16 Network Synthesis

16.1 INTRODUCTION

In the study of electrical networks, broadly there are two topics: 'Network Analysis' and 'Network Synthesis'. Any network consists of excitation, response and network function. In network analysis, network and excitation are given, whereas the response has to be determined. In network synthesis, excitation and response are given, and the network has to be determined. Thus, in network synthesis we are concerned with the realisation of a network for a given excitation-response characteristic. Also, there is one major difference between analysis and synthesis. In analysis, there is a unique solution to the problem. But in synthesis, the solution is not unique and many networks can be realised.

The first step in synthesis procedure is to determine whether the network function can be realised as a physical passive network. There are two main considerations; causality and stability. By *causality* we mean that a voltage cannot appear at any port before a current is applied or vice-versa. In other words, the response of the network must be zero for $t < 0$. For the network to be stable, the network function cannot have poles in the right half of the s-plane. Similarly, a network function cannot have multiple poles on the $j\omega$ axis.

16.2 HURWITZ POLYNOMIALS

A polynomial $P(s)$ is said to be Hurwitz if the following conditions are satisfied:

(a) $P(s)$ is real when s is real.

(b) The roots of $P(s)$ have real parts which are zero or negative.

Properties of Hurwitz Polynomials

1. All the coefficients in the polynomial

$$P(s) = a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0$$

are positive. A polynomial may not have any missing terms between the highest and the lowest order unless all even or all odd terms are missing.

2. The roots of odd and even parts of the polynomial $P(s)$ lie on the $j\omega$-axis only.
3. If the polynomial $P(s)$ is either even or odd, the roots of polynomial $P(s)$ lie on the $j\omega$-axis only.
4. All the quotients are positive in the continued fraction expansion of the ratio of odd to even parts or even to odd parts of the polynomial $P(s)$.

5. If the polynomial $P(s)$ is expressed as $W(s)\,P_1(s)$, then $P(s)$ is Hurwitz if $W(s)$ and $P_1(s)$ are Hurwitz.
6. If the ratio of the polynomial $P(s)$ and its derivative $P'(s)$ gives a continued fraction expansion with all positive coefficients then the polynomial $P(s)$ is Hurwitz.

This property helps in checking a polynomial for Hurwitz if the polynomial is an even or odd function because in such a case, it is not possible to obtain the continued fraction expansion.

Example 16.1 *State for each case, whether the polynomial is Hurwitz or not. Give reasons in each case.*

(a) $s^4 + 4s^3 + 3s + 2$
(b) $s^6 + 5s^5 + 4s^4 - 3s^3 + 2s^2 + s + 3$

Solution (a) In the given polynomial, the term s^2 is missing and it is neither an even nor an odd polynomial. Hence, it is not Hurwitz.

(b) Polynomial $s^6 + 5s^5 + 4s^4 - 3s^3 + 2s^2 + s + 3$ is not Hurwitz as it has a term $(-3s^3)$ which has a negative coefficient.

Example 16.2 *Test whether the polynomial* $P(s) = s^4 + s^3 + 5s^2 + 3s + 4$ *is Hurwitz.*

Solution Even part of $P(s) = m(s) = s^4 + 5s^2 + 4$

Odd part of $P(s) = n(s) = s^3 + 3s$

$$Q(s) = \frac{m(s)}{n(s)}$$

By continued fraction expansion,

$$
\begin{array}{l}
s^3+3s\overline{)\,s^4+5s^2+4\,}\left(s\right.\\
\qquad\quad \underline{s^4+3s^2}\\
\qquad\qquad 2s^2+4\overline{)\,s^3+3s\,}\left(\tfrac{1}{2}s\right.\\
\qquad\qquad\qquad\quad \underline{s^3+2s}\\
\qquad\qquad\qquad\qquad s\overline{)\,2s^2+4\,}\left(2s\right.\\
\qquad\qquad\qquad\qquad\quad \underline{2s^2}\\
\qquad\qquad\qquad\qquad\qquad 4\overline{)\,s\,}\left(\tfrac{1}{4}s\right.\\
\qquad\qquad\qquad\qquad\qquad\quad \underline{s}\\
\qquad\qquad\qquad\qquad\qquad\quad 0
\end{array}
$$

Since all the quotient terms are positive, $P(s)$ is Hurwitz.

Example 16.3 *Test whether the polynomial* $P(s) = s^3 + 4s^2 + 5s + 2$ *is Hurwitz.*

Solution Even part of $P(s) = m(s) = 4s^2 + 2$

Odd part of $P(s)\; n(s) = s^3 + 5s$

The continued fraction expansion can be obtained by dividing $n(s)$ by $m(s)$ as $n(s)$ is of higher order than $m(s)$.

$$Q(s) = \frac{n(s)}{m(s)}$$

$$4s^2+2\,\Big)\,s^3+5s\,\Big(\,\frac{1}{4}s$$

$$\underline{s^3+\frac{2}{4}s}$$

$$\frac{9}{2}s\,\Big)\,4s^2+2\,\Big(\,\frac{8}{9}s$$

$$\underline{4s^2}$$

$$2\,\Big)\,\frac{9}{2}s\,\Big(\,\frac{9}{4}s$$

$$\underline{\frac{9}{2}s}$$

$$0$$

Since all the quotient terms are positive, $P(s)$ is Hurwitz.

Example 16.4 *Test whether the polynomial* $P(s) = s^4 + s^3 + 3s^2 + 2s + 12$ *is Hurwitz.*

Solution Even part of $P(s) = m(s) = s^4 + 3s^2 + 12$

Odd part of $P(s) = n(s) = s^3 + 2s$

$$Q(s) = \frac{m(s)}{n(s)}$$

By continued fraction expansion,

$$s^3+2s\,\Big)\,s^4+3s^2+12\,\Big(\,s$$

$$\underline{s^4+2s^2}$$

$$s^2+12\,\Big)\,s^3+\;2s\,\Big(\,s$$

$$\underline{s^3+12s}$$

$$-10s\,\Big)\,s^2+12\,\Big(\,-\frac{1}{10}s$$

$$\underline{s^2}$$

$$12\,\Big)\,-10s\,\Big(\,-\frac{10}{12}s$$

$$\underline{-10s}$$

$$0$$

Since two quotient terms are negative, $P(s)$ is not Hurwitz.

