

# MATRIX

## MATRIX

## (Matrices)

A matrix is an ordered rectangular array of numbers or functions (real or complex) organised in rows and columns.

↳ **ELEMENTS OR ENTRIES OF THE MATRIX:** The individual numbers or functions within the matrix are called its elements or entities.

↳ **ROW OF THE MATRIX:** The horizontal groupings of elements in a matrix are known as rows.

↳ **COLUMNS OF THE MATRIX:** The vertical groupings of elements in a matrix are referred to as columns.

## GENERAL FORMAT OF A MATRIX

1. **SIMPLE MATRIX:** Consists of only one row and one column ( $1 \times 1$ )  
It is represented by its order as (rows, columns) - eg; (1, 1)

2. **LARGER OR HIGHER-ORDER MATRIX:** contains multiple rows and columns.

Represented in the format:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{bmatrix}$$

Alternatively, it can also be written using parentheses:

$$A = \left( \begin{array}{cccc} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{array} \right)$$

## ORDER OF A MATRIX

The order of a matrix refers to its dimensions, expressed as  $m \times n$

where:

$m$  = number of rows

$n$  = number of columns.

The total number of elements in a matrix is the product of its rows and columns.

Total elements =  $m \times n$

Example: consider the matrix:

$$A = \begin{bmatrix} 3 & 2 & 4 \\ 1 & 5 & 9 \end{bmatrix}$$

Number of rows = 2

Number of columns = 3

Order of matrix A:  $2 \times 3$

Total number of elements =  $2 \times 3 = 6$

GENERAL FORM: A matrix of order  $m \times n$  ( $m$  = rows,  $n$  = columns) is written

as:  $A = [a_{ij}]$ , where  $1 \leq i \leq m$  and  $1 \leq j \leq n$

$i^{\text{th}}$  Row Elements are  $a_{i1}, a_{i2}, a_{i3}, \dots, a_{in}$

$j^{\text{th}}$  Row Elements are  $a_{1j}, a_{2j}, a_{3j}, \dots, a_{nj}$

## TYPES OF MATRIX

1. COLUMN MATRIX: A matrix having only one column and any number of rows.

Eg:  $A = \begin{bmatrix} 0 \\ \sqrt{5} \\ -2 \\ 3/2 \end{bmatrix}$  of order  $4 \times 1$

General Form:  $A = [a_{ij}]_{m \times 1}$ , order of the matrix is  $m \times 1$

2. ROW MATRIX: A matrix having only one row and any number of columns.

Eg:  $A = \left[ -\frac{3}{4}, \sqrt{7}, 3, 5 \right]$  of order  $1 \times 4$

General Form  $\Rightarrow A = [a_{ij}]_{1 \times n}$ , order of matrix is  $1 \times n$

**SQUARE MATRIX:** A matrix of order  $m \times n$ , such that  $m = n$

Eq:  $A = \begin{bmatrix} 3 & -1 & 0 \\ \frac{3}{2} & 3\sqrt{2} & 1 \\ 4 & 3 & -1 \end{bmatrix}$  of order  $3 \times 3$

General Form:  $A = [a_{ij}]_{m \times m}$  order of matrix is  $m$ .

**DIAGONAL MATRIX:** A square matrix is said to be diagonal matrix if all its non-diagonal elements are zero.

Eq:  $A = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$  of order  $3 \times 3$

General Form  $\Rightarrow A = [b_{ij}]_{m \times m}$  order of the matrix is  $m$  where  $b_{ij} = 0$  if  $i \neq j$

**SCALAR MATRIX:** A scalar matrix is a diagonal matrix in which all the diagonal elements are equal.

Eq:  $A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ , order =  $3 \times 3$

General Form:  $A = [b_{ij}]_{m \times n}$

where:  $b_{ij} = 0$  if  $i \neq j$

$b_{ij} = k$  if  $i = j$ , for some constant  $k$ .

