1. Rotational Dynamics

Circular Motion:

Motion of an object along the circumference of circle is called circular motion.

e.g. i) Motion of the earth around the sun
ii) Motion of the moon around the earth
iii) Motion of planets around the sun.
iv) Motion of an object tied at the end of a string & whirled in a circle.

Characteristics of Circular Motion:

i) It is an accelerated motion.
ii) It is a periodic motion.

Kinematics of Circular Motion:

i) Angular displacement $\delta\theta = \frac{\text{Arc}}{\text{radius}} = \frac{\delta s}{r}$

It is a vector quantity.
It's S.I. unit is radian.
It is dimensionless quantity.

ii) Angular velocity $= \frac{\text{Angular displacement}}{\text{time}}$

$$\vec{\omega} = \lim_{\delta t \to 0} \frac{\delta \vec{\theta}}{\delta t} \quad \text{or} \quad \vec{\omega} = \frac{d\vec{\theta}}{dt}$$

It is a vector quantity
Its S.I. unit is rad/s
Its dimensions are $[M^0 L^0 T^{-1}]$

iii) Angular acceleration $= \frac{\text{change in angular velocity}}{\text{time}}$

$$\vec{\alpha} = \lim_{\delta t \to 0} \frac{\delta \vec{\omega}}{\delta t} \quad \text{or} \quad \vec{\alpha} = \frac{d\vec{\omega}}{dt}$$

It is a vector quantity.

It's S.I. unit is rad/s^2
It's dimensions are $[M^0 L^0 T^{-2}]$

iv) The quantities angular displacement, angular velocity and angular acceleration are analogous to linear displacement, linear velocity & linear acceleration of linear motion.

v) The direction of angular displacement, angular velocity and angular acceleration can be determined by the right hand rule.

Uniform Circular Motion:

Motion of an object along the circumference of the circle with constant speed is called uniform circular motion.

e.g. 1) Motion of artificial satellites around the earth
2) Motion of electrons around its nucleus
3) Motion of blades of the windmills
4) Tip of second's hand of a wath with circular dial.

Non-uniform Circular Motion:

In circular motion if speed of an object is not constant then it is called as non-uniform circular motion.

e.g. 1) oscillation of pendulum
2) motion of a train
3) a person jogging in the park

Relation between Linear velocity & Angular velocity:

Linear velocity = ~~An~~ Radius × Angular velocity

$$v = r\omega$$

In vector form, $\vec{v} = \vec{\omega} \times \vec{r}$

Period and Frequency:

i) Period (T) = $\dfrac{\text{Circumference}}{\text{Linear velocity}}$

S.I. unit of period is second (s).

$$T = \frac{2\pi r}{v} = \frac{2\pi}{\omega}$$

ii) Frequency (n) = $\dfrac{1}{\text{period}}$

S.I. unit of frequency is Hertz (Hz).

$$n = \frac{1}{T}$$

Centripetal Force: (CPF)

The force acting on a particle performing UCM, which is along the radius of the circle & directed towards the centre of the circle.

i) Magnitude of CPF $= \dfrac{mv^2}{r} = mr\omega^2$

ii) S.I. unit = N

iii) Dimensions = $[M^1 L^1 T^{-2}]$

Centrifugal Force (CFF):

Centrifugal force is a pseudo force in UCM. It acts along the radius of circle & directed away from the centre of the circle.

i) Magnitude of CFF $= \dfrac{mv^2}{r} = mr\omega^2$

ii) S.I. unit = N

iii) Dimensions = $[M^1 L^1 T^{-2}]$

iv) There are two ways of writing force eq^n for a C.M.

Resultant force $= -m\omega^2\vec{r}$ or $m\omega^2\vec{r} + \Sigma$ (real force) $= 0$

Applications of ucm:

a) Vehicle along a horizontal Circular Track:

For a car moving on a Horizontal circular track of radius r, plane of figure is a vertical plane perpendicular to track.

Forces acting on the car are -

i) It's weight (mg) vertically downwards

ii) Normal reaction (N) vertically upwards that balance the weight (mg)

iii) Force of static friction (f_s) bet^n road & tyres

This is static friction (f_s) because it prevents the vehicle from outward slipping or skidding. This is the resultant force which is centripetal.

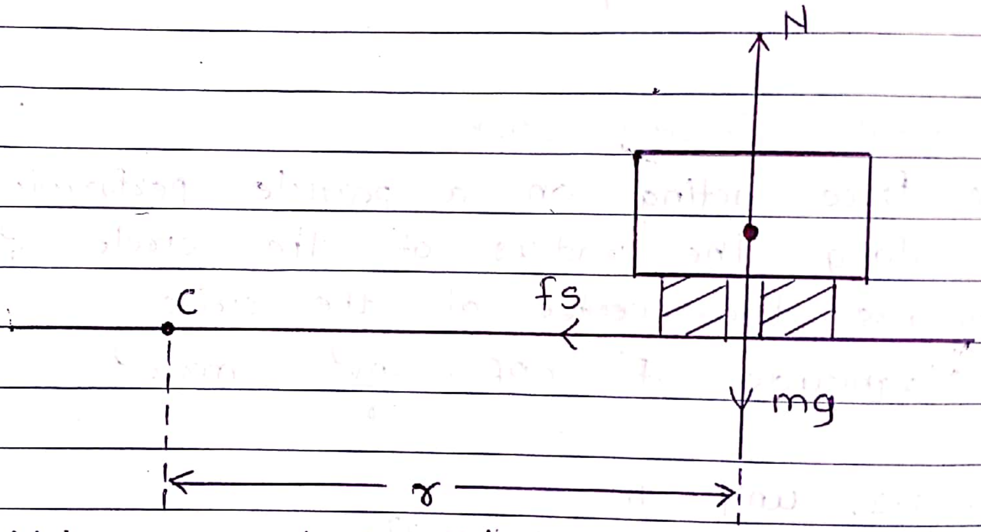

Fig. vehicle on a horizontal road

While working in the frame of reference attached to the vehicle, it balances the centrifugal force.

$$N = mg \quad \text{——— (1)}$$

$$f_s = \frac{mv^2}{r} \quad \text{——— (2)}$$

Dividing (2) by (1),

$$\frac{f_s}{N} = \frac{v^2}{rg}$$

$$v = \sqrt{\mu rg} \qquad \because \text{As } f_s = \mu N$$

$$\therefore v_{max} = \sqrt{\mu rg}$$

b) Well (or wall) of Death :

This is vertical cylindrical wall of radius 'r' inside which a vehicle is driven in horizontal circles. This can be seen while performing stunts.