Example 16.5 *Prove that polynomial* $P(s) = s^4 + s^3 + 2s^2 + 3s + 2$ *is not Hurwitz.*

Solution Even part of $P(s) = m(s) = s^4 + 2s^2 + 2$

Odd part of $P(s) = n(s) = s^3 + 3s$

$$Q(s) = \frac{m(s)}{n(s)}$$

By continued fraction expansion,

$$
\begin{array}{l}
s^3 + 3s\,)\,s^4 + 2s^2 + 2\ (s \\
\quad s^4 + 3s^2 \\
\hline
\quad -s^2 + 2\,)\,s^3 + 3s\,(-s \\
\qquad s^3 - 2s \\
\hline
\qquad 5s\,)\,-s^2 + 2\,\left(-\frac{1}{5}s\right. \\
\qquad\quad -s^2 \\
\hline
\qquad\quad 2\,)\,5s\,\left(\frac{5}{2}s\right. \\
\qquad\qquad 5s \\
\hline
\qquad\qquad 0
\end{array}
$$

Since two quotient terms are negative, $P(s)$ is not Hurwitz.

Example 16.6 *Prove that polynomial* $P(s) = 2s^4 + 5s^3 + 6s^2 + 3s + 1$ *is Hurwitz.*

Solution Even part of $P(s) = m(s) = 2s^4 + 6s^2 + 1$

Odd part of $P(s) = n(s) = 5s^3 + 3s$

$$Q(s) = \frac{m(s)}{n(s)}$$

By continued fraction expansion,

$$
\begin{array}{l}
5s^3 + 3s\,)\,2s^4 + 6s^2 + 1\ \left(\frac{2}{5}s\right. \\
\quad 2s^4 + \frac{6}{5}s^2 \\
\hline
\quad \frac{24}{5}s^2 + 1\,)\,5s^3 + 3s\,\left(\frac{25}{24}s\right. \\
\qquad 5s^3 + \frac{25}{24}s \\
\hline
\end{array}
$$

$$\frac{47}{24}s\Big)\frac{24}{5}s^2+1\Big(\frac{576}{235}s$$

$$\frac{24}{5}s^2$$

$$1\Big)\frac{47}{24}s\Big(\frac{24}{47}s$$

$$\frac{47}{24}s$$

$$0$$

Since all the quotient terms are positive, the polynomial $P(s)$ is Hurwitz.

Example 16.7 *Test whether the polynomial* $P(s) = s^4 + 7s^3 + 6s^2 + 21s + 8$ *is Hurwitz.*

Solution Even part of $P(s) = m(s) = s^4 + 6s^2 + 8$

Odd part of $P(s) = n(s) = 7s^3 + 21s$

$$Q(s) = \frac{m(s)}{n(s)}$$

By continued fraction expansion,

$$7s^3+21s\Big)s^4+6s^2+8\Big(\frac{1}{7}s$$

$$s^4+3s^2$$

$$3s^2+8\Big)7s^3+21s\Big(\frac{7}{3}s$$

$$7s^3+\frac{56}{3}s$$

$$\frac{7}{3}s\Big)3s^2+8\Big(\frac{9}{7}s$$

$$3s^2$$

$$8\Big)\frac{7}{3}s\Big(\frac{7}{24}s$$

$$\frac{7}{3}s$$

$$0$$

Since all the quotient terms are positive, the polynomial $P(s)$ is Hurwitz.

Example 16.8 *Check whether* $P(s) = s^4 + 5s^3 + 5s^2 + 4s + 10$ *is Hurwitz.*

Solution Even part of $P(s) = m(s) = s^4 + 5s^2 + 10$

Odd part of $P(s) = n(s) = 5s^3 + 4s$

$$Q(s) = \frac{m(s)}{n(s)}$$

By continued fraction expansion,

$$5s^3 + 4s\,\big)\, s^4 + 5s^2 + 10 \,\Big(\, \frac{1}{5}s$$
$$\underline{s^4 + \frac{4}{5}s^2}$$
$$\frac{21}{5}s^2 + 10\,\big)\, 5s^3 + 4s \,\Big(\, \frac{25}{21}s$$
$$\underline{5s^3 + \frac{250}{21}s}$$
$$-\frac{166}{21}s\,\big)\, \frac{21}{5}s^2 + 10 \,\Big(\, -\frac{441}{830}s$$
$$\underline{\frac{21}{5}s^2}$$
$$10\,\big)\, -\frac{166}{21}s \,\Big(\, -\frac{166}{210}s$$
$$\underline{-\frac{166}{21}s}$$
$$0$$

Since the last two quotient terms are negative, the polynomial $P(s)$ is not Hurwitz.

Example 16.9 *Test whether the polynomial* $s^5 + 3s^3 + 2s$ *is Hurwitz.*

Solution Since the given polynomial contains odd functions only, it is not possible to perform a continued fraction expansion.

$$P'(s) = \frac{d}{ds}P(s) = 5s^4 + 9s^2 + 2$$
$$Q(s) = \frac{P(s)}{P'(s)}$$

By continued fraction expansion,

$$5s^4 + 9s^2 + 2\,\big)\, s^5 + 3s^3 + 2s \,\Big(\, \frac{1}{5}s$$
$$\underline{s^5 + \frac{9}{5}s^3 + \frac{2}{5}s}$$
$$\frac{6}{5}s^3 + \frac{8}{5}s\,\big)\, 5s^4 + 9s^2 + 2 \,\Big(\, \frac{25}{6}s$$

$$5s^4 + \frac{20}{3}s^2$$

$$\frac{7}{3}s^2 + 2 \,\Big)\, \frac{6}{5}s^3 + \frac{8}{5}s \,\Big(\, \frac{18}{35}s$$

$$\frac{6}{5}s^3 + \frac{36}{35}s$$

$$\frac{20}{35}s \,\Big)\, \frac{7}{3}s^2 + 2 \,\Big(\, \frac{49}{12}s$$

$$\frac{7}{3}s^2$$

$$2 \,\Big)\, \frac{20}{35}s \,\Big(\, \frac{10}{35}s$$

$$\frac{20}{35}s$$

$$0$$

Since all the quotient terms are positive, the polynomial $P(s)$ is Hurwitz.