**IDENTITY MATRIX:** An identity matrix is a square matrix in which all diagonal elements are 1, and all other elements are 0

Eq:  $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ , order =  $3 \times 3$

General Form:  $A = [b_{ij}]_{m \times m}$

where:  $b_{ij} = 0$  if  $i \neq j$

$b_{ij} = 1$  if  $i = j$

**ZERO MATRIX:** A zero matrix (also called a null matrix) is one in which all elements are zero

Eg:  $O = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ , order =  $2 \times 3$

Notation: Denoted by  $O$

**8. EQUAL MATRICES:** Two matrices A and B are said to be equal if:

(i) They have the same order

(ii) Their corresponding elements are equal.

Eg:  $A = \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}$ ,  $B = \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix} \Rightarrow A = B$

### APPLICATION OF EQUAL MATRICES - SOLVING FOR UNKNOWN

Given:  $\begin{bmatrix} x+3 & z+4 & 2y-7 \\ a-1 & 0 & b-3 \end{bmatrix} = \begin{bmatrix} 0 & 6 & 3y-2 \\ -3 & 2c+2 & 2b+4 \end{bmatrix}$

Equating corresponding elements:

•  $x+3=0 \Rightarrow x=-3$

•  $z+4=6 \Rightarrow z=2$

•  $2y-7=3y-2 \Rightarrow y=-5$

•  $a-1=-3 \Rightarrow a=-2$

•  $0=2c+2 \Rightarrow c=-1$

•  $b-3=2b+4 \Rightarrow b=-7$

Solution:

$x=-3, y=-5, z=2, a=-2, b=-7, c=-1.$

## OPERATION ON MATRICES

### • ADDITION OF MATRICES:

Let A and B be two matrices each of order  $m \times n$ . Then, the sum of matrices

$A+B$  is defined only if matrices A and B are of same order.

• If  $A = [a_{ij}]_{m \times n}$  and  $B = [b_{ij}]_{m \times n}$

Then,  $A+B = [a_{ij} + b_{ij}]_{m \times n}$

General format:

$$\text{If: } A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \text{ and } B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix}$$

$$[A+B] = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} & a_{13} + b_{13} \\ a_{21} + b_{21} & a_{22} + b_{22} & a_{23} + b_{23} \end{bmatrix}$$

### • SUBTRACTION OF MATRICES:

Let A and B be two matrices of the same order, then subtraction of matrices

$A-B$ , is defined as  $A-B = [a_{ij} - b_{ij}]_{m \times n}$  where  $A = [a_{ij}]_{m \times n}$ ,  $B = [b_{ij}]_{m \times n}$

$$\text{Eg: If } A = \begin{bmatrix} -1 & 2 & 0 \\ 0 & 3 & 6 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & -4 & 3 \\ 9 & -4 & -3 \end{bmatrix}$$

Here, order of  $A = 3 \times 2$  and order of  $B = 3 \times 2$  therefore, subtraction is possible

$$A-B = \begin{bmatrix} -1 & 2 & 0 \\ 0 & 3 & 6 \end{bmatrix} - \begin{bmatrix} 0 & -4 & 3 \\ 9 & -4 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} -1-0 & 2-(-4) & 0-3 \\ 0-9 & 3-(-4) & 6-(-3) \end{bmatrix} \Rightarrow \begin{bmatrix} -1 & 6 & -3 \\ -9 & 7 & 9 \end{bmatrix}$$

NOTE: When adding two matrices A and B, if the order is not same, then  $A \pm B$  is not defined.