The forces acting on the vehicle are –

i) Normal reaction (N) acting horizontally towards the centre

ii) Weight (mg) acting vertically downwards

iii) Force of static friction (f_s) acting vertically upwards betn walls & tyres. It has to prevent the downward slipping. Its magnitude is equal to mg in upward direction.

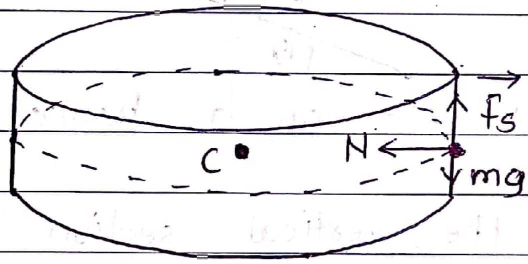

Fig. Well of death

$$\therefore \quad N = \frac{mv^2}{r} \quad \& \quad mg = f_s$$

Force of static friction (f_s) is always less than or equal to $\mu_s N$.

$$\therefore f_s \leq \mu_s N$$

$$\therefore mg \leq \mu_s \left(\frac{mv^2}{r}\right)$$

$$\therefore g \leq \left(\frac{\mu_s v^2}{r}\right)$$

$$v^2 \geq \frac{rg}{\mu_s}$$

$$\therefore V_{min} = \sqrt{\frac{rg}{\mu_s}}$$

c) Vehicle on a Banked Road :

i) Banking of roads : The arrangement of keeping the outer edge of the road surface inclined with the horizontal is called banking of roads.

ii) Angle of banking : The angle made by the inclined road surface with the horizontal is called angle of banking.

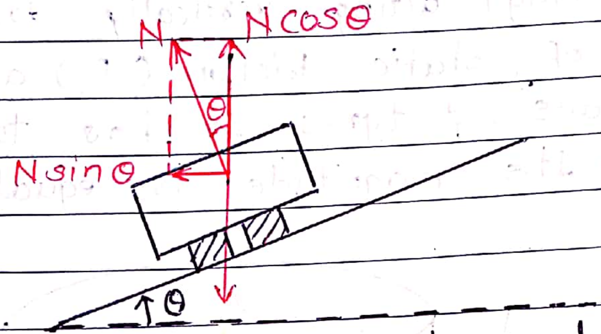

Fig. vehicle on a banked road

Fig. shows the vertical section of vehicle on curved road of radius 'r' banked at an angle θ with the road of radius 'r' horizontal.

Consider, the vehicle to be a point & ignoring friction & non conservative forces like air resistance.

There are two forces acting on a vehicle :-

a) weight (mg) vertically downwards

b) Normal reaction (N) perpendicular to surface of road.

N resolve intwo two components :-

i) N sinθ - horizontal component being resultant force, must be the necessary c.p. force.

ii) N cosθ - vertical component balances weight (mg)

$$\therefore N\sin\theta = \frac{mv^2}{r} \quad \text{--- (1)}$$

$$N\cos\theta = mg \quad \text{--- (2)}$$

Dividing eq^n (1) by (2),

$$\tan\theta = \frac{v^2}{rg} \quad \text{--- (3)}$$

a) Most safe speed -
For a particular road of radius (r) & angle of banking (θ) are fixed.
From eqn ③,
$$v = \sqrt{rg\tan\theta}$$

b) Banking angle -
From eqn ③, $\theta = \tan^{-1}\left(\frac{v^2}{rg}\right)$

c) Speed limits -

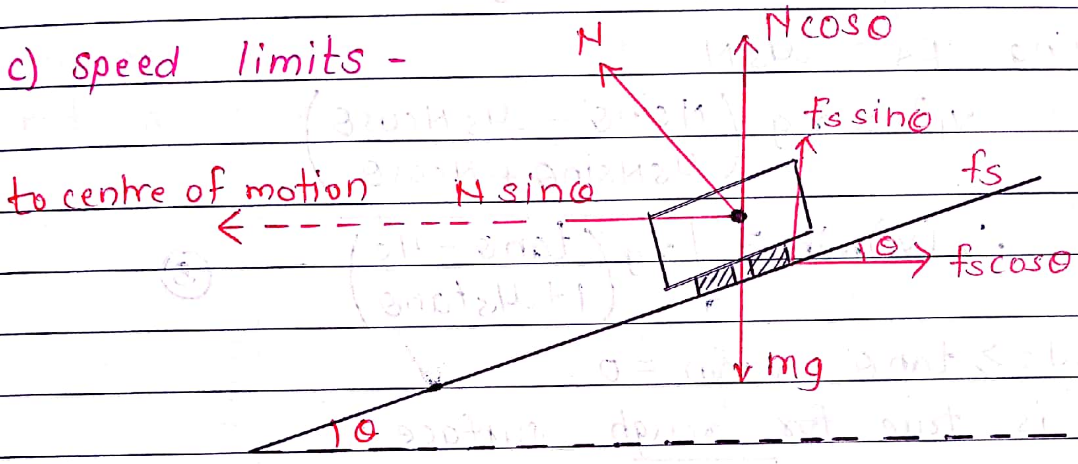

fig. Banked road : lower speed limit

Fig. shows vertical section of a vehicle on a rough curved road of radius r, banked at an angle θ. If the vehicle is running exactly at the speed.
$$V_s = \sqrt{rg\tan\theta}$$
The forces acting on the vehicle are -
i) Weight mg vertically downwards &
ii) Normal reaction N, perpendicular to the surface of road
N resolve into two components $N\cos\theta$ & $N\sin\theta$
In practice, vehicle never travel exactly with this speed. For speeds other than this, the components of force of static friction beth roads & tyres helps us upto a certain limit.
For speed $V_1 < \sqrt{rg\tan\theta}$

$$\frac{mv_1^2}{r} < N\sin\theta$$

From fig,

$$mg = f_s \sin\theta + N\cos\theta \quad \text{--- (3)}$$

$$\frac{mv_1^2}{r} = N\sin\theta - f_s\cos\theta \quad \text{--- (4)}$$

Dividing (4) by (3),

$$\frac{v_1^2}{rg} = \frac{N\sin\theta - f_s\cos\theta}{f_s\sin\theta + N\cos\theta}$$

Putting $F_s = \mu_s N$

$$v_1^2 = rg\left(\frac{N\sin\theta - \mu_s N\cos\theta}{\mu_s N\sin\theta + N\cos\theta}\right)$$

$$\therefore (v_1)_{min} = \sqrt{rg\left(\frac{\tan\theta - \mu_s}{1+\mu_s\tan\theta}\right)} \quad \text{--- (5)}$$

For $\mu_s \geq \tan\theta$, $v_{min} = 0$

This is true for rough surface

For speeds $v_2 > \sqrt{rg\tan\theta}$

In this case, the direction of force f_s is along the inclination of the road as shown in fig. below.