Example 16.10 *Test whether the polynomial P(s) is Hurwitz.*

$$P(s) = s^5 + s^3 + s$$

Solution Since the given polynomial contains odd functions only, it is not possible to perform continued fraction expansion.

$$P'(s) = \frac{d}{ds}P(s) = 5s^4 + 3s^2 + 1$$

$$Q(s) = \frac{P(s)}{P'(s)}$$

By continued fraction expansion,

$$5s^4 + 3s^2 + 1 \,\Big)\, s^5 + s^3 + s \,\Big(\, \frac{1}{5}s$$

$$s^5 + \frac{3}{5}s^3 + \frac{1}{5}s$$

$$\frac{2}{5}s^3 + \frac{4}{5}s \,\Big)\, 5s^4 + 3s^2 + 1 \,\Big(\, \frac{25}{2}s$$

$$5s^4 + 10s^2$$

$$-7s^2 + 1 \,\Big)\, \frac{2}{5}s^3 + \frac{4}{5}s \,\Big(\, -\frac{2}{35}s$$

$$\frac{2}{5}s^3 - \frac{2}{35}s$$

$$\frac{26}{35}s\,\Big)\,-7s^2+1\,\Big(-\frac{245}{26}s$$

$$-7s^2$$

$$1\,\Big)\,\frac{26}{35}s\,\Big(\frac{26}{35}s$$

$$\frac{26}{35}s$$

$$0$$

Since the third and fourth quotient terms are negative, $P(s)$ is not Hurwitz.

Example 16.11 *Test the polynomial P(s) of Hurwitz property.*

$$P(s) = s^6 + 3s^5 + 8s^4 + 15s^3 + 17s^2 + 12s + 4$$

Solution Even part of $P(s) = m(s) = s^6 + 8s^4 + 17s^2 + 4$

Odd part of $P(s) = n(s) = 3s^5 + 15s^3 + 12s$

$$Q(s) = \frac{m(s)}{n(s)}$$

By continued fraction expansion,

$$3s^5+15s^3+12s\,\Big)\,s^6+8s^4+17s^2+4\,\Big(\frac{1}{3}s$$

$$s^6+5s^4+4s^2$$

$$3s^4+13s^2+4\,\Big)\,3s^5+15s^3+12s\,\Big(s$$

$$3s^5+13s^3+4s$$

$$2s^3+8s\,\Big)\,3s^4+13s^2+4\,\Big(\frac{3}{2}s$$

$$3s^4+12s^2$$

$$s^2+4\,\Big)\,2s^3+8s\,\Big(2s$$

$$2s^3+8s$$

$$0$$

The division has terminated abruptly (i.e., the number of partial quotients (that is four) is not equal to the order of polynomial (that is six) with common factor ($s^2 + 4$).

$$P(s) = s^6 + 3s^5 + 8s^4 + 15s^3 + 17s^2 + 12s + 4 = (s^2+4)(s^4+3s^3+4s^2+3s+1)$$

If both the factors are Hurwitz, $P(s)$ will be Hurwitz.

Let $$P_1(s) = s^2 + 4$$

Since it contains only even functions, we have to find the continued fraction expansion of $\dfrac{P_1(s)}{P_1'(s)}$.

$$P_1'(s) = 2s$$

$$\frac{P_1(s)}{P_1'(s)} = \frac{s^2+4}{2s} = \frac{s^2}{2s} + \frac{4}{2s} = \frac{s}{2} + \frac{1}{\frac{s}{2}}$$

Since all the quotient terms are positive, $P_1(s)$ is Hurwitz.

Now, let $$P_2(s) = s^4 + 3s^3 + 4s^2 + 3s + 1$$

$$m_2(s) = s^4 + 4s^2 + 1$$

$$n_2(s) = 3s^3 + 3s$$

By continued fraction expansion,

$$3s^3 + 3s \,\big)\, s^4 + 4s^2 + 1 \,\big(\, \tfrac{1}{3}s$$
$$\underline{s^4 + \; s^2}$$
$$3s^2 + 1)\; 3s^3 + 3s\; (s$$
$$\underline{3s^3 + s}$$
$$2s \,\big)\, 3s^2 + 1 \,\big(\, \tfrac{3}{2}s$$
$$\underline{3s^2}$$
$$1)\; 2s\; (2s$$
$$\underline{2s}$$
$$0$$

Since all the quotient terms are positive, $P_2(s)$ is Hurwitz.
Hence, $P(s) = (s^2 + 4)(s^4 + 3s^3 + 4s^2 + 3s + 1)$ is Hurwitz.

Example 16.12 *Test whether the polynomial* $P(s) = s^7 + 2s^6 + 2s^5 + s^4 + 4s^3 + 8s^2 + 8s + 4$ *is Hurwitz.*

Solution Even part of $P(s) = m(s) = 2s^6 + s^4 + 8s^2 + 4$

Odd part of $P(s) = n(s) = s^7 + 2s^5 + 4s^3 + 8s$

$$Q(s) = \frac{n(s)}{m(s)}$$

By continued fraction expansion,

$$
\begin{array}{l}
2s^6+s^4+8s^2+4\overline{)\,s^7+2s^5+4s^3+8s\,}\left(\frac{1}{2}s\right.\\
\qquad\qquad\underline{s^7+\frac{1}{2}s^5+4s^3+2s}\\
\qquad\qquad\frac{3}{2}s^5\qquad+6s\overline{)\,2s^6+s^4+8s^2+4\,}\left(\frac{4}{3}s\right.\\
\qquad\qquad\qquad\qquad\underline{2s^6\qquad+8s^2}\\
\qquad\qquad\qquad\qquad\quad s^4\qquad+4\overline{)\,\frac{3}{2}s^5+6s\,}\left(\frac{3}{2}s\right.\\
\qquad\qquad\qquad\qquad\qquad\qquad\underline{\frac{3}{2}s^5+6s}\\
\qquad\qquad\qquad\qquad\qquad\qquad\quad 0
\end{array}
$$

Since the division has terminated abruptly it indicates a common factor $s^4 + 4$. The polynomial can be written as

$$P(s) = (s^4+4)(s^3+2s^2+2s+1)$$

If both the factor are Hurwitz, $P(s)$ will be Hurwitz.
In the polynomial $(s^4 + 4)$, the terms s^3, s^2 and s are missing. Hence, it is not Hurwitz.
Therefore, $P(s)$ is not Hurwitz.

