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ADITYA RANJAN SIR
(EXCISE INSPECTOR)

Selected हैं Selection दिलायेंगे

INDEX

यथा शिखा मयूराणा, नागानां मणयो यथा।
तद्वद् वेदाङ्गशास्त्राणां गणितं मूर्धनि संस्थितम्॥

(महर्षि लगध कृत 'वेदांग ज्योतिष')

जैसे मोरों में शिखा और नागों में मणि का स्थान सबसे ऊपर है,
वैसे ही सभी वेदांग और शास्त्रों में गणित का स्थान सबसे ऊपर है।

Just as the crest on the heads of peacocks and the gems on the heads of serpents is in the highest position, in the same way the place of mathematics in the Vedangashastras is at the top.

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CHAPTER 02

MENSURATION 2D

Introduction

The literal meaning of the word Mensuration is 'to measure'. It is a branch of mathematics that deals with the measurement of perimeter, area and volume of the different geometrical figures.

The mensuration is divided in the following parts:

(i) Zero - Dimensional figure

(ii) One - Dimensional figure

(iii) Two - Dimensional figure

(iv) Three - Dimensional figure

Zero – Dimensional figure

- A zero Dimensional figure is a point having
  - No length, No width, No height
- In other words, a point is a dimensionless object that represents a precise location in space

One – Dimensional Mensuration (1D)

This involves the measurement of figures that have only one dimension, which is length.

- ***Line:*** A straight path extending infinitely in both directions. Its measure is its length.
- ***Line Segment:*** A part of line with the distinct end points. Its measure is the distance between the end points (length).
- ***Rays :*** A part of a line that starts at one end point and extends infinitely in one directions. Its measure is its length.

ONE Dimension

Length

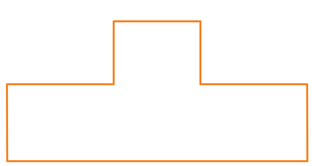
O 1m 1

Length = 1m

$$km \xrightarrow{\times 1000} m \xrightarrow{\times 100} cm \xrightarrow{\times 10} mm$$
$$km \xleftarrow{\div 1000} m \xleftarrow{\div 100} cm \xleftarrow{\div 10} mm$$

Two – Dimensional Mensuration (2D)

- This involves the measurement of flat figures that have two dimensions. Length and width (breadth). These figures lie in a plane, key measurements in 2D mensuration include.
- In two-dimensional mensuration we will study the two-dimensional figures (plane figures), like triangle, quadrilateral, polygon, circle etc.
- ***Perimeter :*** Perimeter can be defined as the path or the boundary that surrounds a figure . It can also be defined as the length of the outline of a shape.

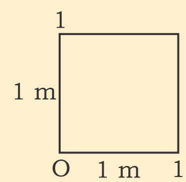


- ***Area*** : The area can be defined as the space occupied by a flat shape or the surface of an object.

The area of a figure is the number of unit squares that cover the surface of a closed figure. Area is measured in square units such as square centimeters, square meter, etc.

TWO Dimension

Area



Area = (1m) × (1m) = 1m²

$$km^2 \xrightarrow{\times 1000^2} m^2 \xrightarrow{\times 100^2} cm^2 \xrightarrow{\times 10^2} mm^2 \qquad hm^2 \xrightarrow{\times 100000^2} mm^2$$
$$km^2 \xleftarrow{\div 1000^2} m^2 \xleftarrow{\div 100^2} cm^2 \xleftarrow{\div 10^2} mm^2 \qquad hm^2 \xleftarrow{\div 100000^2} mm^2$$

- If each corresponding length of a 2D figure is multiplied by k then new perimeter P' is.

  P' = k × P
- If each corresponding length of a 2D figure is multiplied by k then new area A' is

  A' = k × Length × k × width

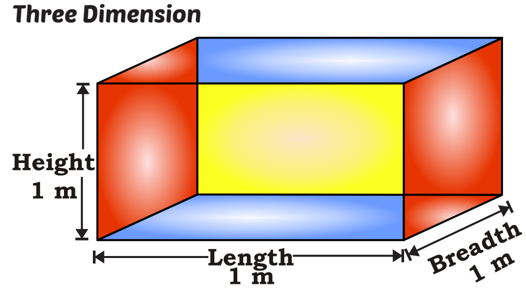
  ⇒ A' = k² (Length × width)

  ⇒ A' = k² × A

Three – Dimensional Mensuration (3D)

In three-dimensional mensuration we will study the three-dimensional figures like cube, cuboid, cylinder, cone, frustum, sphere, hemisphere, Prism, Pyramid etc.

Three Dimension



V = 1m × 1m × 1m = 1m³

$$km^3 \xrightarrow{\times 1000^3} m^3 \xrightarrow{\times 100^3} cm^3 \xrightarrow{\times 10^3} mm^3$$
$$km^3 \xleftarrow{\div 1000^3} m^3 \xleftarrow{\div 100^3} cm^3 \xleftarrow{\div 10^3} mm^3$$

Unit conversion for mass

- 1 Kilogram = 1000g = 10^3
- 1 Hectogram = 100g = 10^2
- 1 Decagram = 10g = 10^1
- 1 Decigram = 0.1g = 10^{-1}
- 1 Centigram = 0.01g = 10^{-2}
- 1 Milligram = 0.001g = 10^{-3}

Unit conversion for length

- 1 Millimeter = 0.001meter
- 1 Centimeter = 0.01 meter
- 1 Decimeter = 0.1 meter
- 1 Hectometer = 100 meter
- 1 Kilometer = 1000 meters
- 1 Inch = 2.54×10^{-2} meters
- 5 Mile = 8 km
- 1 feet = 0.3048 meters

Some Important Conversion For Mensuration

Conversion of units for area

- 1 Sq. mile = 2.5899×10^6 square meter
- 1 Hectare = 1×10^4 square meter
- 1 Acre = 4.0468×10^3 square meter
- 1 Sq. foot = 9.92903×10^{-2} square meter
- 1 Sq. inch = 6.4516×10^{-4} square meter

Some other unit conversion

- 1 Pound = 16 ounces
- 1 Ton = 2000 pounds

Relation between Mass, density & volume

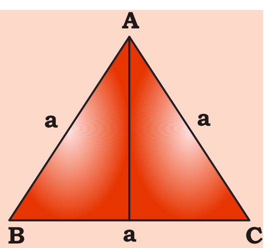
Mass = Density × Volume

$$\text{Density} = \frac{\text{Mass}}{\text{Volume}}$$

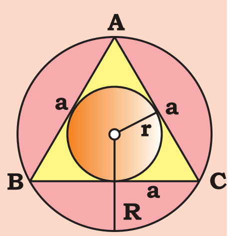
Triangle

Equilateral Triangle

An Equilateral Triangle is a triangle in which three sides are equal in lengths and all interior angles are 60°



- Perimeter = 3a
- Semi-perimeter (s) = $\frac{3a}{2}$
- Height (h) = $\frac{\sqrt{3}a}{2}$
- Area = $\frac{\sqrt{3}}{4}a^2 = \frac{h^2}{\sqrt{3}}$
- In-radius (r) = $\frac{a}{2\sqrt{3}} = \frac{h}{3}$
- Area of the incircle = $\frac{\pi a^2}{12}$
- Circum-radius (R) = $\frac{a}{\sqrt{3}}$
- Area of circum-circle = $\frac{\pi a^2}{3}$
- $\frac{\text{Radius of in-circle (r)}}{\text{Radius of circum-circle (R)}} = \frac{1}{2}$
- $\frac{\text{Area of in-circle}}{\text{Area of circum-circle}} = \frac{1}{4}$
- The ratio of r : a : R = 1 : $2\sqrt{3}$: 2



Side	Height	Area
2	$\sqrt{3}$	$\sqrt{3}$
$\downarrow \times k$	$\downarrow \times k$	$\downarrow \times k^2$
2k	$\sqrt{3}k$	$\sqrt{3}k^2$
$\downarrow$	$\downarrow$	$\downarrow$
$(2 \times 3 = 6)$	$(\sqrt{3} \times 3 = 3\sqrt{3})$	$(\sqrt{3} \times 3^2 = 9\sqrt{3})$

Ex. The length of each side of an equilateral triangle is 22 cm. Find the area (in cm²) of this triangle.

HINTS Area of equilateral triangle = $\frac{\sqrt{3}}{4}a^2$

$= \frac{\sqrt{3}}{4} \times 22 \times 22 = 121\sqrt{3}$ cm²

Alternatively

Side	Area
2	$\sqrt{3}$
$\downarrow \times 11$	$\downarrow \times 11^2$
22	$121\sqrt{3}$

Ex. The altitude of an equilateral triangle is $3\sqrt{3}$ cm. Find its area.

HINTS Height of equilateral Δ = $\frac{\sqrt{3}}{2}a \Rightarrow 3\sqrt{3} = \frac{\sqrt{3}}{2}a$

$\Rightarrow a = 6$

$\therefore$ Area of equilateral Δ = $\frac{\sqrt{3}}{4}a^2 = \frac{\sqrt{3}}{4} \times 36 = 9\sqrt{3}$ cm²

Alternatively

$\therefore$ Area of equilateral Δ = $\frac{h^2}{\sqrt{3}} = \frac{(3\sqrt{3})^2}{\sqrt{3}} = \frac{27}{\sqrt{3}} = 9\sqrt{3}$ cm²

Ex. Height of an equilateral triangle is $12\sqrt{5}$ cm, then find the difference between the areas of the circumcircle and the incircle?

HINTS Height of equilateral triangle = $\frac{\sqrt{3}a}{2}$

$\Rightarrow \frac{\sqrt{3}}{2}a = 12\sqrt{5} \Rightarrow a = \frac{24\sqrt{5}}{\sqrt{3}}$

Difference between area of circumcircle and incircle

$= \pi a^2\left(\frac{1}{3} - \frac{1}{12}\right) = \pi \times \frac{576 \times 5}{3} \times \frac{1}{4} = 240\pi$ cm²

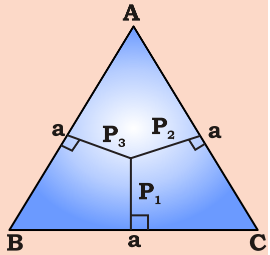
☞ If P_1, P_2 and P_3 are the lengths of the perpendiculars drawn from a point inside an equilateral triangle to its sides then,

(a) $P_1+P_2+P_3=\frac{\sqrt{3}a}{2}=h\text{(height)}$,

$a=\frac{2}{\sqrt{3}}(P_1+P_2+P_3)$

(b) Area of the equilateral triangle

$=\frac{(P_1+P_2+P_3)^2}{\sqrt{3}}$



Ex. 'P' is point inside an equilateral triangle and if lengths of perpendiculars drawn on each side of triangle from 'P' are 2 cm, 3 cm and 4 cm. Find area of equilateral triangle.

HINTS Area of equilateral triangle $=\frac{(P_1+P_2+P_3)^2}{\sqrt{3}}$

$=\frac{(9)^2}{\sqrt{3}}=27\sqrt{3}$ cm²

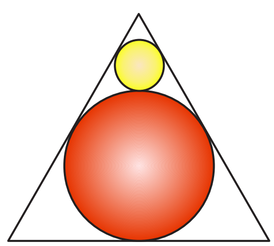
☞ If the length of each side of an equilateral triangle is increased by x then area is found to be increased by y.

$\therefore \frac{\sqrt{3}}{4}(a+x)^2=\frac{\sqrt{3}}{4}a^2+y$

Where $y = x^2 + 2ax$

☞ In equilateral triangle, two circle of different radii are placed inside triangle then

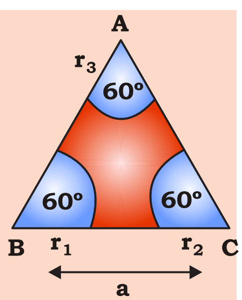
$\frac{\text{Radius of smaller circle}}{\text{Radius of bigger circle}}=\frac{1}{3}$



☞ If 3 cows are tied to each corner of an equilateral triangular field of side 'a' cm with ropes of lengths r_1, r_2 and r_3 cm, then

Area grazed by the cows

$=\frac{\pi}{6}(r_1^2+r_2^2+r_3^2)$



Area ungrazed by the cows $=\frac{\sqrt{3}a^2}{4}-\frac{\pi}{6}(r_1^2+r_2^2+r_3^2)$

Ex. If 3 cows are tied to each corner of an equilateral triangular field with rope length 7cm each then find the total area grazed by the cows?

HINTS

Area grazed by the cows $=\frac{\pi}{6}(r_1^2+r_2^2+r_3^2)$

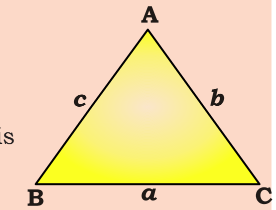
$=\frac{22}{7\times 6}\times(7^2+7^2+7^2)=\frac{22}{42}\times 147=77$ cm²

Scalene triangle

A Scalene triangle is a triangle that has three unequal sides and three unequal angles

- Perimeter = $a + b + c$
- Semi-perimeter (s) = $\frac{a+b+c}{2}$
- Area : The area of a triangle is denoted by the symbol Δ.

  Area = $\sqrt{s(s-a)(s-b)(s-c)}$
- Inradius (r) = $\frac{\Delta}{s}$
- Circumradius (R) = $\frac{abc}{4\Delta}$



Ex. The lengths of the sides of a triangle are 5 cm, 7 cm and 10 cm. Find the area (in cm²) of the triangle.

HINTS

Semi-perimeter (s) = $\frac{a+b+c}{2}=\frac{5+7+10}{2}=11$

By Heron's formula,

Area of triangle = $\sqrt{s(s-a)(s-b)(s-c)}$

$=\sqrt{11(11-5)(11-7)(11-10)}$

$=\sqrt{11\times 6\times 4\times 1}=2\sqrt{66}$ cm²

Ex. The area of triangle is 15 sq cm and the radius of its incircle is 3 cm. Its perimeter is equal to:

HINTS 15 = 3 × s ⇒ s = 5cm

∴ Perimeter = 5 × 2 = 10 cm

☞ ***Special case when sides are 13, 14, 15***

Ex. For a triangle having sides 13cm, 14cm, 15cm.

(i) Area of the triangle.

(ii) Inradius of the triangle.

HINTS

Semi - perimeter (s) = $\frac{13+14+15}{2}=21$cm

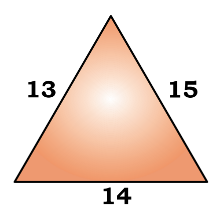
(i) Area of Triangle

$=\sqrt{21(21-13)(21-14)(21-15)}$

$=\sqrt{21\times 8\times 7\times 6}=\sqrt{7\times 3\times 8\times 7\times 6}$

= 7 × 12 = 84 cm²

(ii) Inradius (r) $=\frac{\Delta}{s}=\frac{84}{21}=4$

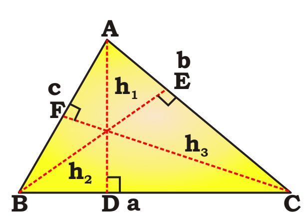


☞ Area = $\frac{1}{2}$ × side × corresponding height

$\Delta=\frac{1}{2}\times BC\times AD=\frac{1}{2}\times AC\times BE=\frac{1}{2}\times AB\times CF$

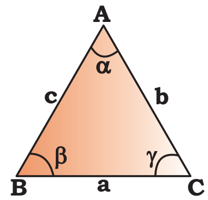
$\therefore ah_1 = bh_2 = ch_3$ = (constant)

$\boxed{\therefore a:b:c=\frac{1}{h_1}:\frac{1}{h_2}:\frac{1}{h_3}}$



☞ When two sides of any triangle and angle between them is given then,

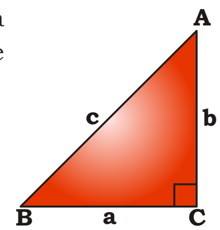
$$\text{Area} = \frac{1}{2}\times bc\sin\alpha = \frac{1}{2}\times ca\sin\beta$$
$$= \frac{1}{2}\times ab\sin\gamma$$



☞ A triangle with two given sides has a maximum area if these two sides are placed at right angles to each other.

If a, b are two sides of a triangle

$$\text{Max. Area} = \frac{1}{2}ab$$

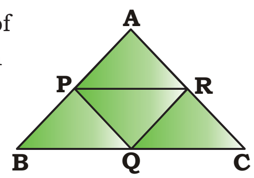


☞ The perimeter and the area of a triangle made by joining the mid-points of the sides will be half of original perimeter and one-fourth of the original area respectively.

If P, Q and R be the mid-point of AB, BC and AC, respectively, then

(a) Perimeter of ΔPQR
$= \frac{1}{2}$ × Perimeter of ΔABC

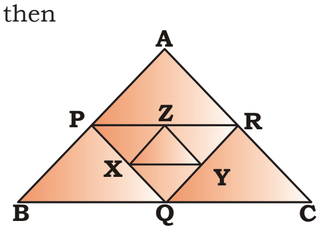
(b) If the area of ΔABC is Δ, then
area of ΔPQR = $\frac{\Delta}{4}$



☞ Let a triangle PQR is formed by joining the mid-points of the sides of ΔABC, then again a ΔXYZ is formed by joining the mid-points of the sides of ΔPQR, if this process continue till infinite, then

(a) Area of all triangles
$= \frac{4}{3}$ × Area of ΔABC

(b) Perimeter of all triangles
= 2 × Perimeter of ΔABC



Ex. Consider an equilateral triangle of a side of unit length. A new equilateral triangle is formed by joining the mid-points of one, then a third equilateral triangle is formed by joining the mid-point of seconds. The process is continued. The perimeter of all triangle, thus formed is:

HINTS Sum of all perimeter

= 2 × perimeter of original triangle

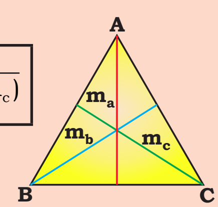
= 2 × 3 × 1 = 6 unit

☞ If three medians of a triangle – m_a, m_b, m_c are given

Area of triangle is

$$\boxed{\Delta = \frac{4}{3}\sqrt{S_m(S_m - m_a)(S_m - m_b)(S_m - m_c)}}$$

Where, $S_m = \dfrac{m_a + m_b + m_c}{2}$



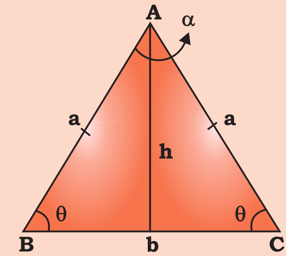
☞ If $m_a^2 = m_b^2 + m_c^2$, then

$$\boxed{\text{Area of } \Delta = \frac{2}{3}m_b\, m_c}$$

Isosceles Triangle

An Isosceles triangle is a triangle that has two sides of equal length and two angles of equal measure.

- Side (AB = AC) = a
- Perimeter = 2a + b
- Area = $\frac{b}{4}\sqrt{4a^2 - b^2} = \frac{1}{2}a^2\sin\theta$
- Height
$$= \sqrt{a^2 - \frac{b^2}{4}} = \frac{\sqrt{4a^2 - b^2}}{2} = \frac{a^2}{2R}$$
- Circum radius (R) = $\dfrac{a^2}{\sqrt{4a^2 - b^2}}$
- Base = $\frac{a}{R}\sqrt{(2R + a)(2R - a)}$



Ex. Find the area of an isosceles triangle whose sides are 8 cm, 5 cm and 5 cm.

HINTS Area = $\frac{8}{4}\sqrt{4\times5^2 - 8^2} = 2\sqrt{(36)} = 12$ cm²

Ex. The perimeter of an isosceles triangle is 100 cm. If the base is 36 cm, then find its semi perimeter (in centimeters).

HINTS Semi-perimeter = $\frac{100}{2}$ = 50 cm

Ex. The perimeter of an isosceles triangle is 91 cm. If one of the equal sides measures 28 cm, then what is the value of the other non-equal side?

HINTS One equal side = 28cm

Perimeter of isosceles triangle = 91

⇒ 28 + 28 + a = 91 ⇒ a = 35 cm

Ex. The ratio of the length of each equal side and the third side of an isosceles triangle is 3:5. If the area of the triangle is 30√11 cm² then the length of the third side (in cm) is:

HINTS Let, length of equal sides = $3x$

Length of Third side = $5x$

$$AD = \sqrt{9x^2 - \frac{25x^2}{4}}$$
$$= \sqrt{\frac{36x^2 - 25x^2}{4}} = \frac{x\sqrt{11}}{2}$$

ATQ,

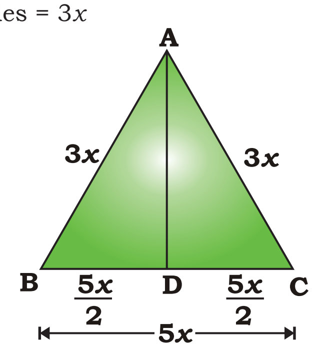
Area of triangle $= 30\sqrt{11}$ cm²

$$\Rightarrow \frac{1}{2}\times 5x \times \frac{x\sqrt{11}}{2} = 30\sqrt{11}$$

$$\Rightarrow x = 2\sqrt{6}$$

∴ BC = $5x = 5\times 2\sqrt{6} = 10\sqrt{6}$ cm

∴ Length of the third side $= 10\sqrt{6}$ cm

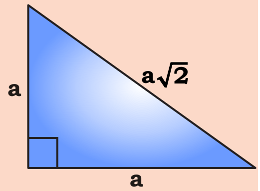


Isosceles-right angle triangle

- Perimeter (P) = $a(2 + \sqrt{2})$

  $= a\sqrt{2}(\sqrt{2}+1)$

- Area = $\dfrac{a^2}{2}$



Ex. If the perimeter of an isosceles right triangle is $8\left(\sqrt{2}+1\right)$ cm , then the length of the hypotenuse of the triangle is:

HINTS $8\left(\sqrt{2}+1\right) = a \times \sqrt{2}(\sqrt{2}+1) \Rightarrow a = 4\sqrt{2}$

$\therefore$ Hypotenuse = $a\sqrt{2} = 4 \times 2 = 8$ cm

Ex. The area of an isosceles right angled triangle is 16 m². Its hypotenuse is _____.

HINTS Area of isosceles right angle triangle = $\dfrac{a^2}{4}$

$\Rightarrow \dfrac{a^2}{4} = 16 \Rightarrow a^2 = 64 \Rightarrow a = 8$

Ex. The perimeter of an isosceles, right-angled triangle is 2p unit. The area of the same triangle is-

HINTS Let, side of isosceles right angle Δ = AB = AC = a

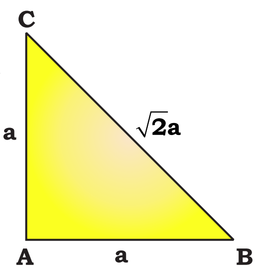
$\Rightarrow BC = \sqrt{2}a$

$\because AB + AC + BC =$ perimeter of ΔABC

$\Rightarrow 2p = 2a + \sqrt{2}a$

$\Rightarrow p = a + \dfrac{a}{\sqrt{2}}$

Area = $\dfrac{1}{2} \times a \times a = \dfrac{1}{2}a^2$



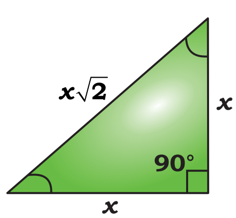
$$= \frac{1}{2}\left(\frac{p}{\left(1+\frac{1}{\sqrt{2}}\right)}\right)^2 = \frac{p^2}{\left(\sqrt{2}+1\right)^2} = \frac{p^2(2+1-2\sqrt{2})}{1} = p^2(3-2\sqrt{2})$$

Ex. In an isosceles right triangle, the perimeter is 30 m. Find its area (in m²). (Rounded off to the nearest integral value)

HINTS Perimeter = $x + x + x\sqrt{2}$

$\Rightarrow 2x + x\sqrt{2} = 30$

$\Rightarrow x = \dfrac{30}{2+\sqrt{2}} = \dfrac{30 \times \left(2-\sqrt{2}\right)}{2}$



$= 15\left(2-\sqrt{2}\right)$

$\Rightarrow$ Area of triangle = $\dfrac{1}{2} \times \left[15(2-1.41)\right]^2$

$= \dfrac{225 \times 0.59 \times 0.59}{2} \approx 39$ m²

Alternatively

$2p = 30m \Rightarrow p = 15m$

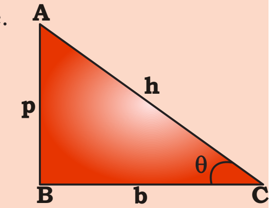
Area of triangle = $p^2(3-2\sqrt{2}) = 15^2(3-2\sqrt{2})$

$= 225 \times 0.172 \sim 39$ m²

Right-Angled Triangle

A right angled triangle is a triangle in which two sides are perpendicular, forming a right angle.

- $p^2 + b^2 = h^2$
- Perimeter = p + b + h
- Area of triangle = $\dfrac{1}{2} \times p \times b$



- Area of right angle triangle

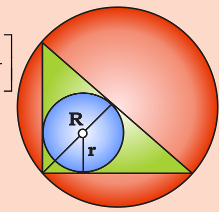
  $= \dfrac{h^2}{4}\sin 2\theta$

- Inradius (r) = $\dfrac{p+b-h}{2}$ 'or'

  $\left[\dfrac{\text{Perpendicular} + \text{Base} - \text{Hypotenuse}}{2}\right]$

- Circumradius (R) = $\dfrac{h}{2}$

  'or' $\left[\dfrac{\text{Hypotenuse}}{2}\right]$



Ex. If the area of a triangle whose base measures 6 cm is 18 cm², then its height is:

HINTS Area of triangle = $\dfrac{1}{2}$ × Base × Height

$\Rightarrow 18 = \dfrac{1}{2} \times 6 \times$ Height

$\Rightarrow$ Height = 6 cm.

Ex. One of the angles of a right-angled triangle is 15° and the hypotenuse is 1 m. The area of the triangle (in sq. cm.) is

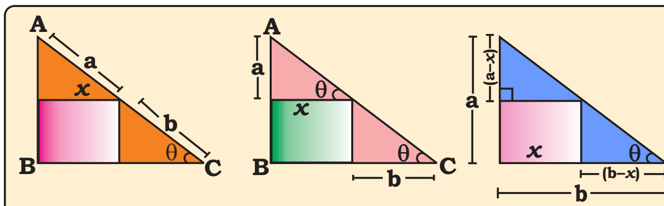
HINTS Area = $\dfrac{h^2}{4}\sin 2\theta$

$= \dfrac{100 \times 100}{4} \times \sin 30° = 1250$ cm²

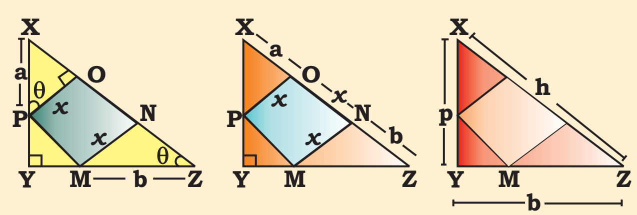
Ex. If hypotenuse of a right angle Δ is 10 cm. What can be its maximum area?

HINTS $\text{Area}_{max} = \dfrac{10 \times 10}{4} = 25$ cm²

Square Inside Right Angle Triangle



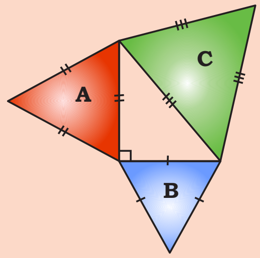
$x = \dfrac{ab}{\sqrt{a^2+b^2}}$ | $x = \sqrt{ab}$ | $x = \dfrac{ab}{a+b}$



$x = \dfrac{ab}{\sqrt{a^2+b^2}}$ | $x = \sqrt{ab}$ | $x = \dfrac{p \times b \times h}{p^2+b^2+h^2}$

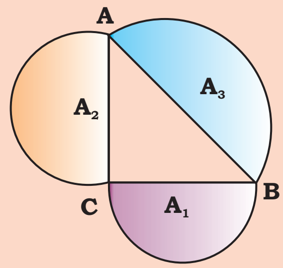
Area of Shaded region

Formation of Equilateral triangle on each sides of right angle triangle



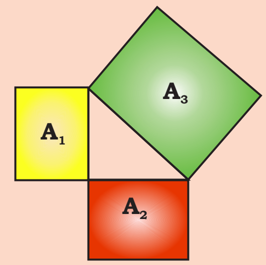
Area (ΔC) = Area (ΔA) + Area (ΔB)

Formation of Semicircle on each sides of right angle triangle



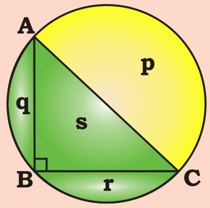
Area (A_3) = Area (A_1) + Area (A_2)

Formation of Square on each sides of right angle triangle

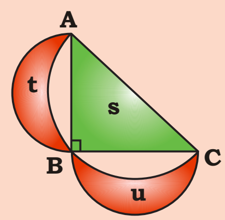


Area(A_1) + Area (A_2) = Area (A_3)

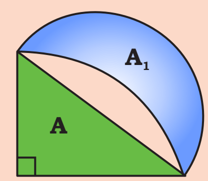
- Area of shaded region (p) = sum of area of region (q), (r) and (s)



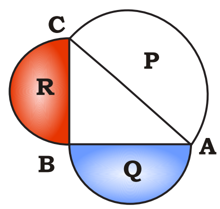
- Area of shaded region (s) = sum of area of region (t) and (u)



- Area of shaded region (A_1) = Area of triangle (A)



Ex. Find the area of shaded region, where AC = 10 cm.

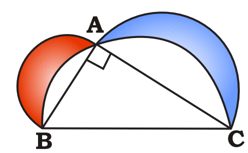


HINTS Total required area

= Area of semi circle formed on hypotenuse

$= \frac{\pi \times 10^2}{4} = 25\pi \text{ cm}^2$

Ex. In the given figure, 3 semicircles are drawn on three sides of ΔABC. If AB = 21 cm, AC = 28 cm and BC = 35 cm. What is the area (in cm²) of the shaded part?



HINTS

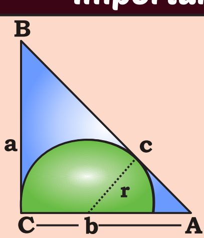
Area of shaded part = Area of ΔABC

$= \frac{1}{2} \times 21 \times 28 = 294 \text{ cm}^2$

Important results

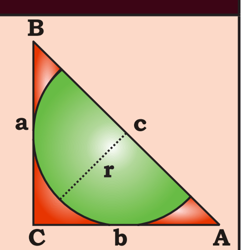
☞ When a semicircle is inscribed in a right angled triangle, as shown in the figure. Then–

$$\boxed{\text{Radius}(r) = \frac{ab}{a+c}}$$



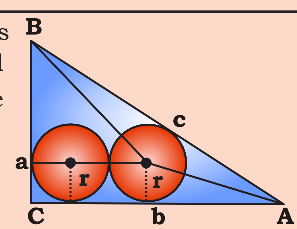
☞ When a semicircle of r-radius is inscribed in a right angled triangle lies on hypotenuse as shown in the figure then

$$\boxed{\text{Radius}(r) = \frac{ab}{a+b}}$$



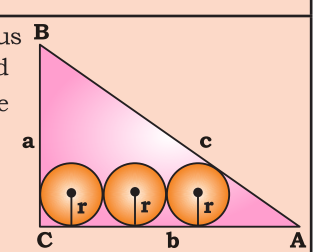
☞ When two circles of equal radius is inscribed in a right angled triangle as shown in the figure then

$$\boxed{\text{Radius}(r) = \frac{ab}{3a+b+c}}$$



☞ When three circles of equal radius is inscribed in a right angled triangle as shown in the figure then

$$\boxed{\text{Radius}(r) = \frac{ab}{5a+b+c}}$$

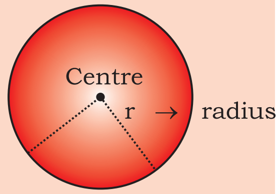


Circle

A circle is closed 2D figure in which the set of all the points in the plane is equidistance from a given point called centre.

- Diameter (d) = 2r = $\sqrt{\frac{4A}{\pi}}$

  Where, r = radius, A = Area
- Area of Circle = πr^2
- Circumference of circle = $2\pi r$



Radius	Circumference = $2\pi r$	Area = πr^2
7	44	154
↓	↓	↓
7k	44k	$154k^2$
↓	↓	↓
(14 = 7×2)	44×2	154×2^2
↓	↓	↓
$\left(3.5 = 7\times\frac{1}{2}\right)$	$44\times\frac{1}{2} = 22$	$154\times\frac{1}{4} = 38.5$

Ex. The area of a circle is 1386 cm². What is the radius of the circle? [Take π = 22/7]

HINTS Area of circle = πr^2

$\Rightarrow 1386 = \frac{22}{7}\times r^2$

$\Rightarrow r^2 = \frac{1386\times7}{22}$

$\Rightarrow r^2 = 441$

$\Rightarrow r = 21$cm

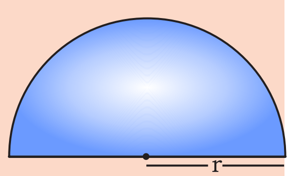
Alternatively

Radius	Area
7	154
↓ ×3	↓ $\times3^2$
21	1386 cm²

Semi-circle

A semi-circle is defined as a half circle formed by cutting the circle into two halves. It is formed when a line passes through the centre and touches the two ends of the circle.

- Area of Semi-circle = $\frac{\pi r^2}{2}$
- Circumference of Semi-circle = $\pi r + 2r$



Radius	Circumference	Area
7	36	77
	↓	↓
	$2r + \pi r$	$\frac{\pi r^2}{2}$
	$= 2\times7+\frac{22}{7}\times7$	$= \frac{22}{7}\times\frac{7\times7}{2}$
	= 36	= 77

Ex. A Semi circle sheet has a radius of 21 cm find its Circumference.

HINTS

Radius	Circumference
7	36
↓ ×3	↓ ×3
21	108

Ex. Find the area of a semicircle with a radius of 35 cm.

HINTS

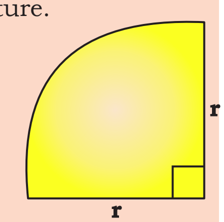
Radius	Area
7	77
↓ ×5	↓ $\times5^2$
35	1925 cm²

Quadrant

It is one fourth section of a circle. It is formed by a set of two lines which are perpendicular in nature.

- Area of quadrant of circle = $\frac{1}{4}\pi r^2$
- Circumference of quadrant of circle

  $= \frac{\pi r}{2} + 2r$



Radius	Circumference	Area
7	25	$\frac{154}{4} = 38.5$
	↓	↓
	$2r + \frac{2\pi r}{4}$	$\frac{\pi r^2}{4}$
	$= 2\times7+2\times\frac{22}{7}\times\frac{7}{4}$	$= \frac{22}{7}\times\frac{7^2}{4}$
	= 25	= 38.5

Ex. What is the area of a Quadrant of a circle with radius 14 cm?

HINTS Area = $\frac{1}{4}\pi r^2 = \frac{1}{4}\times\frac{22}{7}\times14^2 = 154$ cm²

Alternatively

Radius	Area
7	38.5
↓ ×2	↓ $\times2^2$
14	154

Ex. Find the perimeter (circumference) of a quadrant of a circle with radius 21 cm.

HINTS Perimeter of a Quadrant = $\frac{\pi r}{2} + 2r$

$= \frac{22}{7}\times\frac{21}{2} + 2\times21 = 33 + 42 = 75$ cm²

Alternatively

Radius	Circumference
7	25
↓ ×3	↓ ×3
21	75

Sector

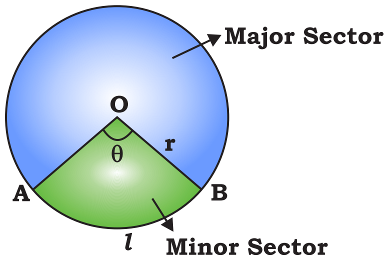
A Sector is a portion of a circle enclosed between two radii and the arc.

- **r** = radius of the circle.
- **θ** = Central angle of the sector (In Degree or radians)
- **Arc length (*l*)** = the length of the curved boundary.
- **Area (A)** = The area covered by the sector.

Major Sector

If $\theta > 180°$

- Area = $\dfrac{\theta}{360°} \times \pi r^2$
- Arc length = $\dfrac{\theta}{360°} \times 2\pi r$
- Area of sector = $\dfrac{1}{2} \times l\, r$



Minor Sector

If $\theta < 180°$

- Area = $1 - \dfrac{\theta}{360°} \times \pi r^2$
- Arc length = $1 - \dfrac{\theta}{360°} \times 2\pi r$

Ex. If length of the arc = 6 cm and radius of circle = 5 cm. Find area of sector of a circle.

HINTS Area = $\dfrac{1}{2} \times 6 \times 5 = 15 \text{ cm}^2$

Ex. Find the area of the sector of a circle of radius 7 cm with a central angle of 90°.

HINTS Area = $\dfrac{\pi r^2}{360°} \times 90° = \dfrac{\pi \times 7 \times 7}{4} = \dfrac{77}{2} = 38.5 \text{ cm}^2$

"**Central angle determines if a sector is major or minor.**"

Segment

A Segment is the area between chrod and the corresponding arc.

Minor Segment

If θ is in degree:-

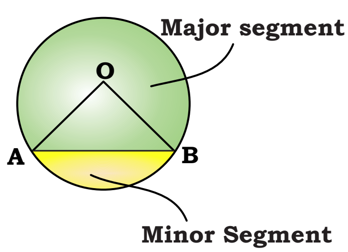
- Area of minor Segment

$= r^2\left[\dfrac{\pi\theta}{360°} - \dfrac{\sin\theta}{2}\right]$

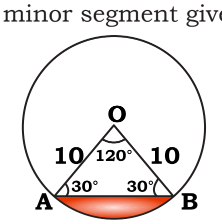
If θ is in radian:-

- Area of minor Segment

$= \dfrac{r^2}{2}(\theta - \sin\theta)$



Ex. Find the area of minor segment given in below figure.



HINTS Area of minor segment = Area of sector – Area of ΔOAB

$= \dfrac{\pi r^2\theta}{360^0} - \dfrac{1}{2}r^2\sin\theta = \pi 10^2 - \dfrac{1}{2} \times 10^2 \times \sin 120°$

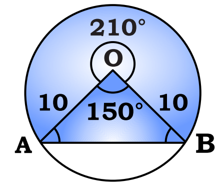
$= \left(\dfrac{100\pi}{3} - 25\sqrt{3}\right) \text{cm}^2$

Major Segment

Area of major segment = $\left(\dfrac{\pi r^2\theta}{360°} + \dfrac{1}{2}r^2\sin\theta\right)$

Perimeter of segment = $2r\left[\dfrac{\pi\theta}{360°} + \dfrac{\sin\theta}{2}\right]$

Ex. Find the area of major segment in the given figure.



HINTS Area of major segment

= area of sector (210°) + Area of ΔOAB

$= \dfrac{\pi r^2\theta}{360^0} + \dfrac{1}{2}r^2\sin\theta$

$= \pi 10^2\,\dfrac{210°}{360°} + \dfrac{1}{2} \times 10^2 \times \sin 150° = \left(\dfrac{175\pi}{3} + 25\right) \text{cm}^2$

Ring

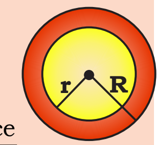
If R and r are radii of two concentric circles, then

- Area enclosed by two circle

$= \pi(R + r)(R - r)$

- Width of path

$= \dfrac{\text{outer circumference} - \text{inner circumference}}{2\pi}$



Ex. The area enclosed between the concentric circles is 770 cm² and the radius of the outer circle is 21 cm, then find the radius of the inner circle.

HINTS Area enclosed between two circle = $\pi(R^2 - r^2)$

$\Rightarrow \dfrac{22}{7}(21^2 - r^2) = 770 \text{ cm}^2 \Rightarrow r = \sqrt{196} = 14 \text{ cm}$

Ex. The area of a circular path enclosed by two concentric circles is 3080 m². If the difference between the radius of the outer edge and that of inner edge of the circular path is 10 m, what is the sum (in m) of the two radii?

HINTS $\pi(R^2 - r^2) = 3080$

$\Rightarrow \dfrac{22}{7}(R + r)(R - r) = 3080 \Rightarrow R + r = 98 \text{ cm}$

Wheel

The number of revolutions completed by a rotating wheel in one minute = $\dfrac{\text{Distance covered in one minute}}{\text{Circumference}}$

Ex. An athlete runs 8 times around a circular field of radius 7 m in 3 minutes 40 seconds. His speed (in km/h) is:

HINTS Total distance covered in 8 rounds

$= 2 \times \dfrac{22}{7} \times 7 \times 8 = 352\text{m} = \dfrac{352}{1000}\text{km}$

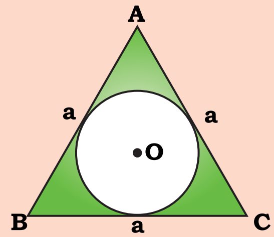
Total time = 3 min 40 sec = 220 sec = $\dfrac{220}{3600}\text{hrs} = \dfrac{11}{180}\text{hrs}$

Speed = $\dfrac{352}{1000} \times \dfrac{180}{11} = \dfrac{144}{25}$ km/hr.

Important results

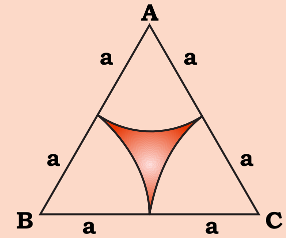
☞ Area of shaded region

$= \frac{\sqrt{3}}{4}a^2 - \pi\left(\frac{a^2}{12}\right)$



☞ Area of shaded region

$= a^2\left(\sqrt{3} - \frac{11}{7}\right)$

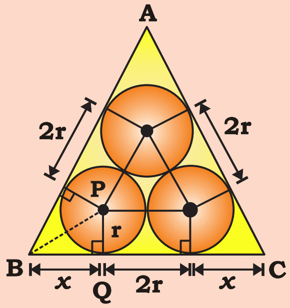


☞ An equilateral triangle circumscribes all the three circles each of radius 'r' then

Side = 2r ($\sqrt{3}$ + 1)

Perimeter of triangle

= 3 × side = 6r($\sqrt{3}$ + 1)

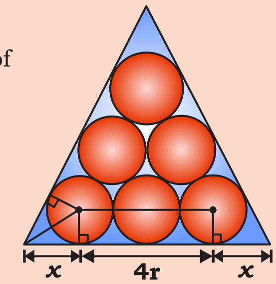


☞ An equilateral triangle circumscribes all the six circles each of radius 'r' then

Side = 2r($\sqrt{3}$ + 2)

Perimeter of triangle

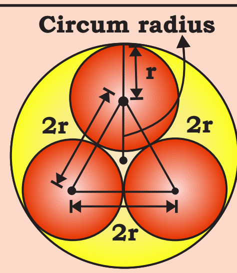
= 6r($\sqrt{3}$ + 2)



☞ Three equal circle each of radius r are circumscribed by a larger circle.

Radius of larger circle

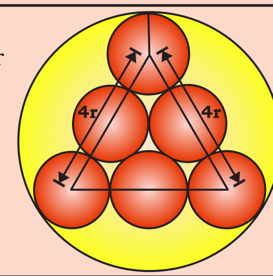
$= r \times \left(\frac{2+\sqrt{3}}{\sqrt{3}}\right)$



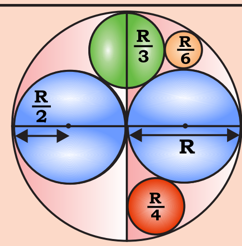
☞ Six equal circle each of radius r are circumscribed by a larger circle.

∴ Radius of larger circle

$= \frac{r}{\sqrt{3}}(\sqrt{3}+4)$



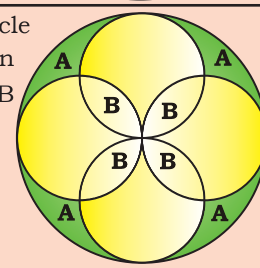
☞ According to figure, if the radius of outer circle is R then the radius of each circle:



☞ As per the given figure, a big circle and 4 small circles are given inside it, then the ratio of A and B will 1:1.

