x + at^2$

$\therefore \boxed{m_T = \frac{1}{t}}$ & $\boxed{m_N = -t}$

$\therefore$ Now, slope-point form

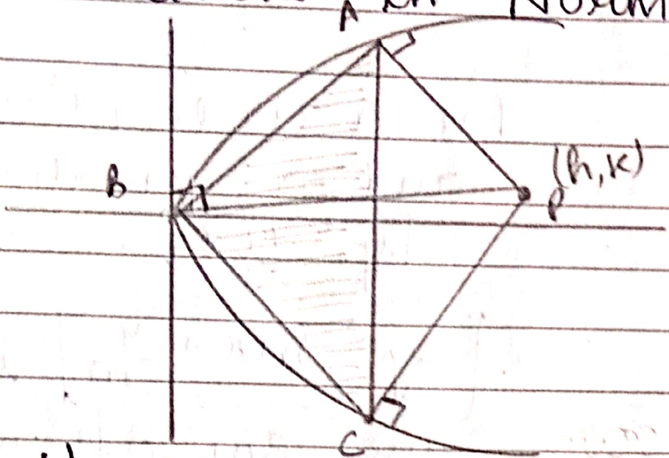
$y - 2at = -t(x - at^2)$

$\boxed{y = -tx + at^3 + 2at}$

Let $\boxed{-t = m}$; thus, $\boxed{y = mx - 2am - am^3}$

$\therefore$ Point of Contact of Normal = $(am^2, -2a$

- Conditions in Normal from a point P :-



Eqn. of Normal $\Rightarrow$

$y = mx - 2am - am^3$

$\Rightarrow k = mh - 2am - am^3$

$\Rightarrow \boxed{am^3 + (2a - h)m + k = 0}$

$m_1 \quad m_2 \quad m_3$

C-1) Max. 3 Normal are possible:

C-2) $\boxed{m_1 + m_2 + m_3 = 0}$

C-3) Points A, B, C $\Rightarrow$ Co-Normal Points and Sum of y coordinates (feet of conormal points) is zero

i.e. $-2am_1 - 2am_2 + (-2am_3) = 0$

C-4) In the $\triangle$ formed by conormal points,

$\Rightarrow G_y = \frac{y_1 + y_2 + y_3}{3} = 0$

$\therefore$ Centroid of $\triangle$ formed by conormal points lies on x-axis.