Example 16.13 *Test whether the polynomial* $2s^6+s^5+13s^4+6s^3+56s^2+25s+25$ *is Hurwitz.*

Solution Even part of $P(s) = m(s) = 2s^6+13s^4+56s^2+25$

Odd part of $P(s) = n(s) = s^5+6s^3+25s$

$$Q(s) = \frac{m(s)}{n(s)}$$

By continued fraction expansion,

$$
\begin{array}{l}
s^5+6s^3+25s)2s^6+13s^4+56s^2+25(2s\\
\qquad\qquad\underline{2s^6+12s^4+50s^2}\\
\qquad\qquad\qquad s^4+6s^2\;\;+25)s^5+6s^3+25s(s\\
\qquad\qquad\qquad\qquad\qquad\underline{s^5+6s^3+25s}\\
\qquad\qquad\qquad\qquad\qquad\qquad 0
\end{array}
$$

The division has terminated abruptly.

$$P(s) = 2s^6+s^5+13s^4+6s^3+56s^2+25s+25 = (s^4+6s^2+25)\,(2s^2+s+1)$$

Let $\qquad P_1(s) = s^4+6s^2+25$

Since $P_1(s)$ contains only even functions, we have to find the continued fraction expansion of $\dfrac{P_1(s)}{P_1'(s)}$.

$$P_1'(s) = 4s^3 + 12s$$

By continued fraction expansion,

$$4s^3 + 12s \,\big)\, s^4 + 6s^2 + 25 \,\big(\, \frac{1}{4}s$$
$$\underline{s^4 + 3s^2}$$
$$3s^2 + 25 \,\big)\, 4s^3 + 12s \,\big(\, \frac{4}{3}s$$
$$\underline{4s^3 + \frac{100}{3}s}$$
$$-\frac{64}{3}s \,\big)\, 3s^2 + 25 \,\big(\, -\frac{9}{64}s$$
$$\underline{3s^2}$$
$$25 \,\big)\, -\frac{64}{3}s \,\big(\, -\frac{64}{75}s$$
$$\underline{-\frac{64}{3}s}$$
$$0$$

Since two of the quotient terms are negative, $P_1(s)$ is not Hurwitz.
We need not test the other factor $(2s^2 + s + 1)$ for being Hurwitz.
Hence, $P(s)$ is not Hurwitz.

There is another method to test a Hurwitz polynomial. In this method, we construct the Routh–Hurwitz array for the required polynomial.

Let $\quad P(s) = a_n s^n + a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + \ldots + a_1 s + a_0$

The Routh–Hurwitz array is given by,

$$\begin{array}{l|llll}
s^n & a_n & a_{n-2} & a_{n-4} & \ldots \\
s^{n-1} & a_{n-1} & a_{n-3} & a_{n-5} & \ldots \\
s^{n-2} & b_n & b_{n-1} & b_{n-2} & \ldots \\
s^{n-3} & c_n & c_{n-1} & & \ldots \\
. & . & & & \\
. & . & & & \\
. & . & & & \\
. & . & & & \\
s^1 & . & & & \\
s^0 & . & & &
\end{array}$$

The coefficients of s^n and s^{n-1} rows are directly written from the given equation.

where
$$b_n = \frac{a_{n-1}a_{n-2} - a_n a_{n-3}}{a_{n-1}}$$

$$b_{n-1} = \frac{a_{n-1}a_{n-4} - a_n a_{n-5}}{a_{n-1}}$$

$$b_{n-2} = \frac{a_{n-1}a_{n-6} - a_n a_{n-7}}{a_{n-1}}$$

$$c_n = \frac{b_n a_{n-3} - a_{n-1}b_{n-1}}{b_n}$$

$$c_{n-1} = \frac{b_n a_{n-5} - a_{n-1}b_{n-2}}{b_n}$$

Hence, for polynomial $P(s)$ to be Hurwitz, there should not be any sign change in the first column of the array.

Example 16.14 *Test whether* $P(s) = s^4 + 7s^3 + 6s^2 + 21s + 8$ *is Hurwitz.*

Solution The Routh array is given by,

$$\begin{array}{c|ccc} s^4 & 1 & 6 & 8 \\ s^3 & 7 & 21 & \\ s^2 & 3 & 8 & \\ s^1 & \frac{7}{3} & 0 & \\ s^0 & 8 & & \end{array}$$

Since all the elements in the first column are positive, the polynomial $P(s)$ is Hurwitz.

Example 16.15 *Determine whether* $P(s) = s^4 + s^3 + 2s^2 + 3s + 2$ *is Hurwitz.*

Solution The Routh array is given by,

$$\begin{array}{c|ccc} s^4 & 1 & 2 & 2 \\ s^3 & 1 & 3 & \\ s^2 & -1 & 2 & \\ s^1 & 5 & 0 & \\ s^0 & 2 & & \end{array}$$

Since there is a sign change in the first column of the array, the polynomial $P(s)$ is not Hurwitz.

Example 16.16 *Test whether* $P(s) = s^5 + 2s^4 + 4s^3 + 6s^2 + 2s + 5$ *is Hurwitz.*

Solution The Routh array is given by,

s^5	1	4	2
s^4	2	6	5
s^3	1	–0.5	
s^2	7	5	
s^1	–1.21		
s^0	5		

Since there is a sign change in the first column of the array, the polynomial is not Hurwitz.

Example 16.17 *Test whether the polynomial* $P(s) = s^5 + s^3 + s$ *is Hurwitz.*

Solution The given polynomial contains odd functions only.

$$P'(s) = 5s^4 + 3s^2 + 1$$

The Routh array is given by,

s^5	1	1	1
s^4	5	3	1
s^3	0.4	0.8	
s^2	–7	1	
s^1	0.86		
s^0	1		

Since there is a sign change in the first column of the array, the polynomial is not Hurwitz.

Example 16.18 *Test whether the polynomial* $P(s) = s^8 + 5s^6 + 2s^4 + 3s^2 + 1$ *is Hurwitz.*

Solution The given polynomial contains even functions only.

$$P'(s) = 8s^7 + 30s^5 + 8s^3 + 6s$$

The Routh array is given by,

s^8	1	5	2	3	1
s^7	8	30	8	6	0
s^6	1.25	1	2.25	1	
s^5	23.6	–6.4	–0.4	0	
s^4	1.33	2.27	1		
s^3	–46.6	–18.14	0		
s^2	1.75	1			
s^1	8.49				
s^0	1				

Since there is a sign change in the first column of the array, the polynomial is not Hurwitz.

Example 16.19 *Test whether* $P(s) = s^5 + 12s^4 + 45s^3 + 60s^2 + 44s + 48$ *is Hurwitz.*

Solution The Routh array is given by,

s^5	1	45	44
s^4	12	60	48
s^3	40	40	
s^2	48	48	
s^1	0	0	
s^0			

Notes: When all the elements in any one row is zero, the following steps are followed:

(i) Write an auxiliary equation with the help of the coefficients of the row just above the row of zeros.

(ii) Differentiate the auxiliary equation and replace its coefficient in the row of zeros.

(iii) Proceed for the Routh test.

Auxiliary equation,

$$A(s) = 48s^2 + 48$$
$$A'(s) = 96s$$

s^5	1	45	44
s^4	12	60	48
s^3	40	40	
s^2	48	48	
s^1	96	0	
s^0	48		

Since there is no sign change in the first column of the array, the polynomial $P(s)$ is Hurwitz.