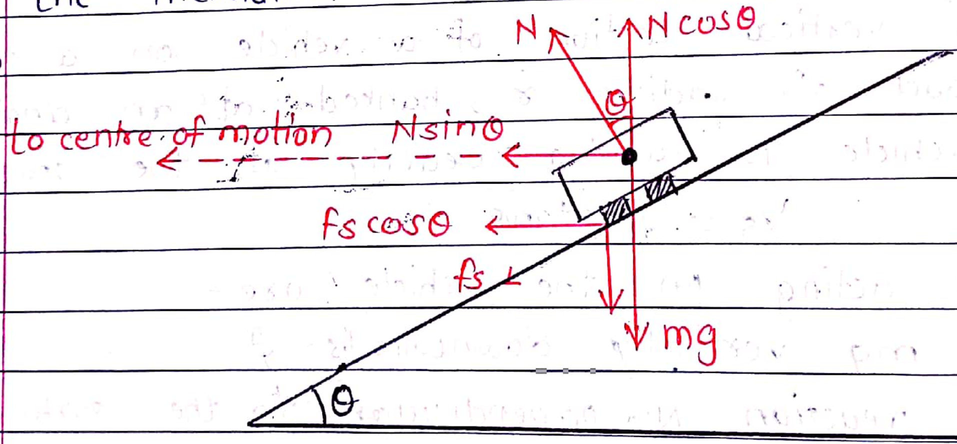

Fig. Banked road : upper speed limit

From fig., $mg = N\cos\theta - f_s\sin\theta \quad \text{--- (6)}$

$$\frac{mv_2^2}{r} = N\sin\theta + f_s\cos\theta \quad \text{--- (7)}$$

Dividing (7) by (6),

$$(v_2)_{max.} = \sqrt{rg\left(\frac{\tan\theta + \mu_s}{1-\mu_s\tan\theta}\right)}$$

for $\mu_s = 0$ both eq^{n}s give

$$v = \sqrt{rg\tan\theta} \quad \text{—— (8)}$$

• Conical Pendulum :

Conical Pendulum is a simple pendulum which gives such a motion that bob describes a horizontal circle & string describes a cone. Its construction is similar to an ordinary pendulum. Instead of swinging back & forth, the bob of a conical pendulum moves at a constant speed in a circle with the string tracing out a cone.

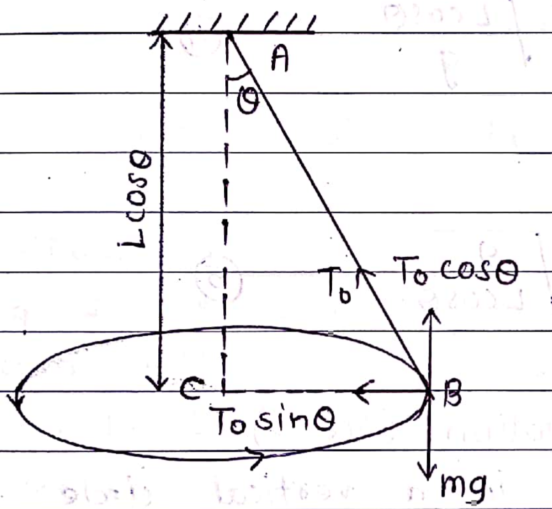

Fig (a) In an inertial frame

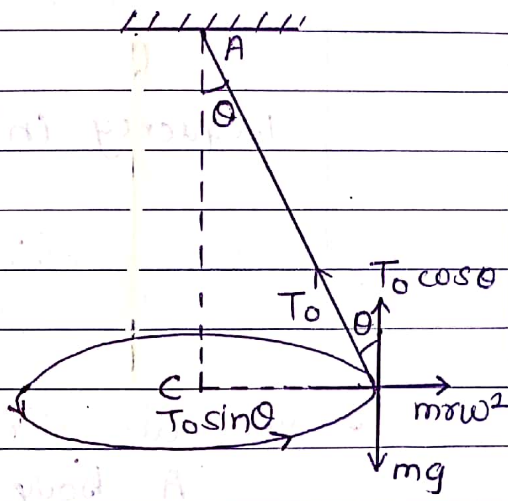

Fig (b). In a non-inertial frame

Above fig. shows the vertical section of a conical pendulum having bob of mass m & string of length 'L'. In given position B, the forces acting on the bob are –

a) It's weight mg vertically downwards

b) The force T_0 – along string

This T_0 resolved into two components

i) $T_0 \cos\theta$ – balances the weight mg

ii) $T_0 \sin\theta$ – becomes the resultant force which is the c.p. force.

$$\therefore \quad T_0 \sin\theta = mr\omega^2 \quad \text{—— (1)}$$

$$T_0 \cos\theta = mg \quad \text{—— (2)}$$

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Dividing eq^n ① by ②,

$$\omega^2 = \frac{g\sin\theta}{r\cos\theta}$$

$$\omega = \sqrt{\frac{g\sin\theta}{r\cos\theta}} \quad \text{—— ③}$$

But, $T = \frac{2\pi}{\omega}$

From eq^n ③, $T = 2\pi\sqrt{\frac{r\cos\theta}{g\sin\theta}}$

From fig. (b),

$$r = L\sin\theta$$

$$T = 2\pi\sqrt{\frac{L\cos\theta}{g}} \quad \text{—— ④}$$

Frequency $(n) = \frac{1}{T}$

$$\therefore n = \frac{1}{2\pi}\sqrt{\frac{g}{L\cos\theta}} \quad \text{—— ⑤}$$

- **Vertical Circular Motion (VCM) :-**

A body revolves in a vertical circle such that its motion at different points is different, then the motion of the body is said to be vertical circular motion.

Ex. i) roller coasters
2) water buckets on a string
3) cars travelling on hilly roads.