C-5). For a point P(h, 0), no. of tangents possible are :-

$\Rightarrow y = mx - 2am - am^3$

$\Rightarrow 0 = mh - 2am - am^3$

$\Rightarrow am^3 + (2a - h)m = 0$

$\Rightarrow m\left[(2a-h)+am^2\right]=0$

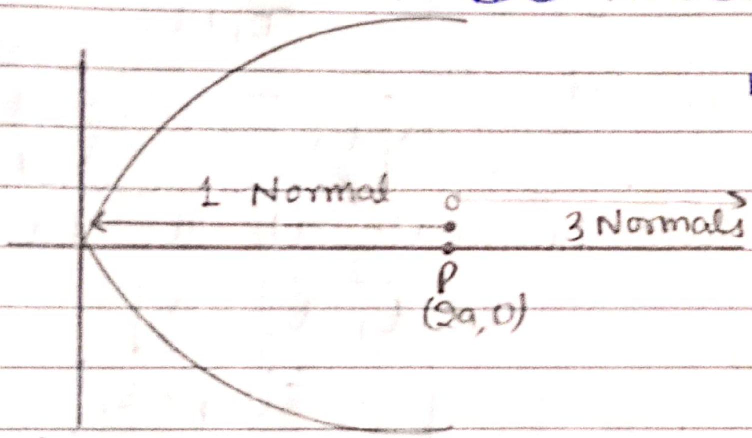
$\boxed{m=0}$; $m_1 ; m_2$

$\hookrightarrow$ If $\boxed{am^2 = h-2a}$

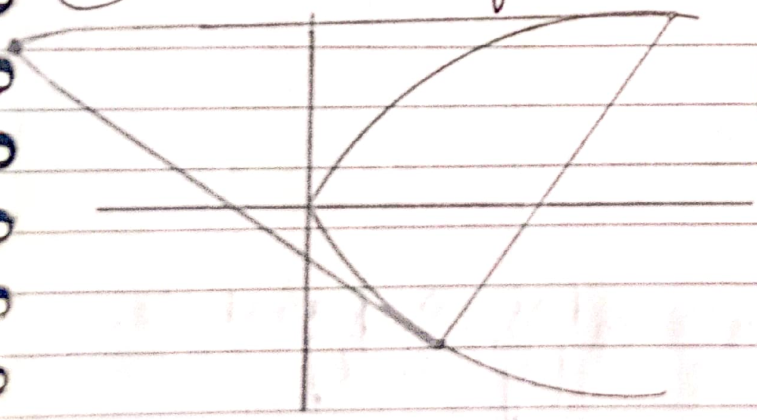
$\Rightarrow m^2 = \dfrac{h-2a}{a} \Rightarrow \boxed{m^2 = \dfrac{h-2a}{a} > 0}$