Example 16.20 *Check whether $P(s) = 2s^6 + s^5 + 13s^4 + 6s^3 + 56s^2 + 25s + 25$ is Hurwitz.*

Solution The Routh array is given by,

s^6	2	13	56	25
s^5	1	6	25	
s^4	1	6	25	
s^3	0	0	0	
s^2				
s^1				
s^0				

$$A(s) = s^4 + 6s^2 + 25$$
$$A'(s) = 4s^3 + 12s$$

Now, the Routh array will be given by,

$$\begin{array}{c|cccc} s^6 & 2 & 13 & 56 & 25 \\ s^5 & 1 & 6 & 25 & \\ s^4 & 1 & 6 & 25 & \\ s^3 & 4 & 12 & & \\ s^2 & 3 & 25 & & \\ s^1 & -21.3 & & & \\ s^0 & 25 & & & \end{array}$$

Since there is a sign change in the first column of the array, the polynomial $P(s)$ is not Hurwitz.

Example 16.21 *Determine the range of values of 'a' so that $P(s) = s^4 + s^3 + as^2 + 2s + 3$ is Hurwitz.*

Solution The Routh array is given by,

$$\begin{array}{c|ccc} s^4 & 1 & a & 3 \\ s^3 & 1 & 2 & \\ s^2 & a-2 & 3 & \\ s^1 & \dfrac{2a-7}{a-2} & & \\ s^0 & 3 & & \end{array}$$

For the polynomial to be Hurwitz, all the terms in the first column of the array should be positive, i.e.,

$$a - 2 > 0$$
$$a > 2$$

and

$$\frac{2a-7}{a-2} > 0$$

$$a > \frac{7}{2}$$

Hence, $P(s)$ will be Hurwitz when $a > \frac{7}{2}$.

Example 16.22 *Determine the range of values of k so that the polynomial $P(s) = s^3 + 3s^2 + 2s + k$ is Hurwitz.*

Solution The Routh array is given by,

$$\begin{array}{c|cc} s^3 & 1 & 2 \\ s^2 & 3 & k \\ s^1 & \dfrac{6-k}{3} & 0 \\ s^0 & k & \end{array}$$

For the polynomial to be Hurwitz, all the terms in the first column of the array should be positive,

i.e.,
$$\frac{6-k}{3} > 0$$
$$6-k > 0$$

i.e., $k < 6$ and $k > 0$
Hence, $P(s)$ will be Hurwitz for $0 < k < 6$.

16.3 POSITIVE REAL FUNCTIONS

A function $F(s)$ is positive real if the following conditions are satisfied:

(a) $F(s)$ is real for real s.
(b) The real part of $F(s)$ is greater than or equal to zero when the real part of s is greater than or equal to zero, i.e.,
Re $F(s) \geq 0$ for Re$(s) \geq 0$

16.3.1 Properties of Positive Real Functions

1. If $F(s)$ is positive real then $\dfrac{1}{F(s)}$ is also positive real.
2. The sum of two positive real functions is positive real.
3. The poles and zeros of a positive real function cannot have positive real parts, i.e., they cannot be in the right half of the s plane.
4. Only simple poles with real positive residues can exist on the $j\omega$-axis.
5. The poles and zeros of a positive real function are real or occur in conjugate pairs.
6. The highest powers of the numerator and denominator polynomials may differ at most by unity. This condition prevents the possibility of multiple poles and zeros at $s = \infty$.
7. The lowest powers of the denominator and numerator polynomials may differ by at most unity. Hence, a positive real function has neither multiple poles nor zeros at the origin.

16.3.2 Necessary and Sufficient Conditions for Positive Real Functions

The necessary and sufficient conditions for a function with real coefficients $F(s)$ to be positive real are the following:

1. $F(s)$ must have no poles and zeros in the right half of the s-plane.
2. The poles of $F(s)$ on the $j\omega$-axis must be simple and the residues evaluated at these poles must be real and positive.
3. Re $F(j\omega) \geq 0$ for all ω.

Testing of the Above Conditions Condition (1) requires that we test the numerator and denominator of $F(s)$ for roots in the right half of the s-plane, i.e., we must determine whether the numerator and denominator of $F(s)$ are Hurwitz. This is done through a continued fraction expansion of the odd to even or even to odd parts of the numerator and denominator.

Condition (2) is tested by making a partial-fraction expansion of $F(s)$ and checking whether the residues of the poles on the $j\omega$-axis are positive and real. Thus, if $F(s)$ has a pair of poles at $s = \pm j\omega_0$, a partial-fraction expansion gives terms of the form

$$\frac{K_1}{s-j\omega_0}+\frac{K_1^*}{s+j\omega_0}$$

Since residues of complex conjugate poles are themselves conjugate, $K_1 = K_1^*$ and should be positive and real.

Condition (3) requires that Re $F(j\omega)$ must be positive and real for all ω.

Now, to compute Re $F(j\omega)$ from $F(s)$, the numerator and denominator polynomials are separated into even and odd parts.

$$F(s)=\frac{m_1(s)+n_1(s)}{m_2(s)+n_2(s)}=\frac{m_1+n_1}{m_2+n_2}$$

Multiplying $N(s)$ and $D(s)$ by $m_2 - n_2$,

$$F(s)=\frac{m_1+n_1}{m_2+n_2}\frac{m_2-n_2}{m_2-n_2}=\frac{m_1m_2-n_1n_2}{m_2^2-n_2^2}+\frac{m_2n_1-m_1n_2}{m_2^2-n_2^2}$$

But the product of two even functions or odd functions is itself an even function, while the product of an even and odd function is odd.

$$\text{Ev } F(s)=\frac{m_1m_2-n_1n_2}{m_2^2-n_2^2}$$

$$\text{Od } F(s)=\frac{m_2n_1-m_1n_2}{m_2^2-n_2^2}$$

Now, substituting $s=j\omega$ in the even polynomial gives the real part of $F(s)$ and substituting $s=j\omega$ into the odd polynomial gives imaginary part of $F(s)$.

$$\text{Ev } F(s)\Big|_{s=j\omega} = \text{Re } F(j\omega)$$

$$\text{Od } F(s)\Big|_{s=j\omega} = j\,\text{Im } F(j\omega)$$

We have to test Re $F(j\omega) \geq 0$ for all ω.

The denominator of Re $F(j\omega)$ is always a positive quantity because

$$m_2^2-n_2^2\Big|_{s=j\omega} \geq 0$$

Hence, the condition that Ev $F(j\omega)$ should be positive requires

$$m_1m_2-n_1n_2\Big|_{s=j\omega} = A(\omega^2)$$

should be positive and real for all $\omega \geq 0$.