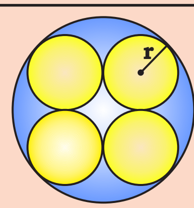
A = B

A : B = 1 : 1



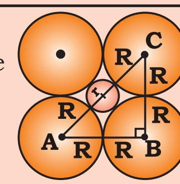
☞ In the given figure all the inner circle have the radius 'r' and then the radius of bigger (outer) circle is.

$R = \left(\sqrt{2}+1\right) r$



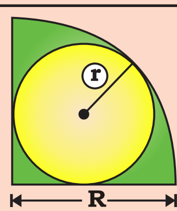
☞ when all the outer circles have radii 'R' then the radius of the inner circle will be

$r = (\sqrt{2}-1)R$



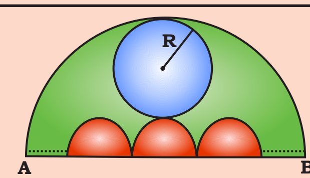
☞ When a circle is inscribed in a quadrant as shown in the figure. Then

$r = (\sqrt{2}-1)R$



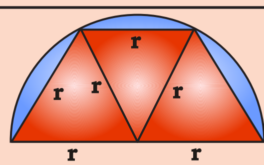
☞ $R = \frac{n(AB)}{4(n+1)}$

n = no of semi circle



☞ Total area of three equilateral triangle inscribed in a semi-circle of radius 'r' cm is

$\text{Area} = 3 \times \frac{\sqrt{3}}{4}r^2 = \frac{3\sqrt{3}}{4}r^2$



☞ According to the figure two semicircles are inscribed in a quadrant.

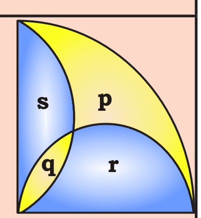
(i) Area of region (p) = Area of region (q)

(ii) Area of shaded region (r)

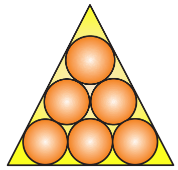
= Area of shaded regions (s)

(iii) Ratio of the area of the region (p), (q), (r), (s) respectively is

p : q : r : s = 4 : 4 : 7 : 7

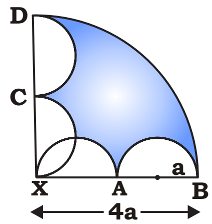


Ex. An equilateral triangle circumscribes all the circles, each with radius 10 cm. What is the perimeter of the equilateral triangle?



HINTS Perimeter = $3 \times 2 \times 10\ (\sqrt{3}+2) = 60\,(\sqrt{3}+2)$ cm

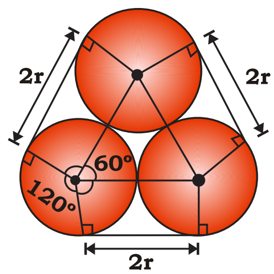
Ex. XBD is quadrant of a circle where, XB = 4a cm, XA = AB = XC = CD. Four semi-circles are drawn taking XA , AB , XC and CD as diameter. Find the area of the shaded region.



HINTS XB = 4a cm

Area of shaded region = Area of quadrant of radius (4a) – 4 × Area of Semicircle of radius (a) + Area of leaf of radius (a) = $\frac{48}{7}a^2$

Ex. Three circular rings each of equal radii r cm are touching each other. A string runs all around the set of rings very trightly. What is the minimum length of string required to bind all the three rings in the given manner?



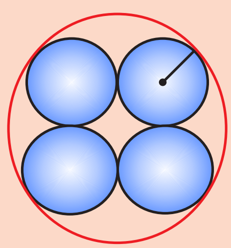
HINTS

Length of three arc = $3 \times \left(\frac{2\pi r}{360°} \times 120°\right) = 2\pi r$

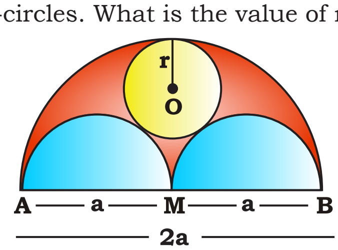
∴ Length of string = $2\pi r$ + Diameter × No. of circles

$= 2\pi r + 3 \times 2r = 2r(\pi + 3)$

☞ Four circular rings each of equal radii r cm are touching each other. A string runs all around the set of rings very tightly. Then the minimum length of string = $8r + 2\pi r$



Ex. In the figure given below, AB is line of length 2a, with M as mid-point. Semi-circles are drawn on one side with AM, MB and AB as diameters. A circle with centre O and radius r is drawn such that this circle touches all the three semi-circles. What is the value of r ?



HINTS $2R = 2a \Rightarrow R = a$

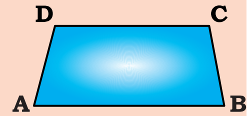
$\therefore r = \frac{R}{3} = \frac{a}{3}$

Quadrilateral

A Quadrilateral is a Polygon with four sides, four vertices and four angle.

- The sum of the interior angles of a Quadrilateral is always 360°.

  $\angle A + \angle B + \angle C + \angle D = 360°$

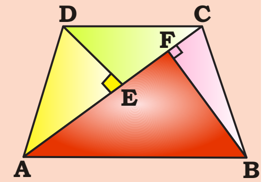


- Area of quadrilateral

  $= \left(\frac{1}{2} \times AC \times DE\right) + \left(\frac{1}{2} \times AC \times BF\right)$

  $= \frac{1}{2} \times AC\ (DE + BF)$

  $= \frac{1}{2} \times$ diagonal × (sum of perpendicular dropped on it)

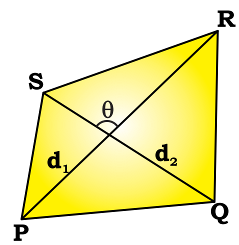


Ex. In a quadrilateral ABCD, AC = 12 cm. If length of the perpendiculars drawn from B and D to line AC are 5 cm and 7 cm, the area of the quadrilateral ABCD is :

HINTS Area = $\frac{1}{2} \times 12 \times (7 + 5) = 72$ cm²

☞ d_1 and d_2 are the diagonals of Quadrilateral PQRS.

$$\boxed{\text{Area} = \frac{1}{2} \times d_1 \times d_2 \sin\theta}$$



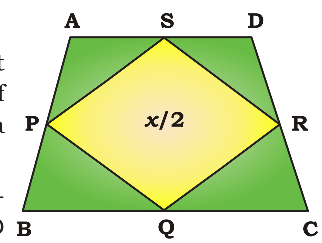
☞ The quadrilateral formed by joining the mid-point of the adjacent side of the quadrilateral will be a parallelogram of half area.

If P, Q , R , S are the mid-point of the side AB, BC, CD and DA respectively, then–

(a) PQRS is a parallelogram

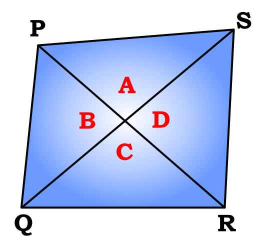
(b) If the area of the quadrilateral is x, then

$$\boxed{\text{Area of the parallelogram PQRS} = \frac{x}{2}}$$



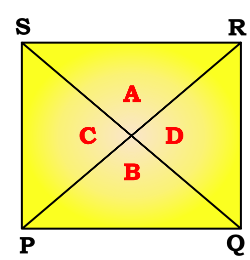
☞ In any Quadrilateral A, B, C and D are the area of respective triangles made by two Diagonals.

$\boxed{A \times C = B \times D}$



☞ PQRS is any Rectangle/ Square. T is a point inside it A, B, C, D are areas, then

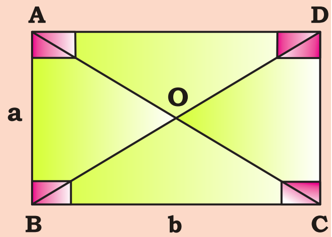
$\boxed{A + B = C + D}$



Types of Quadrilateral

- Rectangle
- Square
- Parallelogram
- Rhombus
- Trapezium

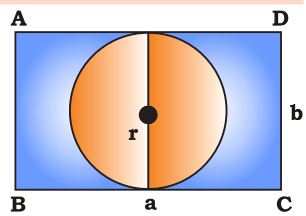
Rectangle



- AB = CD = a and BC = AD = b
- AC = BD = $\sqrt{a^2+b^2}$
- AO = OC = OB = OD $= \dfrac{\sqrt{a^2+b^2}}{2}$
- Perimeter = 2(length + breadth) = 2(a + b)
- Area = length × breadth = ab
- Area of ΔAOB = Area of ΔDOC = Area of ΔAOD = Area of ΔBOC $= \dfrac{ab}{4}$

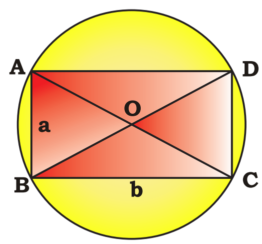
☞ Radius of the maximum large possible circle

$$r = \frac{\text{Breadth}}{2} = \frac{b}{2}$$

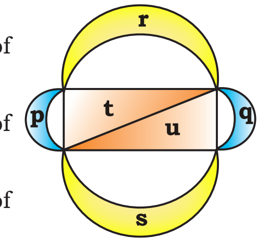


☞ Radius of the circle circumscribed the rectangle ABCD be R, then

$$R = \frac{\sqrt{a^2+b^2}}{2}$$



☞



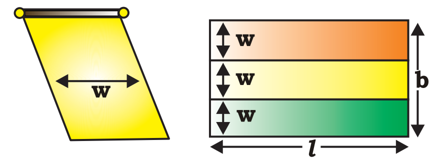
(i) Area of shaded region (p) = Area of shaded region (q)

(ii) Area of shaded region (r) = Area of shaded region (s)

(iii) Area of shaded region (t) = Area of shaded region (u)

(iv) Area of shaded region (p), (q), (r), (s) = Area of rectangular region

☞ A carpet has fix width



(i) Let carpet of width w covers floor of dimension $l \times b$

Area of carpet = Area of floor $\Rightarrow l_c \times w = l \times b$

Length of carpet required $(l_c) = \dfrac{lb}{w}$

(ii) Let rectangular tiles of dimension $(x \times y)$ cover the floor of dimension $(l \times b)$

$\Rightarrow$ Area of n tiles = Area of floor

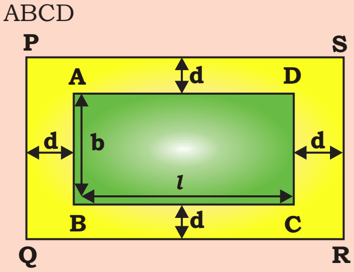
$\Rightarrow n \times x \times y = l \times b \Rightarrow n = \dfrac{lb}{xy}$

(iii) If the floor is covered by the minimum number of square tiles, then the side of square tile is HCF of the length and breadth of floor.

If the length of the rectangle will become x times and breadth will become y times, then area of the rectangle will become xy times.

Path inside / outside a rectangle

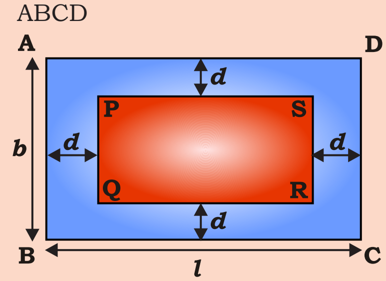
☞ Area of the path of uniform width d all around outside the rectangle ABCD



Area of the path = 2d $(l + b + 2d)$

Perimeter of path = 4 $(l + b + 2d)$

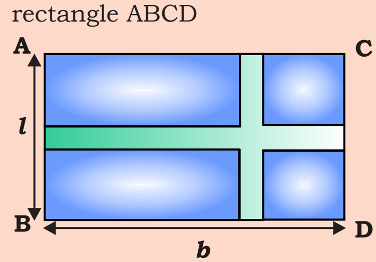
☞ Area of the path of uniform width 'd' all around inside the rectangle ABCD



Area of the path = 2d $(l + b - 2d)$

Perimeter of path = 4 $(l + b - 2d)$

☞ Area of the path of uniform width d along the length and the breadth in rectangle ABCD



Area of the path = $(l + b - d)d$

Perimeter of path = 2 $(l + b - 2d)$

Ex. A street of width 10 metres surrounds from outside a rectangular garden whose measurement is 200m × 180m. The area of the path (in square metres is)

HINTS Area of the path

= 2 × 10 (200 + 180 + 20) = 20 (400) = 8000 m²

Ex. A path around the interior of a rectangular park with dimensions 37m × 30m covering 570 m². What is the width of the path?

(a) 10m (b) 15m (c) 5m (d) 28m

HINTS Area of path = 2d × $(l + b - 2d)$

$\Rightarrow$ 570 = 2d × (37 + 30 – 2d)

$\Rightarrow$ 570 = 2d × (67 – 2d)

With the help of options, d = 5

570 = 2 × 5 × (67 – 10)

$\Rightarrow$ 570 = 570

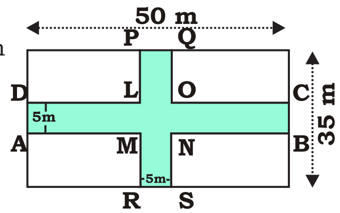
So, width of path = 5 m

Ex. The diagram given below shows two paths drawn inside a rectangular field 50 m long and 35 m wide. The width of each path is 5 metres. Find the area of the shaded portion.

HINTS Area of shaded portion

$= (l + b - d)d$

$= (50 + 35 - 5)5 = 400\ m^2$



Ex. A rectangular plot 20 m long and 14 m wide is to be covered with grass leaving 2 m all around. Find the area to be laid with grass.

HINTS

Length of grassy lawn

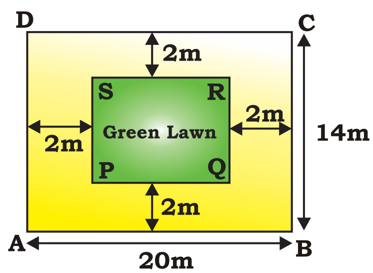
$= 20 - 2 \times 2 = 20 - 4$

$= 16$ m

Breadth of grassy lawn

$= 14 - 2 \times 2 = 14 - 4$

$= 10$m



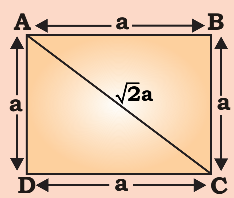
Area of grassy lawn = length × breadth

$= 16 \times 10 = 160\ m^2$

Square

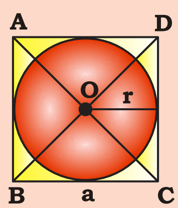
- Perimeter = 4 × Side = 4a
- Area = $(\text{Side})^2 = a^2$
- When diagonal is given then,

$$\text{Area} = \frac{(\text{diagonal})^2}{2} = \frac{d^2}{2}$$



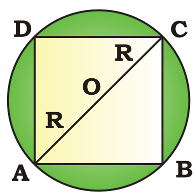
- Area of ΔAOB = Area of ΔBOC = Area of ΔCOD

$= \text{Area of } \Delta DOA = \frac{a^2}{4}$



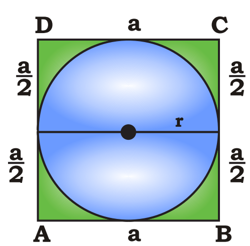
- Radius of the largest circle drawn outside the square of side 'a'

$$\text{Radius of circumcircle (R)} = \frac{a}{\sqrt{2}}$$



- Radius of the largest circle inscribed inside the square of side 'a'

$$\text{Radius of incircle (r)} = \frac{a}{2}$$



- Circum Radius (R) : Inradius (r) $= \sqrt{2} : 1$

$\Rightarrow$ Area = 2 : 1

- If one of the diagonal or the perimeter become x times then the area will become x^2 times or increases by $(x^2 - 1)$ times.

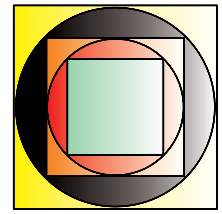
- For two squares

(a) Ratio of sides = Ratio of diagonal = Ratio of perimeter

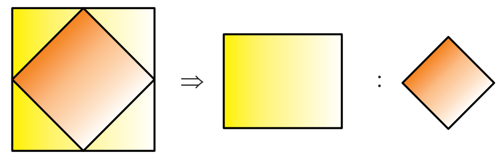
(b) Ratio of area = $(\text{Ratio of sides})^2 = (\text{Ratio of diagonal})^2$

$= (\text{Ratio of perimeter})^2$

☞ If we make circle inside a square and again make a square inside the circle and so on then area will become half and soon.

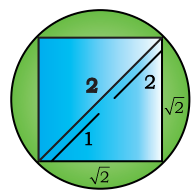


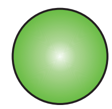
Area of largest square : middle of square : smallest square = 4 : 2 : 1

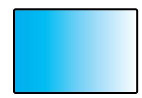


Area = 2 : 1

☞ The ratio of the area of the circle to the area of the square is:

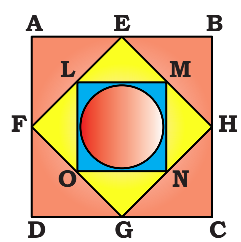




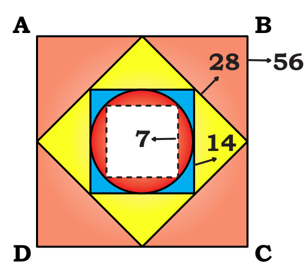


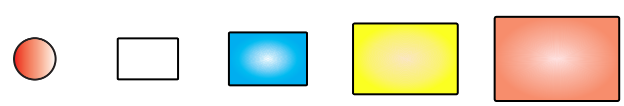
area $= \pi\, 1^2$	:	$(\sqrt{2})^2$
$= \pi$	:	2
$= \frac{22}{7}$	:	2
11	:	7

Ex. In the given figure, ABCD is a square. EFGH is a square formed by joining the mid points of sides of ABCD. LMNO is a square formed by joining the mid points of sides of EFGH. A circle is inscribed inside LMNO. If area of circle is 38.5 cm², then what is the area (in cm²) of square ABCD?



HINTS





11 : 7 : 14 : 28 : 56

38.5 cm² | 196 cm²

When circumference of a circle is equal to perimeter of a square then the ratio of area will be 14 : 11

Ex. When a wire is bent in the form of a square, then the area enclosed by it is 5929 cm². If wire is bent into the form of a circle, then what will be the area enclosed by the wire?

HINTS Given,

$a^2 = 5929\ cm^2$ and $2\pi R = 4a$

$\Rightarrow R = \frac{4a}{2\pi} = \frac{2a}{\pi}$

Area $= \pi R^2 = \pi \times \frac{2a}{\pi} \times \frac{2a}{\pi} = \frac{4a^2}{\pi}$

$= \frac{4 \times 5929 \times 7}{22} = 7546\ cm^2$

Alternatively

	Square	:	Circle
Area ratio →	11	:	14
	↓ ×539		↓ ×539
Area →	5929		$\boxed{7546}$

Area of circle = 7546 cm²

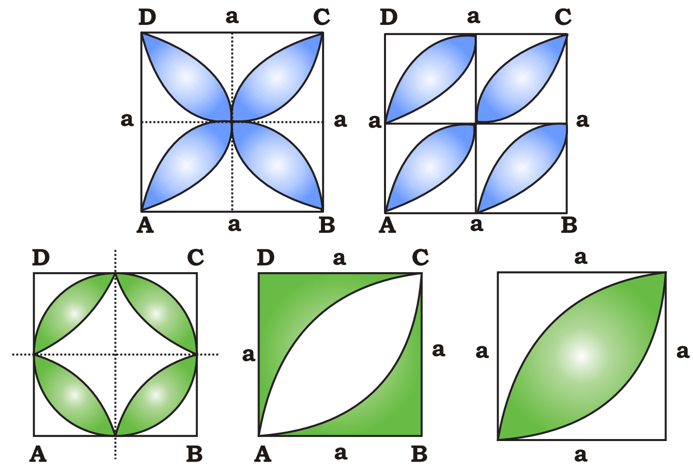
Ex. A copper wire is bent in the form of a square and it encloses an area of 30.25 cm². If the same wire is bent to form a circle, then find the area of the circle.

HINTS

	Square	:	Circle
Area ratio →	11	:	14
	↓ ×2.75		↓ ×2.75
Area →	30.25		$\boxed{38.5}$

Area of circle = 38.5 cm²

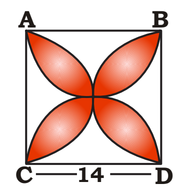
☞ According to the figure area of shaded region



"Area of shaded region is equal in each case"

$\boxed{\text{Area of shaded region in each case} = \frac{4}{7}a^2}$

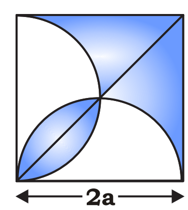
Ex. ABCD is a square whose side is 14 cm, find the area of the shaded region.



HINTS Area of shaded region $= \frac{4}{7}a^2$

$= \frac{4}{7} \times 14 \times 14 = 112\ cm^2$

Ex. If side of square = 2a cm, find the area of shaded region.

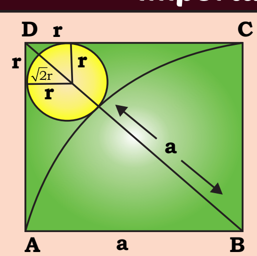


HINTS Area of shaded region = Area of square of side (2a) − 2 × Area of semicircle of radius (a) + 2 Area of leaf of radius (a) = 2a²

Important results

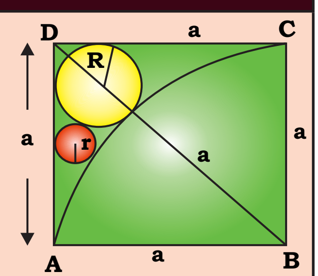
☞ ABCD is square of side a. AC is a curve and a circle of radii r is drawn, then

$\boxed{r = \left(\sqrt{2}-1\right)^2 a}$



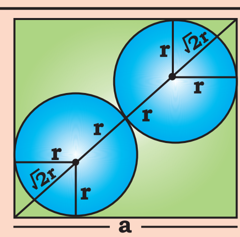
☞ ABCD is square of side a. AC is a curve and 2 circles of radii r and R are drawn, then

$\boxed{\frac{1}{\sqrt{r}} = \frac{1}{\sqrt{R}} + \frac{1}{\sqrt{a}}}$ $\boxed{r = \frac{R}{2}}$



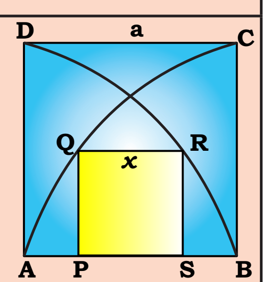
☞ When two circle of equal radii 'r' are placed inside a square of length 'a'

$\boxed{r = \frac{a\left(\sqrt{2}-1\right)}{\sqrt{2}}}$



ABCD and PQRS are squares of side a and x. Curves AC, BD cuts PQRS at Q and R.

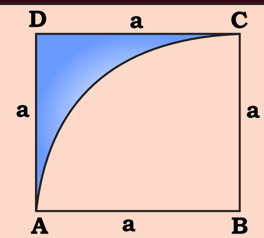
$\boxed{x = \frac{3a}{5}}$



Area of Shaded Region

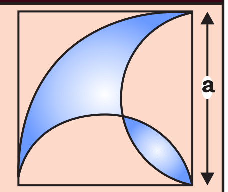
☞ If side of square = a cm then

Area of shaded region $= \frac{a^2}{4}(4-\pi)$



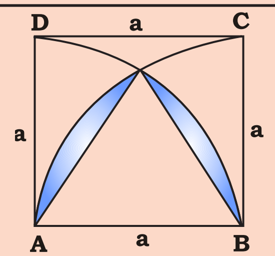
☞ If side of square = a cm then

Area of shaded region $= \frac{a^2(\pi-2)}{4}$



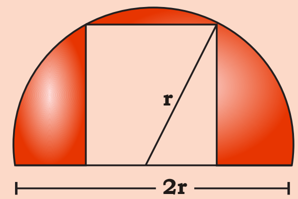
☞ For the following figure, side of square is 'a'.

Area of shaded region $= 2\left(\frac{a^2\pi}{6} - \frac{\sqrt{3}}{4}a^2\right)$



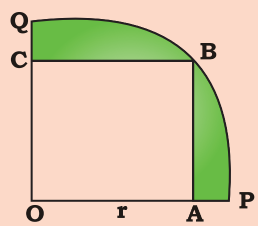
☞ Area of the largest square that can be drawn inside a semi-circle of radius r.

Area of square $= \frac{4}{5}r^2$



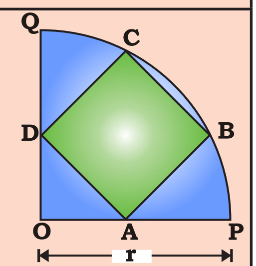
☞ Area of the square inscribed in the quadrant of a circle of radius 'r'(As shown in the figure).

Area of square $= \frac{r^2}{2}$



☞ Area of the square inscribed in the quadrant of a circle of radius 'r'(As shown in the figure).

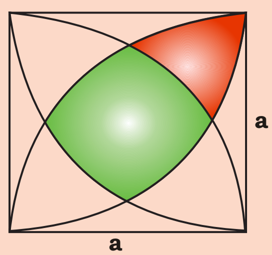
Area of square $= \frac{2}{5}r^2$



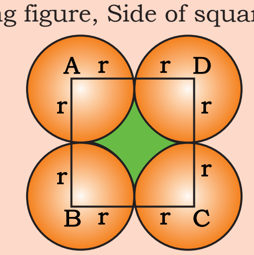
☞ According to the figure area of shaded region

Area of Green region $= \frac{a^2}{3}\{3(1-\sqrt{3})+\pi\}$

Area of Red region $= \frac{a^2}{12}[\pi - 12 + 6\sqrt{3}]$

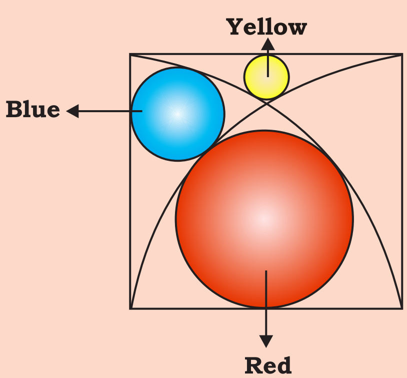


☞ For the follwing figure, Side of square is 'a'



Area of Green region $= \frac{3a^2}{14} = r^2(4-\pi)$

☞ If side of square = 'a'

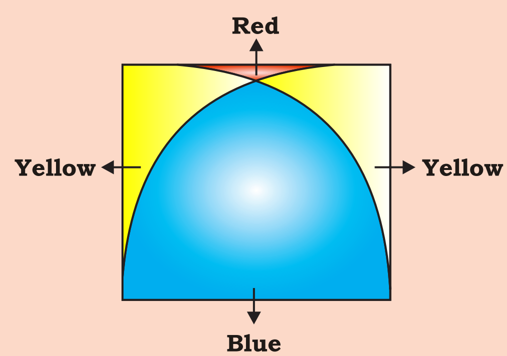


Radius of yellow circle $= \frac{a}{16}$

Radius of blue circle $= \frac{a}{6}$

Radius of red circle $= \frac{3a}{8}$

☞ For the following figure, side of square is 'a'.



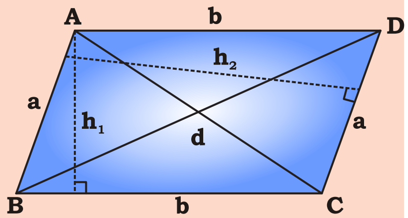
Area of blue Region $= \frac{\pi a^2}{3} - \frac{\sqrt{3}}{4}a^2$

Area of yellow Region $= 2\left(\frac{\sqrt{3}}{4}a^2 - \frac{\pi a^2}{12}\right)$

Area of red Region $= a^2 - \frac{\pi a^2}{6} - \frac{\sqrt{3}}{4}a^2$

Parallelogram

In Parallelogram ABCD, let side AB = a cm and BC = b cm, then



- AB = CD and BC = AD
- Each diagonal AC or BD divides the parallelogram in the congruent triangles.
- $AC^2 + BD^2 = 2(a^2 + b^2)$
- Perimeter = 2(a + b)
- Area = Base × Height = $a \times h_2 = b \times h_1$
- The length of one diagonal is d

  Then, Area of parallelogram ABCD

  $= 2\sqrt{s(s-a)(s-b)(s-d)}$

  Where, s = $\dfrac{a+b+d}{2}$

Types of Parallelogram

1. **Rectangle -** All angles are 90°, diagonals are equal
2. **Rhombus-** All sides are equal, diagonals are perpendi cular
3. **Square -** All sides and angles are equal. It is a rhombus and rectangle.

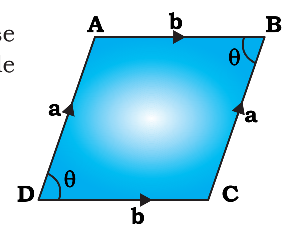
Ex. Side AB of a parallelogram ABCD is 24cm and side AD = 16 cm. The distance between AB and CD is 10 cm, then find the distance between AD and BC.

HINTS Area ⇒ 24 × 10 = 16 × x

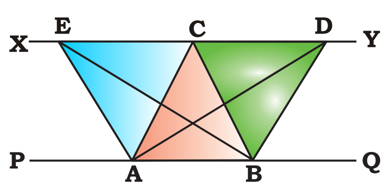
⇒ x = 15 cm

☞ ABCD is a parallelogram whose sides are a and b and angle between them is θ then

$$\boxed{\text{Area} = \frac{1}{2}ab\sin(\theta)}$$



☞ Area of all triangles of same base and between the parallel lines are the same.



If XY | | PQ, then

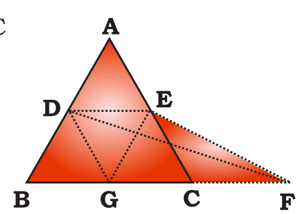
Area ΔABC

= Area ΔABD = Area ΔABE

Ex. In ΔABC, D and E are mid points of sides AB and AC respectively BC is extended up to F such that CF = BC. Then what is relation between Ar.ΔDEF and Ar.ΔABC.

HINTS Let G is mid point of BC

∴ Ar.ΔDEG = $\dfrac{\text{Ar.}\Delta ABC}{4}$



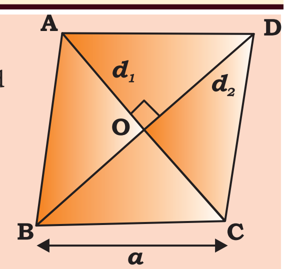
If D and E are mid points of AB and AC respectively then DE | | BC | | BF

Ar.ΔDEF = Ar.ΔDEG (triangles in between | | lines with same base) = $\dfrac{\text{Ar.}\Delta ABC}{4}$

Rhombus

In rhombus ABCD,

Let the side BC = a, AC = d_1 and BD = d_2, then



- AB = BC = CD = DA = a
- Side (a) = $\dfrac{1}{2}\sqrt{d_1^2 + d_2^2}$ or,

  $4a^2 = d_1^2 + d_2^2$
- Perimeter = 4a
- Area = $\dfrac{1}{2} \times d_1 \times d_2$
- Perimeter of a rhombus is 2p unit and sum of the lengths of diagonals is m unit, then the area of the rhombus is-

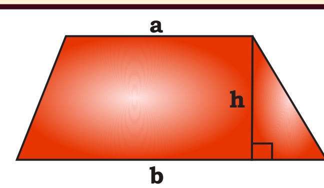
  $= \dfrac{1}{4}(m^2 - p^2)$ sq.unit

Ex. Length of each side of a rhombus is 13 cm and one of the diagonal is 24 cm. What is the area (in cm²) of the rhombus?

HINTS $4 \times 13 \times 13 = 24^2 + d_2^2 \Rightarrow d_2 = 10$ cm

Area = $\dfrac{1}{2} \times 24 \times 10 = 120$ cm²

Trapezium



Area of trapezium = $\dfrac{1}{2}$ × sum of parallel sides × height

Ex. The ratio of the length of the parallel sides of a trapezium is 3 : 2. The shortest distance between them is 15 cm. If the area of the trapezium is 450 cm², the sum of the length of the parallel sides is

HINTS Area of trapezium = $\dfrac{1}{2}$ × sum of parallel sides × height

⇒ 450 = $\dfrac{1}{2}(2x + 3x) \times 15$

⇒ $5x = 30 \times 2 \Rightarrow x = 12$

∴ Sum of the length of the parallel sides

= $5x = 5 \times 12 = 60$ cm

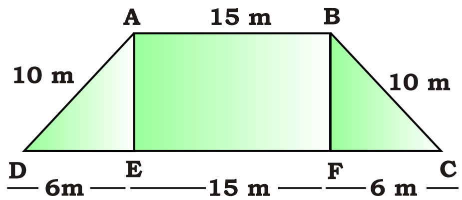
Ex. A wall is in the form of a trapezium with height 4m and parallel sides being 3m and 5m. What is the cost of painting the wall, if the rate of painting is ₹ 25 per sq. m?

HINTS Area of trapezium = $\dfrac{1}{2}(3 + 5) \times 4 = 16$ m²

Total cost of painting = 16 × 25 = ₹400

Ex. In a trapezium ABCD, AB and DC are parallel to each with a perpendicular distance of 8 m between them. Also, (AD) = (BC) = 10 m, and (AB) = 15m < (DC). What is the perimeter (in m) of the trapezium ABCD?

HINTS



Pythagorean theorem DE = $\sqrt{AD^2 - AE^2}$

$= \sqrt{10^2 - 8^2} = \sqrt{36} = 6m$

Perimeter of trapezium

= AD + DC + BC + AB

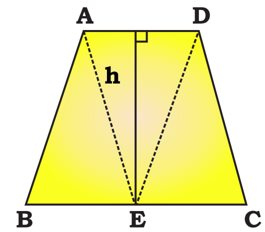
=10 + 27 +10 + 15

= 62 m

Ex. ABCD is a trapezium with AD and BC parallel sides, E is a point on BC. The ratio of the area of ABCD to that of ΔAED is

HINTS $\frac{\text{Area of trapezium ABCD}}{\text{Area } \Delta\text{AED}}$

$= \frac{\frac{1}{2}(AD + BC) \times h}{\frac{1}{2} AD \times h} = \frac{AD + BC}{AD}$



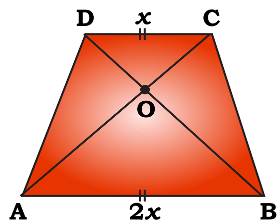
Ex. ABCD is a trapezium in which AB || DC and AB = 2CD. The diagonals AC and BD meet at O. The ratio of area of ΔAOB and ΔCOD is

HINTS

Let CD = x then AB = 2 CD = $2x$

ΔCOD ~ ΔAOB

$\frac{\text{Area } \Delta\text{AOB}}{\text{Area } \Delta\text{COD}} = \frac{AB^2}{CD^2} = \frac{(2x)^2}{x^2} = 4:1$



CHAPTER 03 MENSURATION 3D

Introduction

A 3-D solid is a three-dimensional object that has length, breadth, and height (or thickness), unlike flat 2-D shapes. These solids are enclosed by flat surfaces called faces. Where two faces meet, they form a line segment called an edge, and where edges meet, they form vertices (corners). To describe the structure of any 3-D solid, Euler formulated a mathematical relationship between the number of vertices (V), edges (E), and faces (F). This relationship is known as **Euler's formula**, and it states:

$$\boxed{V + F = E + 2}$$

This means that for every 3-D solid that is a convex polyhedron, the sum of its vertices and faces is always two more than the number of its edges.

Let us consider an example of cube:

Clearly,

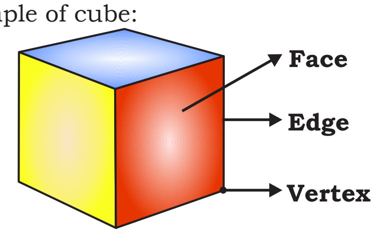
V= 8, F = 6 and E = 12

From Euler's Rule,

$V + F = E + 2$

$\Rightarrow 8 + 6 = 12 + 2$

$\Rightarrow 14 = 14$



Lateral/Curved Surface Area (LSA/CSA)

This refers to the area of the curved surface of a 3D shape. It's commonly used for objects like cylinders, Cones and spheres. The curved surface area of a cylinder is the area of the curved part excluding the circular top and bottom.

Total Surface Area (TSA)

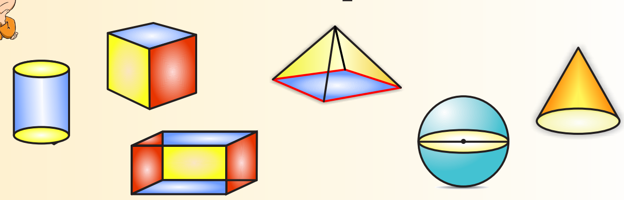
TSA of a solid is the sum of the lateral surface area and the areas of the base and the top.

TSA = LSA + Area of top surface + Area of bottom surface

Volume

It refers to the amount of space that an object occupies, measured in cubic units. Only 3–D objects have volume.

Match the shape to its name.

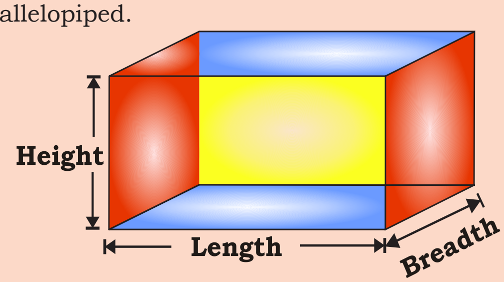


Sphere | Cone | Pyramid

Cuboid | Cylinder | Cube

Cuboid

A cuboid is a rectangular solid object having six rectangular surfaces. It is sometimes also called as rectangular parallelopiped.



For a cuboid with base length *(l)*, breadth *(b)* and height *(h)*

(i) Lateral Surface Area = area of 4 walls = $2(l + b)h$

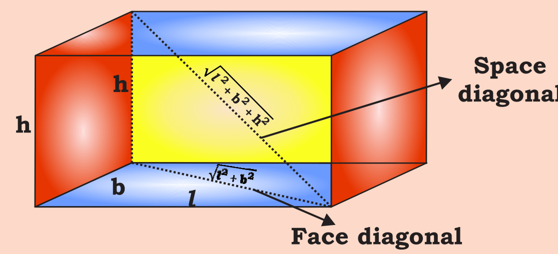
(ii) Total Surface Area = $2(lb + bh + hl)$

(iii) Volume of Cuboid = $l \times b \times h$

(iv) Face Diagonal of Cuboid = $\sqrt{l^2 + b^2}$ (base or top face)

$= \sqrt{l^2 + h^2}$ (front or back face)

$= \sqrt{b^2 + h^2}$ (side faces)



(v) Space diagonal = $\sqrt{l^2 + b^2 + h^2}$ (Longest)

Length of longest rod that can be placed in the room = Space Diagonal = $\sqrt{l^2 + b^2 + h^2}$

Ex. A mosquito sitting at the top corner of a cuboidal room of dimensions 7 × 3 × 2.5 m sees a cat sleeping at the farthest bottom corner of the room. How much distance will the mosquito have to fly to bite the cat?

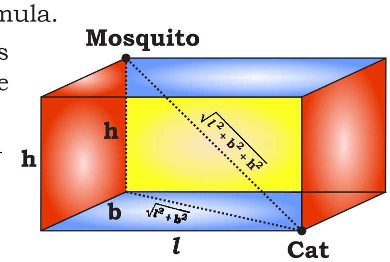
HINTS To find the shortest straight-line distance across a cuboid (from one corner to the opposite corner), use the space diagonal formula.

If the cuboid has sides *l*, b, and h, the distance will be $\sqrt{l^2 + b^2 + h^2}$

$\text{Distance} = \sqrt{7^2 + 3^2 + 2.5^2}$

$= \sqrt{49 + 9 + 6.25}$

$= \sqrt{64.25} \simeq 8.01$ m

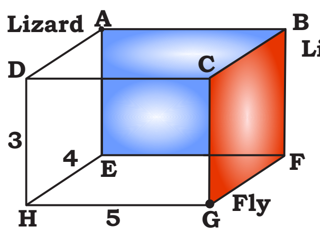


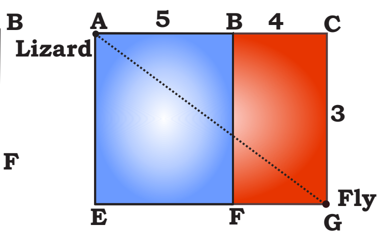
(Because the mosquito can fly freely through the air, it doesn't have to follow the walls or surfaces — it simply takes the shortest straight-line path through 3D space)

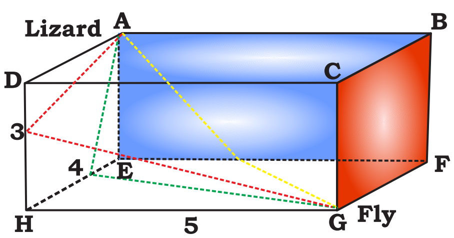
In general if the dimensions of the cuboid are a, b, c where a ≥ b ≥ c, then the shortest path along the surface will be $\sqrt{(b+c)^2+a^2}$
(This can be proved algebraically, if we so desire)

Ex. A lizard sitting at the top corner of a cuboidal room of dimensions 5 × 4 × 3 m sees a fly sleeping at the farthest bottom corner of the room. How much distance will the lizard have to crawl to catch the fly.

HINTS







Unfold two adjacent faces into a flat rectangle, turning the 3D crawl into a straight 2D line.

Route 1 (Green line) = A → E → G

AG = $\sqrt{(4+3)^2+5^2}$ = $\sqrt{74}$ ≈ 8.6 m (Shortest)

Route 2 (Red line) = A → D → H → G

AG = $\sqrt{(5+3)^2+4^2}$ = $\sqrt{80}$ ≈ 8.94 m (Medium)

Route 3 (Purple line) = A → B → F → G

AG = $\sqrt{(5+4)^2+3^2}$ = $\sqrt{90}$ ≈ 9.48 m (Longest)

So, the shortest crawling path along the surfaces = 8.6 m (by unfolding height + breadth sides)

Alternatively

Shortest crawling path = $\sqrt{(b+c)^2+a^2}$ (a ≥ b ≥ c)

= $\sqrt{(4+3)^2+5^2}$ = $\sqrt{74}$ ≈ 8.6 m

(Unlike a mosquito, the lizard cannot fly. It must crawl along the surfaces of the room to reach the fly.So we must find the shortest crawling path along the walls of the cuboid.)

Cuboid with Adjacent Faces

Let a cuboid have dimensions:

Length = l, Breadth = b, Height = h

Let the areas of three adjacent faces meeting at one corner be:

P = lb (base area), Q = bh (side area), R = hl (front area)

Volume (V = l × b × h) using only P, Q, R

P × Q × R = (lb) × (bh) × (hl) = $l^2b^2h^2$

Volume = $l$bh = $\sqrt{PQR}$

If you know the areas of any 3 adjacent faces of a cuboid, you can directly find the volume.

Volume = $\sqrt{PQR}$

Ex. The areas of three adjacent faces of a cuboid are 32 cm², 24cm² and 48cm². Find the volume of cuboid.

HINTS Volume = $\sqrt{32\times24\times48}$ = 192 cm³

☞ **Relation between diagonal and total surface area of a cuboid:**

$(l + b + h)^2 = l^2 + b^2 + h^2 + 2(lb + bh + hl)$

or (Sum of dimensions)² = (Diagonal)² + Total Surface Area

Ex. The sum of length, breadth and height of a cuboid is 20 cm. If the length of the diagonal is 12 cm, then find the total surface area of the cuboid.