$\Rightarrow$ ~~[illegible]~~ $\boxed{m^2 = h-2a>0}$

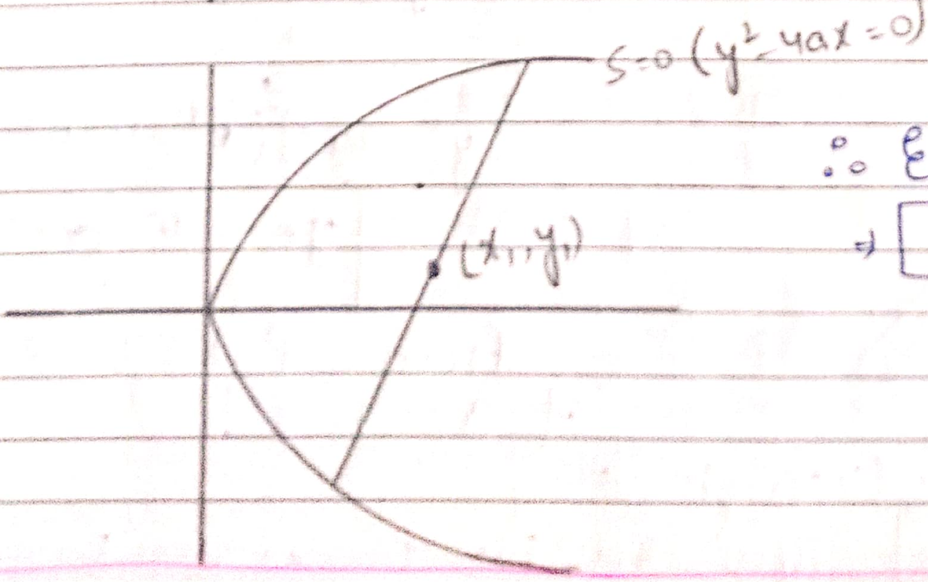
Always Valid



(16) Chord Of Contact :-



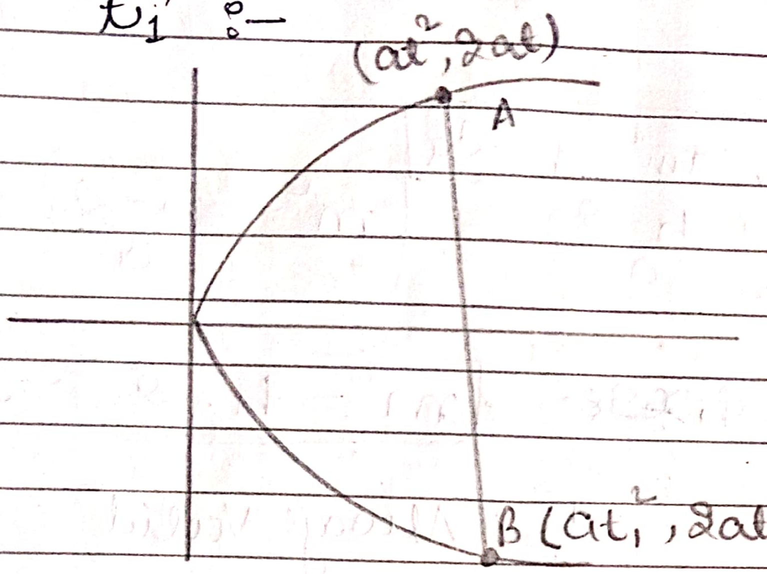
$\boxed{T=0}$ Eqn. Of Chord Of contact.



∴ Eqn. Of Chord

$\Rightarrow \boxed{T = S_1}$

• Normal at 't' meets the parabola again at 't₁' :-



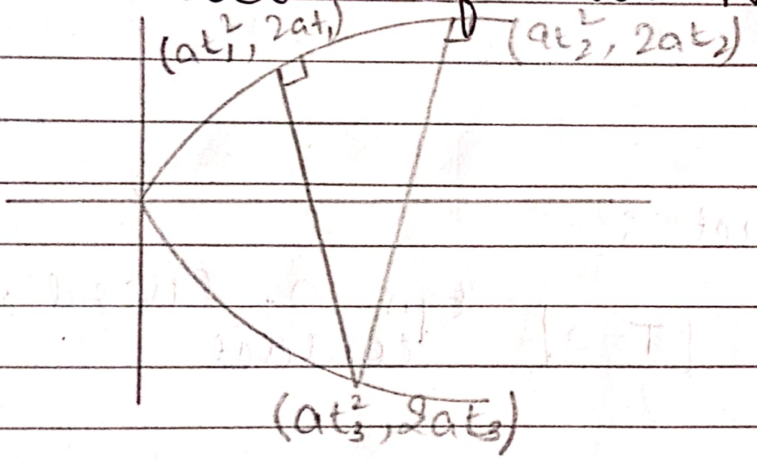
Here, Slope of Normal $= -t$

& $m_{AB} = \frac{2\cancel{a}(t_1 - t)}{\cancel{a}(t_1^2 - t^2)} = -t$

$m_{AB} = \frac{2}{t_1 + t} = -t$

$\Rightarrow \boxed{t_1 = -t - \frac{2}{t}}$

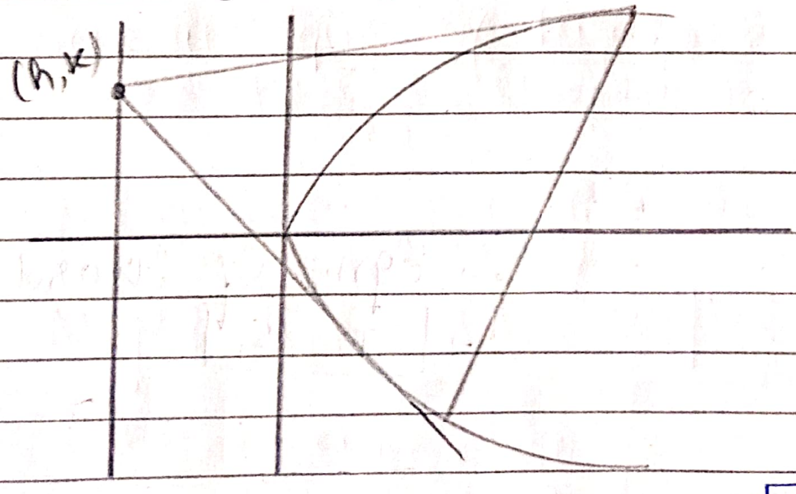
• Intersection of two Normal at the Parabola