Example 16.23 *Test whether* $F(s)=\dfrac{s+3}{s+1}$ *is a positive real function.*

Solution

(a) $F(s)=\dfrac{N(s)}{D(s)}=\dfrac{s+3}{s+1}$

The function $F(s)$ has pole at $s = -1$ and zero at $s = -3$ as shown in Fig. 16.1.

Thus, pole and zero are in the left half of the s-plane.

(b) There is no pole on the $j\omega$ axis. Hence, the residue test is not carried out.

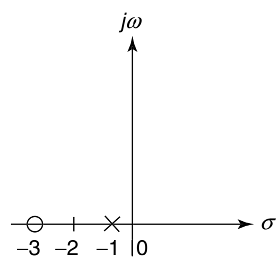


Fig. 16.1

(c) Even part of $N(s) = m_1 = 3$

Odd part of $N(s) = n_1 = s$

Even part of $D(s) = m_2 = 1$

Odd part of $D(s) = n_2 = s$

$$A(\omega^2) = m_1 m_2 - n_1 n_2 \,|_{s=j\omega} = (3)(1) - (s)(s)\,|_{s=j\omega} = 3 - s^2\,|_{s=j\omega} = 3 + \omega^2$$

$A(\omega^2)$ is positive for all $\omega \geq 0$.
Since all the three conditions are satisfied, the function is positive real.

Example 16.24 *Test whether* $F(s) = \dfrac{s^2 + 6s + 5}{s^2 + 9s + 14}$ *is positive real function.*

Solution

(a) $F(s) = \dfrac{N(s)}{D(s)} = \dfrac{s^2 + 6s + 5}{s^2 + 9s + 14} = \dfrac{(s+5)(s+1)}{(s+7)(s+2)}$

The function $F(s)$ has poles at $s = -7$ and $s = -2$ and zeros at $s = -5$ an $s = -1$ as shown in Fig. 16.2.

Thus, all the poles and zeros are in the left half of the s plane.

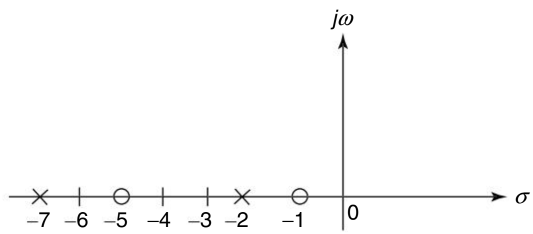


Fig. 16.2

(b) Since there is no pole on the $j\omega$ axis, the residue test is not carried out.

(c) Even part of $N(s) = m_1 = s^2 + 5$

Odd part of $N(s) = n_1 = 6s$

Even part of $D(s) = m_2 = s^2 + 14$

Odd part of $D(s) = n_2 = 9s$

$$A(\omega^2) = m_1 m_2 - n_1 n_2 \,|_{s=j\omega} = (s^2+5)\,(s^2+14) - (6s)(9s)\,|_{s=j\omega} = s^4 - 35s^2 + 70\,|_{s=j\omega} = \omega^4 + 35\omega^2 + 70$$

$A(\omega^2)$ is positive for all $\omega \geq 0$.
Since all the three conditions are satisfied, the function is positive real.

Example 16.25 *Test whether* $F(s) = \dfrac{s(s+3)(s+5)}{(s+1)(s+4)}$ *is positive real function.*

Solution

(a) $F(s) = \dfrac{N(s)}{D(s)} = \dfrac{s(s+3)(s+5)}{(s+1)(s+4)} = \dfrac{s^3 + 8s^2 + 15s}{s^2 + 5s + 4}$

The function $F(s)$ has poles at $s = -1$ and $s = -4$ and zeros at $s = 0$, $s = -3$ and $s = -5$ as shown in Fig. 16.3.

Thus, all the poles and zeros are in the left half of the s plane.

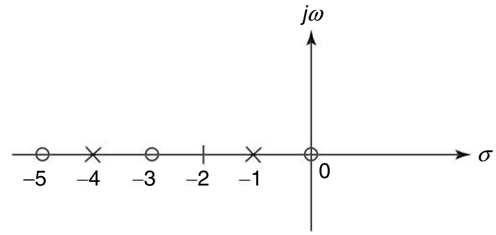


Fig. 16.3

(b) There is no pole on the $j\omega$ axis, hence the residue test is not carried out.

(c) Even part of $N(s) = m_1 = 8s^2$

Odd part of $N(s) = n_1 = s^3 + 15s$

Even part of $D(s) = m_2 = s^2 + 4$

Odd part of $D(s) = n_2 = 5s$

$$A(\omega^2) = m_1 m_2 - n_1 n_2 \,|_{s=j\omega} = (8s^2)(s^2+4) - (s^3+15s)(5s)\,|_{s=j\omega} = 3s^4 - 43s^2\,|_{s=j\omega} = 3\omega^4 + 43\omega^2$$

$A(\omega^2)$ is positive for all $\omega \ge 0$.

Since all the three conditions are satisfied, the function is positive real.

Example 16.26 *Test whether* $F(s) = \dfrac{s^2+1}{s^3+4s}$ *is positive real function.*

Solution

(a) $F(s) = \dfrac{N(s)}{D(s)} = \dfrac{s^2+1}{s^3+4s} = \dfrac{(s+j1)(s-j1)}{s(s+j2)(s-j2)}$

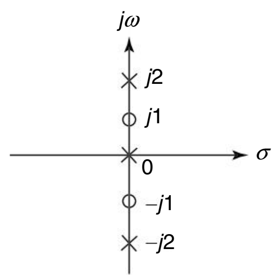


Fig. 16.4

The function $F(s)$ has poles at $s = 0$, $s = -j2$ and $s = j2$ and zeros at $s = -j1$ and $s = j1$ as shown in Fig. 16.4.

Thus, all the poles and zeros are on the $j\omega$ axis.

(b) The poles on the $j\omega$ axis are simple. Hence, residue test is carried out.

$$F(s) = \frac{s^2+1}{s^3+4s} = \frac{s^2+1}{s(s^2+4)}$$

By partial-fraction expansion,

$$F(s) = \frac{K_1}{s} + \frac{K_2}{s+j2} + \frac{K_2^*}{s-j2}$$

The constants K_1, K_2 and K_2* are called residues.

$$K_1 = s\,F(s)\,|_{s=0} = \left.\frac{s^2+1}{s^2+4}\right|_{s=0} = \frac{1}{4}$$

$$K_2 = (s+j2)F(s)\,|_{s=-j2} = \left.\frac{s^2+1}{s(s-j2)}\right|_{s=-j2} = \frac{-4+1}{(-j2)(-j2-j2)} = \frac{3}{8}$$

$$K_2^* = K_2 = \frac{3}{8}$$

Thus, residues are real and positive.