HINTS

(Sum of dimensions)² = (Diagonal)² + Total Surface Area

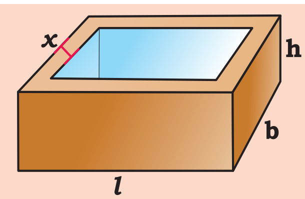
⇒ (20)² = 12 × 12 + Total Surface Area

⇒ Total Surface Area = 256 cm²

☞ **Volume of hollow cuboid**

= $l$bh – (l – 2x) (b – 2x) (h – 2x)

Where, x is the thickness of walls of the cuboid



Ex. A wooden box measures 20 cm by 12 cm by 10 cm. Thickness of wood is 1 cm. Volume of wood to make the box (in cubic cm) is

HINTS Volume of hollow cuboid

= 20 × 12 × 10 – (20 – 2) (12 – 2) (10 – 2) = 960 cm³

Ex. A cuboid has dimensions 30 cm × 25 cm × 20 cm. If the mass of the cuboid is 15 kg. Find its density in g/cm³.

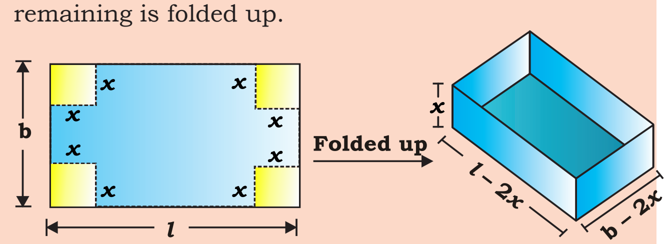
HINTS Volume = 30 × 25 × 20 = 15000 cm³

Mass = 15 kg = 15000g

Density = $\frac{\text{Mass}}{\text{Volume}} = \frac{15000}{15000}$ = 1 g/cm³

Making a box from Rectangular Sheet

We can make an open rectangular box by cutting off equal squares of side x unit at four corners and the remaining is folded up.



Volume of rectangular box = (l – 2x) (b – 2x)x

Ex. From the four corners of a rectangular sheet of dimensions 25 cm × 20 cm, square of side 2 cm is cut off from four corners and a box is made. The volume of the box is.

HINTS Volume of rectangular box = $(l - 2x)(b - 2x)x$

$= (25 - 2 \times 2)(20 - 2 \times 2) \times 2 = 21 \times 16 \times 2 = 672\ cm^3$

Ex. A square of side 3 cm is cut off from each corner of a rectangular sheet of length 24 cm and breadth 18 cm and the remaining sheet is folded to form an open rectangular box. The surface area of box is.

HINTS Surface area of open box

$= (l - 2x)(b - 2x) + 2(b - 2x)(x) + 2(x)(l - 2x)$

$= (24 - 2 \times 3)(18 - 2 \times 3) + 2(18 - 2 \times 3)(3) + 2(3)(24 - 2 \times 3)$

$= 18 \times 12 + 6 \times 12 + 6 \times 18$

$= 216 + 72 + 108 = 396\ cm^2$

Alternatively

Surface area of the open box = Area of rectangle – 4 × Area of square of side 3 cm

$= 24 \times 18 - 4 \times 3^2 = 396\ cm^2$

In case of an open box, The area of upper surface (lb) is excluded. Similarly While calculating the cost of painting a room, the area of floor (lb) is excluded and the remaining surface area ($lb + 2bh + 2hl$) is multiplied by the cost per unit area.

Ex. The length, breadth, and height of a room are 10m, 8m and 6 m respectively. Find the cost of white washing the walls of the room and the ceiling at the rate of ₹7.50 per m².

(a) ₹2,220 (b) ₹1,850
(c) ₹2,150 (d) ₹2,000

HINTS Area = $(10 \times 8 + 2 \times 8 \times 6 + 2 \times 6 \times 10) = 296\ m^2$

∴ Cost of white washing = 296 × 7.5 = ₹2220

In Mensuration, if the cost per unit area is a prime number or multiple of a prime number, then the final answer will also be a multiple of that prime number, if it is 9 or multiple of 9, then the digit sum of the correct answer will be 9.
In the above example, 7.5 is a multiple of the prime number 3, hence the final answer will also be a multiple of 3. Out of the given options, only option (a) is a multiple of 3.

☞ A rectangular tank is 'l' metres long and 'h' metres deep. If 'x' cubic metres of water be drawn off the tank, the level of the water in the tank goes down by 'd' metres, then the amount of water the tank can hold is given by $\left(\frac{x \times h}{d}\right)$ cubic metres and the breadth of the tank is $\left(\frac{x}{ld}\right)$ metres.

Ex. A rectangular tank is 50 metres long and 29 metres deep. If 1000 cubic metres of water be drawn off the tank, the level of the water in the tank goes down by 2 metres. How many cubic metres of water can the tank hold? And also find the breadth of the tank.

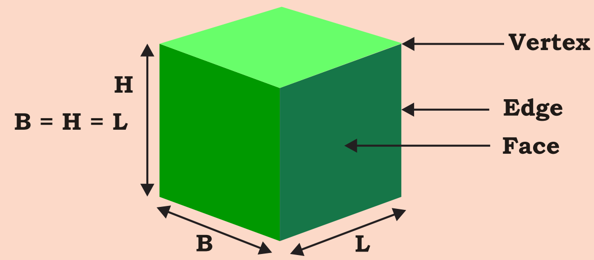
HINTS Volume of the tank = $\frac{1000 \times 29}{2}$ = 14500 cubic meter.

We may conclude that even if the length of the rectangular tank is not given, Volume of the tank can be calculated. To find breadth of the tank length is needed but not the height of the tank.

∴ Breadth of the tank = $\frac{1000}{50 \times 2}$ = 10 metres.

Cube

A solid object having all the six identical square surfaces is known as cube. Thus, length, breadth and height of a cube are equal.



Consider 'a' is the side of the cube.

(i) Lateral Surface Area = $4a^2$

(ii) Total Surface Area = $6a^2$

(iii) Volume = $a^3 = \left(\sqrt{\frac{\text{Total Surface area}}{6}}\right)^3$

(iv) Diagonal = $\sqrt{3}a$

(v) Face diagonal = $\sqrt{2}\,a$

(vi) If diagonal of cube is given then volume of cube

$= \left(\frac{\text{Diagonal}}{\sqrt{3}}\right)^3$

Ex. The total surface area of a cube is 1728 cm². Find its volume.

HINTS Let, side of cube = a

Surface area of cube = $6a^2$

$\Rightarrow 6a^2 = 1728 \Rightarrow a = 12\sqrt{2}$

∴ Volume of cube = $a^3 = (12\sqrt{2})^3 = 3456\sqrt{2}\ cm^3$

Alternatively

Volume $= \left(\sqrt{\frac{\text{Total Surface area}}{6}}\right)^3 = \left(\sqrt{\frac{1728}{6}}\right)^3$

$= (\sqrt{288})^3 = (12\sqrt{2})^3 = 3456\sqrt{2}\ cm^3$

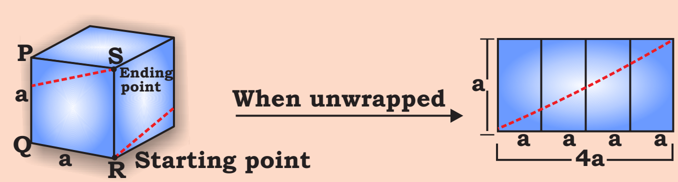
Ex. The length of longest diagonal of a cube is $7\sqrt{3}$ cm. Find its volume (in cm³).

HINTS Volume of cube $= \left(\frac{\text{Diagonal}}{\sqrt{3}}\right)^3$

$= \left(\frac{7\sqrt{3}}{\sqrt{3}}\right)^3 = 343\ cm^3$

The string goes diagonally from the bottom left corner (point R) to the top right corner (Point S) of this rectangle.

Length of the string $= \sqrt{(4a)^2 + a^2} = a\sqrt{17}$



Ex. A string makes to complete turns arround the cube along its sides faces, starting at a point at the bottom and ending at a point directly above the start. Find the length of string?

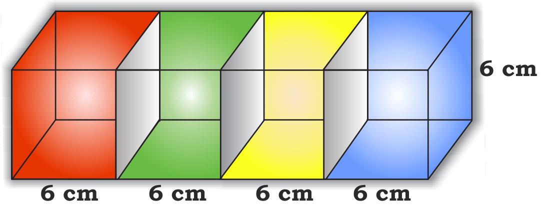
HINTS When unwrapped, it forms a rectangle of width 8a (4 faces × 2 turns) and height a.

Length of the string = $\sqrt{(8a)^2 + a^2} = a\sqrt{65}$

Special case I:-
When n cubes (side = a) are joined to form a cuboid.
(i) Length of cuboid = n × a
(ii) Breadth and height of cuboid = a
(iii) Volume of cuboid = n × a^3
(iv) Surface area of cuboid = $2a^2$ (2n+1)

Ex. Four cubes each of volume 216 cubic cm are joined end to end to form a new solid. The surface area of the new solid is:

HINTS Volume of cube = $a^3 \Rightarrow a^3 = 216 \Rightarrow a = 6$



When 4 cubes are joined together then length = 4 × 6 = 24cm, breadth = height = 6 cm

Total surface area of cuboid = $2(lb + bh + hl)$

= 2 × (24 × 6 + 6 × 6 + 6 × 24)

= 2 × (144 + 36 + 144) = 2 × 324 = 648 cm^2

Alternatively

Volume of cube $(a)^3 = 216 \Rightarrow a = 6$ cm

No of cube (n) = 4

Length of cuboid = n × a = 4 × 6 cm = 24 cm

Height = breadth = 6 cm

Total surface area of cuboid = $2a^2$ (2n + 1)

= 2 × 6^2 (2 × 4 + 1) = 648 cm^2

Special case II:-
When a cuboid is cut up form minimum n cubes.
(i) Side of cube = HCF of sides of cuboid
(ii) $n = \dfrac{\text{Volume of cuboid}}{\text{Volume of cube}}$

Ex. A rectangular block of length 20 cm, breadth 15 cm and height 10 cm is cut up into exact number of equal cubes. The least possible number of cubes will be

HINTS Side of cube = HCF of (20, 15, 10) = 5

$n = \dfrac{20 \times 15 \times 10}{5 \times 5 \times 5} = 24$

Special case III:-
When minimum n cuboids are joined (arranged) to form a cube.
(i) Side of cube = LCM of sides of cuboid
(ii) $n = \dfrac{\text{Volume of cube}}{\text{Volume of cuboid}}$

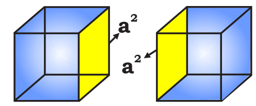
Ex. There is a cuboid of dimension 6 cm by 4 cm by 3 cm. The minimum such cuboids are arranged to make a cube. Find the volume of the cube.

HINTS Side of cube = LCM of (6, 4, 3) = 12

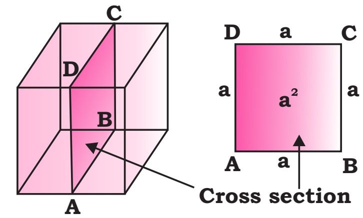
Volume of cube = 12^3 = 1728 cm^3

Did You Know

(i) Joining of cubes: If we join two cubes, two of their surfaces will be lost.

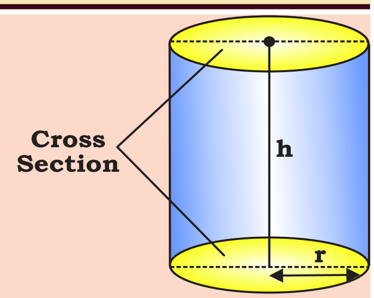


TSA = $6a^2 + 6a^2 - a^2 - a^2 = 10a^2$
(ii) By making n cuts, we obtain (n+1) cuboids.
Now, increase in TSA = $2na^2$
TSA of these (n+1) cuboids = $2a^2$ (n+3)



Right Circular Cylinder

A solid which has uniform circular cross-section is called a cylinder (or, a right circular cylinder)



Let r be the radius of circular cross-section and h be the height of cylinder, then

(i) Area of cross-section = πr^2
(ii) Perimeter (circumference) of cross-section = $2\pi r$
(iii) Curved Surface area = Perimeter of cross-section × height = $2\pi rh$
(iv) Total Surface area = Curved surface area + 2 × Area of cross-section = $2\pi rh + 2\pi r^2 = 2\pi r(r + h)$
(v) Volume = Area of cross-section × height = $\pi r^2 h$
(vi) Ratio of total surface area to curved surface area = $\dfrac{h + r}{h}$

Ex. What is the total surface area of a cylinder having radius of base as 7 cm and height as 5 cm?

HINTS r = 7 cm, h = 5 cm

Total surface Area of a cylinder = $2\pi r(h + r)$

$= 2 \times \dfrac{22}{7} \times 7 \times (7 + 5) = 44 \times 12 = 528\ cm^2$

Ex. If the height and diameter of a cylinder are 12 cm and 28 cm, respectively, then find the ratio of the total surface area to the curved surface area.

HINTS h = 12 cm, $r = \dfrac{28}{2} = 14$ cm

$\dfrac{TSA}{CSA} = \dfrac{h + r}{h} = \dfrac{12 + 14}{12} = \dfrac{26}{12} = \dfrac{13}{6}$

Ex. If the curved surface of a cylinder is 880 cm^2, then the product of its height and radius is.

HINTS Curved surface area = $2\pi rh$

$\Rightarrow 2 \times \dfrac{22}{7} \times rh = 880 \Rightarrow rh = 140\ cm^2$

When volume, CSA and TSA is multiple of 11 then it is necessary that radius or height will be multiple of 7.

(i) If curved surface area (c) and volume (V) are given then ratio of radius to height is.

$$\boxed{\frac{r}{h} = \frac{8\pi V^2}{c^3}}$$

Ex. The curved area of a cylindrical pillar is 264 m^2 and its volume is 924 m^3. $\left(\text{Taking } \pi = \frac{22}{7}\right)$ Find the ratio of its diameter to its height.

HINTS

$$\frac{\text{Diameter}}{\text{Height}} = \frac{2r}{h} = \frac{2\times 8\pi V^2}{(CSA)^3} = \frac{2\times 8\times 22\times 924\times 924}{7\times 264\times 264\times 264} = \frac{7}{3}$$

(ii) If curved surface area (c) and height (h) are given the volume of cylinder is.

$$\boxed{V = \frac{c^2}{4\pi h}}$$

Ex. A right circular cylinder of height 16 cm is covered by a rectangular tin foil of size 16cm × 22 cm. The volume of the cylinder is:

HINTS If tin foil covered cylinder then curved surface area of cylinder is equal to area of tin foil.

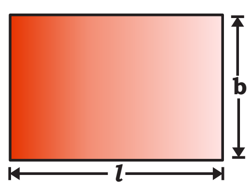
C.S.A. = 16cm × 22cm

h = 16cm

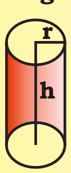
$$\text{Volume} = \frac{c^2}{4\pi h} = \frac{16\times 22\times 16\times 22\times 7}{4\times 22\times 16} = 616\ cm^3$$

Folding and Revolving a Rectangular Sheet to form a Cylinder

Rectangular Sheet to be Fold



Folding along Length

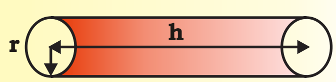


h = b

$2\pi r = l$

$\Rightarrow r = \frac{l}{2\pi}$

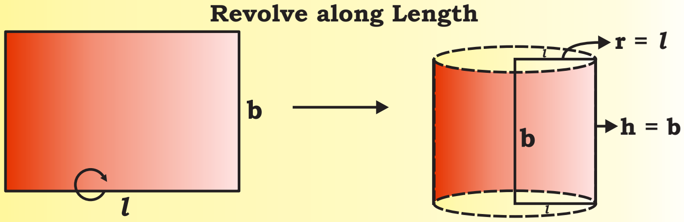
Folding along Breadth



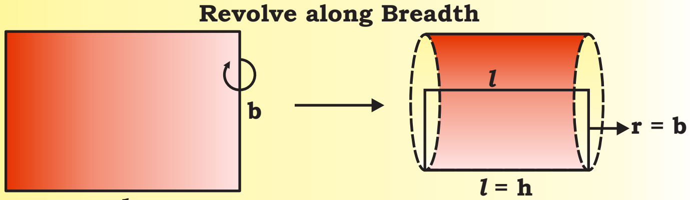
$h = l,\ 2\pi r = b \Rightarrow r = \frac{b}{2\pi}$

Rectangular Sheet to be Revolve

Revolve along Length



Revolve along Breadth



Ex. A rectangular paper sheet of dimensions 22 cm × 12 cm is folded in the form of a cylinder along its length. What will be the volume of this cylinder?

HINTS $2\pi r = 22 \Rightarrow r = \frac{22\times 7}{2\times 22} = \frac{7}{2}$ cm., h = 12 cm

$\therefore$ Volume of cylinder = $\pi r^2 h = \frac{22}{7}\times\frac{49}{4}\times 12 = 462\ cm^3$

Ex. A figure is formed by revolving a rectangular sheet of dimensions 7 cm × 4 cm about its breadth. What is the volume of the figure thus formed?

HINTS h = 7 cm, r = 4 cm

Volume of cylinder = $\pi r^2 h = \frac{22}{7} \times 4\times 4\times 7 = 352\ cm^3$

When a rope is wound spirally (helically) around a cylinder then the cylinder is unwrapped into a rectangle.

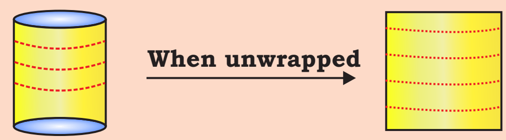
For each turn -

length per turn = $\sqrt{(\text{Circumference})^2 + (\text{Rise per turn})^2}$

Where, circumference = given base perimeter

Rise per turn = $\frac{\text{Total Height}}{\text{Number of turns}}$

Total length of rope = length per turn × no. of turns



Ex. A thread is wound once around the cylinder of radius 10 cm and height 24 cm. Find the length of the thread?

HINTS

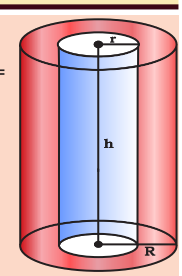
Circumference of cylinder = $2\pi r$ = 2 × 3.14 × 10 = 62.8

h = 24 cm, Number of turn (n) = 1

Length per turn = $\sqrt{(62.8)^2 + (24)^2} = \sqrt{4519.84} \approx 67.2$

Length of the thread = 67.2 cm

Hollow Cylinder



(i) Thickness (T) = (R – r)

(ii) Curved Surface Area = $2\pi rh + 2\pi Rh = 2\pi h(R + r)$

(iii) Total Surface Area

= $2\pi h(R + r) + 2\pi(R^2 - r^2)$

(iv) Volume of material of hollow Cylinder

= $\pi(R^2 - r^2)h$

(v) Mass (weight) of hollow cylinder

= Density × Volume of material

Ex. A hollow cylinder is made up of steel. The difference in its outer and inner CSA is 132 cm^2. Height of cylinder is 21 cm and sum of its inner and outer radius is also 21 cm. Find TSA of the hollow cylinder (in cm^2).

HINTS $2\pi h(R - r) = 132 \Rightarrow R - r = 1$

Also, R + r = 21 cm (given) $\Rightarrow$ R = 11 cm, r = 10 cm

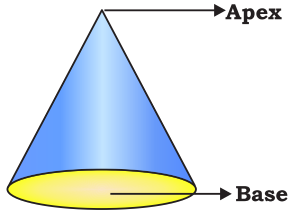
$\therefore$ TSA of hollow cylinder = $2\pi h(R + r) + 2\pi(R^2 - r^2)$

= $2\pi(R + r)[h + R - r] = 2 \times \frac{22}{7} \times 21\ [21 + 1]$

= 6 × 22 × 22 = 2904 cm^2

Cone

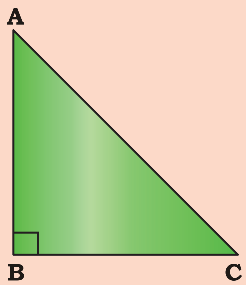
Cone is a three dimensional structure having a circular base where a set of line segments, connect all of the point on the base to a common point called apex.

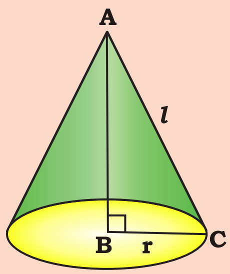


Right Circular Cone

The solid obtained by revolving a right-angled triangle about one of its sides (other than hypotenuse) is called a cone or right circular cone.

Let the right angled triangle ABC be revolved about its side AB to form a cone; then AB is the height (h) of the cone formed, BC is the radius (r) of its base and AC is the slant height(l).





(i) Slant height (l) = $\sqrt{r^2 + h^2}$

(ii) Curved Surface Area = $\pi r l$

(iii) Total Surface area = $\pi r(r + l)$

(iv) Volume = $\frac{1}{3}\pi r^2 h$

Ex. The volume of a conical tent is 1232 m³ and the area of its base is 154 sq.m. Find the length of the canvas required to build the tent, if the width of canvas is 2 m.

HINTS $\pi R^2 = 154\ m^2 \Rightarrow R = 7m$

& $\pi R^2 h = 1232\ m^3 \Rightarrow \frac{h}{3} = \frac{1232}{154} \Rightarrow h = 24$ m

$\therefore l = 25$ m (by triplet of 7, 24, 25)

Area of the canvas required to build the tent = $\pi r l$

Which will equal to the area of the rectangular canvas.

$\Rightarrow \frac{22}{7} \times 7 \times 25 = 2 \times$ Length of the canvas

$\Rightarrow$ Length of the canvas = 275 m

Canvas required to construct a conical tent = Curved surface area of cone

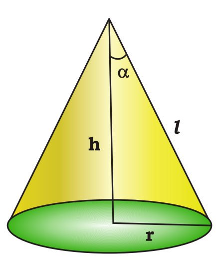
☞ If S denotes the curved surface area of a right circular cone of h height and semivertical angle α then S equals to?

$\tan\alpha = \frac{r}{h} \Rightarrow r = h\tan\alpha$

$\cos\alpha = \frac{h}{l} \Rightarrow l = h\sec\alpha$

Surface area of cone = $\pi r l$

$= \pi h^2 \sec\alpha . \tan\alpha$



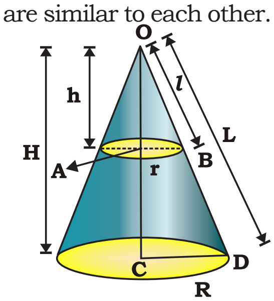
Cutting of Cone

☞ All triangles formed by cutting cone are similar to each other.

(A) (i) $\Delta OCD \sim \Delta OAB$

$(\angle A = \angle C = 90°, \angle O = \angle O)$

$\frac{H}{h} = \frac{R}{r} = \frac{L}{l}$ or $\frac{H}{R} = \frac{h}{r}$



☞ Let V is volume of larger cone and v is volume of smaller cone

(ii) $\frac{V}{v} = \frac{\frac{1}{3}\pi R^2 H}{\frac{1}{3}\pi r^2 h} = \frac{R^2 H}{r^2 h}$

(iii) $\frac{V}{v} = \frac{H^3}{h^3} = \frac{R^3}{r^3} = \frac{L^3}{l^3}$

(B) CSA of 5 parts of cone

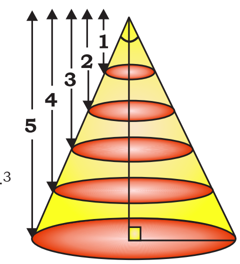
$= 1^2 : 2^2 - 1^2 : 3^2 - 2^2 : 4^2 - 3^2 : 5^2 - 4^2$

$= 1 : 3 : 5 : 7 : 9$

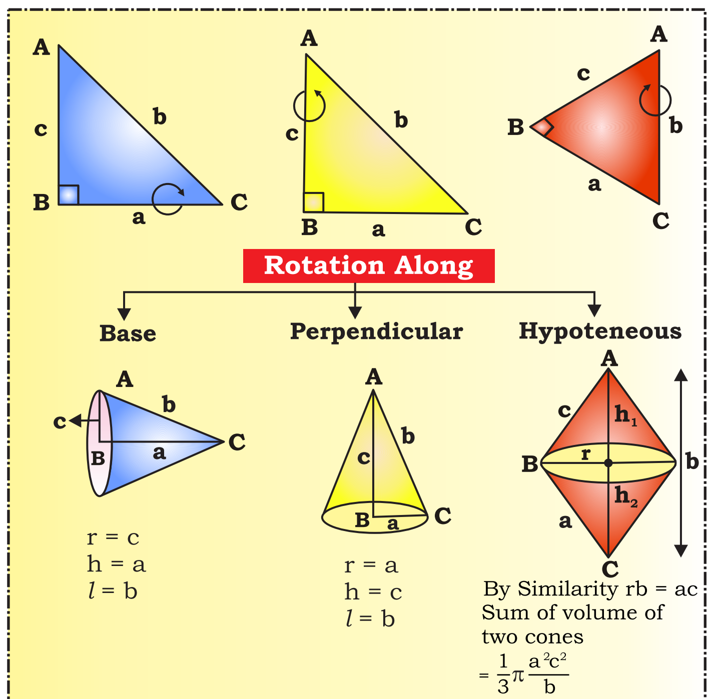
Volume of 5 parts of cone

$= 1^3 : 2^3 - 1^3 : 3^3 - 2^3 : 4^3 - 3^3 : 5^3 - 4^3$

$= 1 : 7 : 19 : 37 : 61$



Cone by Rotating Right Angle Triangle



Ex. A right angle triangle ABC with sides 5 cm, 12 cm and 13 cm is revolved about the side 12 cm. What is the volume of the solid so obtained?

HINTS r = 5 cm, h = 12 cm

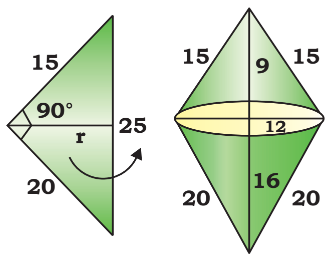
Volume of cone = $\frac{1}{3}\pi r^2 h = \frac{1}{3}\pi \times 5^2 \times 12$

$= 100\ \pi cm^3$

Ex. A right angle triangle whose side are 15cm and 20cm (other than Hypotenuse) is made to revolve about its hypotenuse. Find the volume of double cone so formed?

HINTS

$r = \frac{15 \times 20}{25} = 12\text{cm}$



Volume of cone so formed

$= \frac{1}{3}\pi \times 12^2 \times (9 + 16)$

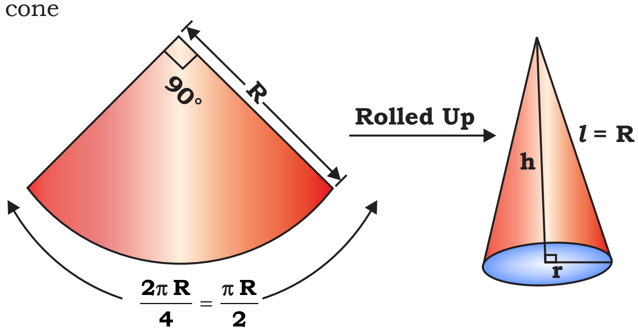
$= 48\pi \times 25$

$= 1200\pi \text{ cm}^3$

Cone Formed by Rolling up a Sector

When a sector is rolled up in such a way that the two binding radii are joined together then a cone formed.

1. A right angled sector of radius R cm is rolled up into a cone



$\Rightarrow 2\pi r = \frac{\pi R}{2} \Rightarrow r = \frac{R}{4}$ & $l = R$

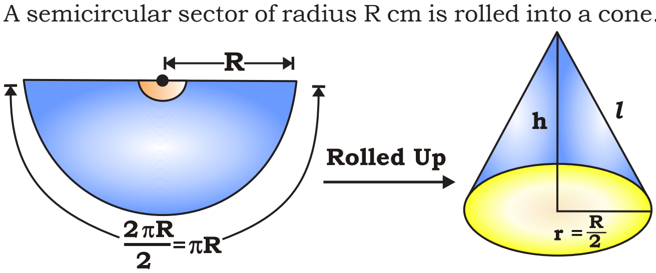
Height of cone (h) $= \sqrt{l^2 - r^2} = \sqrt{R^2 - \frac{R^2}{16}} = \frac{\sqrt{15}R}{4}$

Curved surface area of cone = Area of sector $= \frac{\pi R^2}{4}$

Volume of cone $= \frac{1}{3}\pi \times \left(\frac{R}{4}\right)^2 \times \frac{\sqrt{15}R}{4}$

$= \frac{\sqrt{15}\pi R^3}{192}$

2. A semicircular sector of radius R cm is rolled into a cone.



Height of cone (h)

$= \sqrt{l^2 - r^2} = \sqrt{R^2 - \frac{R^2}{4}} = \frac{\sqrt{3}R}{2}$

Curved surface area of cone = Area of sector $= \frac{\pi R^2}{2}$

Volume of cone $= \frac{1}{3}\pi r^2 h$

$= \frac{1}{3}\pi \times \frac{R^2}{4} \times \frac{\sqrt{3}R}{2} = \frac{\pi R^3}{8\sqrt{3}}$

Circle to Cone

Semicircle ($\frac{1}{2}$ part of circle)	Quadrant ($\frac{1}{4}$ part of circle)
Cone = l = R $r = \frac{R}{2}$	Cone = l = R $r = \frac{R}{4}$

Where R = Radius, l = Slant height of cone
r = radius of cone

Ex. A cone is made of a circle of radius 35 cm and an angle of 90°. What is the total surface area of cone?

HINTS Angle = 90° ($\frac{1}{4}$ part of circle)

Radius of sector (R) = 35 = l

Radius of cone (r) $= \frac{35}{4}$

Total surface area $= \pi r(l + r) = \frac{22}{7} \times \frac{35}{4}\left(35 + \frac{35}{4}\right)$

$= \frac{55}{2} \times \frac{175}{4} = 1203.125 \text{ cm}^2$

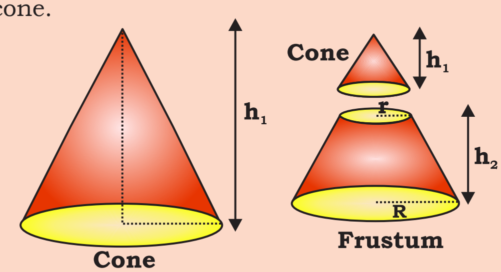
Ex. A semicircular sheet of metal of diameter 28 cm is bent into an open conical cup. The capacity of the cup $\left(\text{use } \pi = \frac{22}{7}\right)$ is:

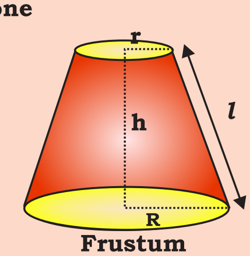
HINTS Capacity of cup $= \frac{\pi r^3}{8\sqrt{3}}$

$= \frac{22 \times 14 \times 14 \times 14}{7 \times 8 \times 1.732} = 622.4 \text{ cm}^3$

Frustum

If a cone is cut by a plane parallel to its base, the portion of solid between this plane and the base is known as frustum of the cone.





(i) Volume of frustum $= \frac{1}{3}\pi(R^2 + r^2 + Rr)\,h$

(Compare with volume of a cone $= \frac{1}{3}\pi r^2 h$)

(ii) Curved surface area $= \pi(R + r)l$

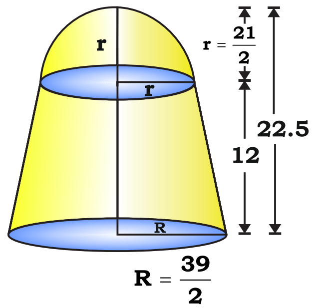
(Compare with curved surface area of a cone $= \pi r l$)

(iii) Total surface area $= \pi l(R + r) + \pi(R^2 + r^2)$

(Compare with total surface area of cone $= \pi r\,(r + l)$)

Where $l = \sqrt{h^2 + (R - r)^2}$

Ex. A 22.5 m high tent is in the shape of a frustum of a cone surmounted by a hemisphere. If the diameters of the upper and the lower circular ends of the frustum are 21 m and 39m, respectively, then find the area of the cloth (in m²) used to make the tent (ignoring the wastage).



HINTS

$l = \sqrt{h^2 + (R - r)^2}$

$= \sqrt{12^2 + \left(\frac{39}{2} - \frac{21}{2}\right)^2} = 15$

Required Area

$= \pi(r + R)l + 2\pi r^2$

$= \frac{22}{7}\left(\frac{21}{2} + \frac{39}{2}\right)15 + 2 \times \frac{22}{7} \times \left(\frac{21}{2}\right)^2$

$= \frac{22}{7} \times \frac{60}{2} \times 15 + 2 \times \frac{22}{7} \times \frac{21}{2} \times \frac{21}{2}$

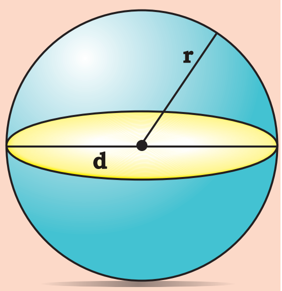
$= \frac{22}{7} \times \frac{1341}{2} = \frac{14751}{7} = 2107\frac{2}{7}\ m^2$

Sphere

A sphere is a solid obtained on revolving a circle about any diameter of it.

(i) Surface area of sphere = $4\pi r^2$

(ii) Volume of sphere = $\frac{4}{3}\pi r^3$



☞ Let V_1 and V_2 be volume and S_1 and S_2 be area of two sphere then

$\frac{V_1}{V_2} = \left(\frac{S_1}{S_2}\right)^{\frac{3}{2}}$ or $\frac{S_1}{S_2} = \left(\frac{V_1}{V_2}\right)^{\frac{2}{3}}$

Ex. Find the diameter of a spherical ball whose volume is 38.808 cm³.

HINTS Volume of sphere = $\frac{4}{3}\pi r^3 \Rightarrow 38.808 = \frac{4}{3} \times \frac{22}{7} \times r^3$

$\Rightarrow r = 2.1$ cm

$\therefore$ Diameter = 2.1 × 2 = 4.2 cm

Ex. The surface area of two spheres are in the ratio 4 : 9. Their volumes will be in the ratio.

HINTS $\frac{V_1}{V_2} = \left(\frac{S_1}{S_2}\right)^{\frac{3}{2}} = \left(\frac{4}{9}\right)^{\frac{3}{2}} = \left(\frac{2^2}{3^2}\right)^{\frac{3}{2}} = \frac{2^3}{3^3} = \frac{8}{27}$

Ex. The ratio of weights of two spheres of different materials is 8 : 17 and the ratio of weights per 1cc of materials of each is 289 : 64. Find the ratio of radii of the two spheres:

HINTS Density = $\frac{\text{Mass}}{\text{Volume}}$

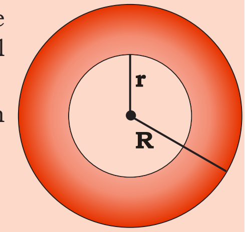
Ratio of the volumes of sphere = $\frac{\frac{8}{289}}{\frac{17}{64}} = \frac{8 \times 64}{289 \times 17}$

$\Rightarrow \frac{\frac{4}{3}\pi r_1^3}{\frac{4}{3}\pi r_2^3} = \frac{8 \times 8 \times 8}{17 \times 17 \times 17} \Rightarrow \frac{r_1^3}{r_2^3} = \left(\frac{8}{17}\right)^3 \Rightarrow \frac{r_1}{r_2} = \frac{8}{17}$

Spherical Shell

It is solid encosed between two concentric spheres.

Let R be the external radius and r be the internal radius of a spherical shell then,



(i) Volume = Volume of material in spherical shell = $\frac{4}{3}\pi(R^3 - r^3)$

(ii) Total surface area = $4\pi(R^2 - r^2)$

Ex. A hollow spherical metallic ball has an external diameter 6cm and is $\frac{1}{2}$ cm thick. The volume of the ball (in cm³) is

HINTS Volume of the ball = $\frac{4}{3}\pi\left[3^3 - \left(3 - \frac{1}{2}\right)^3\right]$

$= \frac{4}{3} \times \frac{22}{7}\left(27 - \frac{125}{8}\right) = \frac{4}{3} \times \frac{22}{7}\left(\frac{216 - 125}{8}\right)$

$= \frac{4}{3} \times \frac{22}{7} \times \frac{91}{8} = \frac{143}{3} = 47\frac{2}{3} cm^3$

Hemisphere

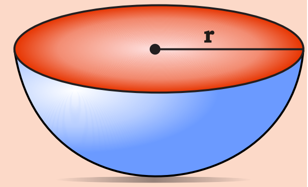
When a solid sphere is cut through its center into two equal (identical) piece, each piece is called a hemisphere.

(i) Curved Surface area = $2\pi r^2$

(ii) Total surface area

= $2\pi r^2 + \pi r^2 = 3\pi r^2$

(iii) Volume = $\frac{2}{3}\pi r^3$



Ex. The volume of a solid hemisphere is 19404 cm³. Its total surface area is:

HINTS $\frac{2}{3}\pi r^3 = 19404 \Rightarrow \frac{2}{3} \times \frac{22}{7} \times r^3 = 19404$

$\Rightarrow r^3 = \frac{19404 \times 3 \times 7}{2 \times 22} = 9261 \Rightarrow r = (9261)^{\frac{1}{3}} = 21cm$

$\therefore$ Total surface area = $3\pi r^2 = 3 \times \frac{22}{7} \times 21 \times 21 = 4158\ cm^2$

Hemispherical Shell

(i) Curved Surface area

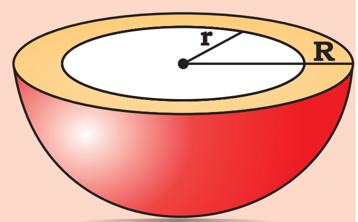
= $2\pi(R^2 + r^2)$

(ii) Total surface area

= $2\pi R^2 + 2\pi r^2 + \pi(R^2 - r^2)$

= $3\pi R^2 + \pi r^2$

(iii) Volume = $\frac{2}{3}\pi(R^3 - r^3)$



Ex. A hemispherical bowl is made of silver and its inner diameter is 4 cm. If the thickness of the silver is 0.5 cm, then calculate the amount of silver used to make the bowl, correct to two decimal places. (Use π = 3.14)

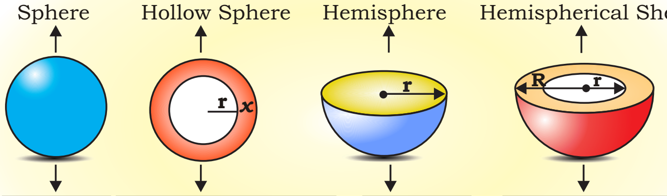
HINTS r = 2 cm, R = 2 + 0.5 = 2.5 cm

Volume of silver used = $\frac{2}{3}\pi(R^3 - r^3)$

$= \frac{2}{3} \times 3.14\,(2.5^3 - 2^3) = \frac{2}{3} \times 3.14 \times 7.625 = 15.96\ cm^3$

Summary

Sphere	Hollow Sphere	Hemisphere	Hemispherical Shell



$V = \frac{4}{3}\pi r^3$	$V = \frac{4}{3}\pi\left[(r+x)^3 - r^3\right]$	$V = \frac{2}{3}\pi r^3$	$V = \frac{2}{3}\pi\left(R^3 - r^3\right)$

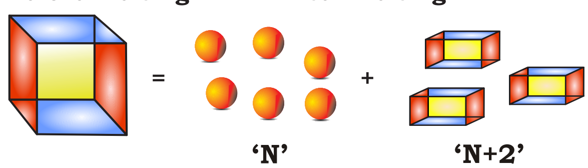
Concept of Melting & Recasting

Volume of Melted object = Volume of recasted object

Ex. A large solid cube is melted and cast into 'N' small solid spheres, each of radius 3 cm, and 'N + 2' small solid cuboids, each of dimensions 4 cm × 4 cm × 6.5 cm. If the length of each side of the large solid cube is 12 cm, then find the value of 'N'.

HINTS

Before Melting **After Melting**



Given, radius of small spheres = 3 cm

Volume of cuboid = (4 × 4 × 6·5) cm³

Side of cube = 12 cm

ATQ,

$$12 \times 12 \times 12 = \left(\frac{4}{3} \times \frac{22}{7} \times 3 \times 3 \times 3 \times N\right) + (N+2)(4 \times 4 \times 6{\cdot}5)$$

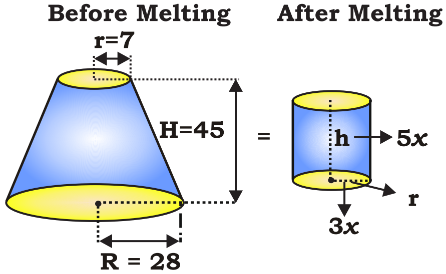
$$\Rightarrow 1728 = \frac{88}{7} \times 9N + 104N + 208$$

$$\Rightarrow 1520 = \frac{792}{7} \times N + 104N \Rightarrow 1520 = \frac{1520}{7}N \Rightarrow N = 7$$

Ex. The radii of the ends of a frustum of a solid right-circular cone 45 cm high are 28 cm and 7 cm. If this frustum is melted and reconstructed into a solid right circular cylinder whose radius of base and height are in the ratio 3 : 5 then find the curved surface area (in cm²) of this cylinder.

HINTS

Before Melting **After Melting**



Volume of frustum = Volume of cylinder

$$\Rightarrow \frac{1}{3}\pi H\left(R^2 + Rr + r^2\right) = \pi r^2 h$$

$$\Rightarrow \frac{45}{3}\left[28^2 + (28 \times 7) + 7^2\right] = 9x^2 \times 5x \Rightarrow 784 + 196 + 49 = 3x^3$$

$$\Rightarrow 1029 = 3x^3 \Rightarrow x^3 = 343 \Rightarrow x = 7$$

∴ Radius of cylinder = 3 × 7 = 21 cm

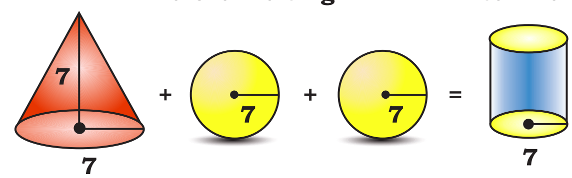
Height of cylinder = 5 × 7 = 35 cm

$$\therefore \text{C.S.A of cylinder} = 2\pi rh = 2 \times \frac{22}{7} \times 21 \times 35 = 4620 \text{ cm}^2$$

Ex. A solid cone of radius 7 cm and height 7 cm was melted along with two solid spheres of radius 7 cm each to form a solid cylinder of radius 7 cm. What is the curved surface area (in cm²) of the cylinder?

HINTS

Before Melting **After Melting**



Volume of (Cone + 2 spheres) = Volume of cylinder

$$\Rightarrow \frac{1}{3} \times \pi \times 7^2 \times 7 + 2 \times \frac{4}{3}\pi \times 7^3 = \pi R^2 h$$

$$\Rightarrow \frac{1}{3}\pi \times 343 \times (1 + 8) = \pi R^2 h$$

$$\Rightarrow R^2 h = 343 \times 3 \Rightarrow 49 \times h = 343 \times 3$$

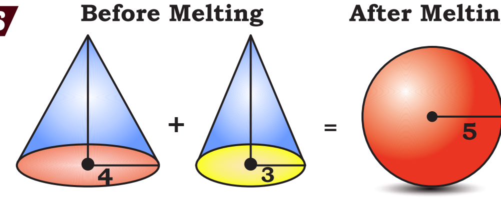
$$\Rightarrow h = 21 \text{ cm}$$

∴ C.S.A. of cylinder = 2πrh

$$= 2 \times \frac{22}{7} \times 7 \times 21 = 44 \times 21 = 924 \text{ cm}^2$$

Ex. Two right circular cones of equal height of radii of base 3 cm and 4 cm are melted together and made to a solid sphere of radius 5cm. The height of a cone is:

HINTS **Before Melting** **After Melting**



Volume of two cones = Volume of sphere

$$\Rightarrow \frac{1}{3}\pi \times 3^2 \times h + \frac{1}{3}\pi \times 4^2 \times h = \frac{4}{3}\pi \times 5^3$$

$$\Rightarrow 9h + 16h = 4 \times 125$$

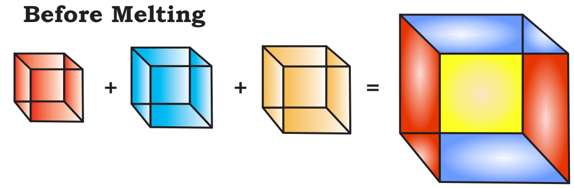
$$\Rightarrow h = \frac{4 \times 125}{25} = 20 \text{ cm}$$

Ex. Three cubes of metal whose edges are 6cm, 8 cm and 10 cm, respectively are melted and a single cube is formed. What is the length of the edge of the newly formed cube?