Two types of VCM are observed in practice
a) A controlled VCM such as a giant wheel not totally controlled only by gravity.
b) VCM controlled only by gravity.

During the motion, there is inter conversion of K.E. & gravitational P.E.

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- **Point mass undergoing vertical circular motion under gravity :**

Case 1 : Mass tied to a string

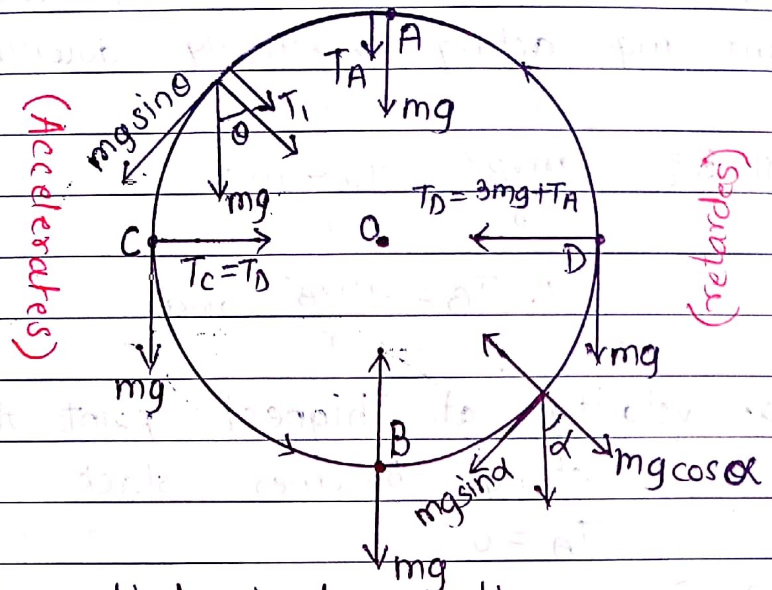


Fig. vertical circular motion.

Fig. shows a bob tied to a massless & inextensible string. It is whirled along vertical circle so that bob perform VCM.

let, m be the mass of an object & r be the radius of vertical circle.

Let, V_A - be the velocity of an object at highest point which is minimum.

V_B - be the velocity of an object at lowest point which is maximum.

V_C - be the velocity at point C.

The forces acting on object at point A are -

i) Tension T_A acting in downward direction

ii) Weight mg acting vertically downward.

At point A : Centripetal Force = Weight + Tension

$$\frac{mV_A^2}{r} = mg + T_A$$

$$\therefore T_A = \frac{m v_A^2}{r} - mg \quad \text{--- (1)}$$

Now, the forces acting at point B are -
i) Tension T_B acting vertically upwards
ii) Weight mg acting vertically downwards.

At point B:
$$\frac{m v_B^2}{r} = T_B - mg$$

$$\therefore T_B = \frac{m v_B^2}{r} + mg \quad \text{--- (2)}$$

a) Linear velocity at highest point A:
At point A string becomes slack
$$T_A = 0$$
$\therefore$ Eqn (1) becomes,
$$\frac{m v_A^2}{r} = mg$$
$$v_A = \sqrt{rg}$$

b) Linear velocity at lowest point B:

As an object moves from point A to B there is decrease in P.E. while K.E. increases according to law of conservation of energy.

$$\therefore (T.E.)_{\text{at } A} = (T.E.)_{\text{at } B}$$

$$\therefore (K.E. + P.E.)_{\text{at } A} = (K.E. + P.E.)_{\text{at } B}$$

$$\therefore \frac{1}{2} m v_A^2 + m g h_A = \frac{1}{2} m v_B^2 + m g h_B$$

Putting $v_A = \sqrt{g r}$, $h_A = 2r$ & $h_B = 0$

$$\therefore \frac{1}{2} m(g r) + 2 m g r = \frac{1}{2} m v_B^2 + 0$$

$$\therefore \frac{1}{2} g r + 2 g r = \frac{1}{2} v_B^2$$

$$\frac{5}{2} g r = \frac{1}{2} v_B^2$$

$$\therefore v_B = \sqrt{5 r g}$$

c) Linear velocity at midpoint C:

According to law of conservation of energy

$$(T.E.)_{\text{at } B} = (T.E.)_{\text{at } C}$$

$$\therefore (K.E. + P.E.)_{\text{at } B} = (K.E. + P.E.)_{\text{at } C}$$

$$\therefore \frac{1}{2} m v_B^2 + m g h_B = \frac{1}{2} m v_C^2 + m g h_C$$

Putting $v_B = \sqrt{g r}$, $h_B = 0$ & $h_C = r$

$$\frac{1}{2}m \times 5gr + 0 = \frac{1}{2}mv_c^2 + mgr$$

$$5gr = v_c^2 + 2gr$$

$$\therefore v_c^2 = 3gr$$

$$\therefore v_c = \sqrt{3rg}$$

Case II : Mass tied to a rod :

i) Consider a bob tied to a massless & rigid rod & whirled along a vertical circle.

ii) The basic difference between the rod & the string is that the string needs some tension at all the points including the uppermost point.

iii) Thus, a certain minimum speed is necessary at the uppermost point in the case of string. In the case of a rod, such a condition is not necessary.

iv) Thus, zero speed is possible at the uppermost point i.e. $V_A = 0$

- **Linear velocity at lowest point :**

According to law of conservation of energy

$$(T.E.)_B = (T.E.)_A$$

$$\therefore \cancel{\frac{1}{2}mv_B^2 + mgh_B = \frac{1}{2}mv_A^2 + mgh_A}$$

$$\frac{1}{2}mv_B^2 + 0 = 0 + mg(2r)$$

$$v_B^2 = 4rg$$

$$v_B = \sqrt{4gr}$$

Similarly, velocity at horizontal position is min. velocity $= \sqrt{2rg}$

Also, $T_B - T_A = 6mg$

Sphere of Death:

1) This is a popular show in a circus. It is similar to the wall of death, but in this act riders can loop vertically as well as horizontally.
2) During this, two-wheeler rider undergo rounds inside a hollow sphere. Starting with small horizontal circles they eventually perform revolutions along vertical circles.
3) This vcm is same as that of the point mass tied to the string, except that the force due to tension T is replaced by the normal reaction force N.
4) The linear speed is more for larger circles but angular speed (frequency) is more for smaller circles (while starting or stopping)

Vehicle at the top of Convex over-bridge:

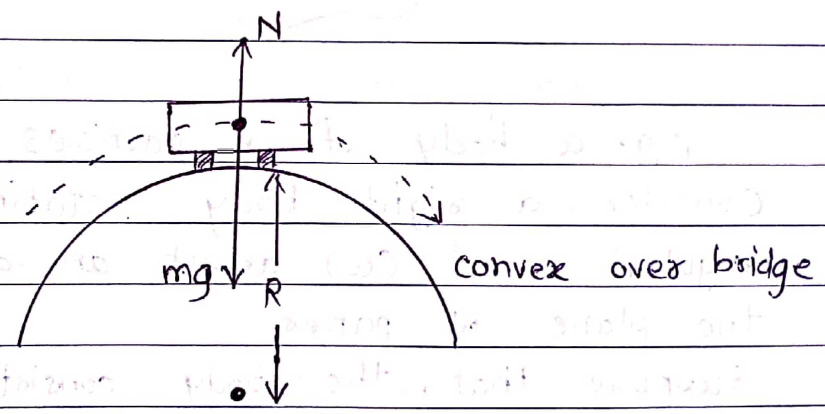


Fig: vehicle on a convex over bridge

Fig. shows a vehicle at the top of a convex over-bridge. During its motion, forces acting on the vehicle are -

a) Weight (mg) in downward direction

b) Normal reaction force (N) in upward direction.

The resultant of these two must provide the necessary centripetal force. $mg - N = \frac{mv^2}{r}$

As speed is increased, N goes on decreasing. Thus for just maintaining contact $N=0$, $mg = \frac{mv^2}{r}$

$\therefore v = \sqrt{rg}$

Moments of Inertia as an analogous quantity for mass :

In expression of linear momentum, force & K.E., mass is a common term. In order to have their rotational analogues we need a replacement for mass. To know this let us derive an expression for rotational K.E.

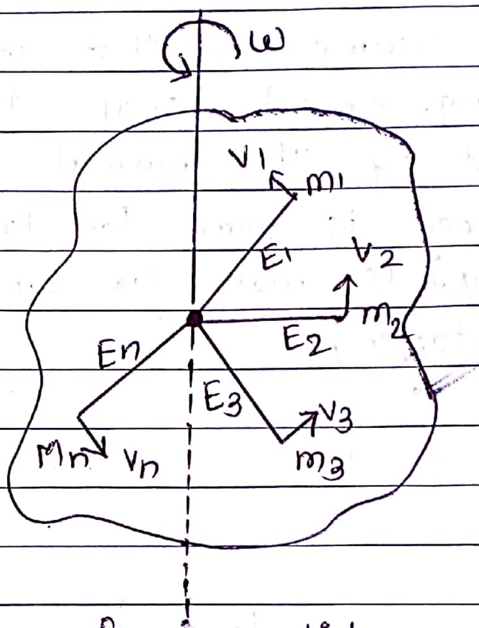


Fig. a body of n particles

Consider, a rigid body rotating with constant angular speed (ω) about an axis perpendicular to the plane of paper.

Suppose that, the body consists of n particles of masses $m_1, m_2, m_3, \dots m_n$ situated at distances $r_1, r_2, r_3, \dots r_n$ respectively from the axis of rotation.

Let, $v_1, v_2, v_3, \dots v_n$ be their linear velocities.

Consider, particle of mass m_1. When the body rotates, this particle revolves in a circular orbit with radius r_1 & velocity v_1.

K.E. of 1st particle E_1 is $= \frac{1}{2} m_1 v_1^2$

$$= \frac{1}{2} m_1 r_1^2 \omega^2 \quad (\text{As } v_1 = r\omega)$$

Similarly, K.E. of 2nd particle is $E_2 = \frac{1}{2} m_2 r_2^2 \omega^2$

K.E. of 3rd particle is $E_3 = \frac{1}{2} m_3 r_3^2 \omega^2$

and so on.

Total K.E. of rotating body is, $E = E_1 + E_2 + E_3 + \cdots + E_n$

$$\therefore E = \frac{1}{2} m_1 r_1^2 \omega^2 + \frac{1}{2} m_2 r_2^2 \omega^2 + \cdots + \frac{1}{2} m_n r_n^2 \omega^2$$

$$E = \frac{1}{2} \omega^2 (m_1 r_1^2 + m_2 r_2^2 + \cdots + m_n r_n^2)$$

$$E = \frac{1}{2} \omega^2 \left[\sum_{i=1}^{n} m_i r_i^2\right]$$

$$E = \frac{1}{2} I \omega^2 \quad \text{--- (1)}$$

where, $I = \sum_{i=1}^{n} m_i r_i^2$

Eqn (1) is analogous to translational K.E. $= \frac{1}{2} m v^2$

Thus, I is defined to be the moment of Inertia,

$$I = \sum_{i=1}^{n} m_i r_i^2$$

Also, I can be expressed in terms of integration

$$I = \int r^2 \, dm$$

- **Moment of Inertia of a uniform ring:**

Consider a uniform ring of mass M & radius R rotating about an axis passing through centre of ring & perpendicular to the plane of the ring. M.I. of uniform ring rotating about an axis passing through point O is $I = \int R^2 \, dm$

$$\therefore I = MR^2$$

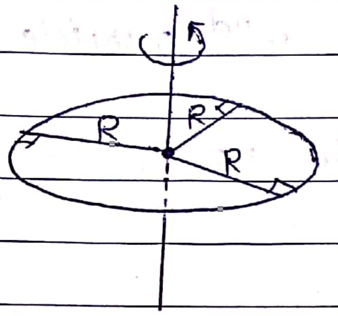


Fig. M.I. of a ring

- **Moment of Inertia of A uniform disc:**

Consider a uniform disc of mass M & radius R rotating about ax axis which is perpendicular to its plane & passing through its centre.

Surface density (σ) $= \frac{\text{Mass}}{\text{area}} = \frac{M}{\pi R^2}$

As it is a uniform circular object, it can be considered to be consisting of a number of concentric rings of radii increasing from zero to R. One of such ring of mass dm is shown by shaded portion.