## PROPERTIES OF ADDITION OF MATRICES:

If  $A = [a_{ij}]$ ,  $B = [b_{ij}]$  and  $C = [c_{ij}]$  are three matrices of order  $m \times n$ , then:

• COMMUTATIVE LAW:  $A+B = B+A$

• ASSOCIATIVE LAW:  $(A+B)+C = A+(B+C)$

• EXISTENCE OF ADDITIVE IDENTITY:

A zero matrix (0) of order  $m \times n$  (same as of A), is additive identity, if  
 $A+0 = A = 0+A$

• EXISTENCE OF ADDITIVE INVERSE:

If A is a square matrix, then the matrix  $(-A)$  is called additive inverse,

$$\text{if } A+(-A) = 0 = (-A)+A$$

*-A is the additive inverse of A.*

• CANCELLATION LAW:

$$A+B = A+C \Rightarrow B=C \text{ (left cancellation law)}$$

$$B+A = C+A \Rightarrow B=C \text{ (right cancellation law)}$$

## MULTIPLICATION OF MATRICES:

• MULTIPLICATION OF A MATRIX BY A SCALAR:

Let  $A = [a_{ij}]_{m \times n}$  be a matrix and  $k$  be any scalar. Then, the matrix obtained by multiplying each element of A by  $k$  is called the scalar multiple of A by  $k$  and is denoted by  $kA$  given as  $kA = [ka_{ij}]$ .

eg: If  $A = \begin{bmatrix} 3 & 1 & 1.5 \\ \sqrt{5} & 7 & -3 \\ 2 & 0 & 5 \end{bmatrix}$  Find  $3A$ .

Here, A is the matrix of order  $3 \times 3$  and  $k=3$ , constant

$$\text{Hence, } 3A = 3 \begin{bmatrix} 3 & 1 & 1.5 \\ \sqrt{5} & 7 & -3 \\ 2 & 0 & 5 \end{bmatrix} = \begin{bmatrix} 9 & 3 & 4.5 \\ 3\sqrt{5} & 21 & -9 \\ 6 & 0 & 15 \end{bmatrix}$$

## PROPERTIES OF SCALAR MULTIPLICATION:

If  $A$  and  $B$  are matrices of order  $m \times n$  then:

$$K(A+B) = KA + KB$$

$$(K_1 + K_2)A = K_1A + K_2A$$

$$K_1K_2A = K_1(K_2A) = K_2(K_1A)$$

$$(-K)A = -(KA) = K(-A) \text{ also called as negative of a matrix.}$$

## MULTIPLICATION OF TWO MATRICES:

If  $A = [a_{ij}]$  of order  $m \times n$  and  $B = [b_{ij}]$  of order  $n \times p$  then the product  $C = [c_{ij}]$  will be a matrix of order  $m \times p$

Eg:  $A = \begin{bmatrix} 1 & 3 & -2 \\ 0 & 3 & -1 \end{bmatrix}$  of order  $3 \times 2$ ,  $B = \begin{bmatrix} 0 & 3 \\ -2 & -1 \\ 0 & 4 \end{bmatrix}$  of order  $2 \times 3$

Then,  $A \times B$

$$\begin{bmatrix} 1 & 3 & -2 \\ 0 & 3 & -1 \end{bmatrix} \begin{bmatrix} 0 & 3 \\ -2 & -1 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 1 \times 0 + 0 \times (-2) + (-2) \times 0 & 1 \times 3 + 0 \times (-1) + (-2) \times 4 \\ 0 \times 0 + 3 \times (-2) + (-1) \times 0 & 0 \times 3 + 3 \times (-1) + (-1) \times 4 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 0 & 0 & 0 & 3 & -8 \\ 0 & -6 & 0 & 0 & -7 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -5 \\ -6 & -7 \end{bmatrix}$$

NOTE: 1) If  $AB$  is defined then  $BA$  need not be defined.

2) If  $A, B$  are respectively  $m \times n, k \times l$  matrices, then both  $AB$  and  $BA$  are defined only if  $n=k$  and  $l=m$

3) If both  $A$  and  $B$  are square matrices of the same order, then both  $AB$  and  $BA$  are defined.

4) If  $AB$  and  $BA$  are both defined, it is not necessary that  $AB=BA$

5) If the product of the two matrices is a zero matrix, it is not necessary that one of the matrices is a zero matrix.