HINTS

After Melting

Before Melting



Volume of recast cube = Sum of volume of three cubes

$$\Rightarrow a^3 = 6^3 + 8^3 + 10^3 = 216 + 512 + 1000 = 1728$$

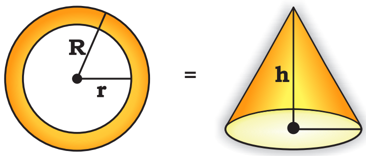
$$\Rightarrow a = (1728)^{\frac{1}{3}} = 12\text{cm}$$

When many cubes integrate into one cube then the side of the new cube

$= \sqrt[3]{\text{Sum of the cubes of sides of all the cubes}}$

Ex. A hollow sphere of internal and external diameters 6cm and 10cm respectively is melted into a right circular cone of diameter 8 cm. The height of the cone is:

HINTS **Before Melting** **After Melting**



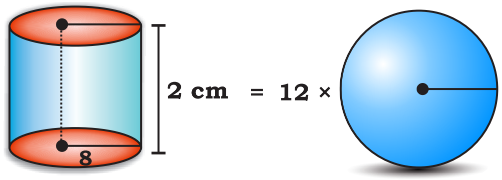
Volume of cone = Volume of hollow sphere

$\Rightarrow \frac{1}{3}\pi \times 4^2 \times h = \frac{4}{3}\pi(5^3 - 3^3) \Rightarrow h = \frac{125-27}{4} = \frac{98}{4} = \frac{49}{2}$

= 24.5cm

Ex. 12 sphere of the same size are made by melting a solid cylinder of 16 cm diameter and 2 cm height. The diameter of each sphere is:

HINTS **Before Melting** **After Melting**



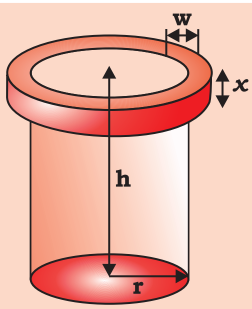
Volume of 12 spheres = Volume of cylinder

$\Rightarrow 12 \times \frac{4}{3}\pi r^3 = \pi \times 8^2 \times 2 \Rightarrow r^3 = \frac{64 \times 2 \times 3}{12 \times 4} = 8 \Rightarrow r = 2\text{cm}$

Concept of Digging & Earth taken out

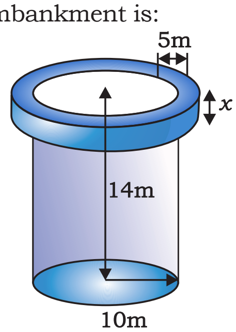
A well of diameter 'D' m or radius 'r' m is dug 'h' m deep. If earth taken out has been spread all around it to a width of 'w' m to form a circular embankment then height of embankment

$= \frac{r^2h}{w(w+D)}$



Ex. A well 20 m in diameter is dug 14 m deep and the earth taken out is spread all around it to a width of 5 m to form an embankment. The height of the embankment is:

HINTS Volume of embankment = Volume of well (Volume of earth taken out)



$\Rightarrow \pi(15^2 - 10^2)\, h = \pi \times 10^2 \times 14$

$\Rightarrow h = \frac{10 \times 10 \times 14}{125} = \frac{56}{5} = 11.2\text{m}$

Alternatively

Height of embankment $= \frac{r^2h}{w(w+D)}$

$= \frac{10 \times 10 \times 14}{5(5+20)} = \frac{1400}{125} = \frac{56}{5} = 11.2 \text{ m}$

Ex. A field is 125m long and 15m wide. A tank 10 m × 7.5 m × 6 m was dug in it and the Earth thus dug out was spread equally on the remaining field. The level of the field thus raised is?

HINTS Remaining field = Area of field – Area of tank

$= 125 \times 15 - 10 \times 7.5 = 1800 \text{ m}^2$

Volume of earth dug $= 10 \times 7.5 \times 6 = 450\text{m}^3$

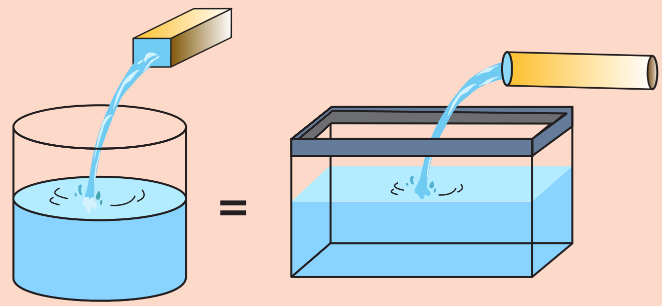
By given condition, $1800 \times h = 450$

$\Rightarrow h = \frac{450}{1800} = \frac{1}{4}\text{m} = \frac{100}{4}\text{cm} = 25\text{cm}$

Filling a Container by Another Shape Container

Volume of water flowing through pipe (Cylindrical / Cuboidal) in t time = Volume of the tank (Cylindrical / Cuboidal)

⇒ Area of base × speed of flow × time = Volume of the tank (Cylindrical / Cuboidal)



(a) $\pi r^2 \times v \times t = \pi R^2 h$ or LBH or $\frac{1}{3}\pi R^2 H$

(b) $l \times b \times v \times t$ = LBH or $\pi R^2 h$

Ex. Water flows through a cylindrical pipe, whose radius is 7cm, at 5 metre per second. The time, it takes to fill an empty water tank, with height 1.54 metres and area of the base (3 × 5) square metres is

HINTS πr^2 (vt) = Area of base × height

$\Rightarrow \frac{22}{7} \times \frac{7}{100} \times \frac{7}{100} \times 5t = (3 \times 5) \times 1.54$

$\Rightarrow t = 300$ seconds = 5 minutes

Ex. Water flows at the rate of 10 metres per minute from a cylindrical pipe 5 mm in diameter. How long it take to fill up a conical vessel whose diameter at the base is 30 cm and depth 24cm?

HINTS $\pi r^2 (vt) = \frac{1}{3}\pi R^2 H$

$\Rightarrow 0.25 \times 0.25 \times 1000t = \frac{1}{3} \times 15 \times 15 \times 24$

$\Rightarrow t = \frac{144}{5} = 28\frac{4}{5}$ minutes = 28 min, $\frac{4}{5} \times 60$ seconds

= 28 min 48 seconds

Ex. Water is flowing at the rate of 3km/hr through a circular pipe of 20cm internal diameter into a circular cistern of diameter 10m and depth 2m. In how much time will the cistern be filled?

HINTS Water flowed in t hours through the pipe = volume of circular cistern

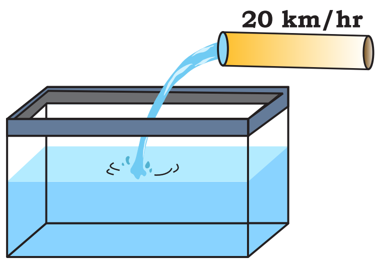
$\Rightarrow \pi r^2 (vt) = \pi R^2 H$

$\Rightarrow t = \frac{R^2H}{r^2v} = \frac{5 \times 5 \times 2 \times 100 \times 100}{10 \times 10 \times 3000} = \frac{5}{3}$

$= 1 \text{ hour}, \frac{2}{3} \times 60$ minutes = 1 hour, 40 minutes

Ex. Water flows into a tank which is 200m long and 150m wide, through a pipe of cross-section 0.3m × 0.2m at 20 km/hour. Then the time (in hours) for the water level in the tank to reach 8m is:

HINTS Let time for the water level in the tank to reach 8m be t hours.



∴ Water supplied by pipe in t hours = Volume of water tank

$\Rightarrow$ Area of cross section × (v × t) = l × b × h

$\Rightarrow$ 0.3 × 0.2 × 20000t = 200 × 150 × 8

$$\Rightarrow t = \frac{200 \times 150 \times 8}{0.3 \times 0.2 \times 20000} = 200 \text{ hours}$$

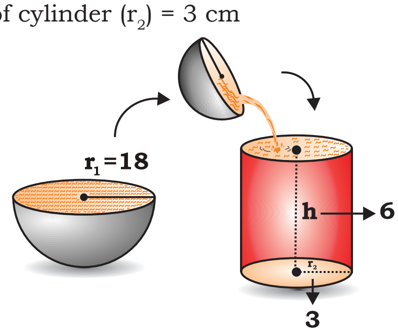
Ex. Some medicine in liquid form is prepared in a hemispherical container of diameter 36 cm. When the container is full of medicine, the medicine is transferred to small cylindrical bottles of diameter 6 cm and height 6 cm. How many bottles are required to empty the container?

HINTS Diameter of hemisphere = 36 cm

Radius of hemisphere (r_1) = 18 cm

Diameter of cylinder = 6 cm

Radius of cylinder (r_2) = 3 cm



Let n cylindrical bottles are required.

∴ Volume of hemisphere = n × volume of cylinder

$$\Rightarrow \frac{2}{3}\pi r_1^3 = n \times \pi r_2^2 \times h$$

$$\Rightarrow \frac{2}{3} \times 18 \times 18 \times 18 = n \times 3 \times 3 \times 6 \Rightarrow n = 72$$

Do It Yourself:

Ex. A hemispherical bowl of internal radius 6 cm contains a liquid. This liquid is to be filled into cylindrical shaped small bottles of diameter 2 cm and height 4 cm. How many bottles will be needed to empty the bowl ?

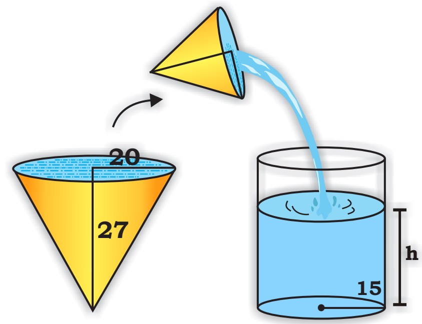
(a) 32 (b) 37 (c) 38 (d) 36

[Correct option (d)]

Concept of Rise in Height

Ex. A conical vessel, whose internal radius is 20 cm and height is 27 cm, is full of water. If this water is poured into a cylindrical vessel with internal radius 15cm, what will be the height to which the water rises in it?

HINTS



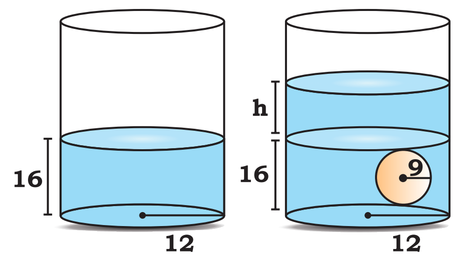
Volume of cone = Volume of cylinder

$$\Rightarrow \frac{1}{3} \times \pi r^2 h = \pi r_1^2 h_1$$

$$\Rightarrow \frac{1}{3} \times \pi \times (20)^2 \times 27 = \pi \times (15)^2 \times h_1 \Rightarrow h_1 = 16 \text{ cm}$$

Ex. A cylinder has some water in it at a height of 16 cm. If a sphere of radius 9 cm is put into it, then find the rise in the height of the water if the radius of the cylinder is 12 cm.

HINTS



Let, the rise in water level be h cm.

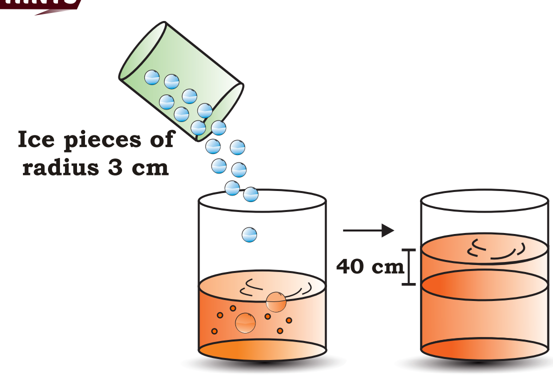
then, Volume of sphere = Volume of water risen.

$$\Rightarrow \frac{4}{3}\pi(9)^3 = \pi \times (12)^2 \times h$$

$$\Rightarrow h = \frac{4}{3} \times \frac{9 \times 9 \times 9}{12 \times 12} = \frac{27}{4} = 6.75 \text{ cm}$$

Ex. Some ice pieces, spherical in shape, of diameter 6 cm are dropped in a cylindrical container containing some juice and are fully submerged. If the diameter of the container is 18 cm and level of juice rises by 40 cm, then how many ice pieces are dropped in the container?

HINTS



Let n Ice pieces are dropped.

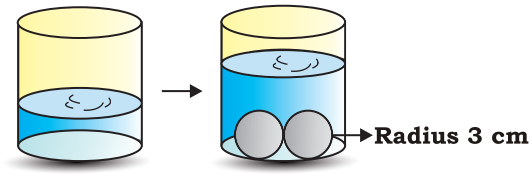
Volume of cylinder = n × Volume of spherical Ice pieces

$\Rightarrow \pi \times 9 \times 9 \times 40 = \frac{4}{3} \times \pi \times (3)^3 \times n$

$\Rightarrow n = \frac{9 \times 9 \times 40 \times 3}{3 \times 3 \times 3 \times 4} = 90$

Ex. Two irons sphere each of diameter 6cm are immeresed in the water contained in a cylindrical vessel of radius 6cm. The level of the water in the vessel will be raised by.

HINTS

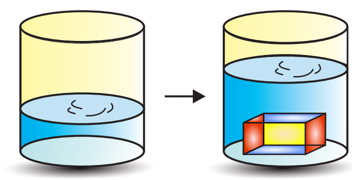


Volume of two iron sphere = volume of cylinder

$\Rightarrow 2 \times \frac{4}{3} \times \pi \times 3^3 = \pi \times 6^2 \times h \Rightarrow h = 2$ cm

Ex. A cylindrical vessel of base radius 14 cm is filled with water to some height. If a rectangular solid of dimensions 22 cm × 7 cm × 5 cm is immersed in it then what is the rise in water level?

HINTS



Volume of cylinder = Volume of rectangular solid

$\Rightarrow \frac{22}{7} \times 14 \times 14 \times h = 22 \times 7 \times 5 \Rightarrow h = \frac{5}{4} = 1.25$ cm

Ex. A cylindrical tank of diameter 35 cm is full of water. If 11 litres of water is drawn off. The water level in the tank will drop by:

$\left(\text{use } \pi = \frac{22}{7}\right)$

HINTS Volume of cylinder = $\pi r^2 h$

$\Rightarrow \pi r^2 h = 11 \times 1000 \text{ cm}^3$

$\Rightarrow \frac{22}{7} \times \frac{35}{2} \times \frac{35}{2} \times h = 11 \times 1000$

$\Rightarrow h = \frac{80}{7} = 11\frac{3}{7}$ cm

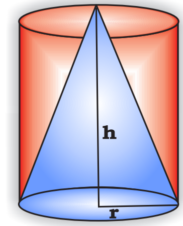
Ex. A rectangular tank whose length and breadth are 2.5 m and 1.5 m, respectively is half fill of water. If 750 L more water is poured into the tank, then what is the height through which water level further goes up?

HINTS Volume of extra cuboid = Volume of water

$\Rightarrow 2.5 \times 1.5 \times h = \frac{750}{1000} \Rightarrow h = 0.2\text{m} = 20\text{cm}$

Combination of 3-D Objects

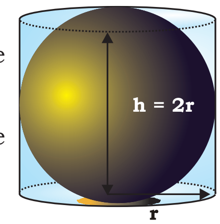
(i) A maximum Cone inside a Cylinder



Volume of cylinder : Volume of cone

$= \pi r^2 h : \frac{1}{3}\pi r^2 h = 3 : 1$

(ii) A Cylinder just encloses a Sphere



Height of cylinder = Diameter of sphere = 2r

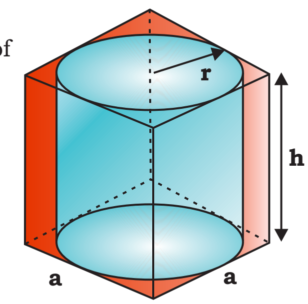
Volume of Cylinder : Volume of Sphere

$= \pi r^2 (2r) : \frac{4}{3}\pi r^3 = 2 : \frac{4}{3} = 3 : 2$

(iii) A maximum Cylinder inside Cube

Radius of cylinder $= \frac{1}{2} \times$ edge of cube $= \frac{a}{2}$

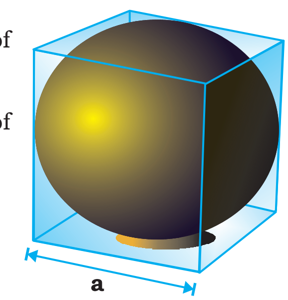
Height of cylinder = edge of cube = a



Volume of cube : Volume of cylinder $= a^3 : \pi\left(\frac{a}{2}\right)^2 a$

$= 1 : \frac{22}{7} \times \frac{1}{4} = 14 : 11$

(iv) A maximum Sphere inside a Cube

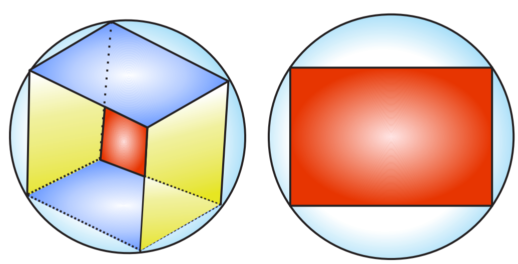


Diameter of sphere (2r) = edge of cube = a

Volume of cube : Volume of sphere $= a^3 : \frac{4}{3}\pi\left(\frac{a}{2}\right)^3$

= 21: 11

(v) A maximum Cube inside a Sphere



Diagonal of cube = Diameter of sphere

$\Rightarrow \sqrt{3}\, a = 2r \Rightarrow a = \frac{2r}{\sqrt{3}}$

Volume of sphere : Volume of cube

$= \frac{4}{3}\pi r^3 : \left(\frac{2r}{\sqrt{3}}\right)^3 = \frac{4}{3} \times \frac{22}{7} : \frac{8}{3\sqrt{3}} = 11\sqrt{3} : 7$

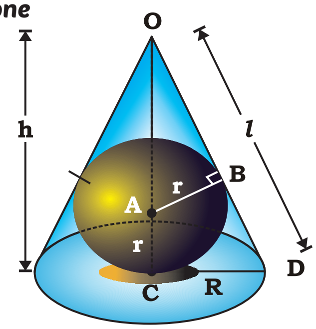
(vi) A maximum Sphere inside a Cone

$\Delta OCD \sim \Delta OBA$

$\Rightarrow \dfrac{OD}{OA} = \dfrac{CD}{AB}$

$\Rightarrow \dfrac{l}{h-r} = \dfrac{R}{r}$

$\Rightarrow l \times r = hR - Rr \Rightarrow r = \dfrac{hR}{l+R}$



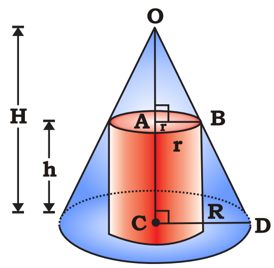
(vii) A maximum Cylinder inside a Cone

$\Delta OCD \sim \Delta OAB$

$(\angle A = \angle C = 90°, \angle O \text{ common})$

$\Rightarrow \dfrac{OC}{OA} = \dfrac{CD}{AB}$

$\Rightarrow \dfrac{H}{H-h} = \dfrac{R}{r}$



(viii) A maximum Cube inside a Cone

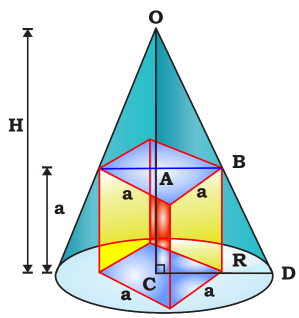
$\Delta OCD \sim \Delta OAB$

$\Rightarrow \dfrac{OC}{OA} = \dfrac{CD}{AB}$

$\Rightarrow \dfrac{H}{H-a} = \dfrac{R}{a/\sqrt{2}}$

$\left[\therefore AB = \dfrac{\sqrt{2}a}{2} = \dfrac{a}{\sqrt{2}}\right]$

$\Rightarrow a = \dfrac{\sqrt{2}RH}{\sqrt{2}R + H}$



(ix) Largest Cube inside a Hemisphere

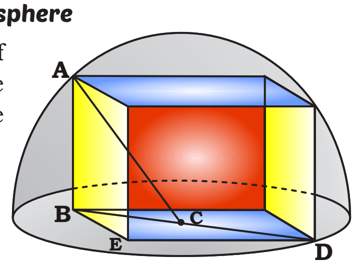
Let, R be the radius of hemisphere and x be the side of cube. C is the centre of hemisphere.

$BD = \sqrt{2}x$

$\therefore BC = \dfrac{x}{\sqrt{2}}$

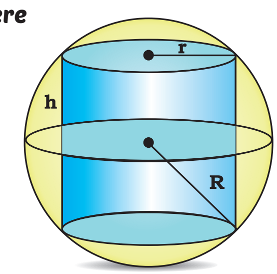
In ΔABC, $AC^2 = AB^2 + BC^2$

$\Rightarrow R^2 = \dfrac{3x^2}{2} \Rightarrow x = \sqrt{\dfrac{2}{3}}R$



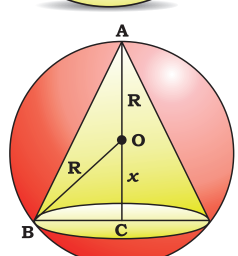
(x) A maximum Cylinder inside a Sphere

$$\frac{\text{Volume of cylinder}}{\text{Volume of Sphere}} = \frac{1}{\sqrt{3}}$$



(xi) A maximum Cone inside a Sphere

$$\frac{\text{Volume of cone}}{\text{Volume of Sphere}} = \frac{8}{27}$$



Ex. The Largest possible sphere is carved out of a cube of side 6 cm. What is the volume of the sphere?

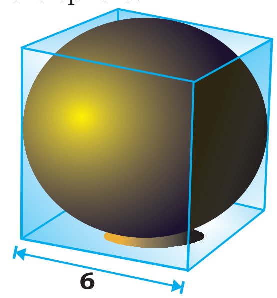
HINTS

Radius of largest possible sphere

$= \dfrac{6}{2} = 3$ cm

Volume of sphere $= \dfrac{4}{3}\pi r^3$

$= \dfrac{4}{3} \times \dfrac{22}{7} \times 3 \times 3 \times 3 = \dfrac{792}{7}$ cm^3



Alternatively

Volume of cube = $a^3 = 6^3 = 216$

Volume of cube : Volume of sphere = 21 : 11

$\downarrow$ 216 $\quad \downarrow \dfrac{792}{7}$

Ex. A sphere passes through the eight corners of a cube of side 14 cm. Find the volume (in cm^3) of the sphere. $\left(\text{use } \pi = \dfrac{22}{7}\right)$

HINTS

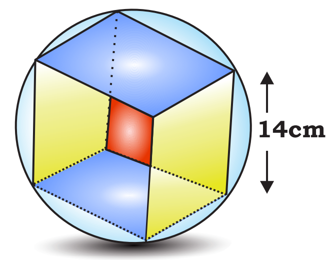
Side of the cube = 14 cm

$\Rightarrow$ Diagonal of cube = $14\sqrt{3}$

$\Rightarrow$ Radius of sphere $= \dfrac{14\sqrt{3}}{2}$

$= 7\sqrt{3}$

$\therefore$ Volume of sphere $= \dfrac{4}{3}\pi r^3$



$= \dfrac{4}{3} \times \dfrac{22}{7} \times 7\sqrt{3} \times 7\sqrt{3} \times 7\sqrt{3} = 88 \times 49\sqrt{3} = 4312\sqrt{3}$ cm^3

Alternatively

Volume of sphere : Volume of cube

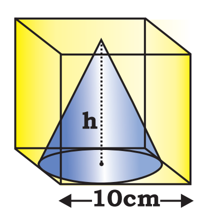
$11\sqrt{3}$: 7

$\downarrow \times 392$ $\quad \downarrow \times 392$

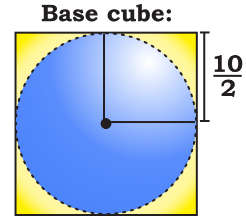
$\boxed{4312\sqrt{3}}$ $\quad 14^3$(Given)

Ex. What is the volume of the largest right circular cone that can be cut out from a cube whose edge is 10 cm?

HINTS



Base cube:



Height of cone = height of cube.

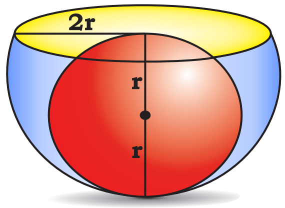
Radius of cone $= \dfrac{1}{2} \times$ width of cube.

Given, edge = 10 cm

$\therefore$ Volume of cone $= \dfrac{1}{3}\pi r^2 h = \dfrac{1}{3}\pi \times 5 \times 5 \times 10 = \dfrac{250}{3}\pi$ cm^3

Ex. A sphere of maximum volume is cut out from a solid hemisphere. What is the ratio of the volume of the sphere to that of the remaining solid?

HINTS



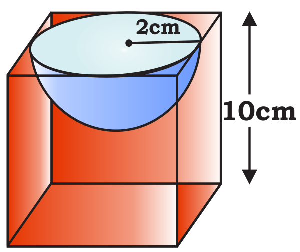
Vol. of sphere	:	Vol. of remaining Solid
$\frac{4}{3}\pi\times r^3$	:	$\frac{2}{3}\pi(2r)^3-\frac{4}{3}\pi r^3$
$\frac{4}{3}\pi r^3$	:	$\frac{2}{3}\pi\times 8r^3-\frac{4}{3}\pi r^3$
$\frac{4}{3}\pi r^3$	:	$\frac{12}{3}\pi r^3$
1	:	3

$\therefore$ Required ratio = 1 : 3

Ex. A hemispherical depression of diameter 4 cm is cut out from each face of a cubical block of sides 10 cm. Find the surface area of the remaining solid (in cm²).

$\left(\text{use } \pi = \frac{22}{7}\right)$

HINTS



Radius of hemisphere

$= \frac{4}{2} = 2$ cm

Required surface area

= T.S.A. of the cube – 6 × area of circle formed on the top of each face of the cube + 6 × C.S.A. of hemisphere formed on the each face of cube

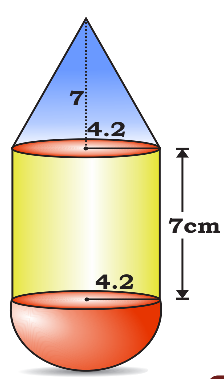
$= 6\times 10^2 - 6\pi\times 2^2 + 6\times 2\pi\times 2^2 = 600 + 24\pi$

$= 600 + 24\times\frac{22}{7} = 600 + \frac{528}{7}$

$= 600 + 75\frac{3}{7} = 675\frac{3}{7}$ cm²

Ex. For making a toy, a hemisphere is attached at one end of a cylinder and a cone is attached at the other end of the cylinder. The cylinder, the cone and the hemisphere have a common radius of 4.2 cm. The height of the cylinder and that of the cone is 7 cm. Find the volume (in cm³) of the toy. (Use $\pi = \frac{22}{7}$)

HINTS



Volume of toy = Volume of cylinder + Volume of hemisphere + Volume of cone

$= \pi r^2 h + \frac{2}{3}\pi r^3 + \frac{1}{3}\pi r^2 h$

$= \pi r^2\left(h + \frac{2}{3}r + \frac{h}{3}\right)$

$= \frac{22}{7}\times 4.2\times 4.2\times\left(7+\frac{2}{3}\times 4.2+\frac{7}{3}\right)$

$= 672.672$ cm³

Concept of Cutting a Solid

(a) Volume of solid does'nt change

Ex. A sphere is cut into hemisphere. One of them is used as bowl. It takes 8 bowlfuls of this to fill a conical vessel of height 12cm and radius 6cm. The radius of the cylinder is:

HINTS Volume of 8 hemishpere = Volume of cone

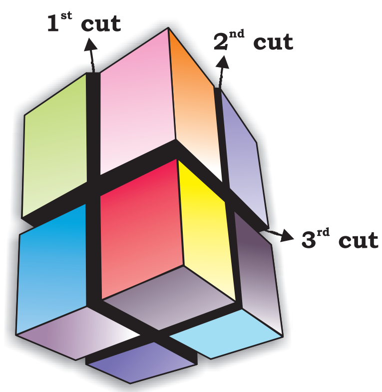
$\Rightarrow 8\times\frac{2}{3}\times\pi\times r^3 = \frac{1}{3}\pi\times 6^2\times 12$

$\Rightarrow r^3 = \frac{6^2\times 12}{8\times 2} = \frac{6^3}{8} = \frac{6^3}{2^3} \Rightarrow r = \frac{6}{2} = 3\text{cm}$

(b) Surface area of solid increases.

Ex. A cuboid of size 50 cm × 40 cm × 30 cm is cut into 8 identical parts by 3 cuts. What is the total surface area (in cm²) of all the 8 parts?

HINTS



Total surface area before cutting = 2(50 × 40 + 40 × 30 + 30 × 50) = 9400 cm²

There are three cuts along length - breadth, breadth height and height - length. So, each cut will produce extra length, breadth and height.

After cutting into 8 equal parts, surface area will be doubled.

$\therefore$ Total surface area after cutting = 2 × 9400 = 18800 cm²

Alternatively

There are three cuts along length - breadth, breadth height and height - length. So, each cut will divide length, breadth and height into two equal part.

New dimension of each part = 25 cm × 20 cm × 15 cm

Total surface area of each part = 2(25 × 20 + 20 × 15 + 15 × 25) = 2350 cm²

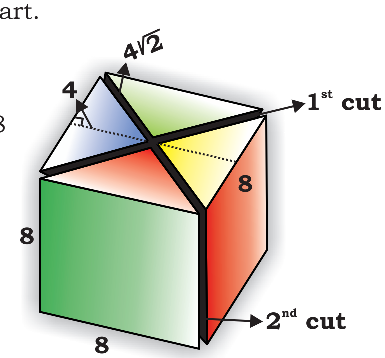
Total surface area of 8 parts = 2350 × 8 = 18800 cm²

Ex. A solid cube has side 8 cm. It is cut along diagonals of top face to get 4 equal parts. What is the total surface area (in cm²) of each part.

HINTS

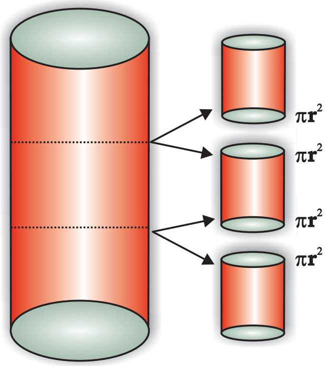
$\text{TSA} = 2\times\frac{1}{2}\times 8\times 4 + 8\times 8 + 8\times 4\sqrt{2}\times 2$

$= 96 + 64\sqrt{2}$



Ex. A right circular cylinder has height 18 cm and radius 7 cm. The cylinder is cut in three equal parts (by 2 cuts parallel to base). What is the percentage increase in total surface area?

HINTS



TSA = $2\pi r(h + r)$

= $44 \times 25 = 1100$ cm²

Increase in surface area = $4\pi r^2$

= $4 \times 154 = 616$ cm²

% increase = $\frac{616}{1100} \times 100\% = 56\%$

Ex. A solid sphere of diameter 17.5cm is cut into two equal halves. What will be the increase (in cm²) in the total surface area?

HINTS Total surface area of solid sphere = $4\pi r^2$

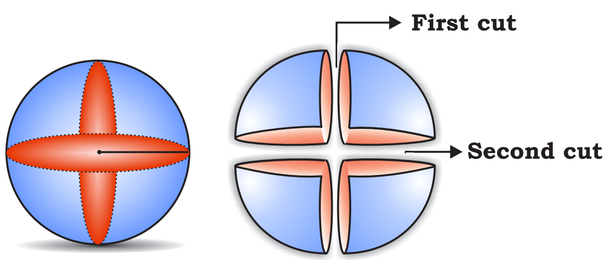
When it cut into half parts in two hemisphere

then, Total surface area = $3\pi r^2 + 3\pi r^2 = 6\pi r^2$

So, increase in area = $6\pi r^2 - 4\pi r^2 = 2\pi r^2$

= $2 \times \frac{22}{7} \times \frac{17.5}{2} \times \frac{17.5}{2}$ = 481.25 cm²

☞



2 cut (4 pieces), 1 cut → 2 circle area (Increase)

4 part TSA = $4\pi r^2 + 4\pi r^2 = 8\pi r^2$

TSA of each part (quarter sphere) = $\frac{8\pi r^2}{4} = 2\pi r^2$

Ex. A spherical ball is first polished and then it was cut into 4 equal pieces. What is the ratio of the polished area to the unpolished area?

HINTS Polished area = $4\pi r^2$

Non-polished area

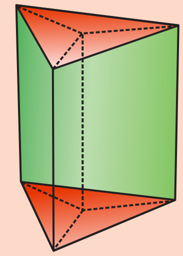
= $4 \times \left(2 \times \frac{\pi r^2}{2}\right) = 4\pi r^2$

$\frac{\text{Ar(Polished)}}{\text{Ar(unpolished)}} = \frac{4\pi r^2}{4\pi r^2}$

= 1 : 1

Prism

A prism is a solid that has two faces that are parallel and congruent and their faces (Polygon) join by vertex to vertex. A prism has a polygon as its base and vertical side perpendicular to the base.



(i) Curved surface area of a prism

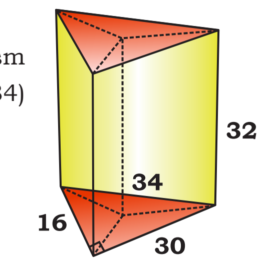
= Perimeter of base × height

(ii) Total surface area of a prism = Curved surface area + 2 × area of base

(iii) Volume of a prism = Area of base × height

Ex. The base of a right prism is a triangle with sides 16 cm, 30 cm and 34 cm. Its height is 32 cm. The lateral surface area (in cm²) and the volume (in cm³) are, respectively:

HINTS



Lateral surface area of prism

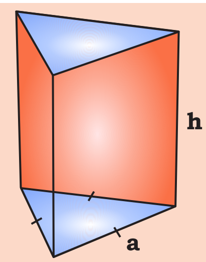
= perimeter × height = (16 + 30 + 34) × 32 = 2560 cm²

Volume of prism

= area of base × height

= $\frac{1}{2} \times 30 \times 16 \times 32 = 7680$ cm³

Equilateral triangular prism



(i) C.S.A. = 3ah

(ii) T.S.A. = $3ah + 2 \times \frac{\sqrt{3}}{4}a^2$

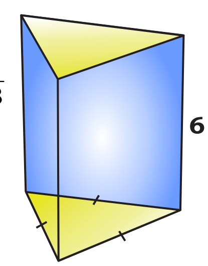
(iii) Volume = $\frac{\sqrt{3}}{4}a^2h$

Where a = side of equilateral triangle,

h = height of prism

Ex. The total surface area of a triangle prism of the height 6 cm is $162\sqrt{3}$ cm². If the base of the prism is an equilateral triangle. Find its volume?

HINTS T.S.A. = $162\sqrt{3}$



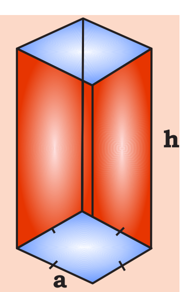
⇒ L.S.A. + 2 × area of base = $162\sqrt{3}$

⇒ $3a \times 6 + \frac{\sqrt{3}}{2}a^2 = 162\sqrt{3} \Rightarrow a = 6\sqrt{3}$

Volume = $\frac{\sqrt{3}}{4}\left(6\sqrt{3}\right)^2 \times h$

= $\frac{\sqrt{3}}{4} \times 6 \times 6 \times 6 \times 3 = 162\sqrt{3}$ cm³

Square Prism



(i) C.S.A = 4ah

(ii) T.S.A = 4ah + 2a²

(iii) Volume = a²h

Where a = side of square,

h = height of prism

Ex. The height of a right prism with a square base is 15 cm. If the total surface of the prism is 608 sq. cm, then its volume is:

HINTS Let, side of base = x

T.S.A = 4ah + 2a²

$\Rightarrow 608 = 4x \times 15 + 2x^2$

$\Rightarrow 2x^2 + 60x - 608 = 0$

On solving, $x = 8$

V = Area of base × height = $8^2 \times 15 = 960$ cm³

Ex. The base of a solid prism of height 10 cm is a square and its volume is 160 cm³, What is its total surface area of the prism (in cm²) ?

HINTS Volume of prism = base area × height

$\Rightarrow 160 = a^2 \times 10$
$\Rightarrow a^2 = 16$
$\Rightarrow a = 4$
side of square (a) = 4 cm.
Perimeter of square = 4a = 4 × 4 = 16 cm
∴ Total surface area of prism
= curved surface area + 2 × area of base
= 16 × 10 + 2 × 4 × 4 = 160 + 32
= 192 cm²

Hexagonal Prism

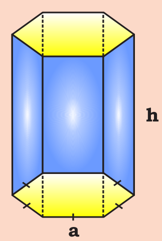
(i) C.S.A = 6ah

(ii) T.S.A = $6ah + 3\sqrt{3}\,a^2$

(iii) Volume = $\dfrac{6 \times \sqrt{3}}{4}a^2h$

Where a = side of hexagon,

h = height of prism



Ex. A prism has a regular hexagonal base with side 6 cm. If the total surface area of the prism is $216\sqrt{3}$ cm², then what is the height of prism?

HINTS Area of equilateral Δ = $\dfrac{\sqrt{3}}{4} \times 6 \times 6 = 9\sqrt{3}$ cm²

Area of hexagonal = $6 \times 9\sqrt{3}$ cm²

Area of two Hexagonal = $2 \times 6 \times 9\sqrt{3} = 108\sqrt{3}$ cm²

TSA = $6ah + 3\sqrt{3}\,a^2$

$\Rightarrow 216\sqrt{3} = 36 \times h + 108\sqrt{3}$

$\Rightarrow h = \dfrac{108\sqrt{3}}{36} = 3\sqrt{3}$ cm

Ex. The base of a right prism is a regular hexagon of side 5 cm. If its height is $12\sqrt{3}$ cm, then its volume (in cm³) is:

HINTS h = $12\sqrt{3}$ cm, a = 5cm

Area of regular hexagon = $\dfrac{3\sqrt{3}a^2}{2}$

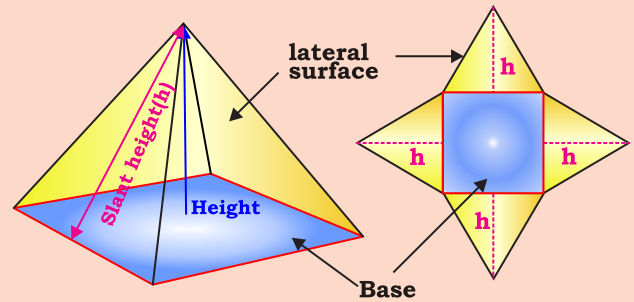
$= \dfrac{3\sqrt{3} \times 5^2}{2} = \dfrac{75\sqrt{3}}{2}$ cm²

∴ Volume = base area × height

$= \dfrac{75\sqrt{3}}{2} \times 12\sqrt{3} = 1350$cm³

Pyramid

A pyramid is a three-dimensional shape. A pyramid has a polygonal base and flat triangular faces, which join at a common point called the apex. A pyramid is formed by connecting the bases to an apex. Each edge of the base is connected to the apex, and forms the triangular face, called the lateral face. If a pyramid has an n-sided base, then it has n + 1 faces, n + 1 vertices, and 2n edges.



(i) Lateral/Curved surface area of Pyramid = Sum of areas of all the lateral triangular faces.

$= \dfrac{1}{2} \times$ Perimeter of base × slant height

(ii) Total surface area of Pyramid = Sum of the areas of all lateral faces + Area of the base.

= Curved surface area + area of base

$= \dfrac{1}{2} \times$ Perimeter of base × slant height + Area of base

(iii) Volume of a Pyramid

$= \dfrac{1}{3} \times$ area of base × height

Ex. If E, F and V are respectively the number of edges, faces and vertices of a square pyramid, then the value of (2E – F + 2V) is :

HINTS In a pyramid, Number of faces = n + 1

Number of vertices = n + 1

Number of edges = 2n

F(faces) = 4 + 1 = 5

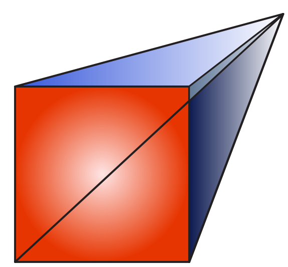
V(vertices) = 4 + 1 = 5

E (edges) = 2 × 4 = 8

∴ (2E – F + 2V)

= (2 × 8 – 5 + 2 × 5)

= 16 – 5 + 10 = 21



Ex. A prism and a pyramid have the same base and the same height. Find the ratio of the volumes of the prism and the pyramid.

HINTS Volume of pyramid = $\dfrac{1}{3} \times$ area of base × h

Volume of prism = Area of base × h

∴ Required ratio

$= \dfrac{Ar \times h}{\frac{1}{3} \times Ar \times h} = \dfrac{3}{1}$

= 3 : 1

Equilateral triangular Pyramid

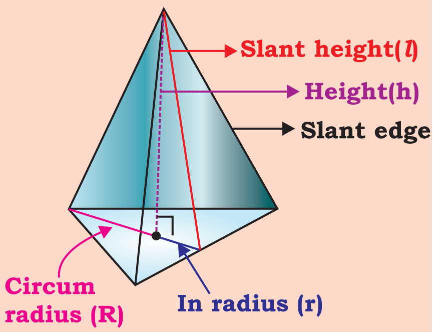
(i) C.S.A. = $\frac{1}{2} \times 3a \times l$

(ii) T.S.A.

$= \frac{1}{2} \times 3al + \frac{\sqrt{3}}{4} a^2$

(iii) Volume

$= \frac{1}{3} \times \frac{\sqrt{3}}{4} a^2 \times h$



(iv) Slant height (l) $= \sqrt{h^2 + r^2} = \sqrt{h^2 + \left(\frac{a}{2\sqrt{3}}\right)^2}$

(v) Slant edge $= \sqrt{h^2 + R^2} = \sqrt{h^2 + \left(\frac{a}{\sqrt{3}}\right)^2} = \sqrt{l^2 + \left(\frac{side}{2}\right)^2}$

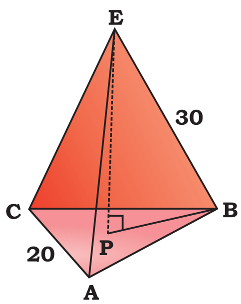
Ex. The base of right pyramid is an equilateral triangle, each side of which is 20 cm. Each slant edge is 30 cm. The vertical height (in cm) of the pyramid is:

HINTS Let, h be the vertical height.

Circum radius of equilateral Δ $= \frac{20}{\sqrt{3}}$

$h = \sqrt{AE^2 - AP^2} = \sqrt{30^2 - \left(\frac{20}{\sqrt{3}}\right)^2}$

$= \sqrt{900 - \frac{400}{3}} = \sqrt{\frac{2300}{3}} = 10\sqrt{\frac{23}{3}}$ cm



Square Pyramid

(i) C.S.A. = $\frac{1}{2} \times 4a \times l$

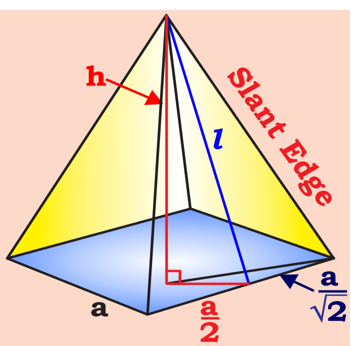
(ii) T.S.A. = $\frac{1}{2} \times 4al + a^2$

(iii) Volume = $\frac{1}{3} \times a^2 \times h$

(iv) Slant height

$= \sqrt{h^2 + \left(\frac{a}{2}\right)^2}$

(v) Slant edge = $\sqrt{h^2 + \left(\frac{a}{\sqrt{2}}\right)^2} = \sqrt{l^2 + \left(\frac{side}{2}\right)^2}$



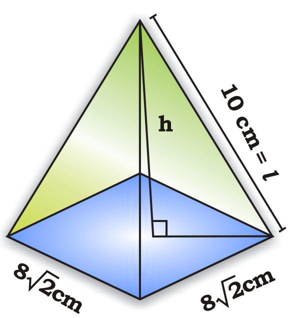
Ex. The base of a right pyramid is a square of side 8√2 cm and each of its slant edge is of length 10 cm. What is the volume (in cm³) of the pyramid?