(c) Even part of $N(s) = m_1 = s^2 + 1$

Odd part of $N(s) = n_1 = 0$

Even part of $D(s) = m_2 = 0$

Odd part of $D(s) = n_2 = s^3 + 4s$

$$A\left(\omega^2\right) = m_1 m_2 - n_1 n_2 \,|_{s=j\omega} = (s^2+1)(0) - (0)(s^3+4s)\,|_{s=j\omega} = 0$$

$A(\omega^2)$ is zero for all $\omega \geq 0$.

Since all the three conditions are satisfied, the function is positive real.

Example 16.27 *Test whether* $F(s) = \dfrac{2s^3 + 2s^2 + 3s + 2}{s^2 + 1}$ *is positive real function.*

Solution

(a) $F(s) = \dfrac{N(s)}{D(s)} = \dfrac{2s^3 + 2s^2 + 3s + 2}{s^2 + 1} = \dfrac{2s^3 + 2s^2 + 3s + 2}{(s + j1)(s - j1)}$

Since numerator polynomial cannot be easily factorized, we will prove whether $N(s)$ is Hurwitz.

Even part of $N(s) = m(s) = 2s^2 + 2$

Odd part of $N(s) = n(s) = 2s^3 + 3s$

By continued fraction expansion,

$$\begin{array}{l}
2s^2+2\;\big)\; 2s^3 + 3s \;\big(\; s \\
\qquad\qquad\; \underline{2s^3 + 2s} \\
\qquad\qquad\qquad\quad s\;\big)\; 2s^2 + 2s \;\big(\; 2s \\
\qquad\qquad\qquad\qquad\;\; \underline{2s^2} \\
\qquad\qquad\qquad\qquad\qquad 2\;\big)\; s \;\big(\; \tfrac{1}{2}s \\
\qquad\qquad\qquad\qquad\qquad\quad\;\; \underline{s} \\
\qquad\qquad\qquad\qquad\qquad\quad\;\; 0
\end{array}$$

Since all the quotient terms are positive, $N(s)$ is Hurwitz. This indicates that zeros are in the left half of the s plane.

The function $F(s)$ has poles at $s = -j1$ and $s = j1$.

Thus, all the poles and zeros are in the left half of the s plane.

(b) The poles on the $j\omega$ axis are simple. Hence, residue test is carried out.

$$F(s) = \frac{2s^3 + 2s^2 + 3s + 2}{s^2 + 1}$$

As the degree of the numerator is greater than that of the denominator, division is first carried out before partial-fraction expansion.

$$
\begin{array}{r|l}
s^2+1 & 2s^3+2s^2+3s+2 \;(\; 2s+2 \\
 & \underline{2s^3 \qquad\quad +2s} \\
 & \quad 2s^2 + \; s+2 \\
 & \quad \underline{2s^2 \qquad +2} \\
 & \quad s
\end{array}
$$

$$F(s) = 2s + 2 + \frac{s}{s^2+1}$$

By partial-fraction expansion,

$$F(s) = 2s + 2 + \frac{K_1}{s+j1} + \frac{K_1^*}{s-j1}$$

$$K_1 = (s+j1)F(s)\,|_{s=-j1} = \frac{-j1}{-j1-j1} = \frac{1}{2}$$

$$K_1^* = K_1 = \frac{1}{2}$$

Thus, residues are real and positive.

(c) Even part of $N(s) = m_1 = 2s^2 + 2$

Odd part of $N(s) = n_1 = 2s^3 + 3s$

Even part of $D(s) = m_2 = s^2 + 1$

Odd part of $D(s) = n_2 = 0$

$$A(\omega^2) = m_1 m_2 - n_1 n_2\,|_{s=j\omega} = (2s^2+2)(s^2+1) - (2s^3+3s)(0)\,|_{s=j\omega} = 2s^4 + 4s^2 + 2\,\;|_{s=j\omega} = 2(\omega^4 - 2\omega^2 + 1)$$
$$= 2(\omega^2 - 1)^2$$

$A(\omega^2) \geq 0$ for all $\omega \geq 0$.

Since all the three conditions are satisfied, the function is positive real.

Example 16.28 *Test whether* $F(s) = \dfrac{s^3 + 6s^2 + 7s + 3}{s^2 + 2s + 1}$ *is positive real function.*

Solution

(a) $F(s) = \dfrac{N(s)}{D(s)} = \dfrac{s^3 + 6s^2 + 7s + 3}{s^2 + 2s + 1} = \dfrac{s^3 + 6s^2 + 7s + 3}{(s+1)(s+1)}$

Since a numerator polynomial cannot be easily factorized, we will test whether $N(s)$ is Hurwitz.

Even part of $N(s) = m(s) = 6s^2 + 3$

Odd part of $N(s) = n(s) = s^3 + 7s$

By continued fraction expansion,

$$
\begin{array}{l}
6s^2+3\,\big)\,s^3+7s\,\big(\,\frac{1}{6}s \\
\qquad\qquad \underline{s^3+0.5s} \\
\qquad\qquad\quad 6.5s\,\big)\,6s^2+3\,\big(\,0.92s \\
\qquad\qquad\qquad\qquad \underline{6s^2\qquad} \\
\qquad\qquad\qquad\qquad\quad 3\,\big)\,6.5s\,\big(\,2.17s \\
\qquad\qquad\qquad\qquad\qquad \underline{6.5s} \\
\qquad\qquad\qquad\qquad\qquad\quad 0
\end{array}
$$

Since all the quotient terms are positive, $N(s)$ is Hurwitz. This indicates that the zeros are in the left half of the s plane.

The function $F(s)$ has a double pole at $s=-1$.

Thus, all the poles and zeros are in the left half of the s plane.

(b) There is no pole on the $j\omega$ axis. Hence, the residue test is not carried out.

(c) Even part of $N(s) = m_1 = 6s^2+3$

Odd part of $N(s) = n_1 = s^3+7s$

Even part of $D(s) = m_2 = s^2+1$

Odd part of $D(s) = n_2 = 2s$

$$A(\omega^2) = m_1 m_2 - n_1 n_2 \,|_{s=j\omega} = (6s^2+3)(s^2+1)-(s^3+7s)(2s)\,|_{s=j\omega} = 4s^4-5s^2+3\,|_{s=j\omega} = 4\omega^4+5\omega^2+3$$

$A(\omega^2)$ is positive for all $\omega \ge 0$.