## PROPERTIES OF MULTIPLICATION OF MATRICES:

If  $A = [a_{ij}]$ ,  $B = [b_{ij}]$  and  $C = [c_{ij}]$  are three matrices of order  $m \times n$  then,

• **COMMUTATIVE LAW:**  $AB \neq BA$

• **ASSOCIATIVE LAW:**  $(AB)C = A(BC)$

• **DISTRIBUTIVE LAW:**  $A(B+C) = AB+AC$

$(A+B)C = AC+BC$ , whenever both sides of equality are defined.

• **EXISTENCE OF MULTIPLICATIVE IDENTITY:**

For every square matrix  $A$ , there exist an identity matrix of same order such that  $IA = AI = A$

• **CANCELLATION LAW:**

If  $A$  is non-singular matrix, then

•  $AB = AC \Rightarrow B = C$  (left cancellation law)

•  $BA = CA \Rightarrow B = C$  (Right cancellation law)

•  $AB = 0$  does not necessarily imply that  $A = 0$  or  $B = 0$  or both  $A$  and  $B = 0$

## TRANSPOSE OF A MATRIX:

If  $A = [a_{ij}]_{m \times n}$ , be a matrix of order  $m \times n$ . Then, the  $n \times m$  matrix obtained by interchanging the rows and columns of  $A$  is called the transpose of  $A$  and is denoted by  $'$  or  $A^T$ .

$$A' = A^T = [a_{ij}]_{n \times m}$$

Eq:

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} \quad A^T = \begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix}$$

## PROPERTIES OF TRANSPOSE:

1.  $(A')' = A$
2.  $(A+B)' = A' + B'$
3.  $(AB)' = B'A'$
4.  $(kA)' = kA'$
5.  $(AN)' = (A')N$
6.  $(ABC)' = C'B'A'$

## SYMMETRIC AND SKEW-SYMMETRIC MATRICES:

A square matrix  $A = [a_{ij}]$  is said to be symmetric if  $A' = A$ , that is  $[a_{ij}] = [a_{ji}]$  for all possible values of  $i$  and  $j$ .

Eg:  $A = \begin{bmatrix} \sqrt{3} & 2 & 3 \\ 2 & -1.5 & -1 \\ 3 & -1 & 1 \end{bmatrix}$  Here,  $A' = A$

A square matrix  $A = [a_{ij}]$  is said to be a skew-symmetric matrix if  $A' = -A$ , that is  $a_{ji} = -a_{ij}$  for all possible values of  $i$  and  $j$ .  
Now, if we put  $i = j$ , we have  $a_{ii} = -a_{ii}$ . Therefore,  $2a_{ii} = 0$  or  $a_{ii} = 0$  for all  $i$ 's.

This means that all the diagonal elements of a skew-symmetric matrix are zero.

For example:

The matrix  $B = \begin{bmatrix} 0 & e & f \\ -e & 0 & g \\ -f & -g & 0 \end{bmatrix}$  is a skew-symmetric matrix as  $B' = -B$

### THEOREM 1:

For any square matrix  $A$  with real number entities,  $A+A'$  is a symmetric matrix and  $A-A'$  is a skew symmetric matrix.

Proof: Let  $B = A+A'$  then:

$$\begin{aligned} B' &= (A+A')' \\ &\Rightarrow A'+(A')' \quad [\text{as } (A+B)' = A'+B'] \\ &= A'+A \quad [\text{as } (A')' = A] \\ &= A+A' \quad [\text{as } A+B = B+A] \\ &= B \end{aligned}$$

Therefore,  $B = A+A'$  is a symmetric matrix

Now let  $C = A-A'$

$$\begin{aligned} C' &= (A-A')' = A'-(A')' \\ &= A'-A \\ &= -(A-A') = -C \end{aligned}$$

$\therefore C = A-A'$  is a skew-symmetric matrix.

### THEOREM 2:

Any square matrix can be expressed as the sum of a symmetric and a skew-symmetric matrix.

Proof: Let  $A$  be a square matrix, then we can write:

$$A = \frac{1}{2}(A+A') + \frac{1}{2}(A-A')$$

From, theorem 1 we know that  $(A+A')$  is a symmetric matrix and  $(A-A')$  is a skew-symmetric matrix.