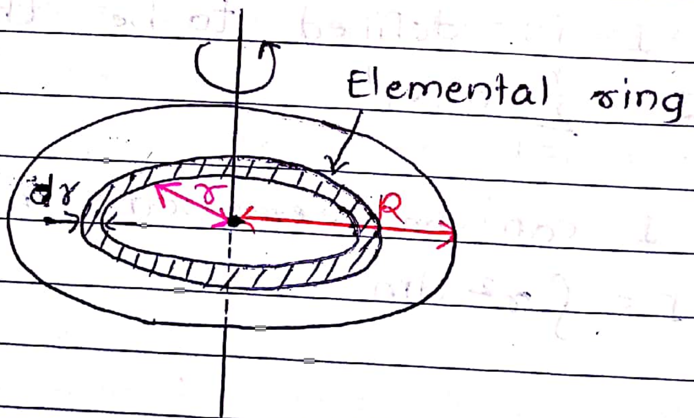


Fig. M.I. of a disc

Let, r & dr be the radius & width of the ring

Area of this ring $A = 2\pi r \cdot dr$

$\therefore \sigma = \frac{\text{Mass}}{\text{area}} = \frac{dm}{2\pi r \cdot dr}$

$\therefore dm = 2\pi \sigma r \cdot dr$

By defn of moment of inertia (with limit 0 to R)

$$I = \int_0^R r^2\,dm$$

$$I = \int_0^R r^2 \cdot 2\pi\sigma r \cdot dr$$

$$I = 2\pi\sigma \int_0^R r^3\,dr$$

$$I = 2\pi\sigma \left[\frac{r^4}{4}\right]_0^R$$

$$I = 2\pi\sigma \left[\frac{R^4}{4}\right]$$

$$\therefore\ I = 2\pi \times \frac{M}{\pi R^2} \times \frac{R^4}{4}$$

$$\therefore\ I = \frac{MR^2}{2}$$

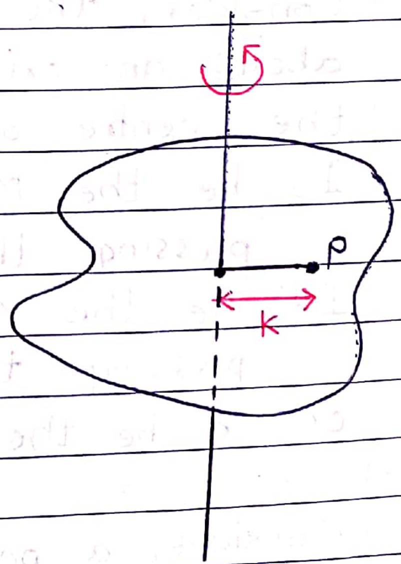
- **Radius of Gyration:**

The radius of gyration of a body about a given axis of rotation is defined as the distance between the axis of rotation & a point at which the whole mass of the body can be supposed to be concentrated so as to possess the same moment of inertia as that of the body.

$$\because\ I = MK^2$$

i.e. $K^2 = \frac{I}{M}$

$$\therefore\ K = \sqrt{\frac{I}{M}}$$



The distance K is called radius of gyration.

S.I. unit of radius of gyration is = m

Dimensions of K = $[M^0L^1T^0]$

Fig. radius of gyration

Theorem of Parallel axes :

Statement:

The moment of inertia (I_0) of a rigid body about any axis is equal to the sum of -

i) it's moment of inertia (I_0) about an axis parallel to the given axis , passing through the centre of mass &

ii) the product of the mass of the object & the square of the distance between the two axes (Mh^2)

$$\therefore \quad I_0 = I_c + Mh^2$$

Proof:

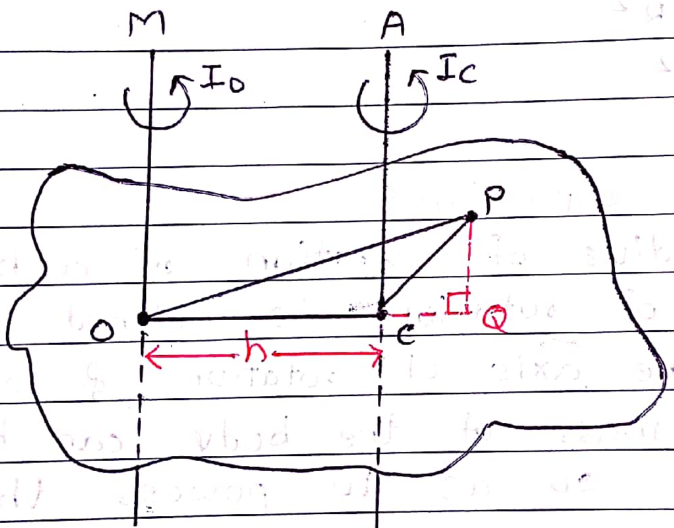


Fig. Theorem of parallel axes

Consider, the rigid body of mass M rotating about an axis passing through 'o'. Let 'c' be the centre of mass of the body.

I_o be the M.I. of the body about an axis passing through O

I_c be the M.I. of the body about parallel axis passing through c

oc = h be the distance betn two parallel axes.

Consider, a particle of mass dm at point 'P'. Join OP & CP. Draw perpendicular PQ to OC.

In ΔOPQ,

$$(OP)^2 = (OQ)^2 + (PQ)^2$$
$$(OP)^2 = (OC + CQ)^2 + (PQ)^2$$
$$(OP)^2 = (OC)^2 + 2\,OC\cdot CQ + (CQ)^2 + (PQ)^2$$
$$= (OC)^2 + 2\,OC\cdot CQ + (CP)^2$$

As $(CQ)^2 + (PQ)^2 = (CP)^2$

Multiplying by dm & integrating we get,

$$\int (OP)^2 dm = \int (OC)^2 dm + \int 2OC\, CQ\, dm + \int (CP)^2 dm \quad \text{--- (1)}$$

By def^n,

$$I_O = (OP)^2 dm \quad \& \text{ putting } OC = h,$$
$$I_C = \int (CP)^2 dm, \quad \int dm = M = \text{total mass}$$

Eq^n (1) becomes,

$$I_O = Mh^2 + 2h \int CQ\, dm + I_C$$
$$I_O = I_C + Mh^2 \qquad \because \text{as } \int CQ\, dm = 0$$

Hence the proof.

Theorem of Perpendicular axes:

Statement:

The moment of inertia of a plane lamina about an axis perpendicular to its plane is equal to the sum of its moment of inertias about two mutually perpendicular axes in the plane of the lamina & intersecting at the point where the perpendicular axes cuts the lamina.

$$I_z = I_x + I_y$$

Proof:

Let, OX & OY be the two mutually perpendicular axes in the plane of the lamina. Let, OZ be the axis passing through their point of intersection 'O' & perpendicular to their plane. Let, I_x, I_y, & I_z be the moment of inertias of the lamina about OX, OY & OZ axes respectively.