HINTS

$h = \sqrt{10^2 - 8^2} = 6$cm

∴ Volume of pyramid $= \frac{1}{3} \times$ area of base × height

$= \frac{1}{3} \times 8\sqrt{2} \times 8\sqrt{2} \times 6 = 256$ cm³



Ex. A right square pyramid having lateral surface area is 624 cm². If the length of the diagonal of the square is $24\sqrt{2}$ cm, then the volume of the pyramid is:

HINTS

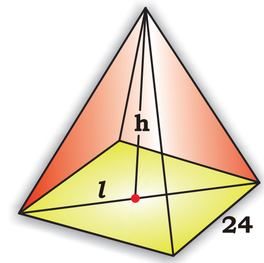
d = $24\sqrt{2}$

∴ Side of square (a) = 24

Lateral surface area

$= 4 \times \frac{1}{2} \times 24 \times l \Rightarrow 624 = 48 \times l$

$\Rightarrow l = \frac{624}{48} = 13$



Now, h = 5 cm [Pythagoras triplet of 5, 12, 13]

∴ Volume of pyramid = $\frac{1}{3}$ × area of base × height

$= \frac{1}{3} \times a^2 \times h = \frac{1}{3} \times 24 \times 24 \times 5$

= 24 × 40 = 960 cm³

Ex. The total surface area of a right pyramid, with base as a square of side 8 cm, is 208 cm². What is the slant height (in cm) of the pyramid?

HINTS Given, a = 8 cm

T.S.A. of pyramid = 208 cm²

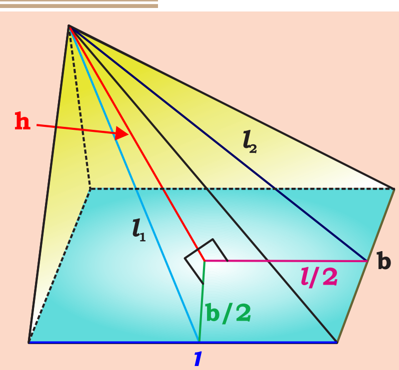
⇒ L.S.A. + area of base = 208 cm²

$\Rightarrow \left[\frac{1}{2} \times \text{perimeter of base} \times l\right]$ + area of base = 208

$\Rightarrow \left[\frac{1}{2} \times 4 \times 8 \times l\right] + (8 \times 8) = 208$

$\Rightarrow 16 \times l + 64 = 208$ cm² $\Rightarrow l = 9$

Rectangular Pyramid



There are two slant height:

- First slant height $(l_1) = \sqrt{h^2 + \left(\frac{b}{2}\right)^2}$
- Second slant height $(l_2) = \sqrt{h^2 + \left(\frac{l}{2}\right)^2}$

(i) C.S.A. = $2 \times \frac{1}{2} l \times l_1 + 2 \times \frac{1}{2} \times b \times l_2$

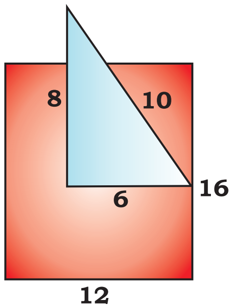
(ii) T.S.A. = C.S.A + lb

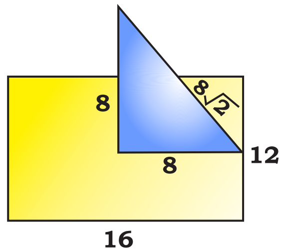
(iii) Volume = $\frac{1}{3} \times lb \times h$

(iv) (Slant edge)² = $h^2 + \left(\frac{d}{2}\right)^2$

Ex. Find TSA of a pyramid with rectangular base whose length is 16 cm and width is 12 cm, if the height of pyramid is 8 cm?

HINTS





C.S.A. = $2\times\frac{1}{2}l\times l_1+2\times\frac{1}{2}\times b\times l_2$

= $\frac{1}{2}\times 16\times 10\times 2+\frac{1}{2}\times 12\times 8\sqrt{2}\times 2 = 160+96\sqrt{2}$

TSA = $160+96\sqrt{2}+16\times 12 = 352+96\sqrt{2}$

Hexagonal Pyramid

(i) C.S.A. = $\frac{1}{2}\times 6al$

(ii) T.S.A.

$= \frac{1}{2}6al+6\times\frac{\sqrt{3}}{4}a^2$

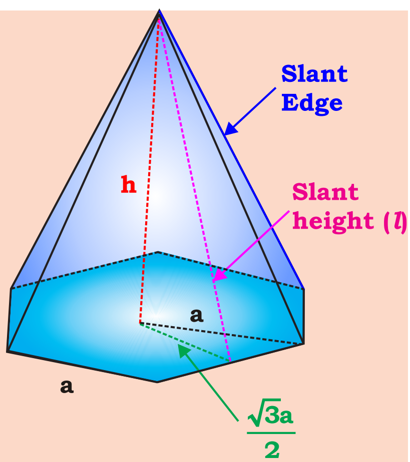
(iii) Volume

$= \frac{1}{3}\times\frac{6\sqrt{3}}{4}a^2\times h$

(iv) Slant height

$(l) = \sqrt{h^2+\left(\frac{\sqrt{3}}{2}a\right)^2}$

(v) Slant edge $= \sqrt{h^2+a^2}$



Ex. There is a pyramid on a base which is a regular hexagon of side 2a cm. If every slant edge of this pyramid is of length $\frac{5a}{2}$ cm, then the volume of this pyramid is:

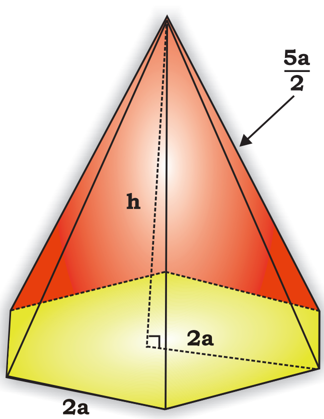
HINTS Height of regular hexagonal pyramid (h)

$= \sqrt{(\text{Slant edge})^2-(\text{Side})^2}$

$= \sqrt{\left(\frac{5a}{2}\right)^2-(2a)^2}$

$= \sqrt{\frac{9}{4}a^2}$

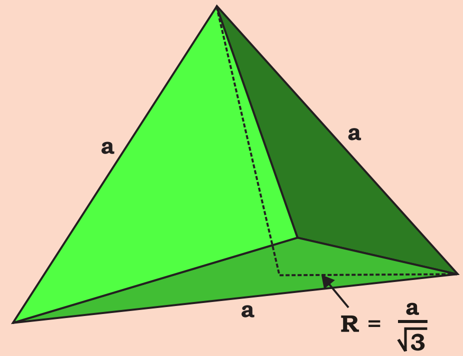
$= \frac{3}{2}a$



Volume = $\frac{1}{3}$ × Area of base × height

$= \frac{1}{3}\times 6\times\frac{\sqrt{3}}{4}\times(2a)^2\times\frac{3}{2}a = 3\sqrt{3}\,a^3\text{cm}^3$

Tetrahedron



Pyramid on a triangular base is a tetrahedron. When a solid is bounded by four triangular faces then it is a tetrahedron. A right tetrahedron is so called when the base of a tetrahedron is an equilateral triangle and other triangular faces are isosceles triangles. When we encounter a tetrahedron that has all its four faces equilateral then it is regular tetrahedron.

(a) There are four equilateral faces.

(b) All edge are equal in length

(c) Slant edge is same as side of base

(i) C. S. A. = Area of 3 equilateral triangle $= 3\times\frac{\sqrt{3}}{4}a^2$

(ii) T.S.A. = Area of 4 equilateral triangle

$= 4\times\frac{\sqrt{3}}{4}a^2 = \sqrt{3}a^2$

(iii) Height $(h) = \sqrt{a^2-\left(\frac{a}{\sqrt{3}}\right)^2} = \sqrt{\frac{2}{3}}a$

(iv) Volume (V) = $\frac{1}{3}$ × Area of base × height

$= \frac{1}{3}\times\frac{\sqrt{3}}{4}a^2\times\sqrt{\frac{2}{3}}a = \frac{\sqrt{2}}{12}a^3$

Ex. The length of the side of a regular tetrahedron is 12 cm. Find the volume.

HINTS Volume of regular tetrahedron = $\frac{\sqrt{2}}{12}a^3$

$= \frac{\sqrt{2}}{12}\times 12^3 = 144\sqrt{2}\text{ cm}^3$

Ex. Find curved surface area of a regular tetrahedron whose side is $6\sqrt{3}$ cm.

HINTS CSA of tetrahedron $= 3\times\frac{\sqrt{3}}{4}a^2$

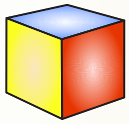
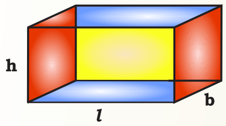
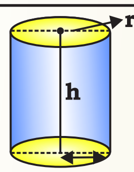
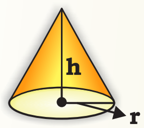
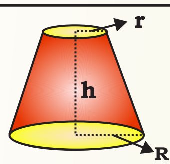
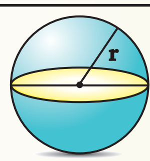
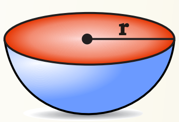
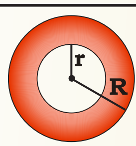
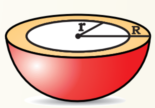
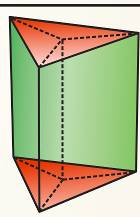
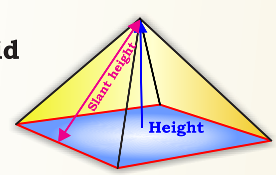
$= 3\times\frac{\sqrt{3}}{4}\times 108 = 81\sqrt{3}\text{ cm}^2$

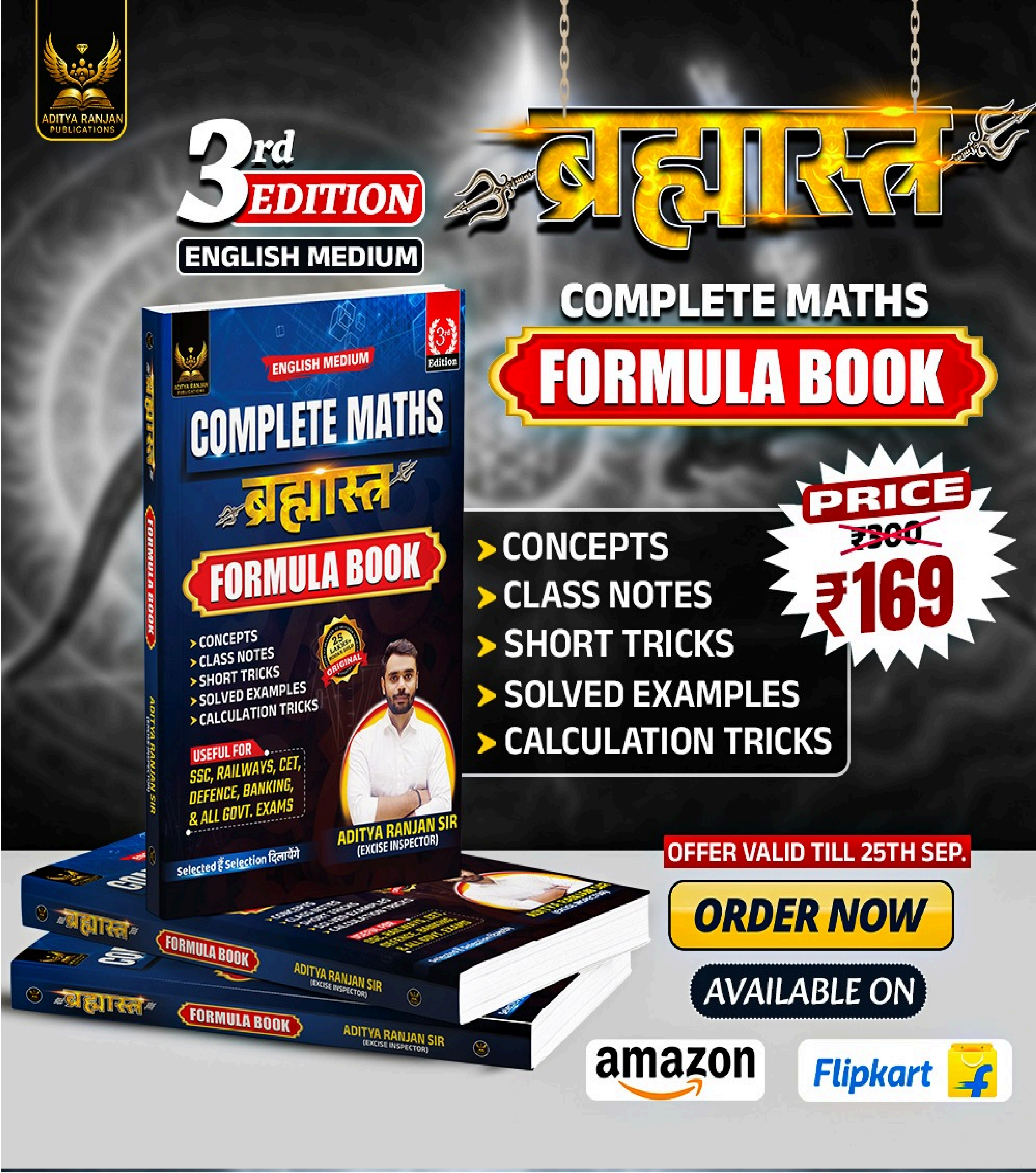
Ex. The volume of regular tetrahedron is $144\sqrt{2}$ cm³. Find the length of its edge.

HINTS Volume of regular tetrahedron = $\frac{\sqrt{2}}{12}a^3$

$\Rightarrow \frac{\sqrt{2}}{12}\times a^3 = 144\sqrt{2}$

$\Rightarrow a = 12$ cm

Solid Figure	Volume	Curved Surface area	Total Surface Area
Cube 	Side^3	$4 \times \text{side}^2$	$6 \times \text{side}^2$
Cuboid 	$l \times b \times h$	$2(lh + bh)$	$2(lb + lh + bh)$
Cylinder 	$\pi r^2 h$	$2\pi rh$	$2\pi r(r + h)$
Cone 	$\left(\frac{1}{3}\right)\pi r^2 h$	$\pi r l$ (where $l = \sqrt{r^2 + h^2}$)	$\pi r(r + l)$
Frustum of cone 	$\frac{1}{3}\pi[R^2 + r^2 + Rr]h$	$\pi(R + r)l$	$\pi(R + r)l + \pi(R^2 + r^2)$
Sphere 	$\left(\frac{4}{3}\right)\pi r^3$	$4\pi r^2$	$4\pi r^2$
Hemisphere 	$\left(\frac{2}{3}\right)\pi r^3$	$2\pi r^2$	$3\pi r^2$
Spherical Shell 	$\frac{4}{3}\pi(R^3 - r^3)$	$4\pi(R^2 - r^2)$	$4\pi(R^2 - r^2)$
Hemispherical Shell 	$\frac{2}{3}\pi(R^3 - r^3)$	$2\pi(R^2 + r^2)$	$3\pi R^2 + \pi r^2$
Prism 	Base area × height	Base peri. × height	LSA+2 × Base area
Pyramid 	$\frac{1}{3} \times$ Base area × H	$\frac{1}{2} \times$ Base peri. × Slant h.	LSA + Base area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CHAPTER 04

NUMBER SYSTEM

NUMBER SYSTEM

- Classification of Numbers
- Unit digit
- Factor
- Divisibility
- Remainder
- Number of Zeroes
- Counting of Numbers
- Binary Number

CLASSIFICATION OF NUMBERS

REAL NUMBER → **COMPLEX NUMBER** ← **IMAGINARY NUMBER**

REAL NUMBER
No. which can be denoted on number line
Ex. $+3, -7, 5, \frac{17}{14}, \frac{5}{8}, \sqrt{7}, \sqrt{5}$

IMAGINARY NUMBER
No. which can't be denoted on number line
Ex. $\sqrt{-7}, \sqrt{-3}, \sqrt{-1} = i$ (iota)

RATIONAL
No. can be expressed in $\frac{P}{Q}$ form $(Q \neq 0)$ P, Q → integer
Ex. $\frac{5}{3}, \frac{-8}{1}, 555, \frac{22}{7}, 4, -3$

IRRATIONAL
No. can't be expressed in $\frac{P}{Q}$ form
Ex. $\sqrt{2}, \sqrt{3}, \frac{\sqrt{2}}{\sqrt{3}}$, ... , 0.1432507.......,
$\pi = 3.141592$ ......

INTEGER

NEGATIVE INTEGER
Ex. –1, –2, –3, –4, ...

WHOLE NUMBER
Ex. 0, 1, 2, 3, 4

ZERO

NATURAL NO.
Ex. 1, 2, 3, 4 ....

FRACTION
Ex. $\frac{7}{9}, \frac{2}{5}, \frac{5}{3}$

DECIMALS NUMBERS

TERMINATING
Ex. $0.5 = \frac{1}{2}$, $0.73 = \frac{73}{100}$

NON-TERMINATING REPEATING DECIMAL
Ex. $0.3333..... = \frac{1}{3}$, $0.565656..... = \frac{56}{99}$

NON-TERMINATING NON-REPEATING DECIMAL
Ex. $\sqrt{2} = 1.414$ .....
↓
Irrational number

ODD NO
No. NOT DIVISIBLE by 2, $(2k \pm 1)$ form
Ex. (1, 3, 5, 7)

EVEN NO
No. DIVISIBLE by 2 (2k form)
Ex. 0, 2, 4, 6

COMPOSITE NUMBER
More than two factor
Ex. 4, 6, 8, 9
4 – Smallest composite no
9 – Smallest odd composite no

PRIME NUMBER
Only two factor 1 and itself
Ex. 2, 3, 5, 7, 11 ...... etc.

- Each prime number can be written in $(6k \pm 1)$, form But every $(6k \pm 1)$ form may not be necessarily prime no.

Ex. 13 → 6 × 2 + 1 (Prime)
25 → 6 × 4 + 1 (Not a prime number)

PERFECT NUMBER
If the sum of all the factors of a number (except that number) is equal to the given number, then that number is called a perfect number.
Ex. 6, 28, 496, 8128 etc.
Factors of 28 → 1, 2, 4, 7, 14
∴ 1 + 2 + 4 + 7 + 14 = 28
Thus, 28 is a perfect number
Note: 6 is a smallest perfect number.

CO-PRIME NUMBERS
If the HCF of two numbers is 1.
Ex. (2, 3), (11, 13),
(16, 9), (25, 19) etc.

TWIN-PRIME NUMBER
When two consecutive prime numbers are with an interval of 2, then they are called twin prime numbers.
Ex. (3, 5), (5, 7), (11, 13)

• 2 is the only even prime no. & smallest prime number • 3, 5, 7 only pair of consecutive odd prime number • 101 is smallest 3 digit prime number • 997 is largest 3 digit prime number • 1 is neither a prime nor a composite number	2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83,89, 97 $0 \xrightarrow{15} 50 \xrightarrow{10} 100$ $1 \longrightarrow 200 = 46$ Prime number Prime number $1 \longrightarrow 500 = 95$ $1 \longrightarrow 1000 = 168$ (1 – 100) → 25 Prime number

☞ **How to check the given number is prime or not?**

To check whether a number is prime number or not, first take the square root of the number. Round of the square root to the immediately lower integer. Then check divisibility of number by all prime number below it. If number is not divisible by any prime number then number is prime number.

Ex. 137 is prime number or not?

HINTS $\sqrt{137} \approx 11 \Rightarrow$ prime number less than or equal to 11 are 2, 3, 5, 7 and 11. 137 is not divisible by any of them. Hence it is prime number.

Ex. What is average of prime number from 80 to 100?

HINTS $\sqrt{100} = 10 \Rightarrow$ Prime number less than 10 are 2, 3, 5, 7 Hence, even number and ending with 5 will not prime numbers so only check divisibility of 3 and 7.

Prime number between 80 to 100 are 83, 89 and 97.

Required Average $= \frac{83+89+97}{3} = \frac{269}{3} = 89.67$

Ex. x, y and z are distinct prime numbers where $x < y < z$. If $x + y + z = 70$, then what is the value of z?

HINTS Sum is 70, means, at least one of the number is even, because odd + odd+ odd = odd

As we know, only one even prime number exists, and that is 2.

2 is also the smallest prime number. Thus, $x = 2$

Now, 70 – 2 = 68 = y + z

Now, let's use the options

Option (a): z = 29 ⇒ y = 68 – 29 = 39 (y is non prime)

Option (b): z = 43 ⇒ y = 68 – 43 = 25 (y is non prime)

Option (c): z = 31 ⇒ y = 68 – 31 = 37 (z < y)

Option (d): z = 37

⇒ y = 68 – 37 = 31 (y < z , satisfies the given condition)

Hence, z = 37

Ex. x, y and z are prime numbers such that $x + y + z = 38$. What is the maximum value of x?

HINTS If all the numbers are even, then, ⇒ $x + y + z = 6$

So, two of them are odd and one of them is 2

Let, z be 2 ⇒ $x + y = 36$

The closest prime number to 36 is 31. If $x = 31$ then we get y = 5, an odd prime number.

∴ 31 is the maximum value of x

Unit Digit

To find the units digit of a given expression, we do not have to solve the entire expression, but by operating on the units digit of all the numbers, the units digit of the resulting expression is known.

Let us consider N = a × b, N = a + b, N = a – b, to calculate the unit digit of N, we only consider the unit digit of number a and b.

CYCLICITY

0, 1, 2, 3, 4, 5, 6, 7, 8, 9

Cyclicity of 1	Cyclicity of 2		Cyclicity of 4			
0, 1, 5, 6	4, 9		2	3	7	8
$(0)^n = 0$	odd	even	$2^1 = 2$	$3^1 = 3$	$7^1 = 7$	$8^1 = 8$
$(1)^n = 1$			$2^2 = 4$	$3^2 = 9$	$7^2 = 49$	$8^2 = 64$
$(5)^n = 5$	(4) = 4	(4) = 6	$2^3 = 8$	$3^3 = 27$	$7^3 = 343$	$8^3 = 512$
$(6)^n = 6$	(9) = 9	(9) = 1	$2^4 = 16$	$3^4 = 81$	$7^4 = 2401$	$8^4 = 4096$

Ex. Find the unit digit of 232 × 235.

HINTS 232 × 235 = unit digit (0)

Ex. Find the unit digit of 628 + 493 + 589.

HINTS 628 + 493 + 589 = 8 + 3 + 9 = unit digit (0)

Ex. Find the unit digit of 2383 – 1689.

HINTS 2383 – 1689 = – 6 + 10 = 4

Let us consider $N = x^y$

To calculate the unit digit of N, we only consider the unit digit of number x.

The unit's digit of an expression can be calculated by getting the remainder while power of the expression is divided by 4.

Ex. What will be the unit's digit of $(382)^{575}$?

HINTS Step-1: Divide last 2 digits of power by 4 and find out remainder.

Remainder = 75/4 = 3

Step- 2: Put remainder as a power of unit place number and find out answer. $[2^3 = 8]$

In step 1, if remainder is 0 then put power equal to 4.

Ex. Find the units digit in each of the following cases.

(i) $(187)^{282} \times (529)^{321} \times (343)^{236}$

(ii) $(789)^{315} + (232)^{644} + (528)^{253}$

(iii) $(982)^{481} - (219)^{241}$

HINTS (i) Divide last 2 digits of power by 4 and put remainder as a power of unit place digit.

$7^2 \times 9^1 \times 3^4 \Rightarrow 9 \times 9 \times 1 = 1$

(ii) $9^3 + 2^4 + 8^1 \Rightarrow 9 + 6 + 8 = 3$

(iii) $2^1 - 9^1 \Rightarrow 12 - 9 = 3$

[If first number is less then add 10 in it]

Ex. Find the unit digit of the expression.

1! + 2! + 3! + 4! + ... + 100!

HINTS 1 + 1 × 2 + 1 × 2 × 3 + 1 × 2 × 3 × 4 + 1 × 2 × 3 × 4 × 5 + ... + 100!

= 1 + 2 + 6 + 4 + 0 + 0 + 0 ..... + 0 = 13

Hence unit digit of the expression is 3.

Factor:- Any number N can be expressed in the form of $a^p \times b^q \times c^r \times d^s \times$ ..... so no, Where, a, b, c, and d are prime number.

Ex. $15 = 3 \times 5$, $45 = 3^2 \times 5$, $100 = 2^2 \times 5^2$ and so on

FACTOR

Total number of factors

If $N = a^p \times b^q \times c^r \times d^s \times ....$, Then
Total Number of factors
$= (p + 1) \times (q + 1) \times (r + 1) \times (s + 1) ....$

Ex. Find the total number of factors of 360.

HINTS Prime Factorisation of 360 $= 2^3 \times 3^2 \times 5^1$

Total number of factors
$= (3 + 1) \times (2 + 1) \times (1 + 1) = 24$

Ex. Find the total number of factors of 480.

HINTS Prime Factorisation of 480
$= 2^5 \times 3^1 \times 5^1$

Total number of factors
$= (5 + 1) \times (1 + 1) \times (1 + 1) = 24$

Total number of even factors

If $N = a^p \times b^q \times c^r \times d^s \times ....$, Then
Total Number of Even factor
$= p \times (q + 1) \times (r + 1) \times (s + 1)$.......

Ex. Find the total number of even factors of 360.

HINTS Prime Factorisation of 360
$= 2^3 \times 3^2 \times 5^1$

Total number of even factors,
$= 3 \times (2 + 1) \times (1 + 1) = 3 \times 3 \times 2 = 18$

Alternatively

Even factors = (Total – Odd) Factors

Total number of Even Factors
$= 4 \times 3 \times 2 - 3 \times 2 = 18$

Total number of odd factors

If $N = a^p \times b^q \times c^r \times d^s \times ....$, Then
Total Number of odd factor
$= (q + 1) \times (r + 1) \times (s + 1)$ ....

Note: When we find the number of odd factors, we ignore the exponent of a, where (a = 2)

Ex. Find the total number of odd factors of 360.

HINTS Prime Factorisation of 360
$= 2^3 \times 3^2 \times 5^1$

Total number of odd factors,
$= (2 + 1) \times (1 + 1) = 3 \times 2 = 6$

Sum of all factors

If $N = a^p \times b^q \times c^r \times d^s \times ....$, Then
Sum of all factors
$= (a^0 + a^1 + ... + a^p)(b^0 + b^1 + ... + b^q)$
$(c^0 + c^1 + ... + c^r)$ .....

'OR'

Sum of all factors
$$= \frac{a^{p+1}-1}{a-1} \times \frac{b^{q+1}-1}{b-1} \times \frac{c^{r+1}-1}{c-1}$$

Ex. Find the sum of all factors of 360.

HINTS Prime Factorisation of 360
$= 2^3 \times 3^2 \times 5^1$

Sum of all factors
$= (2^0 + 2^1 + 2^2 + 2^3)(3^0 + 3^1 + 3^2)$
$(5^0 + 5^1)$
$= (1 + 2 + 4 + 8)(1 + 3 + 9)$
$(1 + 5)$
$= 15 \times 13 \times 6 = 1170$

Alternatively

Sum of all factors
$$= \frac{a^{p+1}-1}{a-1} \times \frac{b^{q+1}-1}{b-1} \times \frac{c^{r+1}-1}{c-1}$$
$$= \frac{2^{3+1}-1}{2-1} \times \frac{3^{2+1}-1}{3-1} \times \frac{5^{1+1}-1}{5-1}$$
$= 15 \times 13 \times 6 = 1170$

Sum of even factors

If $N = a^p \times b^q \times c^r \times d^s \times ....$, Then
Sum of all even factors
$= (a^1 + a^2 + ... + a^p)(b^0 + b^1 + ..b^q)$
$(c^0 + c^1 + ... + c^r)$
$$= \left(\frac{a^{p+1}-1}{a-1} - 1\right) \times \frac{b^{q+1}-1}{b-1} \times \frac{c^{r+1}-1}{c-1}$$

'OR'

Sum of all even factors
$$= \frac{a^{p+1}-a}{a-1} \times \frac{b^{q+1}-1}{b-1} \times \frac{c^{r+1}-1}{c-1}$$

Ex. Find the sum of all even factors of 360.

HINTS Prime Factorisation of 360
$= 2^3 \times 3^2 \times 5^1$

Sum of all even factors
$= (2^1 + 2^2 + 2^3) \times (3^0 + 3^1 + 3^2) \times (5^0 + 5^1)$
$= 14 \times 13 \times 6 = 1092$

Alternatively

Sum of all even factors
$$= \left(\frac{2^{3+1}-2}{2-1}\right)\left(\frac{3^{2+1}-1}{3-1}\right)\left(\frac{5^{1+1}-1}{5-1}\right)$$
$= 14 \times 13 \times 6 = 1092$

Do It Yourself:

Ex. Find the sum of all even factors of 720. **[Ans. 2340]**

Sum of odd factors

If $N = a^p \times b^q \times c^r \times d^s \times ....$, Then
Sum of all odd factors
$= a^0 (b^0 + b^1 + ... + b^q)$
$(c^0 + c^1 + ... + c^r)$, Where a = 2

'OR'

Sum of all odd factors
$$= \frac{b^{q+1}}{b-1} \times \frac{c^{r+1}-1}{c-1}$$

Ex. Find the sum of all odd factors of 360.

HINTS Prime Factorisation of 360
$= 2^3 \times 3^2 \times 5^1$

Sum of all odd factors
$= 2^0 (3^0 + 3^1 + 3^2)(5^0 + 5^1)$
$= 1 \times 13 \times 6 = 78$

Alternatively

Sum of all odd factors
$$= \frac{3^{2+1}-1}{3-1} \times \frac{5^{1+1}-1}{5-1} = \frac{26}{2} \times \frac{24}{4} = 78$$

Do It Yourself:

Ex. Find the sum of all odd factors of 720. **[Ans. 78]**

Prime and Composite Factor of a Natural Number

We know that,
Total numbers of factors of any natural number = 1 + prime factors + composite factors.

Note: 1 is a factor of all natural numbers.
When we do prime factorization, the number of prime factors can be given by just counting the number of prime factors present in it.
Therefore, Total number of composite factors = Total number of factors – Prime factors –1

To find factors which are co-prime to each other

- If $N = a^p \times b^q$ [a, b are prime factors of N]

Then Number of co-prime factors will be
$= [(p + 1)(q + 1) + pq]$

- If $N = a^p \times b^q \times c^r$

Then number of co-prime factors will be
$= [(p + 1)(q + 1)(r + 1) + pq + qr + rp + 3pqr]$
OR take two at a time and then calculate.

Important Note: Let $N = 56 = 2^3 \times 7^1$
Number of distinct prime factors = 2(namely 2 and 7)
Total number of prime factors (repetition allowed)
= 3 + 1 = 4 (sum of powers)

Ex. $N = 56 = 2^3 \times 7^1$, find total number of composite factor.

HINTS Here, Total number of prime factors = 2(namely 2 and 7) Total number of composite factor
$= [(3 + 1)(1 + 1)] – 2 – 1 = 8 – 3 = 5$

Ex. If N = 720 find total no. of prime factors of N.

HINTS Given that, $N = 720 = 2^4 \times 3^2 \times 5^1$
Clearly, there are three prime factors namely 2, 3 and 5.

Ex. N = 720 find total number of composite factors of N.

HINTS Given that, $N = 720 = 2^4 \times 3^2 \times 5^1$
Total number of composite factors = Total number of factors – Number of prime factors – 1 = 30 – 3 – 1 = 26

Ex. How may sets of two factors of N = 56 will be co-prime to each other?

HINTS $N = 56 = 2^3 \times 7^1$
∴ Number of co-prime factors
$= [(3 + 1)(1 + 1) + 3 \times 1] = 11$

Ex. Given that $N = 720 = 2^4 \times 3^2 \times 5^1$ find number of sets of factors which are co-prime to each other.

HINTS
Given that
$N = 720 = 2^4 \times 3^2 \times 5^1$
Number of sets of factors which are co-prime to each other = (p + 1) (q + 1) (r + 1) + pq + qr + rp + 3pqr
= (4 + 1) (2 + 1) (1 + 1) + 8 + 2 + 4 + 3 × 8
= 5 × 3 × 2 + 38 = 30 + 38 = 68

No. of factor of Perfect square

Ex. Given, $N = 720 = 2^4 \times 3^2 \times 5^1$ find total number of factors which are perfect square.

HINTS
$N = 720 = 2^4 \times 3^2 \times 5^1$

Power of 2	Power of 3	Power of 5
2^0	3^0	5^0
2^2	3^2	
2^4		

∴ Total number of factors of N = 720 that are perfect square = 3 × 2 × 1 = 6

Alternatively

$a^{2n} = \frac{\text{power}}{2} = \text{integer} + 1$

$= \left(\frac{4}{2}+1\right)\times\left(\frac{2}{2}+1\right)\times\left(\frac{1}{2}+1\right)$

= (2 + 1) × (1 + 1) × 1 = 3 × 2 × 1= 6

No. of factor of Perfect cube

Ex. Given that $N = 720 = 2^4 \times 3^2 \times 5^1$ find total number of factors which are perfect cubes.

HINTS
$N = 720 = 2^4 \times 3^2 \times 5^1$

Power of 2	Power of 3	Power of 5
2^0	3^0	5^0
2^3		

∴ Total number of factors of N = 720 that are perfect cube = 2 × 1 × 1 = 2

Alternatively

$a^{3n} = \frac{\text{power}}{3} = \text{integer} + 1$

$= \left(\frac{4}{3}+1\right)\times\left(\frac{2}{3}+1\right)\times\left(\frac{1}{3}+1\right)$

= 1 + 1 × 1 × 1 = 2 × 1 × 1 = 2

No.of factor of perfect square & cube

Ex. Given that $N = 720 = 2^4 \times 3^2 \times 5^1$ find total number of factors which are perfect square and perfect cube both

HINTS
$N = 720 = 2^4 \times 3^2 \times 5^1$

Power of 2	Power of 3	Power of 5
2^0	3^0	5^0

∴ Total number of factors of N = 720 for which the factors are perfect square and cube simultaneously
= 1 × 1 × 1 = 1

Alternatively

$a^{6n} = \frac{\text{power}}{6} = \text{integer} + 1$

$= \left(\frac{4}{6}+1\right)\times\left(\frac{2}{6}+1\right)\times\left(\frac{1}{6}+1\right)$

= 1 × 1 × 1 = 1

Ex. How many factor of 720 are divisible by 10.

HINTS $N = 720 = 2^4 \times 3^2 \times 5$

For the factors to be divisible by 10, minimum power of 2 to be used = 1, and minimum power of 5 used is 1.

Hence all the factors of 720 that are divisible by 10 will be of the format $= 2^1 \times 5^1 (2^3 \times 3^2)$

∴ Number of factors = 4 × 3 × 1 = 12

Ex. How many factors of 14,400 are divisible by 18 but not by 36?

HINTS Factors of $14400 = 2^6 \times 3^2 \times 5^2$

Factors of $18 = 2 \times 3^2$, Factors of $36 = 2^2 \times 3^2$

Factors which are divisible by $18 = \frac{2^6 \times 3^2 \times 5^2}{2 \times 3^2} = 2^5 \times 5^2$

Total number of factors which are divisible by 18 = (5 + 1)(2 + 1) = 18

Factors which are divisible by $36 = \frac{2^6 \times 3^2 \times 5^2}{2^2 \times 3^2} = 2^4 \times 5^2$

Total number of factors which are divisible by 36
= (4 + 1)(2 + 1) = 15
∴ Required factors = 18 – 15 = 3

Ex. When 732 is divided by a positive integer x, the remainder is 12. How many values of x are there?

HINTS

Remainder is 12 it means divisor must be greater than 12.
732 = 720 + 12

$\because$ 720 must be divisible by x.

$720 = 2^4 \times 3^2 \times 5^1$

Total factors of 720

$= (4 + 1) \times (2 + 1) \times (1 + 1) = 30$

Factors till 12

= 1, 2, 3, 4, 5, 6, 8, 9, 10, 12 = 10

$\therefore$ Required factor = 30 – 10 = 20

- **Sum of reciprocal of factors** $= \dfrac{\text{Sum of factors}}{\text{Given number}}$

Ex. Find the sum of reciprocal of factor of 16.

HINTS Factors of 16 = 1, 2, 4, 8, 16

Sum of reciprocal of factors $= 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = \frac{31}{16}$

Alternatively

Prime Factorization of 16 = 2^4

Sum of factors of 16 = $2^0 + 2^1 + 2^2 + 2^3 + 2^4$

= 1 + 2 + 4 + 8 + 16 = 31

Sum of reciprocal of factors = $\frac{31}{16}$

- **Product of factors** = $(N)^{\frac{n}{2}}$

Where N = Given number

n = Total number of factors

Ex. What is the product of all factors of 360.

HINTS

Factors of 360 = $2^3 \times 3^2 \times 5^1$

Number of factors = 4 × 3 × 2 = 24

Product of factors = $(N)^{\frac{n}{2}} = (360)^{\frac{24}{2}} = 360^{12}$

DIVISIBILITY

NUMBER	DIVISIBILITY CONDITION	EXAMPLE
2^n: 2	A no. is divisible by 2 when last digit is even (0, 2, 4, 6, 8)	***Ex.*** 224, 112, 264
4	A no. is divisible by 4 when last 2 digits are divisible by 4	***Ex.*** 144, 156
8	A no. is divisible by 8 when last 3 digits are divisible by 8	***Ex.*** 48512, 35480
16	A no. is divisible by 16 when last 4 digits are divisible by 16 & so on.	***Ex.*** 324096, 153248
3, 9	A number is divisible by 3 or 9 when the sum of digits of the number is divisible by 3 or 9	523872 = sum of digit = 5 + 2 + 3 + 8 + 7 + 2 = 27
6	A number "N" is divisible by 6 only when "N" is divisible by both 2 and 3	***Ex.*** 56934 is divisible by both 2 & 3. Therefore, it is divisible by 6.
7	A number is divisible by 7 if the difference between sum of the triplets at odd places and sum of the triplets at even places is divisible by 7.	***Ex.*** 12348 = <u>012</u> <u>348</u> = 348 – 012 = 336/7 = 48
5^n: 5	A number is divisible by 5^1 when the last digit is 0 or 5.	***Ex.*** 10, 45, 105, 300 ....
25	A number is divisible by 5^2 when the last two digit is 00 or divisible by 25.	***Ex.*** 75, 150, 300, 425 ....
125	Similarly, a number is divisible by 5^3 when the last three digit is 000 or divisible by 125 and so on.	***Ex.*** 125, 250, 375, 1000 ....
11	Sum at even place digit, Sum at odd place digit → Take diff. → If diff is 0 or multiple of 11 then no. will be divisible by 11	***Ex.*** 166452 = 1 + 6 + 5 = 12 = 6 + 4 + 2 = 12 Diff. = 12 – 12 = 0 is divisible by 11
13	Adding 4 times the last digit from the rest, If the resulting number is divisible by 13 then "N" will be divisible by 13.	***Ex.*** 169 → 16 + 9 × 4 = 52 So, 169 is divisible by 13.
17	Subtract 5 times the last digit from the rest, If the resulting number is divisible by 17 then "N" will be divisible by 17.	***Ex.*** 391 → 39 – 1 × 5 = 34 So, 391 is divisible by 17.
7, 11, 13	When any 3 digit number multiplied by 1001 then it repeats itself and always completely divisible by 7, 11 and 13.	***Ex.*** 123123, 147147, 164164, 574574
3, 7, 13, 37	When any number multiplied by 10101 then it always completely divisible by 3, 7, 13 and 37. 3 × 7 × 13 × 37 = 10101 $xy \times 10101 = xyxyxy$ i.e. 73 × 10101 = 737373	***Ex.*** 353535, 414141
101	• Any number of the form of $xyxy$ is divisible by 101. i.e. xy × 101 = 2525 • And xyz × 101 = $(xyz + x)yz$	***Ex.*** 25 × 101 = 2525 ***Ex.*** 234 × 101 = (234 + 2) 34 = 23634

Ex. If the six - digit number 479xyz is exactly divisible by 7,11 and 13 then {(y + z) × x} is equal to:

HINTS LCM of 7, 11, 13 = 1001

When 3 digits of a number are repeated twice then that number will be divisible by 7, 11 and 13.

479479 is divisible by 7, 11 and 13

$\Rightarrow x = 4, y = 7, z = 9$

$\therefore (y + z) \times x = (7 + 9) \times 4 = 64$

Ex. If the seven - digit number 35345xy is divisible by 40, then the minimum value of $(4x + 5y)$ is:

HINTS We know, 40 = 5 × 8.

Given Number = 35345xy

ATQ, 35345xy is divisible by 40, Then last digit must be 0.So y = 0.

As we know, If any number is divisible by 8. Last 3 digit must be divisible by 8. In other words, 5x0 must be divisible by 8.

Thus, Minimum value of $x = 2$

$\therefore 4x + 5y = 4 \times 2 + 0 = 8$

Ex. If the 8 - digit number 7y9745x2 is divisible by 72, then the value of $(2x - y)$ for the greatest value of x is:

HINTS The number 7y9745x2, is divisible by 72

Divisibility by 8 - Check last 3 digit

Divisibility by 9 - Check sum of digit

5x2 is divisible by 8 $\rightarrow$ if x = 1,5, 9

the maximum value of x is 9

The number 7y9745x2 is divisible by 9 if y = 2

Thus $x = 9$, y = 2

$\therefore (2x - y) = 2 \times 9 - 2 = 16$

Ex. If the 5-digit number 535ab is divisible by 3, 7 and 11, then what is the value of $(a^2 - b^2 + ab)$?

HINTS LCM of 3, 7, 11 = 231

Divide 53599 (take the maximum possible values of a & b) by 231.

We get 7 as remainder

Now, subtract 7 from 53599 & get the correct number = 53592

$\Rightarrow a = 9, b = 2$

$\therefore a^2 - b^2 + ab = 81 - 4 + 18 = 95$

Ex. $2^{25} + 2^{26} + 2^{27}$ is divisible by

HINTS $2^{25}(2^0 + 2^1 + 2^2) = 2^{25}(1 + 2 + 4) = 2^{25} \times 7$

$\therefore 2^{25} + 2^{26} + 2^{27}$ is divisible by 7

Ex. How many numbers between 300 and 700 are divisible by 5, 6 and 8?

HINTS

LCM of 5, 6 and 8 = 120

Numbers between 300 and 700 which are divisible by 120 = 360, 480, 600

Required number = 3.

Ex. How many numbers are there from 1 to 100 which are neither divisible by 3 nor by 5?

HINTS $100 \times \frac{2}{3} \times \frac{4}{5}$ (since 100 is not divisible by 3 × 5 = 15, let's take 90 instead of 100) = $90 \times \frac{2}{3} \times \frac{4}{5} = 48$

Between 91 and 100 numbers 95 & 100 are divisible by 5 whereas 93, 96 & 99 are divisible by 3. Hence there are 5 numbers which are neither divisible by 3 nor by 5.

Total numbers = 48 + 5 = 53

Alternatively

Neither divisible by 3 nor by 5 = Total – (Divisible by 3 or 5) = Total – [N(3) + N(5) – N(15)]

Total number from 1 to 100 = 100

Divisible by 3 = $\frac{100}{3}$ ~ 33

Divisible by 5 = $\frac{100}{5}$ = 20

Divisible by 15 = $\frac{100}{15}$ ~ 6

Required number = 100 – (33 + 20 – 6) = 53

Ex. How many numbers are there from 700 to 950 which are neither divisible by 3 nor by 7?