Since, for any matrix  $A$ ,  $(KA)' = KA'$ , it follows that  $\frac{1}{2}(A+A')$  is a symmetric matrix and  $\frac{1}{2}(A-A')$  is a skew-symmetric matrix. Thus, any square matrix can be expressed as the sum of a symmetric and a skew-symmetric matrix.

## ELEMENTARY OPERATIONS (TRANSFORMATION) OF A MATRIX:

- The six operations (transformations) on a matrix, three of which are due to rows and three due to columns.
- Interchanging any two rows (or columns), denoted by  $R_i \Leftrightarrow R_j$  or  $C_i \Leftrightarrow C_j$
- Multiplication of the element of any row (or column) by a non-zero quantity and denoted by  $R_i \Leftrightarrow kR_j$  or  $C_i \Leftrightarrow kC_j$
- Addition of constant multiple of the elements of any row to the corresponding element of any other row, denoted by  $R_i \rightarrow R_i + kR_j$  or  $C_i \rightarrow C_i + kC_j$   
OR  $R_i \Leftrightarrow R_i + kR_j$  or  $C_i = C_i + kC_j$

## INVERTIBLE MATRICES:

If  $A$  is a square matrix of order  $m$ , and if there exists another square matrix  $B$  of the same order  $m$ , such that  $AB = BA = I$ , then  $B$  is called the inverse matrix of  $A$  and is denoted by  $A^{-1}$ . In that case  $A$  is said to be invertible.

Eq:  $A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$  and  $B = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$  be two matrices

$$AB = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 4-3 & -6+6 \\ 2-2 & -3+4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$$BA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

## NOTE:

- A rectangular matrix (i.e. a matrix that is not square) does not have an inverse. This is because for the matrix products  $AB$  and  $BA$  to be defined and equal, both  $A$  and  $B$  must be square matrices of the same order.

- If  $B$  is the inverse of  $A$ , then  $A$  is also the inverse of  $B$  i.e.

$$AB = BA = I \Rightarrow B = A^{-1} \text{ and } A = B^{-1}$$

### INVERSE OF SQUARE MATRIX

Let  $A$  be the square matrix of order ' $n$ ', then a square matrix  $B$ , such that  $AB = BA = I$ , is called inverse of  $A$ , denoted by  $A^{-1}$

- $AA^{-1} = A^{-1}A = I$

### THEOREM 3 (UNIQUENESS OF INVERSE)

Inverse of square matrix, if it exists, is unique

i.e. let  $A = [a_{ij}]$  be a square matrix of order  $n$  and  $B$  and  $C$  be two inverses of  $A$ .

To prove:  $B = C$

Proof: Since  $B$  is the inverse of  $A$

$$AB = BA = I \dots \dots \textcircled{1}$$

Since,  $C$  is also the inverse of  $A$

$$AC = CA = I \dots \dots \textcircled{2}$$

$$\text{Thus; } B = BI = B(AC) = (BA)C = IC = C$$

# If  $A$  and  $B$  are invertible matrices of the same order, then  $(AB)^{-1} = B^{-1}A^{-1}$

Proof: From the definition of inverse of matrix, we have

$$(AB)(AB)^{-1} = I$$

$$\Rightarrow A^{-1}(AB)(AB)^{-1} = A^{-1}I \text{ (Pre multiplying both sides by } A^{-1}\text{)}$$

$$\Rightarrow (A^{-1}A)B(AB)^{-1} = A^{-1} \text{ (since: } A^{-1}I = A^{-1}\text{)}$$

$$\Rightarrow IB(AB)^{-1} = A^{-1}$$

$$\Rightarrow B(AB)^{-1} = A^{-1}$$

$$\Rightarrow B^{-1}B(AB)^{-1} = B^{-1}A^{-1}$$

$$\Rightarrow I(AB)^{-1} = B^{-1}A^{-1}$$

$$\text{Hence: } (AB)^{-1} = B^{-1}A^{-1}$$