Consider a particle of mass dm situated at point P with co-ordinates (x, y). Join OP. Draw PM $\perp$ to X-axis & PN $\perp$ to Y-axis.
Let, $PN = OM = x$ & $PM = ON = y$

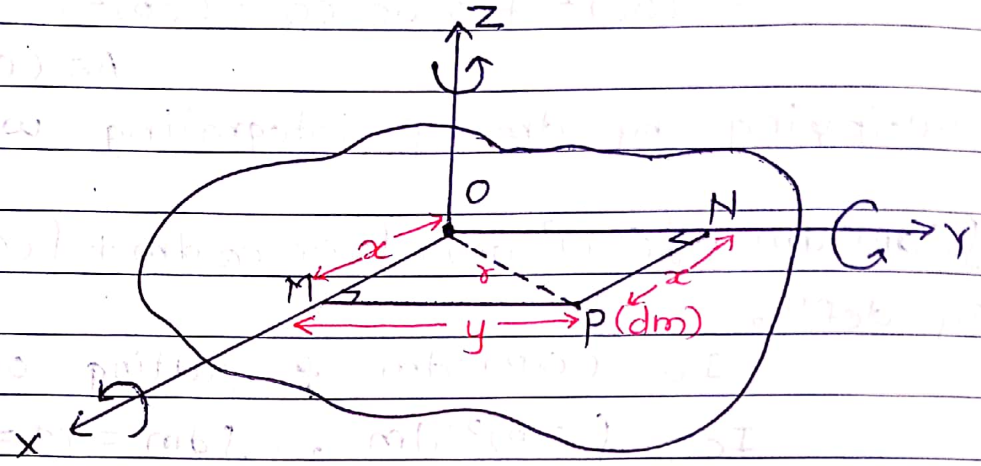


Fig. Theorem of perpendicular axes

From fig, $(OP)^2 = (OM)^2 + (PM)^2$

$$r^2 = x^2 + y^2$$

Multiplying by dm & integrating we get,

$$\int r^2 dm = \int x^2 dm + \int y^2 dm \quad \text{——} (1)$$

By defn of moment of inertia about X, Y & Z axes

$$I_x = \int y^2 dm$$
$$I_y = \int x^2 dm$$
$$I_z = \int r^2 dm$$

Eqn (1) becomes,

$$I_z = I_x + I_y$$

Hence, the proof.

- **Angular momentum or moment of linear momentum**

The quantity angular momentum is analogous to linear momentum.
If $\vec{p}$ is the instantaneous linear momentum of a particle undertaking circular motion, its angular momentum is given by,

$$\vec{L} = \vec{r} \times \vec{p}$$

$L = rp\sin\theta$

where θ is the smaller angle betn $\vec{p}$ & $\vec{r}$.

- Expression for angular momentum in terms of moment of inertia:

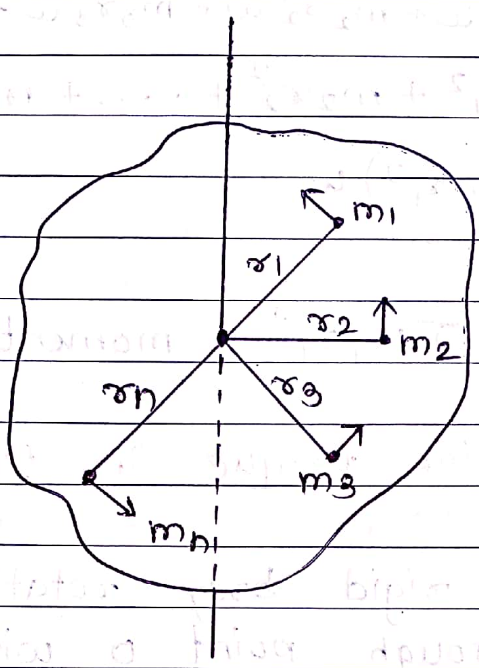


Fig. Expression for angular momentum

Consider a rigid body rotating with a uniform angular velocity (ω) about a fixed axis passing through point O.

Suppose that, the body consists of n particles of masses $m_1, m_2, m_3, \ldots m_n$ situated at distances $r_1, r_2, r_3, \ldots r_n$ respectively from axis of rotation.

As the body rotates, all the particles perform UCM with same angular velocity ω.

Consider, 1st particle of mass m_1. Its linear momentum is,

$p_1 = m_1 v_1 = m_1 r_1 \omega$ $\quad \because v = r\omega$

Angular momentum of first particle is,

$L_1 = p_1 r_1 = m_1 r_1 \omega \times r_1$

i.e. $L_1 = m_1 r_1^2 \omega$

Similarly, for 2nd particle angular momentum is,

$L_2 = m_2 r_2^2 \omega$

For 3rd particle it is,

$$L_3 = m_3 r_3^2 \omega \quad \text{and so on.}$$

$\therefore$ Total angular momentum (L) of rotating body is,

$$L = L_1 + L_2 + L_3 + \cdots + L_n$$

$$L = m_1 r_1^2 \omega + m_2 r_2^2 \omega + m_3 r_3^2 \omega + \cdots + m_n r_n^2 \omega$$

$$L = [m_1 r_1^2 + m_2 r_2^2 + \cdots + m_n r_n^2]\,\omega$$

$$\therefore L = \left(\sum_{i=1}^{n} m_i r_i^2\right) \omega$$

$$\therefore L = I\omega$$

where, $I = \sum m_i r_i^2$ = moment of inertia

- **Expression for torque in terms of moment of inertia:**

Consider, a rigid body rotating about a fixed axis passing through point O with uniform angular acc^n (α).

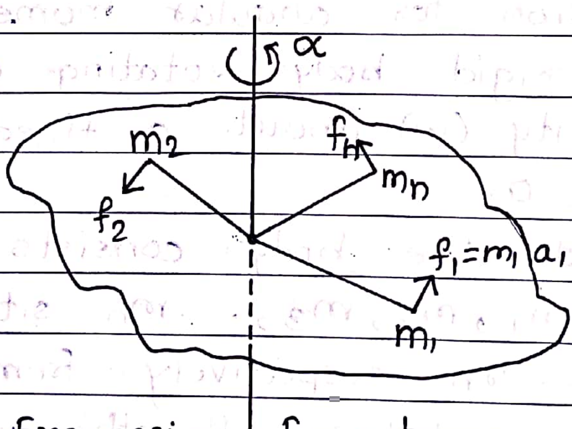


Fig. Expression for torque

Suppose that the body consists of n-particles of masses $m_1, m_2, m_3 \ldots m_n$ situated at distances $r_1, r_2, r_3, \ldots r_n$ respectively from the axis of rotation.