HINTS Numbers from 1 to 950 that are neither divisible by 3 nor by 7 = 950 × $\frac{2}{3} \times \frac{6}{7}$ (since 950 is not divisible by 3 × 7 = 21, let's take 945 instead of 950) = 945 × $\frac{2}{3} \times \frac{6}{7}$ = 540

Between 946 and 950 number 948 is divisible by 3 whereas no number is divisible by 7. Hence there are 4 numbers which are neither divisible by 3 nor by 7.

Total numbers = 540 + 4 = 544

Numbers from 1 to 699 that are neither divisible by 3 nor by 7 = 699 × $\frac{2}{3} \times \frac{6}{7}$ (since 699 is not divisible by 3 × 7 = 21, let's take 693 instead of 699)

693 × $\frac{2}{3} \times \frac{6}{7}$ = 396

Between 694 and 699 numbers 696 & 699 are divisible by 3 whereas no number is divisible by 7. Hence there are 4 numbers which are neither divisible by 3 nor by 7.

Total numbers = 396 + 4 = 400

Numbers between 700 & 950 which are neither divisible by 3 nor by 7

= numbers between 1 & 950 which are neither divisible by 3 nor by 7 – numbers between 1 & 699 which are neither divisible by 3 nor by 7

= 544 – 400 = 144

Alternatively

Neither divisible by 3 nor by 7 = Total – (Divisible by 3 or 7) = Total – [N(3) + N(7) – N(21)]

Total number from 700 to 950 = 251

Divisible by 3 = 251/3 ~ 83

Divisible by 7 = 251/7 ~ 35

Divisible by 21 = 251/21 ~ 11

Required number = 251 – (83 + 35 – 11) = 251 – 107

= 144

Remainder

Remainder is the amount left over after division when one divisor does not divide the dividend exactly.

Let us suppose number N when divided by divisor D, leaves the remainder as R and quotient as Q.

$$\text{Divisor} \rightarrow D\overline{)N} \begin{array}{l} Q \rightarrow \text{Quotient} \\ N \rightarrow \text{Dividend} \\ R \rightarrow \text{Remainder} \end{array}$$

The number "N" can also be expressed as below:

$$\boxed{N = DQ + R}$$

Ex. On dividing a number by 38, the quotient is 24 and the remainder is 13, then the number is:

HINTS

We know that,

Dividend = Divisor × Quotient + Remainder

Here, Divisor = 38, Quotient = 24 and Remainder = 13

∴ Required number (Dividend) = 38 × 24 + 13 = 925

REMAINDER

REMAINDERS ARE ADDITIVE

Let, us suppose N_1, N_2, N_3, ... gives quotients Q_1, Q_2, Q_3, ... leaves the remainders R_1, R_2, R_3, ... respectively, when divided by common divisor D.

Therefore,

$N_1 = D \times Q_1 + R_1$, $N_2 = D \times Q_2 + R_2$

$N_3 = D \times Q_3 + R_3$, ... and so on.

Let S be the sum of N_1, N_2, N_3, ...

then, $S = (D \times Q_1 + R_1) + (D \times Q_2 + R_2) + (D \times Q_3 + R_3) + ...$

$= D \times (Q_1 + Q_2 + Q_3 + ...) + (R_1 + R_2 + R_3 + ...)$

$= D \times K + R_1 + R_2 + R_3$ ......

Where K is some number.

Therefore, the remainder when S is divided by D is the remainder when $R_1 + R_2 + R_3 + ...$ is divided by D.

For Example:

$$\left.\frac{\overset{1}{\uparrow}\,6}{5}\right|_R = \left.\frac{\overset{1}{\uparrow}\,36}{5}\right|_R = \left.\frac{\overset{0}{\uparrow}30+\overset{1}{\uparrow}6}{5}\right|_R = \left.\frac{\overset{1}{\uparrow}21+\overset{4}{\uparrow}9+\overset{1}{\uparrow}6}{5}\right|_R = ...$$

REMAINDERS ARE MULTIPLICATIVE

Let, us suppose N_1, N_2, N_3, ... gives the quotients Q_1, Q_2, Q_3,.... leaves the remainders R^1, R^2, R^3, ... respectively, when divided by common divisor D.

Therefore,

$N_1 = D \times Q_1 + R_1$, $N_2 = D \times Q_2 + R_2$

$N_3 = D \times Q_3 + R_3$, ... and so on.

Let P be the sum of N_1, N_2, N_3, ...

then, $P = N_1\ N_2\ N_3$ ....

$= (D \times Q_1 + R_1) + (D \times Q_2 + R_2) + (D \times Q_3 + R_3) + ...$

$= D \times K + R_1\ R_2\ R_3$ ......

Where K is some number.

Clearly, product of R_1, R_2, R_3,... is free of D, therefore the remainder when P is divided by D is the remainder when the product $R_1\ R_2\ R_3$ ...... is divided by D.

For Example.

$$\left.\frac{\overset{1}{\uparrow}361 \times \overset{3}{\uparrow}363}{12}\right|_R = \left.\frac{1 \times 3}{12}\right|_R = 3$$

CONCEPT OF NEGATIVE REMAINDERS

Remainders by definition are non-negative in nature. Hence, even when we divide -26 by 6 we say remainder is 4 (and not-2). But, sometimes to make the calculation easy we look at the negative value of remainder.

Let us take a example to explain the above concept more clearly. Let us write 109 as the difference of two numbers and then divide it by 11.

For Example.

$$\frac{109}{11} = \frac{118-9}{11} = \frac{\overset{+8}{\downarrow}118}{11} - \frac{\overset{-9}{\downarrow}9}{11}$$

Therefore, Net remainder = 8 – 9 = –1 + 11 = 10

Ex. What is remainder of $\frac{89}{9}$.

HINTS

$$\frac{89}{9} = \frac{93-4}{9} = \frac{93}{9} - \frac{4}{9} = \frac{3-4}{9} = \frac{-1}{9}$$

∴ Net remainder = – 1 + 9 = 8

Ex. What is the rem. when the sum of 335 608 and 853 is divided by 13 ?

HINTS

$$\frac{\overset{10}{\uparrow}335 + \overset{10}{\uparrow}608 + \overset{8}{\uparrow}853}{13} = \frac{10 + 10 + 8}{13}$$

$= \frac{28}{13}$ R = 2 Rem.

Do It Yourself:

Ex. What is the rem. when the sum of 47, 69 and 85 is divided by 9 ?

[Ans. 3]

Ex. When positive number x, y and z are divided by 31, the reminders are 17, 24 and 27 respectively. When $(4x – 2y + 3z)$ is divided by 31, the remainder will be:

HINTS

$x = 31 \times 1 + 17 = 48$

$y = 31 \times 1 + 24 = 55$

$z = 31 \times 1 + 27 = 58$

Now, $(4x – 2y + 3z)$

$= 4 \times 48 – 2 \times 55 + 3 \times 58 = 256$

∴ Remainder = 256 ÷ 31 = 8

Do It Yourself:

Ex. Find the rem. when 179 × 172 × 173 is divided by 17.

[Ans. 3]

Ex. What is remainder of $\frac{111}{12}$

HINTS

$$\frac{111}{12} = \frac{120-9}{12} = \frac{120}{12} - \frac{9}{12}$$

$$= \frac{0-9}{12} = \frac{-9}{12} = -9 + 12 = 3$$

∴ Net remainder = 3

Do It Yourself:

Ex. What is remainder of $\frac{116}{8}$

[Ans. 4]

SOME IMPORTANT CONCEPTS OF REMAINDER

- $\text{Rem}\left\{\frac{(a^n + b^n)}{(a+b)}\right\} \to 0$; Where n is odd number.

Ex. Find the remainder of $\left(\frac{8^{371}+5^{371}}{13}\right)$.

HINTS Here, 8 + 5 = 13.
Thus, remainder of the expression is 0.

- $\text{Rem}\left\{\frac{(a^n + b^n + c^n)}{(a+b+c)}\right\} \to 0$; Where n is odd number.

Ex. Find the remainder of $\left(\frac{3^{61}+2^{61}+4^{61}}{9}\right)$.

HINTS Here, 3 + 2 + 4 = 9.
Thus, remainder of the expression is 0.

- $\text{Rem}\left\{\frac{a^n + b^n + c^n + ...}{(a+b+c)}\right\} \to 0$; Where, a + b + c + ... and so on in Arithmetic Progression and n is odd number.

Ex. Find the rem. of the following expression $\left(\frac{16^{73}+17^{73}+18^{73}+19^{73}}{70}\right)$.

HINTS Here, 16 + 17 + 18 + 19 are in A.P and 73 is odd number.
Thus, rem. of the expression is 0.

- $\text{Rem}\left\{\frac{a^n - b^n}{(a-b)}\right\} \to 0$; For all value of n.

Ex. Find the remainder of $\left(\frac{8^{36}-2^{36}}{6}\right)$.

HINTS Using concept defined above, Rem. of the expression is 0.

- $\text{Rem}\left\{\frac{a^n - b^n}{(a+b)}\right\} \to 0$; Where n is even number.

Ex. Find the remainder of $\left(\frac{7^{24}-4^{24}}{11}\right)$.

HINTS Using concept defined above, Rem.of the expression is 0.

- $(a^n + b^n)$ where n = even

∴ $(a^n + b^n)$ is not divisible by both (a + b) and (a – b)

- $\text{Rem}\left\{\frac{(a+1)^n}{a}\right\} \to 1$; For all value of n.

Ex. Find the remainder of $\frac{16^{13}}{15}$.

HINTS $= \frac{(15+1)^{13}}{15} = \frac{15^{13}}{15} + \frac{1^{13}}{15}$

Remainder $= \frac{1^{13}}{15} = 1$

- $\text{Rem}\left\{\frac{(a-1)^n}{a}\right\} \to 1$; **When n is even number.**

Ex. Find the remainder of the expression $\frac{72^{282}}{73}$.

HINTS $\frac{72^{282}}{73} = \frac{(73-1)^{282}}{73} = 1$

- $\text{Rem}\left\{\frac{(a-1)^n}{a}\right\} \to (a-1) \text{ or } -1$ **; When n is odd number.**

Ex. Find the remainder of the expression $\frac{72^{281}}{73}$.

HINTS $\frac{72^{281}}{73} = \frac{(73-1)^{281}}{73}$ = – 1 or 73 – 1 = 72

Fermat's Remainder Theorem:

If $\frac{a^{P-1}}{P}$ Then remainder = 1

P = prime number
a, P → co-prime

Ex. Find the remainder of $\frac{82^{54}}{19}$.

HINTS

$\frac{82^{54}}{19} = \frac{(82^{18})^3}{19} = 1^3 = 1$

Ex. Find the remainder of $\frac{93^{51}}{11}$.

HINTS

$\frac{93^{51}}{11} = \frac{(93^{10})^5 \times 93}{11} = 1 \times 5 = 5$

Do It Yourself:

Ex. Find the remainder of $\frac{9^{111}}{13}$. ***[Ans. 1]***

Euler's Theorem of Remainder:

$\frac{a^{\phi(N)}}{N} = 1$ (Rem.)

N = any natural number
ϕ (N) = Totient function of N
a, N co-prime

How to find ϕ (N):

$\phi(72) \Rightarrow 72 = 2^3 \times 3^2$

$= 72 \times \left(1-\frac{1}{2}\right) \times \left(1-\frac{1}{3}\right)$

$= 72 \times \frac{1}{2} \times \frac{2}{3} = 24$

Note: For all the prime numbers, Euler number will be the number one less then the given prime number.

$E_2 = 2 \times \left(1-\frac{1}{2}\right) = 1$, $E_3 = 3 \times \left(1-\frac{1}{3}\right) = 2$

Do It Yourself:

Ex. Find the totient of 100. ***[Ans. 40]***

Wilson's Remainder Theorem:

If P is a prime number then (P – 1)! + 1 is divisible by P. In other words, remainder obtained when (P – 1)! is divided by P is –1.

Ex. Find the remainder when 40! is divided by 41

HINTS

Using the Wilson's theorem,

Remainder $= \left[\frac{(41-1)!}{41}\right] = -1$

= 41 + (– 1) = 40

Do It Yourself:

Ex. Find the remainder of $\frac{96^{132}}{97}$. ***[Ans. 1]***

Concept of Successive Division

Lets take an example to explain the concept of Successive Division

Suppose a number 'N' successively divided by 3, 4 and 7 the remainder obtained is 2, 1 and 4 respectively.

If we need to find the value of N, we need to follow the below steps.

Divisor	53	Rem.
3	17	2
4	4	1
7	0	4

$1 \rightarrow 7 \times 0 + 4 = 4$
$2 \rightarrow 4 \times 4 + 1 = 17$
$3 \rightarrow 17 \times 3 + 2 = 53$

Thus, the value of N = 53

Now, Let's look at how successive division worked.

Quotient	$\frac{53}{3} = 17$	$\frac{17}{4} = 4$	$\frac{4}{7} = 0$
Remainder	2	1	4

The above example clearly explain the concept of successive Division.

Ex. When a number (N) is successively divided by 3, 4 and 7, the remainders obtained are 2, 3 and 5, respectively. What will be the number (N)?

HINTS

Divisor	71	Remainder
3	23	2
4	5	3
7	0	5

$1 \rightarrow 7 \times 0 + 5 = 5$
$2 \rightarrow 4 \times 5 + 3 = 23$
$3 \rightarrow 23 \times 3 + 2 = 71$

Thus, the value of N = 71

Quotient	$\frac{71}{3} = 23$	$\frac{23}{4} = 5$	$\frac{5}{7} = 0$
Remainder	2	3	5

Least number to be added or subtracted to given number so it becomes divisible by a divisor.

Ex. What is least number to be added to 42072 to get a number which is divisible by 93?

HINTS 42072 ÷ 93

We get quotient = 452 and Remainder = 36

So least number to be added = 93 – 36 = 57

Ex. What is the least number to be substracted from 25809 to get a number exactly divisibly by 139 is.

HINTS 25809 ÷ 139

We get quotient = 185 and Remainder = 94

So, If we subtract 94 from the given number 25809 i.e. = 25809 – 94 = 25715 is exactly divisible by 139.

Do It Yourself:

Ex. After the division of a number successively by 2, 3 and 5, the remainders are 1, 2 and 3, respectively. What will be the remainder, if 13 divides the same number (if the last quotient is 1)? ***[Ans. 1]***

Number of Zeros

Number of zeros in an expression

Let us assume we have to find the number of zeroes in a product 24 × 13 × 52 × 27 which can also be written as $2^5 \times 3^4 \times 13^2$. Clearly, this product will have no zeroes because it has no 5 in it.

- However, if we have an expression like:

8 × 15 × 24 × 13

The expression can be rewritten as

$2^6 \times 3^2 \times 5 \times 13$

We know that zeroes can be formed by combination of 2 and 5 i.e. (2 × 5).

In the above expression, there are 6 two's and one five.

Hence, we can only form one pair of (2 × 5).

Therefore, there will be 1 zero in the product.

Finding the number of zeroes in a factorial

Let us assume that we have to find the number of zeroes in 7!

$7! = 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$

$= 7 \times (3 \times 2) \times 5 \times 2 \times 2 \times 3 \times 2 \times 1$

$= 7 \times 5 \times 3^2 \times 2^4 \times 1$

The above expression have one pair of 5 × 2. Clearly, there is only one 5 and an abundance of 2's.

It is clear that in any factorial value, the number of 5's will be always lesser than the number of 2's. Hence, the count of 5's in a factorial will give the total number of zeroes.

Power of number contained in a factorial

Highest power of a prime number p contained in n! is given by

$$\left[\frac{n}{p}\right] + \left[\frac{n}{p^2}\right] + \left[\frac{n}{p^3}\right] + \ldots$$

Where $[x]$ denotes the greatest integer less than or equal to x.

Ex. Find the number of zeroes in 47!

HINTS For finding the number of zeroes in 47!, we need to find the maximum power of 5 in 47!

Thus, Highest power of 5 in 47!

$$= \left[\frac{47}{5}\right] + \left[\frac{47}{25}\right] + \left[\frac{47}{125}\right] + \ldots$$

= 9 + 1 + 0 + 0 ... = 10

Hence, the number of zeroes in 47! will be 10

Ex. Find the number of zeroes in 300!

HINTS For finding the number of zeroes in 300!, we need to find the maximum power of 5 in 300!

Thus, Highest power of 5 in 300!

$$= \left[\frac{300}{5}\right] + \left[\frac{300}{25}\right] + \left[\frac{300}{125}\right] + \left[\frac{300}{625}\right] + \ldots$$

= 60 + 12 + 2 + ... = 74

Hence, the number of zeroes in 300! will be 74

Counting of Numbers

Counting a Digit

Ex. How many times 5 will come from 350 to 600.

HINTS

5	600		5	350
5	120		5	70
5	24		5	14
	4			2

Frequency of 5 = 120 + 24 + 4 = 148 | Frequency of 5 = 70 + 14 + 2 = 86

Frequency of 5 from 350 to 600 = 148 – 86 = 62

Ex. How many numbers are there from 400 to 700 in which the digit 6 occurs exactly twice?

HINTS

Range	Numbers	Count
400 – 499	466	1
500 – 599	566	1
660 – 669	660-669 (excluding 666)	9
606 – 696	Every 10^{th}, 606, 616, ... 696 (excluding 666)	9
	Total = 1+ 1 + 9 + 9	= 20

Page counting and key strokes

1 to 9 → 9 key strokes
1 to 99 → 189 key strokes
1 to 999 → 2889 key strokes
1 to 9999 → 38889 key strokes

> Total number of digit = $n(N + 1) - 1_n$
> where n = no. of digits in given no.
> N = larger number
> 1_n = No. of digits in given no. will be counted as 1.

Ex. What is the number of digits required for numbering a book with 428 pages?

HINTS

Digits	Number	Total digits
Single	1 to 9	1 × 9
Two	10 to 99	2 × 90 = 180
Three	100 to 428	3 × 329 = 987

Required number of digits = 9 +180 + 987 = 1176

Alternatively

Total no of digits = $n (N + 1) - 1_n$
= 3 (428 + 1) – 111 = 1287 – 111 = 1176

Ex. A printer number the page of a book starting with 1 and use 3189 digit in all. How many pages does the book have.

HINTS

Total no of digits = $n (N + 1) - 1_n$
⇒ 3189 = 4 (N + 1) – 1111
⇒ 4300 = 4 (N + 1) ⇒ N + 1 = 1075
⇒ N = 1074

Sum of digits

Ex. Find out sum of all digits from 1 to 100.

HINTS

Single digit no. (1 – 9):-
Sum = 1 + 2 ..... + 9 = 45
Two digit no. (10 – 99):-
(Each digit 1 – 9 appears 10 times)
Tens place :- (1 + 2 +..... + 9) × 10 = 45 × 10 = 450
(Each digit 0 – 9 appears 9 times)
Unit digit:- (0 + 1 + 2 +..... + 9) × 9 = 45 × 9 = 405
Three digit (100):- (1 Hundred place)
Total sum = 45 + 450 + 405 + 1 = 901

Concept of Number as a Difference of Squares

If a number N can be written as the product of two number (a × b) then $\boxed{N = ab = \left(\frac{a+b}{2}\right)^2 - \left(\frac{a-b}{2}\right)^2}$ Thus, $\boxed{N = x^2 - y^2}$	a + b = odd a – b = odd $a = \frac{odd + odd}{2}$ = Even natural no.	a + b = even a – b = even $a = \frac{even + even}{2}$ = Even natural no.	a + b = even a – b = odd $a = \frac{even + odd}{2} = \frac{odd}{2}$ ¹ Natural no.

Ex. How many pairs of natural number are there. Such that the diff. of their square is 35.

HINTS Let, the number be x and y.

$x^2 - y^2 = N$
$x^2 - y^2 = 35$

$(x + y)$ $(x - y)$
7 × 5 = 35
35 × 1 = 35

Hence, 2 pair is possible.

∴ x and y are natural number, $(x - y)$. and $(x + y)$ must also be integers. also $x^2 - y^2 = 35 > 0$ Therefore $(x - y)$ and $(x + y)$ must be a positive integer.

Ex. How many pairs of natural number are there. Such that the diff. of their square is 36.

HINTS Let, the number be x and y.

$x^2 - y^2 = 36$

$(x + y)$		$(x - y) = 36$	
1	×	36 = ×	$\longrightarrow x = \frac{1 + 36}{2} \neq$ Natural no.
2	×	18 = ✓	$\longrightarrow \frac{x + y}{2} = \frac{2 + 18}{2} = 10$
3	×	12 = ×	Natural no.
4	×	9 = ×	$\frac{x - y}{2} = \frac{18 - 2}{2} = 8$
6	×	6 = ✓	Hence, 2 pair possible.

> For x and y to be positive integers, both x and y must either be odd or both must be even.If one is even and the other is odd, then the value of x and y will be in decimal.

2-Digit number and its Reverse form

Let, two digit original number = $10x + y$, Reversed number $10y + x$

Sum of original and reversed number	Diff. of original and reversed number
$(10x + y) + (10y + x) = 11(x + y)$	$(10x + y) - (10y + x) = 9(x - y)$

Ex: The sum of two-digit number and the number obtained by inter-changing the digit is 77. If the difference of digits is 1, then the number is:

HINTS

Let, number = $10x + y$

$(10x + y) + (10y + x) = 77$

$\Rightarrow x + y = 7$ .... (i)

$x - y = 1$ ....(ii)

On solving equation (i) and (ii),

$x = 4$ and $y = 3$

Hence, Number = $10x + y = 10 \times 4 + 3 = 43$

Ex: The sum of the digits of a two-digit number is 9. The number obtained by interchanging its digits exceeds the given number by 45, then the original number is:

HINTS

Let, two digit no. is $10x + y$

ATQ, $x + y = 9$ .....(i)

$10y + x - (10x + y) = 45$

$\Rightarrow 9y - 9x = 45$

$\Rightarrow y - x = 5$ .....(ii)

From eqn. (i) and (ii), $x = 2$, $y = 7$

$\therefore$ Original no. = $10x + y = 10 \times 2 + 7 = 27$

3-Digit number and its Reverse form

Let, Hundred digit = x, Tenth digit = y, Unit digit = z, So, original number = $100x + 10y + z$

ORIGINAL NUMBER $(100x + 10y + z)$

Hundreds and unit digits interchange	Hundreds and tenths digits interchange
New number = $100z + 10y + x$	New number = $100y + 10x + z$
After subtraction = original number – new number	After subtraction = original number – new number
= $100x + 10y + z - (100z + 10y + x)$	= $100x + 10y + z - (100y + 10x + z)$
= $99(x - z)$	= $90(x - y)$
= $99(z - x)$ if reversed	= $90(y - x)$ if reversed

Ex: If the hundreds and unit digits of a three-digit number are interchanged, the resulting number is 198 less than the original number. What is the difference between the hundreds and unit digits?

HINTS

Let, the original number be, $100x + 10y + z$

After interchanging hundreds and unit digits, number

= $100z + 10y + x$

Given:

$(100x + 10y + z) - (100z + 10y + x) = 198$

$\Rightarrow 99x - 99z = 198$

$\Rightarrow 99(x - z) = 198$

$\Rightarrow x - z = 2$

$\therefore$ Difference between hundreds and unit digits = 2

Ex: If the hundreds and tens digits of a three-digit number are interchanged, the resulting number is 360 less than the original number. What is the difference between the hundreds and tens digits?

HINTS

Let, the original number be, $100x + 10y + z$

After interchanging hundreds and tens digits, number

= $100y + 10x + z$

Given:

$(100x + 10y + z) - (100y + 10x + z) = 360$

$\Rightarrow 90x - 90y = 360$

$\Rightarrow 90(x - y) = 360$

$\Rightarrow x - y = 4$

$\therefore$ Difference between hundreds and tens digits = 4

Interchange	*New number*	*Difference*	*Result*
Hundreds and unit	$100z + 10y + x$	$99(x - z)$	Multiple of 99
Hundreds and tenths	$100y + 10x + z$	$90(x - y)$	Multiple of 90

BINARY NUMBER

Binary → Base 2 (0, 1)

Ex. Convert $(101010)_2 \rightarrow (?)_8$

HINTS

1	0	1
2^2	2^1	2^0
4	+	1
	5	

0	1	0
2^2	2^1	2^0
	2	
	2	

= 52

Ex. Convert $(1010101)_2 \rightarrow (?)_{10}$

HINTS

1	0	1	0	1	0	1
2^6	2^5	2^4	2^3	2^2	2^1	2^0
64		16		4		1

64 + 16 + 4 + 1 = 85

Ex. Convert $(1010101)_2 \rightarrow (?)_{16}$

HINTS

0	1	0	1	0	1	0	1
2^3	2^2	2^1	2^0	2^3	2^2	2^1	2^0
	4		1		4		1
	5				5		

= 55

Octal → Base 8 (0, 1,2,3,4,5,6,7)

Ex. Convert $(675)_8 \rightarrow (?)_2$

HINTS

	4	2	1
6	1	1	0
7	1	1	1
5	1	0	1

= 1 1 0 1 1 1 1 0 1

Ex. Convert $(175)_8 \rightarrow (?)_{10}$

HINTS

	1	7	5
×	8^2	8^1	8^0
	64	56	5

64 + 56 + 5 = 125

Ex. Convert $(675)_8 \rightarrow (?)_{16}$

HINTS

$(675)_8 \rightarrow (\ )_2 \rightarrow (?)_{16}$

	4	2	1
6	1	1	0
7	1	1	1
5	1	0	1

= 1 1 0 1 1 1 1 0 1

Now, $\rightarrow (\ .\ )_{16}$

0 0 0 1	1 0 1 1	1 1 0 1
8 4 2 1	8 4 2 1	8 4 2 1
1	11(B)	13(D)

= 1BD

Decimal → Base 10 (0, 1,2,3,4,5,6,7,8,9)

Ex. Convert $(63)_{10} \rightarrow (?)_2$

HINTS

2	63	
2	31	1
2	15	1
2	7	1
2	3	1
	1	1

$= (1\ 1\ 1\ 1\ 1\ 1)_2$

Ex. Convert $(67)_{10} \rightarrow (?)_8$

HINTS

8	67	
8	8	3
	1	0

$= (103)_8$

Ex. Convert $(425)_{10} \rightarrow (?)_{16}$

HINTS

16	425	
16	26	9
	1	10 (A)

= 1A9

Hexadecimal → Base 16 (0,1,2,3,4,5,6,7,8,9,A(10), B(11),C(12),D(13),E(14),F(15)

Ex. Convert $(D\ 9)_{16} \rightarrow (?)_2$

HINTS

	8	4	2	1
D(13)	1	1	0	1
9	1	0	0	1

= 1 1 0 1 1 0 0 1

Ex. Convert $(6\ FD)_{16} \rightarrow (?)_8$

HINTS

$(6\ FD)_{16} \rightarrow (\ )_2 \rightarrow (\ )_8$

	8	4	2	1
6	0	1	1	0
F (15)	1	1	1	1
D (13)	1	1	0	1

= 0 1 1 0 1 1 1 1 1 1 0 1

Now, $\rightarrow (\ .\ )_8$

0 1 1	0 1 1	1 1 1	1 0 1
4 2 1	4 2 1	4 2 1	4 2 1
3	3	7	5

= 3 3 7 5

Ex. Convert $(1\ AD)_{16} \rightarrow (?)_{10}$

HINTS

	1	A(10)	D(13)
×	16^2	16^1	16^0
	256	160	13

256 + 160 + 13 = 429

Do It Yourself:

Ex. Convert $(4\ AE)_{16} \rightarrow (?)_8$

Ans. $(2256)_8$

Ex. Convert $(10101)_2 \rightarrow (?)_{16}$

Ans. $(15)_{16}$

Note:

- When decimal presents in left side then divide the given number by right side base.
- When decimal presents in right side then multiple by the series of power of left base.
- Binary to Octal or Octal to Binary → 421/3 digit pair
- Binary to Hexadecimal or Hexadecimal to Binary → 8421/4 digit pair
- Direct conversion of Octal to Hexadecimal or Hexadecimal to Octal is not possible. Firstly you have to convert it into Binary.

Ex. Convert $(17.\ 125)_{10} = (?)_2$

HINTS

First, convert the integer part

2	17	
2	8	1
2	4	0
2	2	0
	1	0

Convert the fractional part

$0.125 \times 2 = 0.250$ 0

$0.250 \times 2 = 0.500$ 0

$0.500 \times 2 = 1.00$ 1

$(17.125)_{10} = (10001.001)_2$

CHAPTER 09

ALGEBRA

Algebra is the branch of mathematics in which abstract symbols, rather than numbers, are manipulated or operated with arithmetic.

Topic Included

- Inverse Function
- Square Formula for 2V & 3V
- Cube Formula for 2V & 3V
- Concept of Perfect Square
- Value Putting and Symmetricity
- Quadratic Equation

Inverse Function

(a) Basic Formulae

$$x \pm \frac{1}{x}$$

If $x+\frac{1}{x}$ = n, then

- $x^2+\frac{1}{x^2}=n^2-2$
- $x^4+\frac{1}{x^4}=(n^2-2)^2-2$
- $x^8+\frac{1}{x^8}=\left[(n^2-2)^2-2\right]^2-2$
- $x^3+\frac{1}{x^3}=n^3-3n$
- $x^6+\frac{1}{x^6}=(n^3-3n)^2-2$

If $x-\frac{1}{x}$ = n, then

- $x^2+\frac{1}{x^2}=n^2+2$
- $x^4+\frac{1}{x^4}=(n^2+2)^2-2$
- $x^8+\frac{1}{x^8}=\left[(n^2+2)^2-2\right]^2-2$
- $x^3-\frac{1}{x^3}=n^3+3n$
- $x^6+\frac{1}{x^6}=(n^3+3n)^2+2$

Ex. If $x+\frac{1}{x}=3$, then

HINTS $x^2+\frac{1}{x^2}=3^2-2=7$

- $x^4+\frac{1}{x^4}=\left(3^2-2\right)^2-2=47$
- $x^8+\frac{1}{x^8}=\left[\left(3^2-2\right)^2-2\right]^2-2=2207$
- $x^3+\frac{1}{x^3}=3^3-3\times3=18$
- $x^6+\frac{1}{x^6}=\left(3^3-3\times3\right)^2-2=322$

Ex. If $x-\frac{1}{x}=3$, then

HINTS $x^2+\frac{1}{x^2}=3^2+2=11$

- $x^4+\frac{1}{x^4}=\left(3^2+2\right)^2-2=119$
- $x^8+\frac{1}{x^8}=\left[\left(3^2+2\right)^2-2\right]^2-2=(119)^2-2=14159$
- $x^3-\frac{1}{x^3}=3^3+3\times3=36$
- $x^6+\frac{1}{x^6}=(3^3+3\times3)^2+2=1298$

$$x \pm \frac{1}{x}$$

If $x + \frac{1}{x} = n$, then

- $x^5 + \frac{1}{x^5} = \left(x^2 + \frac{1}{x^2}\right)\left(x^3 + \frac{1}{x^3}\right) - \left(x + \frac{1}{x}\right)$
  $= (n^2 - 2)(n^3 - 3n) - n$
- $x^7 + \frac{1}{x^7} = \left(x^4 + \frac{1}{x^4}\right)\left(x^3 + \frac{1}{x^3}\right) - \left(x + \frac{1}{x}\right)$
  $= \left[(n^2 - 2)^2 - 2\right](n^3 - 3n) - n$

If $x - \frac{1}{x} = n$, then

- $x^5 - \frac{1}{x^5} = \left(x^2 + \frac{1}{x^2}\right)\left(x^3 - \frac{1}{x^3}\right) - \left(x - \frac{1}{x}\right)$
  $= (n^2 + 2)(n^3 + 3n) - n$
- $x^7 - \frac{1}{x^7} = \left(x^4 + \frac{1}{x^4}\right)\left(x^3 - \frac{1}{x^3}\right) + \left(x - \frac{1}{x}\right)$
  $= \left[(n^2 + 2)^2 - 2\right](n^3 + 3n) + n$

- $x^5 \pm \frac{1}{x^5}$ and $x^7 \pm \frac{1}{x^7}$ must be a multiple of n.
- If $x \pm \frac{1}{x}$ =n (unit digit), then unit digit of $x^5 \pm \frac{1}{x^5}$ and $x^7 + \frac{1}{x^7}$ must be n.

Ex. If $x + \frac{1}{x} = 3$, then

HINTS $x^5 + \frac{1}{x^5} = (3^2 - 2)\ (3^3 - 3 \times 3) - 3 = 123$

- $x^7 + \frac{1}{x^7} = [(3^2 - 2)^2 - 2]\ (3^3 - 3 \times 3) - 3 = 843$

Ex. If $x - \frac{1}{x} = 3$, then

HINTS $x^5 - \frac{1}{x^5} = (3^2 + 2)\ (3^3 + 3 \times 3) - 3 = 393$

- $x^7 - \frac{1}{x^7} = [(3^2 + 2)^2 - 2]\ (3^3 + 3 \times 3) + 3 = 4287$

(B) Quadratic equation

To solve, first convert the quadratic equation ($ax^2 + bx + c = 0$) into $\left(x + \frac{1}{x}\right)$ or $\left(x - \frac{1}{x}\right)$

Ex. If $\frac{2p}{p^2 - 5p + 1} = \frac{1}{10}$, $p \neq 0$, then the value of $\left(p + \frac{1}{p}\right)$ is:

HINTS $\frac{2p}{p^2 - 5p + 1} = \frac{1}{10}$

$\Rightarrow \frac{2}{p - 5 + \frac{1}{p}} = \frac{1}{10} \Rightarrow 20 = p + \frac{1}{p} - 5$

$\Rightarrow 25 = p + \frac{1}{p}$

Ex. If $p^2 - 4p - 1 = 0$, then the value of $p^2 + 3p + \frac{1}{p^2} - \frac{3}{p}$ is:

HINTS $p^2 - 4p - 1 = 0, \Rightarrow p - \frac{1}{p} = 4 \quad \Rightarrow p^2 + \frac{1}{p^2} = 4^2 + 2 = 18$

$\therefore \quad p^2 + \frac{1}{p^2} + 3\left(p - \frac{1}{p}\right)$

$= 18 + 3 \times 4 = 18 + 12 = 30$

(C) Reverse case

If $x^4 + \frac{1}{x^4} = n$, then

$x^2 + \frac{1}{x^2} = \sqrt{n + 2}$ $\qquad$ $x^2 - \frac{1}{x^2} = \sqrt{n - 2}$

$x + \frac{1}{x} = \sqrt{\sqrt{n + 2} + 2}$ $\qquad$ $x - \frac{1}{x} = \sqrt{\sqrt{n + 2} - 2}$

To solve, first convert the given expression into $x \pm \frac{1}{x}$ then proceed further according to requirement of questions.

Ex. If $x^2 + \frac{1}{x^2} = 119$ ($x > 0$), then the value of $\left(x^3 + \frac{1}{x^3}\right)$ is:

HINTS $x^2 + \frac{1}{x^2} = 119$

$\Rightarrow x + \frac{1}{x} = \sqrt{119 + 2} = \sqrt{121} = 11$

$= x^3 + \frac{1}{x^3} = (11)^3 - 3 \times 11 = 1331 - 33 = 1298$

Ex. If $k^4 + \frac{1}{k^4} = 194$, then what is the value of $k^3 + \frac{1}{k^3}$?

HINTS $k^4 + \frac{1}{k^4} = 194$

$\Rightarrow \quad k^2 + \frac{1}{k^2} = \sqrt{194 + 2} = 14 \quad \Rightarrow \quad k + \frac{1}{k} = \sqrt{14 + 2} = 4$

$\therefore \ k^3 + \frac{1}{k^3} = 4^3 - 12 = 52$

Ex. If $x > 1$ and $x^2 + \frac{1}{x^2} = 83$ then, $x^3 - \frac{1}{x^3}$ is:

HINTS $x^2 + \frac{1}{x^2} = 83$

$\Rightarrow \ x - \frac{1}{x} = \sqrt{83 - 2} = 9$

$\therefore \ x^3 - \frac{1}{x^3} = 9^3 + 3 \times 9 = 729 + 27 = 756$

Ex. Given that $x^8 - 34x^4 + 1 = 0$, $x > 0$. What is the value of $x^3 - \frac{1}{x^3}$?

HINTS $x^8 - 34x^4 + 1 = 0$

$\Rightarrow x^4 - 34 + \frac{1}{x^4} = 0 \Rightarrow x^4 + \frac{1}{x^4} = 34$

$\Rightarrow x^2 + \frac{1}{x^2} = \sqrt{34+2} = 6 \qquad x - \frac{1}{x} = \sqrt{6-2} = 2$

$\therefore x^3 - \frac{1}{x^3} = 2^3 + 3 \times 2 = 14$

(D) Concept of bridging

$$x \pm \frac{1}{x} = n$$

If $x + \frac{1}{x} = n$, then

- $x - \frac{1}{x} = \pm\sqrt{n^2 - 4}$
- $x^2 - \frac{1}{x^2} = \pm n\sqrt{n^2 - 4}$

If $x - \frac{1}{x} = n$, then

- $x + \frac{1}{x} = \pm\sqrt{n^2 + 4}$
- $x^2 - \frac{1}{x^2} = \pm n\sqrt{n^2 + 4}$

Ex. If $x + \frac{1}{x} = 3$, then

HINTS

- $x - \frac{1}{x} = \pm\sqrt{3^2 - 4} = \pm\sqrt{5}$
- $x^2 - \frac{1}{x^2} = 3 \times (\pm\sqrt{5}) = \pm 3\sqrt{5}$

Ex. If $x - \frac{1}{x} = 3$, then

HINTS

- $x + \frac{1}{x} = \pm\sqrt{3^2 + 4} = \pm\sqrt{13}$
- $x^2 - \frac{1}{x^2} = 3 \times (\pm\sqrt{13}) = \pm 3\sqrt{13}$

Ex. If $x + \frac{1}{x} = -14$, and $x < -1$ what will be the value of $x^2 - \frac{1}{x^2}$

HINTS $x + \frac{1}{x} = -14$

- $x - \frac{1}{x} = \sqrt{(-14)^2 - 4} = \sqrt{192} = \pm 8\sqrt{3}$

$x + \frac{1}{x}$ or $x - \frac{1}{x} = \pm n$ gives two values.

- Choose the sign based on the range of x.
- If $x < -1$, then the value is negative.

Here, We take $-8\sqrt{3}$ because $x < -1$,

$x^2 - \frac{1}{x^2} = (-14) \times (-8\sqrt{3}) = 112\sqrt{3}$

General Form:-

- If $x^m + \frac{1}{x^m} = a$, then $x^m - \frac{1}{x^m} = \pm\sqrt{a^2 - 4}$
- If $x^m - \frac{1}{x^m} = a$, then $x^m + \frac{1}{x^m} = \pm\sqrt{a^2 + 4}$

Ex. If $= x^3 + \frac{1}{x^3} = 18$, then $x^3 - \frac{1}{x^3} = ?$

HINTS $x^3 - \frac{1}{x^3} = \sqrt{18^2 - 4} = \sqrt{324 - 4} = \sqrt{320} = 8\sqrt{5}$

Ex. If $x^5 - \frac{1}{x^5} = 82$, then $= x^5 + \frac{1}{x^5}$?

HINTS $x^5 + \frac{1}{x^5} = \sqrt{82^2 + 4} = \sqrt{6724 + 4} = \sqrt{6728} = 58\sqrt{2}$

Ex. If $x > 1$ and $x^2 + \frac{1}{x^2} = 2\sqrt{5}$, what is the value of $x^4 - \frac{1}{x^4}$?

HINTS $x^2 + \frac{1}{x^2} = 2\sqrt{5}$

$x^2 - \frac{1}{x^2} = \sqrt{(2\sqrt{5})^2 - 4} = 4$

$\therefore x^4 - \frac{1}{x^4} = \left(x^2 + \frac{1}{x^2}\right)\left(x^2 - \frac{1}{x^2}\right) = 2\sqrt{5} \times 4 = 8\sqrt{5}$

Ex. If $\left(x + \frac{1}{x}\right) = 2\sqrt{2}$ and $x > 1$, what is the value of $\left(x^6 - \frac{1}{x^6}\right)$?

HINTS Given, that

$x + \frac{1}{x} = 2\sqrt{2} \Rightarrow x - \frac{1}{x} = \sqrt{8 - 4} = 2$

$\Rightarrow x^3 - \frac{1}{x^3} = 2^3 + 3 \times 2 = 14$

$\Rightarrow x^3 + \frac{1}{x^3} = (2\sqrt{2})^3 - 3 \times 2\sqrt{2} = 10\sqrt{2}$

$\Rightarrow x^6 - \frac{1}{x^6} = (x^3 - \frac{1}{x^3})(x^3 + \frac{1}{x^3})$

$= 14 \times 10\sqrt{2} = 140\sqrt{2}$

Ex. If $\left(x - \frac{1}{x}\right) = \sqrt{6}$ and $x > 1$, what is the value of $\left(x^8 - \frac{1}{x^8}\right)$?

HINTS $\left(x - \frac{1}{x}\right) = \sqrt{6}$

$\Rightarrow x + \frac{1}{x} = \sqrt{6 + 4} = \sqrt{10} \Rightarrow x^2 + \frac{1}{x^2} = \left(\sqrt{10}\right)^2 - 2 = 8$

$\Rightarrow x^4 + \frac{1}{x^4} = 8^2 - 2 = 62$

$x^4 - \frac{1}{x^4} = \sqrt{3844 - 4} = \sqrt{3840}$

$\therefore x^8 - \frac{1}{x^8} = \left(x^4 + \frac{1}{x^4}\right)\left(x^4 - \frac{1}{x^4}\right) = 62\sqrt{3840} = 992\sqrt{15}$

(E) Unequal Co-efficient

$$ax \pm \frac{b}{x} = k$$

If $ax + \frac{b}{x} = k$, then

- $ax - \frac{b}{x} = \pm\sqrt{k^2 - 4ab}$
- $a^2x^2 + \frac{b^2}{x^2} = k^2 - 2ab$
- $a^3x^3 + \frac{b^3}{x^3} = k^3 - 3kab$

If $ax - \frac{b}{x} = k$, then

- $ax + \frac{b}{x} = \pm\sqrt{k^2 + 4ab}$
- $a^2x^2 + \frac{b^2}{x^2} = k^2 + 2ab$
- $a^3x^3 - \frac{b^3}{x^3} = k^3 + 3kab$

Ex. If $\left(5a + \frac{4}{a} - 2\right) = 13$ and $a > 0$, then

HINTS $5a + \frac{4}{a} - 2 = 13 \Rightarrow 5a + \frac{4}{a} = 15$

- $5a - \frac{4}{a} = \pm\sqrt{15^2 - 4\times5\times4} = \pm\sqrt{225-80} = \pm\sqrt{145}$
- $25a^2 + \frac{16}{a^2} = 225 - 2\times5\times4 = 225 - 40 = 185$
- $125a^3 + \frac{64}{a^3} = 3375 - 3\times15\times5\times4 = 3375 - 900 = 2475$

Ex. If $\left(5a - \frac{4}{a}\right) = 13$ and $a > 0$, then

HINTS $5a - \frac{4}{a} = 13$

- $5a + \frac{4}{a} = \pm\sqrt{13^2 + 4\times5\times4} = \pm\sqrt{169+80} = \pm\sqrt{249}$
- $25a^2 + \frac{16}{a^2} = 169 + 2\times5\times4 = 169 + 40 = 209$
- $125a^3 + \frac{64}{a^3} = (13)^3 + 3\times13\times5\times4 = 2197 + 780 = 2977$

(F) Special Case

- If $x + \frac{1}{x} = 2$, then $x = 1$
- If $x + \frac{1}{x} = -2$, then $x = -1$
- If $x + \frac{1}{x} = 1$, then $x^3 = -1$
- If $x + \frac{1}{x} = -1$, then $x^3 = 1$
- If $x + \frac{1}{x} = \pm\sqrt{2}$, then $x^2 + \frac{1}{x^2} = 0 \Rightarrow x^4 = -1$
- If $x + \frac{1}{x} = \pm\sqrt{3}$, then $x^3 + \frac{1}{x^3} = 0 \Rightarrow x^6 = -1$

Ex. If $x + \frac{1}{x} = 2$ then find the value of $x^{121} + \frac{1}{x^{121}}$.