$\boxed{t_1 \cdot t_2 = 2}$

$\boxed{t_3 = -(t_2 + t_1)}$

(17) Director Circle :-



For a parabola, $\boxed{y^2 = 4ax}$,

$\boxed{SS_1 = T^2} \Rightarrow$

$(y^2 - 4ax)(k^2 - 4ah) = \left[yk - 4a\left(\frac{x+h}{2}\right)\right]^2$

From P.O.S.L.,

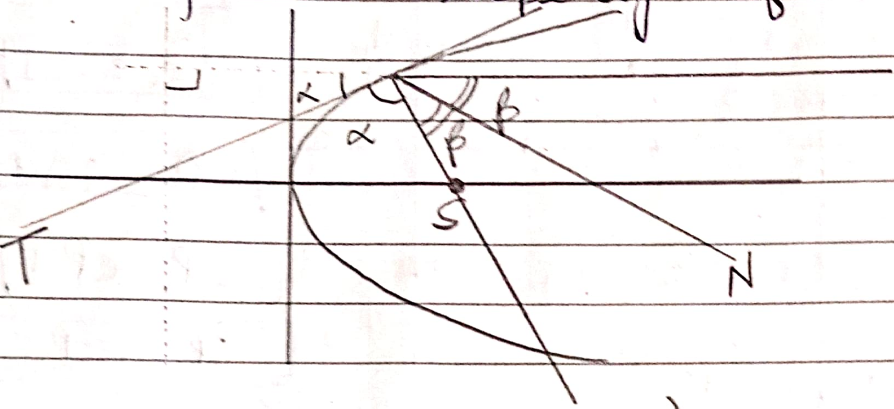
$\Rightarrow \boxed{a + b = 0}$

$\therefore \; 4a^2 + k^2 + (4ah - k^2) = 0$

$\Rightarrow \boxed{x = h = -a}$. Thus, the Directrix is the Director Circle of Parabola.

(18) Some Special Properties of Parabola :-

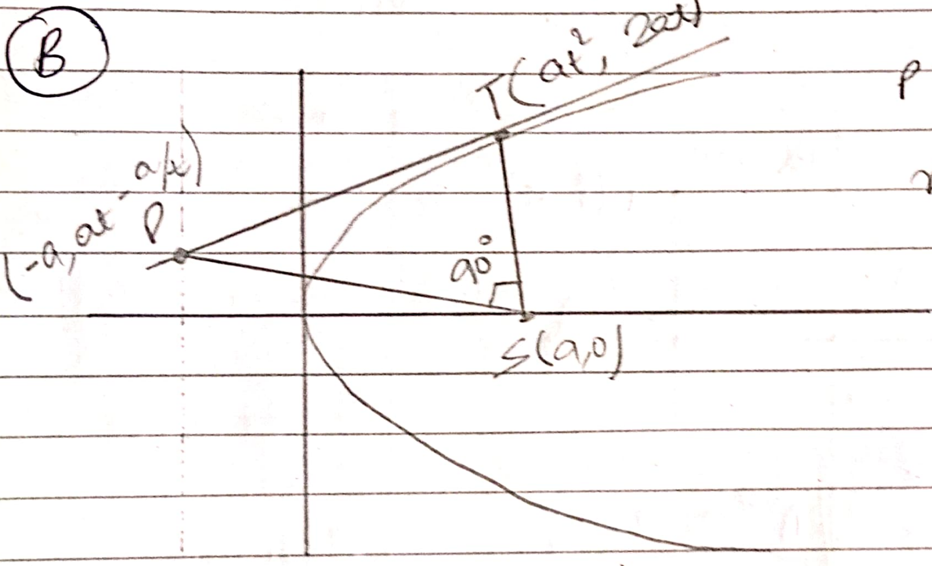
(A) Reflection Property of Parabola :-



~~Tangent bisects the~~

$\boxed{\alpha + \beta = 90^\circ}$

(B)



PT = portion of tangent

$m_{SP} \times m_{ST} = -1$

(C)