Let $f_1, f_2, f_3, \ldots f_n$ be the forces acting on the particles.

Consider, first particle of mass m_1. The linear acc^n of this particle is

$$a_1 = r_1 \alpha$$

Force acting on 1st particle is,

$$f_1 = m_1 a_1$$

$$\therefore f_1 = m_1 r_1 \alpha$$

$$\text{Torque } (\tau_1) = f_1 r_1$$

$$= m_1 r_1^2 \alpha$$

Similarly torque acting on 2nd particle is,

$$\tau_2 = m_2 r_2^2 \alpha$$

For 3rd particle,

$$\tau_3 = m_3 r_3^2 \alpha \text{ \& so on.}$$

$\therefore$ Total torque acting on body is,

$$\tau = \tau_1 + \tau_2 + \tau_3 + \dots + \tau_n$$

$$\tau = m_1 r_1^2 \alpha + m_2 r_2^2 \alpha + m_3 r_3^2 \alpha + \dots + m_n r_n^2 \alpha$$

$$\tau = (m_1 r_1^2 + m_2 r_2^2 + \dots + m_n r_n^2) \alpha$$

$$\tau = \left(\sum_{i=1}^{n} m_i r_i^2\right) \alpha$$

$$\tau = I \alpha$$

where, $I = \sum_{i=1}^{n} m_i r_i^2$ = moment of inertia

The relation $\tau = I\alpha$ is analogous to $f = ma$ for the translational motion.
Moment of inertia (I) is analogous to mass which is its physical significance.

Conservation of angular momentum:

Statement:

If no external torque acts on a body then angular momentum of the body remains constant.

Proof: Angular momentum of system is given by,

$$\vec{L} = \vec{r} \times \vec{p}$$

Differentiating w.r.t. time we get,

$$\frac{d\vec{L}}{dt} = \frac{d}{dt}(\vec{r} \times \vec{p})$$

$$\frac{d\vec{L}}{dt} = \vec{r} \frac{d\vec{p}}{dt} + \frac{d\vec{r}}{dt}\vec{p}$$

But we know that, $\frac{d\vec{r}}{dt} = v$ & $\frac{d\vec{p}}{dt} = \vec{F}$

$$\therefore \frac{d\vec{L}}{dt} = \vec{r} \times \vec{F} + \vec{v} \times m\vec{v} \qquad \because \vec{p} = m\vec{v}$$

$$= \vec{r} \times \vec{F} + m \times (\vec{v} \times \vec{v})$$

But, $\vec{v} \times \vec{v} = 0$ & $\vec{r} \times \vec{F} = \vec{\tau}$

$$\therefore \frac{d\vec{L}}{dt} = \vec{\tau}$$

Thus if $\vec{\tau} = 0$, $\frac{d\vec{L}}{dt} = 0$

$\therefore$ L = constant

i.e. angular momentum is conserved.

- Examples of conservation of angular momentum:
  1) Ballet dancers
  2) Diving in a swimming pool
  3) Acrobat in a circus
  4) Ice skating

- **Rolling Motion:**

i) The objects like a cylinder, sphere, wheel etc are quite often seen to perform rolling motion

ii) When body rolls it has both translational motion as well as rotational motion.

iii) Total K.E. of rolling body is,

Total K.E. = Translational K.E + Rotational K.E.

$$\therefore \quad K.E. = \frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2$$

Putting $I = MK^2$ & $\omega = \frac{v}{R}$

$$E = \frac{1}{2}Mv^2 + \frac{1}{2}MK^2\left(\frac{v}{R}\right)^2$$

$\therefore$ Total $K.E. = \frac{1}{2}Mv^2\left(1 + \frac{K^2}{R^2}\right)$

- **Linear acceleration & speed while pure rolling down an inclined plane:**

Consider, a rigid object of mass M & radius R rolling down on an inclined plane without slipping. Let, θ be the angle betⁿ inclined plane & horizontal surface.

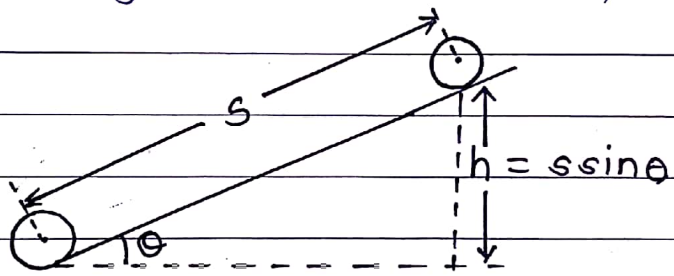


Fig. Rolling along a plane

As the object starts rolling down, its gravitational P.E. converted into K.E.

$$\therefore \quad K.E. = P.E.$$

$$\therefore \quad \frac{1}{2}Mv^2\left(1 + \frac{K^2}{R^2}\right) = Mgh$$

$$\therefore \quad v = \sqrt{\frac{2gh}{\left(1 + \frac{K^2}{R^2}\right)}}$$

Linear distance travelled is,

$$s = \frac{h}{\sin\theta}$$

By 3rd kinematical eqⁿ,

$$v^2 - u^2 = 2as$$ $\because$ If a is linear acc^n along the plane.

Putting $s = \frac{h}{\sin\theta}$, $u = 0$ & $v = \sqrt{\frac{2gh}{\left(1+\frac{K^2}{R^2}\right)}}$

We get,

$$2a\left(\frac{h}{\sin\theta}\right) = \frac{2gh}{\left(1+\frac{K^2}{R^2}\right)} \quad -0$$

$$\therefore \quad a = \frac{g\sin\theta}{\left(1+\frac{K^2}{R^2}\right)}$$

For pure sliding without friction, the acc^n is $g\sin\theta$ & final velocity is $\sqrt{2gh}$. Thus, during pure rolling, the factor $\left(1+\frac{K^2}{R^2}\right)$ is effective for both the expressions.

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