HINTS We know that, if $x + \frac{1}{x} = 2$ then $x = 1$

$\therefore x^{121} + \frac{1}{x^{121}} = 1 + 1 = 2$

Ex. If $x + \frac{1}{x+2} = -4$ then find the value of $(x+2)^{231} + \frac{1}{(x+2)^{231}}$.

HINTS Given, $x + \frac{1}{x+2} = -4$

$\Rightarrow (x+2) + \frac{1}{(x+2)} = -2 \Rightarrow x + 2 = -1$

$\therefore (x+2)^{231} + \frac{1}{(x+2)^{231}} = (-1)^{231} + \frac{1}{(-1)^{231}}$

$= -1 - 1 = -2$

Ex. If $\frac{r}{13} + \frac{13}{r} = 1$ then the value of r^3 is:

HINTS $\frac{r}{13} + \frac{13}{r} = 1$ Let, $x = \frac{r}{13}$

If $x + \frac{1}{x} = 1 \Rightarrow x^3 = -1 \Rightarrow \left(\frac{r}{13}\right)^3 = -1 \Rightarrow r^3 = -2197$

Ex. If $x + \frac{1}{x} = -1$ then find the value of $x^{51} + x^{45} + x^{21} + x^{15} + x^3 + 4$.

HINTS We know that, if $x + \frac{1}{x} = -1$ then $x^3 = 1$

$\therefore x^{51} + x^{45} + x^{21} + x^{15} + x^3 + 4$

$= (x^3)^{17} + (x^3)^{15} + (x^3)^7 + (x^3)^5 + x^3 + 4$

$= 1 + 1 + 1 + 1 + 1 + 4 = 9$

Ex. If $x + \frac{1}{x} = \sqrt{2}$; then find the value of $x^{96} + x^{100} + x^{112} + x^{116} + x^4 + 3$.

HINTS $x + \frac{1}{x} = \sqrt{2}$ then $x^4 = -1$

$\therefore x^{96} + x^{100} + x^{112} + x^{116} + x^4 + 3$

$= x^{96}(x^4 + 1) + x^{112}(x^4 + 1) + x^4 + 3$

$= 0 + 0 + (-1) + 3 = 2$

Ex. If $x + \frac{1}{x} = \sqrt{3}$ then find the value of $x^{1012} + x^{1006} + x^{506} + x^{500} + x^{400} + x^{406} + x^{206} + x^{200} + x^6 + 5$

HINTS We know that, $x + \frac{1}{x} = \sqrt{3}$ then $x^6 + 1 = 0$ or $x^6 = -1$.

$\therefore x^{1012} + x^{1006} + x^{506} + x^{500} + x^{400} + x^{406} + x^{206} + x^{200} + x^6 + 5$

$= x^{1006}(x^6 + 1) + x^{500}(x^6 + 1) + x^{400}(x^6 + 1) + x^{200}(x^6 + 1) + x^6 + 5 = -1 + 5 = 4$

(G) Miscellaneous Concept

☞ If $x = \frac{\sqrt{a}+\sqrt{b}}{\sqrt{a}-\sqrt{b}}$ then, $x + \frac{1}{x} = \frac{\sqrt{a}+\sqrt{b}}{\sqrt{a}-\sqrt{b}} + \frac{\sqrt{a}-\sqrt{b}}{\sqrt{a}+\sqrt{b}} = \frac{2(a+b)}{(a-b)}$

Ex. If $x = \frac{\sqrt{5}-\sqrt{4}}{\sqrt{5}+\sqrt{4}}$ and $y = \frac{\sqrt{5}+\sqrt{4}}{\sqrt{5}-\sqrt{4}}$ then the value of $\frac{x^2 - xy + y^2}{x^2 + xy + y^2}$.

HINTS $xy = \frac{\sqrt{5}-\sqrt{4}}{\sqrt{5}+\sqrt{4}} \times \frac{\sqrt{5}+\sqrt{4}}{\sqrt{5}-\sqrt{4}} = 1$

When two numbers x and y are reciprocal then $x + y = \frac{2(a+b)}{(a-b)} = \frac{2(5+4)}{(5-4)} = 18 \Rightarrow x^2 + y^2 = 18^2 - 2 = 322$

$\therefore \frac{x^2 - xy + y^2}{x^2 + xy + y^2} = \frac{322 - 1}{322 + 1} = \frac{321}{323}$

Ex. If $x^2 - 11x + 1 = 0$, what is the value of $x^8 - 14159x^4 + 11$?

HINTS $x^2 - 11x + 1 = 0$

$\Rightarrow x + \frac{1}{x} = 11 \Rightarrow x^2 + \frac{1}{x^2} = 121 - 2 = 119$

$\Rightarrow x^4 + \frac{1}{x^4} = 119^2 - 2 = 14161 - 2$

$\Rightarrow x^4 + \frac{1}{x^4} = 14159 \Rightarrow x^8 + 1 = 14159x^4$

$\Rightarrow x^8 - 14159x^4 = -1$

$\therefore x^8 - 14159x^4 + 11 = -1 + 11 = 10$

Ex. If $x(x - 5) = -1$, then the value of $x^3(x^3 - 110)$?

HINTS Given, $x(x - 5) = -1$

$\Rightarrow x - 5 = \frac{-1}{x} \Rightarrow x + \frac{1}{x} = 5$

$\Rightarrow x^3 + \frac{1}{x^3} = 5^3 - 3 \times 5$

$\Rightarrow x^3 + \frac{1}{x^3} = 110 \Rightarrow x^6 + 1 = 110x^3$

$\Rightarrow x^6 - 110x^3 = -1$

$\therefore x^3(x^3 - 110) = -1$

Ex. If $8k^6 + 15k^3 - 2 = 0$, then the positive value of $\left(k + \frac{1}{k}\right)$ is:

HINTS Let, $k^3 = x$

So,

$8x^2 + 15x - 2 = 0$

$\Rightarrow 8x^2 + 16x - x - 2 = 0$

$\Rightarrow 8x(x + 2) - 1(x + 2) = 0$

$\Rightarrow (8x - 1)(x + 2) = 0$

$\Rightarrow 8x - 1 = 0 \Rightarrow x = \frac{1}{8}$

or $x + 2 = 0 \Rightarrow x = -2$

[Not possible because of negative value]

Now,

$k^3 = \frac{1}{8} \Rightarrow k = \frac{1}{2}$

$\therefore (k + \frac{1}{k}) = (\frac{1}{2} + 2) = \frac{5}{2} = 2\frac{1}{2}$

Ex. If $x^2 + 6x + 1 = 0$ then the value of $(x + 6)^3 + \frac{1}{(x+6)^3}$.

HINTS Let, $x + 6 = t \Rightarrow x = t - 6$

$x^2 + 6x + 1 = 0$

$\Rightarrow (t - 6)^2 + 6(t - 6) + 1 = 0$

$\Rightarrow t^2 + 36 - 12t + 6t - 36 + 1 = 0$

$\Rightarrow t^2 - 6t + 1 = 0 \Rightarrow t + \frac{1}{t} = 6$

$t^3 + \frac{1}{t^3} = 6^3 - 3 \times 6 = 198$

$\therefore (x + 6)^3 + \frac{1}{(x+6)^3} = 198$

Alternatively

$x^2 + 6x + 1 = 0 \Rightarrow x + 6 = \frac{-1}{x} \Rightarrow x + \frac{1}{x} = -6$

$x^3 + \frac{1}{x^3} = -216 + 18 = -198$

$\therefore (x + 6)^3 + \frac{1}{(x+6)^3} = \left(\frac{-1}{x}\right)^3 + (-x)^3 = -\left(\frac{1}{x^3} + x^3\right)$

$= -(-198) = 198$

Ex. If $x^2 - 16x + 59 = 0$, then what is the value of $(x - 6)^2 + \frac{1}{(x-6)^2}$.

HINTS Let $y = x - 6$

$\Rightarrow x = y + 6$

Now, $(y + 6)^2 - 16(y + 6) + 59 = 0$

$\Rightarrow y^2 + 36 + 12y - 16y - 96 + 59 = 0$

$\Rightarrow y^2 - 4y = 1$

$\Rightarrow y - \frac{1}{y} = 4$

$\Rightarrow y^2 + \frac{1}{y^2} = 18$

$\therefore (x - 6)^2 + \frac{1}{(x-6)^2} = 18$

Square Formulae (Two Variables)

$(a + b)^2$ = $a^2 + b^2 + 2ab$ = $(a - b)^2 + 4ab$

$(a - b)^2$ = $a^2 + b^2 - 2ab$ = $(a + b)^2 - 4ab$

- $(a + b)^2 + (a - b)^2 = 2(a^2 + b^2)$
- $(a + b)^2 - (a - b)^2 = 4ab$
- $ab = \left(\frac{a+b}{2}\right)^2 - \left(\frac{a-b}{2}\right)^2$
- $a^2 - b^2 = (a + b)(a - b)$

$a^4 - b^4$ = $(a^2 + b^2)(a^2 - b^2)$ = $(a^2 + b^2)(a + b)(a - b)$

$a^4 + b^4$ = $(a^2 - b^2)^2 + 2a^2b^2$ = $(a^2 + b^2)^2 - 2a^2b^2$

Ex. The simplified form of $(7x + 4y)^2 + (7x - 4y)^2$

HINTS $(a + b)^2 + (a - b)^2 = 2(a^2 + b^2)$

$(7x + 4y)^2 + (7x - 4y)^2 = 2[(7x)^2 + (4y)^2]$

$= 2(49x^2 + 16y^2)$

$= 98x^2 + 32y^2$

Ex. $\left(\frac{2x+3y}{2}\right)^2 - \left(\frac{2x-3y}{2}\right)^2 = ?$

HINTS $\left(\frac{2x+3y}{2}\right)^2 - \left(\frac{2x-3y}{2}\right)^2 = 2x \times 3y = 6xy$

Cube Formulae (Two Variables)

- $(a + b)^3 = a^3 + b^3 + 3ab(a + b)$
- $(a - b)^3 = a^3 - b^3 - 3ab(a - b)$
- $(a + b)^3 + (a - b)^3 = 2a^3 + 6ab^2$
- $(a + b)^3 - (a - b)^3 = 2b^3 + 6a^2b$

$a^3 + b^3$	$a^3 - b^3$
$(a + b)(a^2 - ab + b^2)$	$(a - b)(a^2 + ab + b^2)$
$(a + b)[(a + b)^2 - 3ab]$	$(a - b)[(a - b)^2 + 3ab]$
$(a + b)^3 - 3ab(a + b)$	$(a - b)^3 + 3ab(a - b)$

- If $\frac{a}{b} + \frac{b}{a} = \pm 1$, then $a^3 \pm b^3 = 0$
- If $\frac{1}{a} \pm \frac{1}{b} = \frac{1}{a \pm b}$, then $a^3 \mp b^3 = 0$
- If $ab(a + b) = 1$, then $\frac{1}{a^3b^3} - a^3 - b^3 = 3$

Ex. If $x + y = 1$, then what is the value of $x^3 + 3xy + y^3$?

HINTS $x + y = 1$

cube on both sides

$x^3 + y^3 + 3xy(x + y) = 1^3$

$\Rightarrow x^3 + y^3 + 3xy = 1$

Ex. If a = 999, then the value of $\sqrt[3]{a(a^2+3a+3)+1}$ is.

HINTS we know that, $(a + 1)^3 = a^3 + 3a^2 + 3a + 1$

Here, a = 999

$\therefore \sqrt[3]{a(a^2+3a+3)+1} = (a + 1) = 999 + 1 = 1000$

Ex. Simplify $(x + y)^3 - (x - y)^3 - 6y(x^2 - y^2)$

HINTS $(x + y)^3 - (x - y)^3 = 2y^3 + 6x^2y$

$\therefore (x + y)^3 - (x - y)^3 - 6y(x^2 - y^2)$

$= 2y^3 + 6x^2y - 6x^2y + 6y^3 = 8y^3$

Ex. What is the simplified form of the following expression?

$$\left(x - \frac{1}{y}\right)^3 + \left(x + \frac{1}{y}\right)^3$$

HINTS $(a + b)^3 + (a - b)^3 = 2a^3 + 6ab^2$

$$\left(x - \frac{1}{y}\right)^3 + \left(x + \frac{1}{y}\right)^3 = 2(x)^3 + 6(x)\left(\frac{1}{y}\right)^2 = 2x^3 + \frac{6x}{y^2}$$

Ex. If $(2x - 5y)^3 - (2x + 5y)^3 = y[Ax^2 + By^2]$, then what is the value of (2A – B)?

HINTS $(a - b)^3 - (a + b)^3 = -2b(b^2 + 3a^2)$

$(2x - 5y)^3 - (2x + 5y)^3 = -2 \times 5y[(5y)^2 + 3 \times (2x)^2]$

$= -10y(25y^2 + 12x^2) = y(-120x^2 - 250y^2)$

$\Rightarrow A = -120, B = -250$

$\therefore (2A - B) = 2 \times (-120) + 250 = 10$

Ex. If a + b = 10 and ab = 6, then the value of $a^3 + b^3$ is:

HINTS a + b = 10, ab = 6

$a^3 + b^3 = (a + b)\{(a + b)^2 - 3ab\}$

$= 10\{(10)^2 - 3 \times 6\} = 10 \times 82$

$= 820$

Ex. If $x - y = 25$ and $xy = 444$, compute the value of $x^3 - y^3$.

HINTS $x^3 - y^3 = (x - y)[(x - y)^2 + 3xy]$

$= 25[25^2 + 3 \times 444]$

$= 25[625 + 1332]$

$= 48925$

Square Formulae (Three Variables)

- $(a + b + c)^2 = a^2 + b^2 + c^2 + 2(ab + bc + ca)$
- $(a - b - c)^2 = a^2 + b^2 + c^2 - 2(ab - bc + ca)$
- $(a + b - c)^2 = a^2 + b^2 + c^2 + 2(ab - bc - ca)$
- $(a + b + c)^2 + (a - b - c)^2 = 2[a^2 + (b + c)^2]$
- $(a + b + c)^2 - (a - b - c)^2 = 4a(b + c)$
- $(a - b)^2 + (b - c)^2 + (c - a)^2 = 2(a^2 + b^2 + c^2 - ab - bc - ac)$

Ex. If a + b + c = 10 and ab + bc + ca = 30, then the value of $a^2 + b^2 + c^2$ is:

HINTS $a^2 + b^2 + c^2 = (a + b + c)^2 - 2(ab + bc + ca)$

$= (10)^2 - 2(30) = 40$

Ex. If $x + y + z = 13$, $x^2 + y^2 + z^2 = 91$ and $xz = y^2$, then the difference between z and x is:

HINTS $(x + y + z)^2 = x^2 + y^2 + z^2 + 2(xy + yz + zx)$

$\Rightarrow 169 = 91 + 2(xy + yz + y^2)$

$\Rightarrow 78 = 2y(x + z + y)$

$\Rightarrow 39 = y \times 13$

$\Rightarrow y = 3$

$(z - x)^2 = x^2 + z^2 - 2xz$

$\Rightarrow (z - x)^2 = 82 - 2 \times 9 \Rightarrow (z - x)^2 = 64$

$\therefore z - x = 8$

Alternatively

Put $x = 1$, y = 3 and z = 9 satisfy all the equation.

$\therefore z - x = 9 - 1 = 8$

Ex. If a + b + c = 10; $a^2 + b^2 + c^2 = 38$, what is the value of $(a - b)^2 + (b - c)^2 + (c - a)^2$?

HINTS $ab + bc + ca = \frac{(a+b+c)^2 - (a^2+b^2+c^2)}{2}$

$= \frac{100-38}{2} = \frac{62}{2} = 31$

$\therefore (a - b)^2 + (b - c)^2 + (c - a)^2$

$= 2(38 - 31) = 14$

Alternatively

Put a = 5, b = 3 and c = 2 satisfy all the equation.

$\therefore (a - b)^2 + (b - c)^2 + (c - a)^2 = 2^2 + 1^2 + (-3)^2 = 4 + 1 + 9 = 14$

Cube Formulae (Three Variables)

$a^3+b^3+c^3-3abc$

- **F-1** $(a+b+c)(a^2+b^2+c^2-ab-bc-ca)$
- **F-2** $(a+b+c)[(a+b+c)^2-3(ab+bc+ca)]$
- **F-3** $\frac{1}{2}(a+b+c)[3(a^2+b^2+c^2)-(a+b+c)^2]$
- **F-4** $\frac{1}{2}(a+b+c)[(a-b)^2+(b-c)^2+(c-a)^2]$

Ex. If a + b + c = 6, $a^2 + b^2 + c^2 = 14$ and ab + bc + ca = 11, then what is the value of $a^3 + b^3 + c^3 - 3abc$?

HINTS a + b + c = 6,

$a^2 + b^2 + c^2 = 14$,

ab + bc + ca = 11

$\therefore a^3 + b^3 + c^3 - 3abc$

$= 6(14 - 11) = 6 \times 3 = 18$

Ex. If $x + y + z = 7$, $xy + yz + zx = 8$, then what is the value of $x^3 + y^3 + z^3 - 3xyz$?

HINTS $x^3 + y^3 + z^3 - 3xyz = 7(49 - 3 \times 8)$

$= 7(49 - 24) = 7 \times 25 = 175$

Ex. If a + b + c = 6, $a^2 + b^2 + c^2 = 32$ and $a^3 + b^3 + c^3 = 189$, then the value of abc – 3 is:

HINTS $a^3 + b^3 + c^3 - 3abc$

$= \frac{(a+b+c)}{2}[3(a^2 + b^2 + c^2) - (a + b + c)^2]$

$\Rightarrow 189 - 3abc = 3[3 \times 32 - 36]$

$\Rightarrow 189 - 3abc = 3 \times 60 \Rightarrow abc = 3$

$\therefore abc - 3 = 3 - 3 = 0$

Ex. If $x = 32$, $y = 33$ and $z = 35$, then evaluate the expression $x^3 + y^3 + z^3 - 3xyz$.

HINTS $x - y = -1$, $y - z = -2$, $z - x = 3$

$x^3 + y^3 + z^3 - 3xyz = \frac{100}{2}(1 + 4 + 9) = 700$

Special cases

Case-I	Case-II	Case-III	Case-IV
If $a^3 + b^3 + c^3 - 3abc = 0$ a + b + c = 0 (When, a, b, c are distinct integer) a = b = c (When, a, b, c are positive integer)	If a, b and c are in A.P. and common difference is d then, $a^3 + b^3 + c^3 - 3abc = 9bd^2$ Where b is the middle term.	If a, b and c are in A.P. and common difference is d then, $a^2 + b^2 + c^2 - ab - bc - ca = 3d^2$	If two numbers are equal and third number is one more than the numbers then, $a^3 + b^3 + c^3 - 3abc = a + b + c$

Ex. If $(3x + 1)^3 + (x - 3)^3 + (4 - 2x)^3 + 6(3x + 1)(x - 3)(x - 2) = 0$, then what is the value of x?

HINTS If $a^3 + b^3 + c^3 - 3abc = 0$

Then, a + b + c = 0

$\Rightarrow 3x + 1 + x - 3 + 4 - 2x = 0$

$\Rightarrow 2x + 2 = 0$

$\Rightarrow x = -1$

Ex. If $(4x - 3)^3 + (2x + 5)^3 + (5x - 7)^3 = (4x - 3)(6x + 15)(5x - 7)$ and $x \neq \frac{5}{11}$ then, find x?

HINTS **Case 1**:- If $a^3 + b^3 + c^3 - 3abc = 0$ then a + b + c = 0

$\Rightarrow 4x - 3 + 2x + 5 + 5x - 7 = 0$

$\Rightarrow 11x - 5 = 0$

$\Rightarrow x = \frac{5}{11}$ (It is not possible)

Case 2 :- If $a^3 + b^3 + c^3 - 3abc = 0$ then a = b = c

$\Rightarrow 4x - 3 = 2x + 5$

$\Rightarrow 2x = 8$

$\Rightarrow x = 4$

Ex. If a = 199, b = 200, c = 201 then find the value of $a^3 + b^3 + c^3 - 3abc$

HINTS Here, a = 199 b = 200 and c = 201 a, b, c are in A.P. Because common difference (d) = 200 – 199 = 1,

b = 200

$\therefore a^3 + b^3 + c^3 - 3abc = 9bd^2$

$= 9 \times 200 \times 1 = 1800$

Ex. If a = 2001, b = 2002 and c = 2003 then find the value of $a^2 + b^2 + c^2 - ab - bc - ca$.

HINTS Here, a = 2001, b = 2002 and c = 2003 a, b and c are in A.P because common difference (d)

= 2002 – 2001 = 1

$\therefore a^2 + b^2 + c^2 - ab - bc - ca = 3d^2$

$= 3 \times (1)^2 = 3$

Ex. If a = b = 336 and c = 337 then find the value of $a^3 + b^3 + c^3 - 3abc$

HINTS a = b = 336 and c = 337

$a^3 + b^3 + c^3 - 3abc = a + b + c$

= 336 + 336 + 337 = 1009

Concept of Perfect Square

If $x + y = 0$ then, either x or y should be negative but if $x^2 + y^2 = 0$ then, both x and y should be 0 because neither x nor y can be negative

(i) If $x^2 + y^2 + z^2 = 0$ then $x = y = z = 0$

(ii) If $(x - a)^2 + (y - b)^2 + (z - c)^2 = 0$

then $x - a = 0 \Rightarrow x = a$

$y - b = 0 \Rightarrow y = b$

$z - c = 0 \Rightarrow z = c$

Whenever the sum of square of n–terms is equal to zero then individually all terms is zero:-

If $(x - a_1)^2 + (x - a_2)^2 + (x - a_3)^2 + ... (x - a_n)^2 = 0$

then, $x - a_1 = 0$, $x - a_2 = 0$ .... $x - a_n = 0$

Ex. If $(a - 18)^2 + (b - 12)^2 + (c - 6)^2 = 0$, then find the value of $(a + b + c)^{\frac{1}{2}}$

HINTS $(a - 18)^2 + (b - 12)^2 + (c - 6)^2 = 0$

$\Rightarrow a = 18$; $b = 12$; $c = 6$

$\therefore (a + b + c)^{½} = (36)^{½} = \pm 6$

Ex. If the value of $(a + b - 2)^2 + (b + c - 5)^2 + (c + a - 5)^2 = 0$ then the value of $\sqrt{(b+c)^a + (c+a)^b - 1}$ is:

HINTS $(a + b - 2)^2 + (b + c - 5)^2 + (c + a - 5)^2 = 0$

$a + b = 2$ ...... (i)

$b + c = 5$ ...... (ii)

$c + a = 5$ ..... (iii)

Adding equation (i), (ii) & (iii)

$\therefore 2(a + b + c) = 12$

$\Rightarrow a + b + c = 6$ ...... (iv)

$\Rightarrow a = 1, b = 1, c = 4$

$\therefore \sqrt{(b+c)^a + (c+a)^b - 1} = \sqrt{(5)^1 + (5)^1 - 1} = \sqrt{9} = 3$

Ex. If $x^2 + 4y^2 + 2x + 1 = 0$, then find the value of $x^{39} + y^{36}$.

HINTS $x^2 + 4y^2 + 2x + 1 = 0$

$\Rightarrow (x + 1)^2 + (2y)^2 = 0$

$\Rightarrow x + 1 = 0 \Rightarrow x = -1$

$2y = 0 \Rightarrow y = 0$

$\therefore x^{39} + y^{36} = (-1)^{39} + (0)^{36} = -1$

Ex. If $a^2 + b^2 + 49c^2 + 18 = 2(b - 28c - a)$ then the value of $(a + b - 7c)$ is:

HINTS $a = \frac{\text{Coefficient of a}}{\text{Coefficient of } a^2} = \frac{-1}{1} = -1$

$b = \frac{\text{Coefficient of b}}{\text{Coefficient of } b^2} = 1$

$c = \frac{\text{Coefficient of c}}{\text{Coefficient of } c^2} = \frac{-28}{49} = \frac{-4}{7}$

$\therefore a + b - 7c = -1 + 1 - 7 \times \left(\frac{-4}{7}\right) = 4$

Concept of Value Putting

If $x + y = 7$ find $5(x + y)$

↓	↓
7	0
6	1
5	2
4	3

We can't find exact value of x and y

But $5(x + y) = 5 \times 7 = 35$

$\Rightarrow$ The value of $5(x + y)$ depends upon $(x + y)$ not upon x & y

If an equation contains two variable, then two equations are required to find the values of both variables.

- If only one equation is given (e.g. $x + y = 7$) then we can fix the value of either one of the variables. The other variable will remain unfixed and we can assume its value according to our convenience, as long as it satisfies the given condition.

Cautions:

(i) When putting a value in the equation, make sure the equation ***doesn't become undefined*** such as $\frac{0}{0}$, infinity or any other undefined form.

(ii) First try putting 0 for any one of the variable. If that doesn't work, try putting another variable to 0.

(iii) If putting 0 doesn't satisfied, try other values like +1, –1, –2, –6 etc depending on what fits the equation.

(iv) Any value we assume must ***not violate any condition*** mentioned in the question

- If two/three equations are given then the value of two/three variables will be fixed and remaining all other variables will be unfix which we can assume their values as long as the equations are satisfied.

Ex. If $2a + 2b + c = 0$, then find the value of.

$\frac{4a^2 + 4b^2 + 4c^2}{5c^2 - 8ab}$

HINTS Put, $a = b = 1$

$\Rightarrow c = -4$

$\therefore \frac{4a^2 + 4b^2 + 4c^2}{5c^2 - 8ab} = \frac{4 + 4 + 64}{80 - 8} = \frac{72}{72} = 1$

Ex. If $\alpha + \beta + \gamma = 0$, then $\frac{3\beta^2 + \alpha^2 + \gamma^2}{2\beta^2 - \alpha\gamma} = ?$

HINTS Put, $\alpha = \beta = 1$; $\gamma = -2$

$\therefore \frac{3\beta^2 + \alpha^2 + \gamma^2}{2\beta^2 - \alpha\gamma}$

$= \frac{3 \times (1)^2 + (1)^2 + (-2)^2}{2 \times (1)^2 - 1 \times (-2)} = 2$

Ex. If $x^2 + y^2 + z^2 = xy + yz + zx$, then the value of $\left(\frac{17x^4 + 9y^4 + 16z^4}{8x^2y^2 + 6y^2z^2 + 10z^2x^2}\right)$ is:

HINTS Put $x = y = z = 1$

$\therefore \frac{17 + 9 + 16}{8 + 6 + 10} = \frac{42}{24} = 1.75$

Concept of Symmetricity

Whenever the degree of each term in an expression is same & they come in same numbers in the given expression, Apply the concept of symmetricity. i.e equate the each variable/terms to another and simplify the expression.

- An algebraic expression is symmetric in its variables of interchanging any variables does not changes the expression.

  If $a^2 + b^2 + c^2 = ab + bc + ca$, then $a = b = c$

- Degree of each term in both side should be same.

Ex. If $a^2 = b + c$, $b^2 = c + a$ and $c^2 = a + b$ then find the value of $\frac{1}{1+a}+\frac{1}{1+b}+\frac{1}{1+c}$?

HINTS Check the value of $a = b = c = 2$ in $a^2 = b + c$, $b^2 = c + a$ and $c^2 = a + b$ and these value satisfying the equation.

$$\Rightarrow \frac{1}{1+a}+\frac{1}{1+b}+\frac{1}{1+c}=\frac{1}{1+2}+\frac{1}{1+2}+\frac{1}{1+2}=1$$

Ex. If $\frac{x^2}{by+cz}=\frac{y^2}{ax+cz}=\frac{z^2}{ax+by}=1$ then find the value of $\frac{x}{x+a}+\frac{y}{y+b}+\frac{z}{z+a}$

HINTS Check the value of $x = y = z = 2$ and $a = b = c = 1$ in above eqn and these value satisfying the equation.

Put $x = y = z = 2$ & $a = b = c = 1$

$$\Rightarrow \frac{x}{x+a}+\frac{y}{y+b}+\frac{z}{z+c}$$

$$=\frac{2}{2+1}+\frac{2}{2+1}+\frac{2}{2+1}=2$$

Concept of Degree

In algebra, the degree refers to the highest power (exponent) of the variables in a polynomial expression (maximum power of variable is called degree)

Key points:-

(i) Linear polynomial: $ax + b$ (degree = 1)

(ii) Quadratic polynomial: $ax^2 + bx + c$ (degree = 2)

(iii) Cubic polynomial: $ax^3 + bx^2 + cx + d$ (degree = 3)

1. ***For a single-variable polynomial:*** The degree is the highest exponent of that variable:

Ex. In $4x^3 + 2x^2 - x + 7$ The degree is 3 (because of highest power of x^3)

2. ***For a multivariable polynomial:*** The degree is the highest sum of exponent in any term:

$$\underbrace{3x^2y}_{2+1=3} + \underbrace{2xy^3}_{1+3=4} + \underbrace{y}_{1}$$

So, the degree of the equation is 4.

Ex. $7x^8 - 9x^5 + 13x^4 + 12x^2 - 37 \rightarrow$ Degree = 8

3. The degree of a constant (like 5 or –2) is 0, and the degree of the zero polynomial (just 0) is undefined or sometimes defined as negative infinity, depending on the context.

- If the variable is in multiple then power will be added

Ex. $a^1b^1 + bc + ca \Rightarrow$ Degree $(a^1b^1) = 1 + 1 = 2$

- If variable is in division then power will be subtracted

Ex. $\frac{b^2}{a^1} \Rightarrow$ degree $= 2 - 1 = 10$

Ex. $\frac{ab+bc+ca}{a-b-c} \Rightarrow \frac{a^1b^1}{a^1}=\frac{2}{1}=2-1=1$

- In any Homogenous expression degree of question must be equal to degree of answer.

Ex. $\frac{a(b-c)^2}{(c-a)(a-b)}+\frac{b(c-a)^2}{(a-b)(b-c)}+\frac{c(a-b)^2}{(b-c)(c-a)}$ has degree ____.

HINTS Here,

$(b - c)^2$ has degree = 2

$a(b - c)^2$ has degree = 2 + 1 = 3

Also, $(c - a)$ has degree = 1

$(a - b)$ has degree = 1

So, $(c - a)(a - b)$ has degree = 1 + 1 = 2

Hence

$\frac{a(b-c)^2}{(c-a)(a-b)}$ has degree = 3 – 2 = 1

Ex. The value of $\frac{\{(m^2+n^2)(m-n)-(m-n)^3\}}{(m^2n-mn^2)}$ is:

(a) m + n (b) m – n

(c) 2 (d) mn

HINTS Here, the degree of polynomial $=\frac{2+1}{2+1}=\frac{3}{3}=3-3=0$

So, the value of polynomial will also be of 0 degree i.e. the value is constant.

Since only option (c) contains the constant value.

∴ Option (c) is the correct answer.

Componendo & Dividendo Rule

If $\frac{a}{b}=\frac{x}{y}$ then $\frac{a+b}{a-b}=\frac{x+y}{x-y}$

☞ If $x=\frac{2ab}{a+b}$ then $\frac{x+2a}{x-2a}+\frac{x+2b}{x-2b}=2$

Where, $\frac{2ab}{2a}=b$ and $\frac{2ab}{2b}=a$

Ex. If $x=\frac{4\sqrt{15}}{\sqrt{5}+\sqrt{3}}$ then find the value of $\frac{x+\sqrt{20}}{x-\sqrt{20}}+\frac{x+\sqrt{12}}{x-\sqrt{12}}$.

HINTS Using the concept explained above, we will get the result of the expression is 2.

$$\therefore \frac{x+\sqrt{20}}{x-\sqrt{20}}+\frac{x+\sqrt{12}}{x-\sqrt{12}}=2$$

☞ If $\frac{1}{(\sqrt[3]{a^2}-\sqrt[3]{a}+1)}=A\sqrt[3]{a^2}+B\sqrt[3]{a}+C$, then

$A = 0$; $B=\frac{1}{a-1}$; $C=\frac{-1}{a-1}$

Ex. If $\dfrac{1}{\sqrt[3]{4}+\sqrt[3]{2}+1} = a\sqrt[3]{4} + b\sqrt[3]{2} + c$ and a, b, c is a prime number, then find the value of a + b + c.

HINTS Using the above result, a = 0, b = 1, c = –1

Hence, a + b + c = 0 + 1 – 1 = 0

Identity Based Algebric Equation

Ex. If $a^3 + 3a^2 + 9a = 1$, then what is the value of $a^3 + \dfrac{3}{a}$?

HINTS $a^3 + 3a^2 + 9a = 1$ ......(i) × 3

$3a^3 + 9a^2 + 27a = 3$ ......(ii)

Multiply by 'a' in equation (i),

$a^4 + 3a^3 + 9a^2 = a$ .....(iii)

Subtract (ii) from (iii),

$a^4 - 27a = a - 3$

$\Rightarrow a^4 + 3 = 28a$

$\Rightarrow a^3 + \dfrac{3}{a} = 28$

Ex. If $a^2 + 3a + 3 = 0$ find the value of $a^3 + 6a^2 + 12a + 10 = ?$

HINTS $a^2 + 3a + 3 = 0$ .....(i) × a

$a^3 + 3a^2 + 3a = 0$ ....(ii)

Multiply by '3' in equation (i)

$3a^2 + 9a + 9 = 3$ ... (iii)

Add equation (ii) and (ii),

$a^3 + 6a^2 + 12a + 9 = 0$

$\therefore a^3 + 6a^2 + 12a + 10 = 1$

Ex. If $x^2 + 2 = 2x$, then the value of $x^4 - x^3 + x^2 + 2$ is:

HINTS $x^2 - 2x + 2 = 0$ .... (i) × x

$x^3 - 2x^2 + 2x = 0$ .... (ii) × x

$x^4 - 2x^3 + 2x^2 = 0$ .... (iii)

Add equation (ii) and (iii),

$x^4 - x^3 + 2x = 0$

$\Rightarrow x^4 - x^3 + x^2 + 2 = 0$

Some Important Formulae & Results

Formulae 1:-

- $1 + A + B + AB = (1 + A)(1 + B)$
- $(a^2 - ab + b^2)(a^2 + ab + b^2) = a^4 + a^2b^2 + b^4$
- $(a^2 + b^2)(x^2 + y^2) = (ax + by)^2 + (ay - bx)^2$

Formulae 2:-

$$ab + bc + ca = \frac{a^3(b^2-c^2)+b^3(c^2-a^2)+c^3(a^2-b^2)}{a^2(b-c)+b^2(c-a)+c^2(a-b)}$$

$$ab + bc + ca = \frac{a^3(b+c)}{(a-b)(a-c)} + \frac{b^3(c+a)}{(b-c)(b-a)} + \frac{c^3(a+b)}{(c-a)(c-b)}$$

Formulae 3:-

- $\dfrac{a\times(b-c)^2}{(c-a)(a-b)} + \dfrac{b\times(c-a)^2}{(a-b)(b-c)} + \dfrac{c\times(a-b)^2}{(b-c)(c-a)} = a + b + c$
- $(a + b + c + d)^2 = a^2 + b^2 + c^2 + d^2 + 2a(b + c + d) + 2b(c + d) + 2cd$
- $a^2(b - c) + b^2(c - a) + c^2(a - b) = -(b - c)(c - a)(a - b)$
- $a^2(b + c) + b^2(c + a) + c^2(a + b) = (a + b + c)(ab + bc + ca) - 3abc$
- $a(b^2 - c^2) + b(c^2 - a^2) + c(a^2 - b^2) = (b - c)(c - a)(a - b)$
- $ab(a - b) + bc(b - c) + ca(c - a) = (b - a)(b - c)(c - a)$
- $(b + c)(c + a)(a + b) + abc = (a + b + c)(ab + bc + ca)$
- $(x + a)(x + b)(x + c) = x^3 + x^2(a + b + c) + x(ab + bc + ca) + abc$

Ex. If $a^4 + a^2b^2 + b^4 = 24$ and $a^2 + ab + b^2 = 8$, then find ab.

HINTS $a^2 + ab + b^2 = 8$

$a^4 + a^2b^2 + b^4 = 24 \Rightarrow a^2 - ab + b^2 = \dfrac{24}{8} = 3$

$a^2 - ab + b^2 = 3$ ...(i)

$a^2 + ab + b^2 = 8$ ...(ii)

Subtracting eqn (ii) from eq^n (i)

$\Rightarrow -2ab = -5 \Rightarrow ab = 2.5$

Ex. If $ax + by = 8$, $ay - bx = 6$, $a^2 + b^2 + x^2 + y^2 = 29$ find $\dfrac{a^2+b^2}{x^2+y^2}$; $a^2 + b^2 > x^2 + y^2$.

HINTS $(a^2 + b^2)(x^2 + y^2) = (ax + by)^2 + (ay - bx)^2 = 8^2 + 6^2$

$= 100$

25 4

$\therefore \dfrac{a^2+b^2}{x^2+y^2} = \dfrac{25}{4}$

Result 1:-

If $xy + yz + zx = 0$, then

$\dfrac{1}{x^2-yz} + \dfrac{1}{y^2-zx} + \dfrac{1}{z^2-xy} = 0$ $(x, y, z) \neq 0$

$\dfrac{x^2}{x^2-yz} + \dfrac{y^2}{y^2-zx} + \dfrac{z^2}{z^2-xy} = 1$

Result 2:-

- If $xy = 1$ or $x = \dfrac{1}{y}$ then $\dfrac{1}{1+x^n} + \dfrac{1}{1+y^n} = 1$
- If $x = a + \dfrac{1}{a}$ and $y = a - \dfrac{1}{a}$ then $\sqrt{x^4 + y^4 - 2x^2y^2} = 4$
- If $x \pm \dfrac{1}{y} = a$, $y \pm \dfrac{1}{z} = b$, $z \pm \dfrac{1}{x} = c$ then, $xyz \pm \dfrac{1}{xyz} = abc \mp (a+b+c)$
- If $(a^2 + 1)(b^2 + 1) + N^2 = 2N(a + b)$ then $a + \dfrac{1}{a} = N$
- If $a + \dfrac{1}{b} = b + \dfrac{1}{c} = c + \dfrac{1}{a}$ (where $a \neq b \neq c$) then abc is equal to ±1
- If $bc + ca + ab = abc$, then $\dfrac{b+c}{bc(a-1)} + \dfrac{c+a}{ca(b-1)} + \dfrac{a+b}{ab(c-1)} = 1$
- If $x^2 + y^2 = z + 1$, $y^2 + z^2 = x + 1$, $z^2 + x^2 = y + 1$, then $xyz = 1$ or $-\dfrac{1}{8}$
- If $a + b + c = 2s$, then $\dfrac{(s-a)^2 + (s-b)^2 + (s-c)^2 + s^2}{a^2 + b^2 + c^2} = 1$
- If $\left[\sqrt{(a^2 + b^2 + ab)} + \sqrt{(a^2 + b^2 - ab)}\right] = 1$ then $(1 - a^2)(1 - b^2) = \dfrac{3}{4}$

Ex. If $x = \dfrac{\sqrt{87}-\sqrt{71}}{\sqrt{55}+\sqrt{39}}$ and $y = \dfrac{\sqrt{87}+\sqrt{71}}{\sqrt{55}-\sqrt{39}}$ then

$\dfrac{1}{1+x}+\dfrac{1}{1+y} = ?$

HINTS Here, $xy = \dfrac{\sqrt{87}-\sqrt{71}}{\sqrt{55}+\sqrt{39}} \times \dfrac{\sqrt{87}+\sqrt{71}}{\sqrt{55}-\sqrt{39}} = \dfrac{87-71}{55-39} = \dfrac{16}{16} = 1$

$\therefore \dfrac{1}{1+x}+\dfrac{1}{1+y} = 1$

Ex. If $x(x+y+z) = 9$, $y(x+y+z) = 16$ and $z(x+y+z) = 144$ then what is the value of x?

HINTS $x(x+y+z) + y(x+y+z) + z(x+y+z)$

$= 9 + 16 + 144$

$\Rightarrow (x+y+z)(x+y+z) = 169$

$\Rightarrow (x+y+z) = 13$

$\therefore x(x+y+z) = 9$

$\Rightarrow x = \dfrac{9}{13}$

Quadratic Equation

A Quadratic equation is a polynomial equation of degree 2. Its standard form is:

$ax^2 + bx + c = 0$

where a, b, c, are real numbers, and $a \neq 0$

- The values of variable x which satisfy the quadratic equation is called roots of quadratic equation.

Relation between roots and coefficients

☞ If α and β are the roots of the equation $ax^2 + bx + c = 0$

(i) Sum of roots

$$(\alpha+\beta) = \frac{-b}{a} = \frac{-\text{Coefficient of } x}{\text{Coefficient of } x^2}$$

(ii) Product of roots

$$(\alpha.\beta) = \frac{c}{a} = \frac{\text{Constant term}}{\text{Coefficient of } x^2}$$

☞ If the roots α and β are known then the equation is given by $x^2 - (\alpha + \beta)x + (\alpha.\beta) = 0$

☞ If α and β are rational numbers and one roots of quadratic equation is $a + \sqrt{b}$ then other roots is $(a - \sqrt{b})$ i.e (vice – versa)

Ex. Find quadratic equation whose one roots is $2 + \sqrt{3}$.

HINTS as $\alpha = 2 +$, $\beta = 2 - \sqrt{3}$

Sum of roots $(\alpha + \beta) = (2 + \sqrt{3}) + (2 - \sqrt{3}) = 4$

Product of root $(\alpha . \beta) = (2 + \sqrt{3})(2 - \sqrt{3}) = 1$

Using Formula,

$x^2 - (\alpha + \beta)x + \alpha.\beta = 0 \Rightarrow x^2 - 4x + 1 = 0$

Methods to Solve Quadratic Equation

Factorisation
- Simplest Method
- Not conventionally applicable for all quadratic equations

Completing square
- Move constant to the other side
- Make coefficient of $x^2 = 1$
- Take half of coefficient of x, square it and add to both sides
- Take square root

Quadratic formula
- Formula:- $x = \dfrac{-b \pm \sqrt{b^2-4ac}}{2a}$
- Applicable for all equations

(1) Factorisation Method:

Ex. $x^2 - 15x + 56 = 0$

Step 1: Find factors

Find two factor of 56 that add up to 15. Factor are 8 and 7 because 8 × 7 = 56 and 8 + 7 = 15

Step 2: Change sign

Quadratic equation	Positive/Negative roots
• $ax^2 + bx + c = 0$	Both value of x is negative
• $ax^2 - bx + c = 0$	Both value of x is positive
• $ax^2 + bx - c = 0$	One value of x is –ve and one is +ve
• $ax^2 - bx - c = 0$	One value of x is +ve and one is –ve

Change the sign of the factors. So, factors = 8, 7

Step 3: Divide by coefficient of x^2

Since the coefficient of x^2 is 1. So, dividing by 1 doesnot change the values.

Step 4: Roots

The roots of the quadratic equation are 8 and 7.

(2) Completing square

Ex. $x^2 - 15x + 56 = 0$

HINTS $x^2 - 15x = -56$

$\Rightarrow x^2 - 15x + \left(\dfrac{15}{2}\right)^2 = -56 + \left(\dfrac{15}{2}\right)^2$

$\Rightarrow = \left(x - \dfrac{15}{2}\right)^2 = \dfrac{1}{4} \Rightarrow x - \dfrac{15}{2} = \pm\dfrac{1}{2} \Rightarrow x = 8, 7$

(3) Quadratic formula (Sridharacharya formula)

☞ Two Roots of the equation $ax^2 + bx + c = 0$ are

$= \alpha = \dfrac{-b+\sqrt{b^2-4ac}}{2a} = \dfrac{-b+\sqrt{D}}{2a}$

$= \beta = \dfrac{-b-\sqrt{b^2-4ac}}{2a} = \dfrac{-b-\sqrt{D}}{2a}$

Ex. $x^2 - 15x + 56 = 0$

HINTS $D = b^2 - 4ac = 15^2 - 4 \times 1 \times 56 = 1$

$\alpha = \dfrac{-b+\sqrt{b^2-4ac}}{2a} = \dfrac{15+1}{2} = 8$

$= \beta = \dfrac{-b-\sqrt{b^2-4ac}}{2a} = \dfrac{15-1}{2} = 7$

Nature of Roots

The nature of roots of a quadratic equation depend on the discriminant, which is the expression under the square root in the quadratic formula.

$ax^2 + bx + c = 0$

Discriminant (D) = $b^2 - 4ac$

Nature of Roots

- **If D = 0** → Roots are real and equal
- **If D < 0** → Roots are imaginary
- **If D > 0**
  - D = Perfect square → Roots are real, unequal and rational
  - D = Not Perfect Square → Roots are real, unequal and irrational

Ex. For what value of m, the roots of the equation $4x^2 + 6mx + 9 = 0$ are equal.

HINTS Standard form of quadratic equation

$ax^2 + bx + c = 0$ and $4x^2 + 6mx + 9 = 0$

Comparing standard equation then we get

a = 4, b = 6m and c = 9

If roots are equal $\Rightarrow b^2 - 4ac = 0$

$\Rightarrow 36m^2 = 144 \Rightarrow m = \pm 2$

Ex. If $x^2 - 5x + 6 = 0$, find the nature of roots.

HINTS a = 1, b = –5, c = 6

$D = (-5)^2 - 4 \times 1 \times 6$

= 25 – 24 = 1 (perfect square)

So, roots are real, rational and distinct.

Ex. If $\sqrt{2}\,t^2 - 3t + 3\sqrt{2} = 0$, find the nature of roots.

HINTS Here, $a = \sqrt{2}$, b = – 3, c = 3

$D = b^2 - 4ac = 9 - 24 = -15$

$D < 0$

$\therefore$ Roots are imaginary

Conditions for Common Roots

Consider two quadratic equations: $a_1x^2 + b_1x + c_1 = 0$

$a_2x^2 + b_2x + c_2 = 0$

(i) Condition for both root common :

(ii) Condition for one root common :

$(a_1b_2 - a_2b_1)(b_1c_2 - b_2c_1) = (c_1a_2 - c_2a_1)^2$

Ex. If the equations $x^2 + 2x - 3 = 0$ and $x^2 + 3x - m = 0$ have a common root, then non-zero value of m is.

HINTS $(a_1b_2 - a_2b_1)(b_1c_2 - b_2c_1) = (c_1a_2 - c_2a_1)^2$

$\Rightarrow [1 \times 3 - 1 \times 2]\,[2 \times (-m) - 3 \times (-3)] = [(-3) \times 1 - (-m) \times 1]^2$

$\Rightarrow (3 - 2)(-2m + 9) = (-3 + m)^2$

$\Rightarrow -2m + 9 = 9 + m^2 - 6m$

$\Rightarrow m^2 - 4m = 0 \Rightarrow m(m - 4) = 0$

$\Rightarrow m = 4\ (\because m \neq 0)$

Cubic Equation

- If α , β and γ are the roots of cubic equation

  $ax^3 + bx^2 + cx + d = 0$

  Then,

  Sum of roots $(\alpha + \beta + \gamma) = -\dfrac{b}{a} = -\dfrac{\text{Coefficient of } x^2}{\text{Coefficient of } x^3}$

  Sum of product of two roots $(\alpha\beta + \beta\gamma + \gamma\alpha) = \dfrac{c}{a}$

  $= \dfrac{\text{Coefficient of } x}{\text{Coefficient of } x^3}$

  Product of roots $(\alpha\beta\gamma) = -\dfrac{d}{a} = -\dfrac{\text{Constant term}}{\text{Coefficient of } x^3}$

- If α , β and γ are the roots of cubic equation, then the equation is:

  $x^3 - (\alpha + \beta + \gamma)x^2 + (\alpha\beta + \beta\gamma + \gamma\alpha)x - \alpha\beta\gamma = 0$

Factor Formulae

- $(x + a)(x + b) = x^2 + (a + b)x + ab$
- $(x - a)(x - b) = x^2 - (a + b)x + ab$
- $(x - a)(x + b) = x^2 - (a - b)x - ab$
- $(x + a)(x - b) = x^2 + (a - b)x - ab$

Quadratic Equation Factor & Remainder Theorem

☞ $(x + k)$ is the factor of a polynomial $f(x)$ then $f(-k) = 0$

☞ $(ax + k)$ is the factor of a polynomial $f(x)$ then $f\left(-\dfrac{k}{a}\right) = 0$

☞ $(ax - k)$ is the factor of a polynomial $f(x)$ then $f\left(\dfrac{k}{a}\right) = 0$

☞ $(x - a)(x - b)$ is the factor of a polynomial $f(x)$ then $f(a) = 0$ and $f(b) = 0$

☞ If a polynomial $f(x)$ is divided by $(x + k)$, the remainder is the value of $f(x)$ at $x = -k$ i.e. $f(-k)$

☞ If a polynomial $f(x)$ is divided by $(k - ax)$, the remainder is equal to the value of $f(x)$ at $x = \dfrac{k}{a}$ i.e. $f\left(\dfrac{k}{a}\right)$

Ex. If $2x^2 + kx + 8$ is divisible by $(x + 2)$ leaves remainder 3k find k?

HINTS $x + 2 = 0 \Rightarrow x = -2$

$R = 2(-2)^2 + (-2)k + 8 = 8 - 2k + 8 = 16 - 2k$

$\Rightarrow 3k = 16 - 2k \Rightarrow 5k = 16 \Rightarrow k = \dfrac{16}{5}$

Ex. Find the remainder when we divide $3x^4 - 2x^2 + 4x - 1$ by $2x - 1$.

HINTS For finding the required remainder

put $2x - 1 = 0 \Rightarrow x = \dfrac{1}{2}$

Required remainder $= 3\left(\dfrac{1}{2}\right)^4 - 2\left(\dfrac{1}{2}\right)^2 + 4\left(\dfrac{1}{2}\right) - 1$

$= \dfrac{3}{16} - \dfrac{1}{2} + 2 - 1 = \dfrac{3}{16} + \dfrac{1}{2} = \dfrac{11}{16}$

Ex. When $f(x) = 15x^3 - 14x^2 - 4x + 10$ is divided by $(3x + 2)$, then the remainder is:

HINTS $f(x) = 15x^3 - 14x^2 - 4x + 10$

$3x + 2 = 0 \Rightarrow 3x = -2$

$R = 3x \times 5x^2 - 14x^2 - 4x + 10$

$= -2 \times 5x^2 - 14x^2 - 4x + 10$

$= -24x^2 - 4x + 10$

$= -3x \times 8x - 4x + 10$

$= 16x - 4x + 10 = 12x + 10$

$= 3x \times 4 + 10 = -2 \times 4 + 10 = 2$

Concept of Maximum & Minimum Values in Algebra

Concept-01

	Maximum value	Minimum value
Odd power (x)	$+\infty$	$-\infty$
Even power (x^2)	$+\infty$	0

Ex. (i) $18 + x^2$ → Maximum $\rightarrow +\infty$; Minimum $\rightarrow 0$

Minimum value of $(18 + x^2) = 18 + 0 = 18$

Maximum value of $(18 + x^2) = 18 + \infty = \infty$

(ii) $18 + x^3$ → Maximum $\rightarrow +\infty$; Minimum $\rightarrow -\infty$

Minimum value of $(18 + x^3) = 18 - \infty = -\infty$

Maximum value of $(18 + x^3) = 18 + \infty = \infty$

Concept-02

In quadratic equation $ax^2 + bx + c = 0$

- If $a > 0$, quadratic expression has least value at $x = \frac{-b}{2a}$.

  This least value is given by $\frac{4ac - b^2}{4a} = \frac{-D}{4a}$ but there is no greatest value.
- If $a < 0$, quadratic expression has greatest value at $x = \frac{-b}{2a}$. This greatest value is given by

  $\frac{4ac - b^2}{4a} = \frac{-D}{4a}$ but there is no least value.

Ex. $f(x) = x^2 - 4x + 3$

HINTS Here, $a > 0$,

$a = 1, b = -4, c = 3$

Minimum value $= \frac{4ac - b^2}{4a} = \frac{4 \times 3 \times 1 - (-4)^2}{4 \times 1} = -1$

Ex. $f(x) = -2x^2 + 8x + 1$

HINTS Here, $a < 0$,

$a = -2, b = 8, c = 1$

Maximum value $= \frac{4ac - b^2}{4a} = \frac{4 \times (-2) \times 1 - 8^2}{4 \times (-2)} = 9$

Concept-03

If a, b are +ve number then,

AM ≥ GM

$\Rightarrow \frac{a + b}{2} \geq \sqrt{ab}$, $(a, b > 0) \Rightarrow (a + b) \geq 2\sqrt{ab}$

Ex. If x is a real number, then minimum value of $x^2 + \frac{1}{x^2} = ?$

HINTS Let, $a = x^2$, $b = \frac{1}{x^2}$

AM ≥ GM

$\Rightarrow \frac{x^2 + \frac{1}{x^2}}{2} \geq \sqrt{x^2 \times \frac{1}{x^2}} \Rightarrow x^2 + \frac{1}{x^2} \geq 2$

Equality holds when $x^2 = \frac{1}{x^2} \Rightarrow x^4 = 1$

Since x is real, $x = 1$ or $x = -1$

For these values, $x^2 + \frac{1}{x^2} = (1)^2 + \frac{1}{(1)^2} = 2$

$x^2 + \frac{1}{x^2} = (-1)^2 + \frac{1}{(1)^2} = 2$

> If a is +ve number, then $a + \frac{1}{a} \geq 2 \rightarrow$ Minimum

If a is real number, then $a^2 + \frac{1}{a^2} \geq 2 \rightarrow$ Minimum

Maximum value $\rightarrow \infty$

Concept-04

If $x + y = a$, then the value of $x \times y$ will be maximum at $x = y$,

'OR'

> If $a + b + c$ ......(n numbers) $= k$
>
> then, maximum $(abc) = \left(\frac{k}{n}\right)^n$

Ex. If $x + y = 100$, what is the maximum value of xy?

HINTS $x + y = 100 \Rightarrow x = y = 50$

$\therefore xy = 50 \times 50 = 2500$

Concept-05

If $abc = k$, then minimum value of $a + b + c$ is possible only when $a = b = c$, **'OR'**

> If abc ......(n numbers) $= k$
>
> then minimum $(a + b + c + ...) = n\sqrt[n]{k}$

Ex. If a, b, c are +ve and $abc = 125$, then find minimum value of $a + b + c$

HINTS $abc = 125 \Rightarrow a = b = c = 5$

$\therefore a + b + c = 5 + 5 + 5 = 15$

Ex. If x, y and z are positive and $x + y + z = 1$ then find the min value of $\frac{1}{x} + \frac{1}{y} + \frac{1}{z}$

HINTS $x + y + z = 1 \Rightarrow x = y = z = \frac{1}{3}$

$\therefore$ Minimum value of $= \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{1}{\frac{1}{3}} + \frac{1}{\frac{1}{3}} + \frac{1}{\frac{1}{3}}$

$= 3 + 3 + 3 = 9$

CHAPTER 15

PROFIT & LOSS

Some Common Terms

- Cost Price (CP) : The price at which we buy things.
- Selling Price (SP) : The price at which we sell things.
- If SP > CP then:

$$\boxed{\text{Profit} = \text{Selling Price (SP)} - \text{Cost Price (CP)}}$$

- If CP > SP then:

$$\boxed{\text{Loss} = \text{Cost Price (CP)} - \text{Selling Price (SP)}}$$

- $$\text{Profit \%} = \left(\frac{\text{Profit}}{\text{Cost Price}}\right) \times 100\%$$

- $$\text{Loss \%} = \left(\frac{\text{Loss}}{\text{Cost Price}}\right) \times 100\%$$

- Selling price when profit % is given:

$$SP = CP\left(\frac{100 + P\%}{100}\right)$$

- Selling price when loss % is given:

$$SP = CP\left(\frac{100 - L\%}{100}\right)$$

Ex. A product is bought at ₹20 and sold at ₹50. Find the profit percent.

HINTS Profit = 50 – 20 = ₹30

$\therefore$ Profit % = $\frac{30}{20} \times 100\%$ = 150%

Ex. The selling price of a mobile phone is ₹59,620 and it was sold at 8.4% profit. Find the cost price (in ₹) of the mobile phone.

HINTS SP = ₹59,620, P = 8.4%, CP = ?

CP	**SP**
100	108.4
↓ ×550	↓ ×550
$\boxed{₹55{,}000}$	₹59,620

$\therefore$ Cost price of the mobile phone = ₹55,000

Ex. If an article is sold for ₹355, there is a loss of 29%. At what price (in ₹) should it be sold to gain 31% profit?

HINTS Let, CP of the article = 100 unit

SP_1	–29%	CP	+31%	SP_2
71		100		131
↓ ×5				↓ ×5
₹355				$\boxed{₹655}$

$\therefore$ The article should be sold at ₹655.

Ex. The difference in selling prices of an article when sold at 15% profit and 17% loss is ₹96. If it is sold at 10% profit, then what is the selling price?

HINTS

Let, CP of the article = 100 unit

SP_1	–17%	CP	+15%	SP_2
83		100		115
		diff. 32 unit		

ATQ,

32 unit → ₹96

110 unit → $\frac{96}{32} \times 110 = \boxed{₹330}$

$\therefore$ Selling price = ₹330

Difference Between SP & CP

Ex. The difference between the cost price and the selling price of an article is ₹1,800. If there is a profit of 20%, then find the cost price of the article.

HINTS

20% = $\frac{1}{5}$ +1 ⟶ $\frac{6 \rightarrow SP}{5 \rightarrow CP}$

$\because$ (6 – 5) = 1 unit → ₹1,800

5 unit → ₹9,000

$\therefore$ Cost price of the article = ₹9,000

CP of X = SP of Y

If cost price of 'x' articles is equal to the selling price of 'y' articles, then Selling Price = x, Cost Price = y.

$$\boxed{\text{Profit or Loss \%} = \frac{x - y}{y} \times 100\%}$$

Ex. The cost price of 15 articles is equal to the selling price of 10 articles. Find the profit or loss percentage.

HINTS x = 15, y = 10

$\therefore$ Profit % = $\frac{x - y}{y} \times 100\ \% = \frac{15 - 10}{10} \times 100\% = 50\%$

Alternatively

ATQ,

CP × 15 = SP × 10

$\Rightarrow \frac{CP}{SP} = \frac{2}{3}$ +1

$\therefore$ Profit % = $\frac{1}{2} \times 100\% = 50\%$

Ex. The cost price of 44 items is equal to the selling price of 'x' items. If the profit is 10%, then what is the value of 'x'?

HINTS $10\% = \frac{1}{10} \xrightarrow{+} \frac{11 \to SP}{10 \to CP}$

ATQ,

$44CP = xSP$

$\Rightarrow \frac{SP}{CP} = \frac{44}{x} \Rightarrow \frac{11}{10} = \frac{44}{x} \Rightarrow \boxed{x = 40}$

Ex. If 70% of the cost price of an article is equal to 40% of its selling price, then what is the profit percentage?

HINTS ATQ,

$70CP = 40SP$

$\Rightarrow \frac{CP}{SP} = \frac{4}{7}$

$\therefore$ Profit % = $\frac{7-4}{4} \times 100\% = \frac{300}{4}\% = 75\%$

Ex. The cost price of 36 articles is the same as the selling price of N articles. If the profit is 20%, then find the value of N.

HINTS $20\% = \frac{1}{5} \xrightarrow{+} \frac{6 \to SP}{5 \to CP}$

ATQ,

$36 \times CP = N \times SP$

$\Rightarrow \frac{SP}{CP} = \frac{36}{N} \Rightarrow \frac{6}{5} = \frac{36}{N} \Rightarrow \boxed{N = 30}$

Profit/Loss = CP/SP of Some Articles

Ex. By selling two articles for ₹800, a person gains the cost price of 5 articles. Find the profit percent.

HINTS ATQ,

$2SP - 2CP = 5CP \Rightarrow 2SP = 7CP \Rightarrow \frac{SP}{CP} = \frac{7}{2}$

$\therefore$ Profit % = $\frac{5}{2} \times 100\% = 250\%$

Ex. By selling 42m cloth, Vijay gains the selling price of 7m cloth. Find the profit percentage.

HINTS ATQ,

$42SP - 42CP = 7SP$

$\Rightarrow 35SP = 42CP$

$\Rightarrow \frac{SP}{CP} = \frac{6}{5} \xrightarrow{+1}$

$\therefore$ Profit % = $\frac{1}{5} \times 100\% = 20\%$

Ex. In a medical transaction, 17 times the cost price is equal to 8 times the sum of the cost price and the selling price. What is the gain or loss percentage?

HINTS ATQ,

$17CP = 8(CP + SP)$

$\Rightarrow 17CP - 8CP = 8SP$

$\Rightarrow 9CP = 8SP \Rightarrow \frac{CP}{SP} = \frac{8}{9}$

$\therefore$ Profit % = $\frac{1}{8} \times 100\% = 12.5\%$

Profit/Loss on Selling Price

Ex. If a person calculates his profit % on selling price and according to him, his profit is 20%. Find his actual profit %.

HINTS $20\% = \frac{1 \to P}{5 \to SP} \longrightarrow \frac{4 \to CP}{5 \to SP}$

$\therefore$ Actual profit % = $\frac{1}{4} \times 100\% = 25\%$

Ex. If loss is $\frac{1}{8}$ of the selling price, then find the actual loss percentage.

HINTS $\frac{1 \to Loss}{8 \to SP} \longrightarrow \frac{9 \to CP}{8 \to SP}$

$\therefore$ Actual loss % = $\frac{1}{9} \times 100\% = 11\frac{1}{9}\%$

Ex. Aditi sold her scooter for ₹40,620, gaining $\frac{1}{5}$ of the selling price. Find the gain percentage.

HINTS $\frac{1 \to P}{5 \to SP} \longrightarrow \frac{4 \to CP}{5 \to SP}$

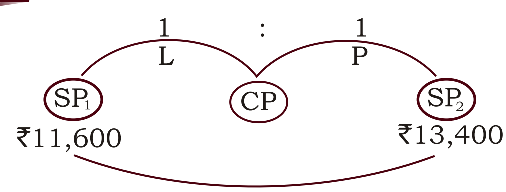
$\therefore$ Gain % = $\frac{1}{4} \times 100\% = 25\%$

BUTTERFLY CONCEPT

$CP = SP_1 + L = SP_2 - P$

Ex. The profit made by selling an article for ₹13,400 is equal to the loss incurred on selling the same article at ₹11,600. What will be the profit (in ₹) if it is sold for ₹14,750?

HINTS ATQ,

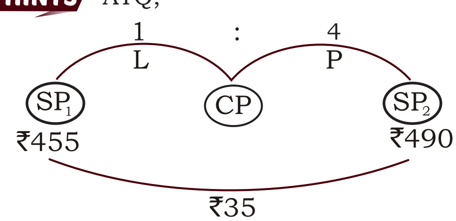


CP = $\frac{11600 + 13400}{2}$ = ₹12,500 ⇒SP = ₹14,750 (Given)

$\therefore$ Required profit = 14,750 – 12,500 = ₹2,250

Ex. A shopkeeper sold an article for ₹455 at a loss. If he sells it for ₹490, then he would gain an amount which is equal to four times the loss. At what price (in ₹) should he sell the article to gain 25%?

HINTS ATQ,



$\because$ (4 + 1) = 5 unit → ₹35

1unit → ₹7

CP = 455 + (1 × 7) = ₹462

OR

CP = 490 – (4 × 7) = ₹462

$\therefore$ Required SP = 462 × $\frac{5}{4}$ = ₹577.5

Ex. The profit obtained by selling an article for ₹86 is twice the loss obtained by selling the same article for ₹53. Find the cost price.

HINTS ATQ,

1 : 2

L P

SP_1 CP SP_2

₹53 ₹86

₹33

$\because$ (2 + 1) = 3 unit → ₹33

1 unit → ₹11

$\therefore$ Cost price = 53 + (1 × 11) = ₹64

Ex. A man sells a mobile phone for ₹680 and loses something. If he had sold it for ₹1,070, his gain would have been double the former loss. Find the cost price (in ₹) of the mobile phone.

HINTS ATQ,

1 : 2

L P

SP_1 CP SP_2

₹680 ₹1,070

₹390

$\because$ (2 + 1) = 3 unit → ₹390

1 unit → ₹130

CP = SP + Loss = 680 + 130 = ₹810

$\therefore$ Cost price of the mobile phone = ₹810

Number of Articles

Ex. A person bought some articles at the rate of 11 for ₹10 and sold them at the rate of 10 for ₹11. Find the profit/loss %.

HINTS

	Price		Article		Price		Article
Buying →	$10_{\times10}$	:	$11_{\times10}$	=	100	:	110
Selling →	$11_{\times11}$	:	$10_{\times11}$		121	:	110

$\therefore$ Profit % = $\frac{21}{100} \times 100\% = 21\%$

Alternatively

	Buying	Selling
Price →	10	11
Article →	11	10
	100	121

+21

$\therefore$ Profit % = 21%

Ex. A trader bought some oranges at 7 for ₹11. He sold all at 2 for ₹3. He incurred a loss of ₹30 in this transaction. Find the number of oranges that he sold.

HINTS

	Price		Orange		Price		Orange
Buying →	$11_{\times2}$	:	$7_{\times2}$	=	22	:	14
					↓ −1		
Selling →	$3_{\times7}$	:	$2_{\times7}$		21	:	14

$\because$ ₹1 loss → 14 oranges

₹30 loss → 14 × 30 = 420 oranges

$\therefore$ Number of oranges that he sold = 420

Ex. A shopkeeper bought 60 pencils at a rate of 4 for ₹5 and another 60 pencils at a rate of 2 for ₹3. He mixed all the pencils and sold them at a rate of 3 for ₹4. Find his gain or loss percentage.

HINTS

	Price	:	Pencil
Case-I →	5	:	4
Case-II →	$3_{\times2}$	:	$2_{\times2}$
Total →	11	:	8

⇒

	Price	:	Pencil
Buying →	$11_{\times3}$	:	$8_{\times3}$
Selling →	$4_{\times8}$	:	$3_{\times8}$
	33	:	24
	↓ −1		
	32	:	24

$\therefore$ Loss % = $\frac{1}{33} \times 100\% = 3\frac{1}{33}\%$

Alternatively

CP	:	SP
$\frac{5}{4}\times 60+\frac{3}{2}\times 60$	:	$\frac{4}{3}\times 120$
$75+90$	:	160
165	:	160

−5

$\therefore$ Loss % = $\frac{5}{165}\times 100\% = 3\frac{1}{33}\%$

In such type of questions, we make the number of items equal in number while buying and selling.

Average Concept (When CP is Same)

Ex. A man bought three articles for ₹6,000 each. He sold the articles respectively at 15% profit, 12% profit and 15% loss. Find the total percentage profit/loss earned by him.

HINTS

Article →	1st	2nd	3rd
Profit →	15%	12%	−15%

$\therefore$ Profit % = $\frac{15+12-15}{3} = 4\%$

Ex. A fruit merchant bought some bananas. One-fifth of them got rotten and was thrown away. He sold two -fifth of the remaining of the bananas with him at 15% profit and the remaining bananas at 10% profit. Find his overall loss or profit percent.

HINTS

Let, total number of bananas be 50 unit.

50 unit

$\times\frac{1}{5}$ → Rotten; $\times\frac{4}{5}$ → 40 unit

40 unit: $\times\frac{2}{5}$ → 16; Remaining → 24

	Rotten		Remaining
	10	16	24
Profit/Loss % =	−100%	+15%	+10%
Total =	−1000%	+240%	+240% = −520%

$\therefore$ Loss % = $\frac{520}{50}\% = 10.4\%$

Ex. A vegetable vendor bought 100kg of potatoes at the rate of ₹19 per kg and spent ₹100 as cartage. He sold 60kg of potatoes with a 50% profit and half of the remaining stock with a 40% profit. He sold half of the still remaining potatoes with a 25% profit. What profit percentage should he aim for when selling the ultimate remaining potatoes to achieve an overall profit of 42%?

HINTS

$$\begin{array}{lccccc} & 60 & : 20 & : 10 & : 10 & = 100 \\ & \downarrow 50\% & \downarrow 40\% & \downarrow 25\% & \downarrow x\% & \downarrow 42\% \\ \textbf{Profit} = & 30 & 8 & 2.5 & 10x\% & = 42 \end{array}$$

$\Rightarrow 10x\% = 42 - 40.5 \Rightarrow 10x\% = 1.5 \Rightarrow \frac{10x}{100} = 1.5 \Rightarrow x = 15$

$\therefore$ Required profit % = 15%

Ex. A shopkeeper purchased 1600 mangoes at the rate of ₹120 per dozen. Out of these, he sold 900 mangoes at the rate of ₹15 per mango and the remaining mangoes at the rate of ₹14 per mango. Find his gain percentage.

HINTS

Total CP = $1600 \times \frac{120}{12}$ = ₹16,000

Total SP = (900 × 15) + (14 × 700) = ₹23,300

Gain = 23,300 – 16,000 = ₹7,300

$\therefore$ Gain % = $\frac{7300}{16000} \times 100\%$ = 45.625%

Alternatively

CP of 1 mango = $\frac{120}{12}$ = ₹10

Net profit = 900 × (15 – 10) + 700 × (14 – 10)

= 4500 + 2800 = ₹7,300

$\therefore$ Gain % = $\frac{7300}{16000} \times 100\%$ = 45.625%

When SP is Same

- If two articles were sold at same price each. First was sold at x% profit and second was sold at y% loss, then overall profit or loss % is given by:

$$\text{Profit/Loss \%} = \frac{100(x+y)+2xy}{200+x+y}\%$$

- If two articles were sold at same price each. First was sold at x% profit and second was sold at x% loss, then overall loss % is given by:

$$\text{Loss \%} = \frac{x^2}{100}\%$$

Ex. Two horses were sold at ₹1,599 each. First was sold at 25% profit and second at 20% loss. Find the overall profit or loss %.

HINTS Profit/Loss % = $\frac{100(x+y)+2xy}{200+x+y}\%$

$= \frac{100(25-20)+2\times 25\times(-20)}{200+25-20}\% = \frac{500-1000}{205}\%$

$= \frac{-500}{205}\% = \frac{-100}{41}\% = -2.43\%$

$\therefore$ Overall loss % = 2.43%

Alternatively

$25\% = \frac{+1}{4}$, $20\% = \frac{-1}{5}$

	CP	SP		CP		SP
I →	$4_{\times 4}$	: $5_{\times 4}$	=	16	:	20
II →	$5_{\times 5}$	: $4_{\times 5}$		25	:	20
				41		40

−1

$\therefore$ Loss % = $\frac{1}{41} \times 100\%$ = 2.43%

To make the SP same in both cases, take the LCM of 5 and 4.

Ex. Two horses were sold for ₹1,920 each. First was sold at 20% loss and second at 20% profit. Find overall profit or loss %.

HINTS Loss % = 20%, Profit % = 20%

$\therefore$ Loss % = $\frac{x^2}{100}\% = \frac{20^2}{100}\% = 4\%$

Ex. A man buys two watches, 'A' and 'B' at a total cost of ₹800. He sells both watches at the same price and earn a profit of 18% on watch 'A' and incurs a loss of 22% on watch 'B'. What are the cost prices of the two watches (upto two places after decimal)?

HINTS

Overall profit/loss % = $\frac{100(x+y)+2xy}{200+x+y}\%$

$= \frac{100(18-22)+2\times 18\times(-22)}{200+18-22}\%$

$= \frac{-400-792}{196}\% = -\frac{298}{49}\% = -6.08\%$

Now,

A		B
18%		−22%
	−6.08%	
15.92%		24.08%

ATQ,

40% → ₹800

1% → ₹20

$\therefore$ CP_A = 15.92 × 20 = ₹318.4

CP_B = 24.08 × 20 = ₹481.6

Alternatively

$18\% = \frac{+9}{50}$, $22\% = \frac{-11}{50}$

	CP	SP	CP		SP
A →	$50_{\times 39}$	$59_{\times 39}$	1950	:	2301
B →	$50_{\times 59}$	$39_{\times 59}$	2950	:	2301

Total CP = (1950 + 2950) = 4900 unit

$\because$ 4900 unit → ₹800

1950 unit → ₹318.37

2950 unit → ₹481.63

$\therefore$ CP_A = ₹318.37, CP_B = ₹481.63

CP of 1st Article = SP of 2nd Article

Ex. A shopkeeper sold two items. The selling price of the first item is equal to the cost price of the second item. He sold the first item at a profit of 20% and the second item at a loss of 10%. What is his overall profit/loss percent?

HINTS Let, two items be 'A' and 'B'.

Let, SP_A = 100 unit

$\therefore$ CP_B = 100 unit

$$CP_A = 100 \times \frac{100}{120} = \frac{1000}{12} \text{ unit}$$

$$SP_B = 100 \times \frac{90}{100} = 90 \text{ unit}$$

$$\therefore \text{Profit } \% = \frac{190 - \left(\frac{1000}{12} + 100\right)}{\left(\frac{1000}{12} + 100\right)} \times 100\%$$

$$= \frac{\frac{80}{12}}{\frac{2200}{12}} \times 100\% = 3\frac{7}{11}\%$$

Alternatively

$$20\% = \frac{+1}{5}, \ 10\% = \frac{-1}{10}$$

Let, two items be 'A' and 'B'.

Given, $SP_A = CP_B$

	CP	SP		CP	SP
A →	(5	$6)_{\times 5}$	=	25	30
B →	(10	$9)_{\times 3}$		30	27
				55	57

$55 \xrightarrow{+2} 57$

$$\therefore \text{Profit } \% = \frac{2}{55} \times 100\% = 3\frac{7}{11}\%$$

Dishonest Shopkeeper

Dishonesty

During **Buying** | During **Selling**

A dishonest shopkeeper sells his goods at CP but uses false weight, then his profit:

$$\boxed{\text{Profit }\% = \frac{\text{True weight} - \text{False weight}}{\text{False weight}} \times 100\%}$$

Ex. A dishonest shopkeeper promises to sell his goods at CP but he uses 30% less weight. Find his profit %.

HINTS Let, true weight = 100 unit, false weight = 70 unit

$70 : 100$

$70 \xrightarrow{+30} 100$

$$\therefore \text{Profit } \% = \frac{30}{70} \times 100\% = \frac{300}{7}\% = 42\frac{6}{7}\%$$

Alternatively

$$30\% = \frac{3}{10} \longrightarrow \frac{7 \rightarrow CP}{10 \rightarrow SP} \qquad 7 : 10$$

$7 \xrightarrow{+3} 10$

$$\therefore \text{Profit } \% = \frac{3}{7} \times 100\% = \frac{300}{7}\% = 42\frac{6}{7}\%$$

Ex. A dishonest shopkeeper promises to sell his goods at 44% loss but he uses 910g weight instead of 1 kg. Find his actual loss percent (approx.).

HINTS

CP	:	SP
100	:	56
910	:	1000
91	:	56

$91 \xrightarrow{-35} 56$

$$\therefore \text{Loss } \% = \frac{35}{91} \times 100\% = 38.4\%$$

Ex. A shopkeeper marks up his goods at 35% above the CP and gives 23% discount to the customer. At the time of buying, he uses 1120g weight instead of 1kg and at the time of selling the goods, he gives 880g weight instead of 1kg. Find his profit %.

HINTS

CP	:	SP
100	:	135
100	:	77
1000	:	1120
880	:	1000
1000		1323

$1000 \xrightarrow{+323} 1323$

$$\therefore \text{Profit } \% = \frac{323}{1000} \times 100\% = 32.3\%$$

CHAR MINAR CONCEPT

In these type of questions, two situations are given. In the first case, an article is sold at some profit/loss % and in the second case, it is given that if the article was bought at a certain percent less or more than the actual CP and sold at a certain amount more or less than the actual SP, then buyer would have made a certain profit/loss %. CP of the article is to be found.

Ex. A man bought an article and sold it at a gain of 10%. If he had bought the article at 20% less and sold it for ₹1,000 more, he would have made a profit of 40%. Find the cost price of the article (in ₹).

HINTS

Let, CP of the article be 100 unit.

$$\begin{array}{ccc} CP_1 & & SP_1 \\ 100 & \xrightarrow{+10\%} & 110 \\ \downarrow{\scriptstyle -20\%} & & \downarrow{\scriptstyle +2 \text{ unit}} \\ CP_2 & & SP_2 \\ 80 & \xrightarrow{+40\%} & 112 \end{array}$$

$\because$ 2 unit → ₹1,000

100 unit → ₹50,000

$\therefore$ Cost price of the article = ₹50,000

Ex. A person sold a house at 15% profit. If he had purchased it for 25% less and sold it for ₹60 less, then there would have been a profit of 32%. Find the cost price of the house.

HINTS

Let, CP of the house be 100 unit.

$$\begin{tikzcd} \underset{100}{CP_1} \arrow[r, "+15\%"] \arrow[d, "-25\%"'] & \underset{115}{SP_1} \arrow[d, "-16 \text{ unit}"] \\ \underset{75}{CP_2} \arrow[r, "+32\%"] & \underset{99}{SP_2} \end{tikzcd}$$

$\because$ 16 unit $\rightarrow$ ₹60

100 unit $\rightarrow$ ₹375

$\therefore$ Cost price of the house = ₹375

Based on Alligation

Ex. A person buys 5 tables and 9 chairs for ₹15,400. He sells tables at 10% profit and chairs at 20% profit. If his total profit on selling all the tables and chairs is ₹2,080, then find the cost price of 3 chairs.

HINTS

Overall profit % = $\frac{2080}{15400} \times 100\,\% = \frac{1040}{77}\,\%$

Table		**Chair**
$10\%_{\times 77} = 770$		$20\%_{\times 77} = 1540$
	$\frac{1040}{77}\% \times 77$	
500	:	270
Price/quantity $\rightarrow$ 50	:	27

$\because$ 77 unit $\rightarrow$ ₹15,400

1 unit $\rightarrow$ ₹200

27 unit $\rightarrow$ ₹5,400

Cost price of 9 chairs = ₹5,400

$\therefore$ Cost price of 3 chairs = ₹1,800

Alternatively

	5 Tables	**9 Chairs**
Profit % $\rightarrow$	10%	10% + 10%

Profit = 15,400 × 10%
= ₹1,540

Remaining profit = 2,080 – 1,540 = ₹540

$\because$ 10% $\rightarrow$ ₹540

100% $\rightarrow$ ₹5,400

Cost price of 9 chairs = ₹5,400

$\therefore$ Cost price of 3 chairs = ₹1,800

$$\frac{\mathbf{P_1}}{\mathbf{Q_1 \times (100 \pm P_1 / L_1)\%}} = \frac{\mathbf{P_2}}{\mathbf{Q_2 \times (100 \pm P_2 / L_2)\%}}$$

Ex. By selling 3 dozen oranges for ₹405, a trader loses 25%. How many oranges should he sell for ₹288 if he needs to earn a profit of 20% in the transaction?

HINTS Let the required number of oranges be 'x'.

$\because \frac{P_1}{Q_1 \times (100 \pm P_1 / L_1)\%} = \frac{P_2}{Q_2 \times (100 \pm P_2 / L_2)\%}$

$\Rightarrow \frac{405}{12 \times 3 \times (100 - 25)\%} = \frac{288}{x \times (100 + 20)\%}$

$\Rightarrow \frac{405}{12 \times 3 \times 75\%} = \frac{288}{x \times 120\%}$

$\Rightarrow x = 16$

$\therefore$ Required number of oranges = 16

Ex. By selling 18 table fans for ₹11,664, a man incurs a loss of 10%. How many table fans should he sell for ₹17,424 to earn 10% profit?

HINTS Let the required number of table fans be 'x'.

$\because \frac{P_1}{Q_1 \times (100 \pm P_1 / L_1)\%} = \frac{P_2}{Q_2 \times (100 \pm P_2 / L_2)\%}$

$\Rightarrow \frac{11664}{18 \times (100 - 10)\%} = \frac{17424}{x \times (100 + 10)\%}$

$\Rightarrow \frac{11664}{18 \times 90\%} = \frac{17424}{x \times 110\%}$

$\Rightarrow x = 22$

$\therefore$ Required number of table fans = 22

Ex. By selling 90 ball pens for ₹160, a person incurs a loss of 20%. Find the number of ball pens, which should be sold for ₹96 to gain a profit of 20%.

HINTS Let the required number of ball pens be 'x'.

$\because \frac{P_1}{Q_1 \times (100 \pm P_1 / L_1)\%} = \frac{P_2}{Q_2 \times (100 \pm P_2 / L_2)\%}$

$\Rightarrow \frac{160}{90 \times 80\%} = \frac{96}{x \times 120\%}$

$\Rightarrow x = 36$

$\therefore$ Required number of ball pens = 36

Ex. On selling 70 articles for ₹160, a person incurred a loss of 20%. How many articles should he sell for ₹96 to earn a profit of 20%?

HINTS Let the required number of articles be 'x'.

$\because \frac{P_1}{Q_1 \times (100 \pm P_1 / L_1)\%} = \frac{P_2}{Q_2 \times (100 \pm P_2 / L_2)\%}$

$\Rightarrow \frac{160}{70 \times 80\%} = \frac{96}{x \times 120\%}$

$\Rightarrow x = 28$

$\therefore$ Required number of articles = 28

If P% = CP, then CP = $10\left(\sqrt{25+SP}-5\right)$

Ex. A wrist watch is sold for ₹1,200 at a profit percent equal to its cost price. Find the cost price of the wrist watch.

HINTS $CP = 10\left(\sqrt{25+1200}-5\right) = 10(35-5) = ₹300$

$\therefore$ Cost price of the wrist watch = ₹300

Alternatively

When profit % is equal to CP then we take two numbers whose product is equal to SP and difference is 10. The CP is equal to the product of the smallest number and 10.

$SP = 1200 = 40 \times 30$, $\underbrace{40 - 30}_{\text{diff.}} = 10$

$CP = \underset{\substack{\uparrow \\ \text{Smallest no.}}}{30} \times 10 = ₹300$

$\therefore$ Cost price of the wrist watch = ₹300

Miscellaneous

Ex. If a shopkeeper sells sugar at ₹44.8 per kg, he is able to make a 12% profit. Due to water seepage, $\frac{1}{5}$ of the sugar is damaged. What should now be the selling price per kg of the rest of the sugar to make a 5% profit?

HINTS Let the initial quantity of sugar be 5 kg.

$\because \frac{1}{5}$ of the sugar is damaged.

$\therefore$ Remaining quantity of sugar = $5 \times \frac{4}{5} = 4$ kg

CP of 5 kg sugar = $\left(44.8 \times \frac{100}{112}\right) \times 5$

SP of remaining 4 kg sugar to make a 5% profit

$= \left(44.8 \times \frac{100}{112} \times 5\right) \times \frac{105}{100}$

$\therefore$ Required SP of 1 kg sugar

$= \left(44.8 \times \frac{100}{112} \times 5 \times \frac{105}{100}\right) \times \frac{1}{4} = ₹52.5$

Alternatively

Selling price per kg = $44.8 \times \frac{100}{112} \times \frac{5}{4} \times \frac{105}{100} = \frac{105}{2}$

$= ₹52.5$

Ex. A reduction of 7.5% in the cost price of a commodity enables a shopkeeper to purchase 15 kg more than what he previously purchased for a sum of ₹7,400. In order to make a profit of 32.5% on the pre-reduction cost price of the commodity, at what price (in ₹) per kg should the commodity be sold?

HINTS

$7.5\% \downarrow = \frac{3}{40}$

	Old		New
Price →	40	:	37
Quantity →	37	:	40

+3 unit

$\because$ 3 unit → 15 kg

1 unit → 5 kg

37 unit → 185 kg

$CP_{Old} = \frac{7400}{185} = ₹40/kg$

$\therefore$ Required SP = $40 \times \frac{132.5}{100} = ₹53/kg$

Ex. A man sells a table at 12% loss and a book at 19% profit. He earns a profit of ₹160. If he sells the table at 12% profit and the book at 16% loss, then he bears a loss of ₹40. Find the difference between the price of a table and a book.

HINTS

Let, CP of the table be '100x' and CP of the book be '100y'.

ATQ,

$-12x + 19y = 160$

$+12x - 16y = -40$

$3y = 120$

$\Rightarrow y = 40$

$\therefore CP_{Book} = 100y = 100 \times 40 = ₹4,000$

$-12x + 760 = 160$

$\Rightarrow 600 = 12x$

$\Rightarrow x = 50$

$\therefore CP_{Table} = 100x = 100 \times 50 = ₹5,000$

$\therefore$ Required difference = 5,000 – 4,000 = ₹1,000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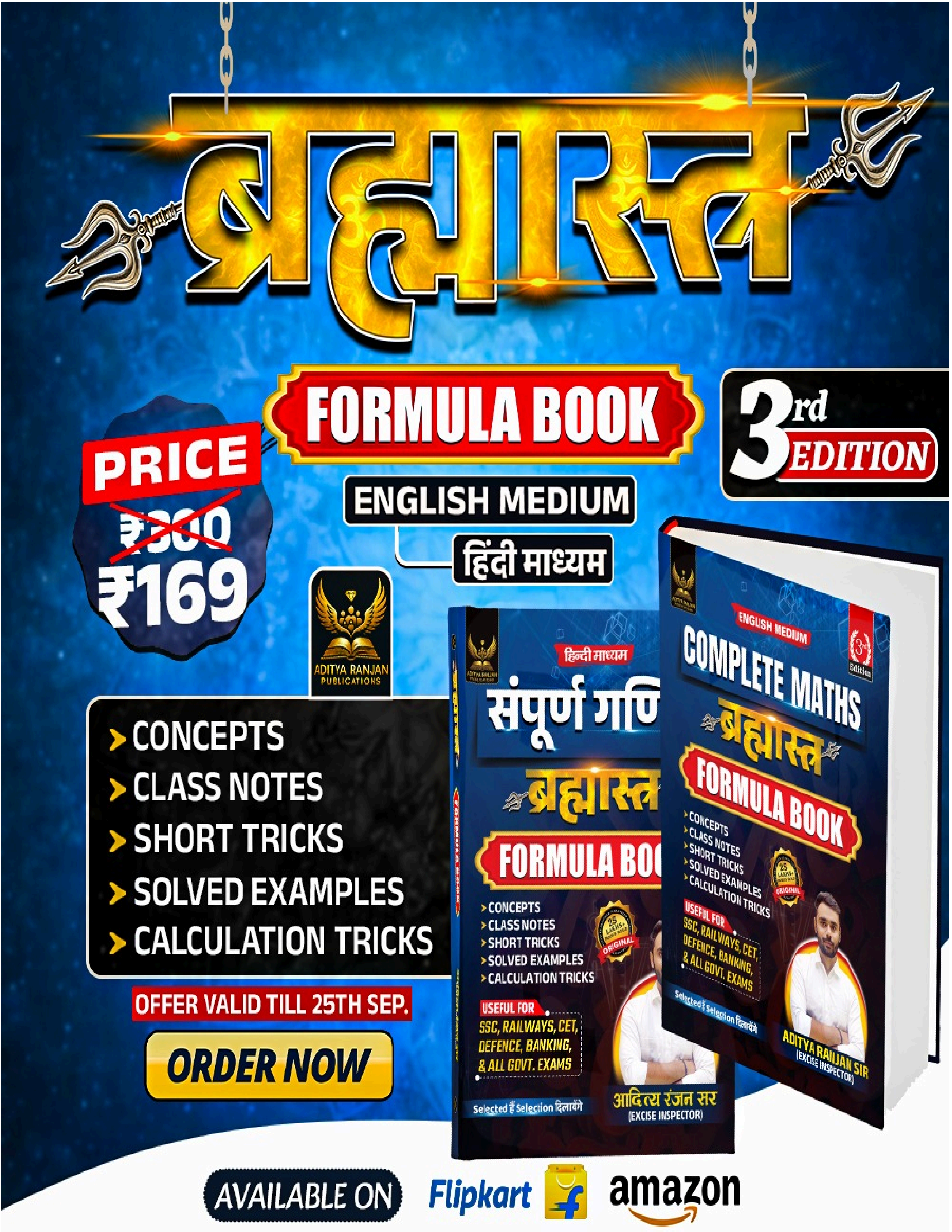
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