Since all the three conditions are satisfied, the function is positive real.

Example 16.29 *Test whether* $F(s) = \dfrac{s^2+s+6}{s^2+s+1}$ *is a positive real function.*

Solution

(a) $$F(s) = \frac{N(s)}{D(s)} = \frac{s^2+s+6}{s^2+s+1} = \frac{\left(s+\frac{1}{2}+j\frac{\sqrt{23}}{2}\right)\left(s+\frac{1}{2}-j\frac{\sqrt{23}}{2}\right)}{\left(s+\frac{1}{2}+j\frac{\sqrt{3}}{2}\right)\left(s+\frac{1}{2}-j\frac{\sqrt{3}}{2}\right)}$$

The function $F(s)$ has zeros at $s=-\frac{1}{2}\pm j\frac{\sqrt{23}}{2}$ and poles at $s=-\frac{1}{2}\pm j\frac{\sqrt{3}}{2}$.

(b) There is no pole on the $j\omega$ axis. Hence, the residue test is not carried out.

(c) Even part of $N(s) = m_1 = s^2+6$

Odd part of $N(s) = n_1 = s$

Even part of $D(s) = m_2 = s^2+1$

Odd part of $D(s) = n_2 = s$

$$A(\omega^2) = m_1 m_2 - n_1 n_2 \,|_{s=j\omega} = (s^2+6)(s^2+1) - (s)(s)\,|_{s=j\omega} = s^4 + 6s^2 + 6\,|_{s=j\omega} = \omega^4 - 6\omega^2 + 6$$

For $\omega = 2$, $A(\omega^2) = 16 - 24 + 6 = -2$
This condition is not satisfied.
Hence, the function $F(s)$ is not positve real.

Example 16.30 *Test whether* $F(s) = \dfrac{s^2+4}{s^3+3s^2+3s+1}$ *is positive real function.*

Solution

(a) $F(s) = \dfrac{N(s)}{D(s)} = \dfrac{s^2+4}{s^3+3s^2+3s+1} = \dfrac{(s+j2)(s-j2)}{(s+1)^3}$

The function $F(s)$ has two zeros at $s = \pm j2$ and three poles at $s = -1$.
Thus, all the poles and zeros are in the left half of the s plane.

(b) There is no pole on the $j\omega$ axis. Hence, the residue test is not carried out.

(c) Even part of $N(s) = m_1 = s^2 + 4$
Odd part of $N(s) = n_1 = 0$
Even part of $D(s) = m_2 = 3s^2 + 1$
Odd part of $D(s) = n_2 = s^3 + 3s$

$$A(\omega^2) = m_1 m_2 - n_1 n_2 \,|_{s=j\omega} = (s^2+4)(3s^2+1) - (0)(s^3+3s)\,|_{s=j\omega} = 3s^4 + 13s^2 + 4\,|_{s=j\omega} = 3\omega^4 - 13\omega^2 + 4$$

For $\omega = 1$, $A(\omega)^2 = 3 - 13 + 4 = -6$
This condition is not satisfied.
Hence, the function $F(s)$ is not positive real.

Example 16.31 *Test whether* $F(s) = \dfrac{s^3+5s}{s^4+2s^2+1}$ *is positive real function.*

Solution

(a) $F(s) = \dfrac{N(s)}{D(s)} = \dfrac{s^3+5s}{s^4+2s^2+1} = \dfrac{s(s^2+5)}{(s^2+1)^2} = \dfrac{s(s+j\sqrt{5})(s-j\sqrt{5})}{(s\pm j1)(s\pm j1)}$

The function $F(s)$ has zeros at $s = 0$, $s = \pm j\sqrt{5}$ and two poles at $s = j1$ and two poles at $s = -j1$.

Thus, poles on the $j\omega$ axis are not simple.
Hence, the function $F(s)$ not positive real.

Example 16.32 *Test whether* $F(s) = \dfrac{s^4+3s^3+s^2+s+2}{s^3+s^2+s+1}$ *is positive real function.*

Solution

$$F(s) = \frac{N(s)}{D(s)} = \frac{s^4+3s^3+s^2+s+2}{s^3+s^2+s+1}$$

Here, it is easier to prove that $N(s)$ and $D(s)$ are Hurwitz.

By Routh array,

$$\begin{array}{c|ccc} s^4 & 1 & 1 & 2 \\ s^3 & 3 & 1 & \\ s^2 & \frac{2}{3} & 2 & \\ s^1 & -8 & & \\ s^0 & 2 & & \end{array}$$

Since there is a sign change in the first column of the array, $N(s)$ is not Hurwitz. Thus, all the zeros are not in the left half of the s plane. The remaining two tests need not be carried out.
Hence, the function $F(s)$ is not positive real.

16.4 ELEMENTARY SYNTHESIS CONCEPTS

We know that impedances and admittances of passive networks are positive real functions. Hence, addition of impedances of the two passive networks gives a function which is also a positive real function. Thus, $Z(s) = Z_1(s) + Z_2(s)$ is a positive real function, if $Z_1(s)$ and $Z_2(s)$ are positive real functions. Similarly, $Y(s) = Y_1(s) + Y_2(s)$ is a positive real function, if $Y_1(s)$ and $Y_2(s)$ are positive real functions. There is a special terminology for synthesis procedure. We have,

$$Z(s) = Z_1(s) + Z_2(s)$$

$$Z_2(s) = Z(s) - Z_1(s)$$

Here, $Z_1(s)$ is said to have been removed from $Z(s)$ in forming the new function $Z_2(s)$ as shown in Fig. 16.5. If the removed network is associated with the pole or zero of the original network impedance then that pole or zero is also said to have been removed.
There are four important removal operations.

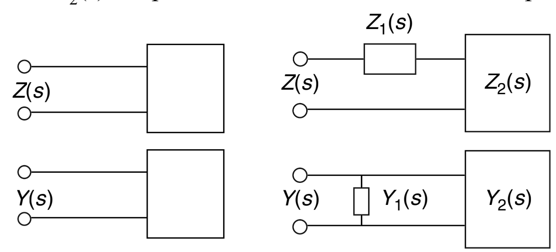


Fig. 16.5 *Network interpretation of the removal of impedance and admittance*

16.4.1 Removal of a Pole at Infinity

Consider an impedance function $Z(s)$ having a pole at infinity which means that the numerator polynomial is one degree greater than the degree of the denominator polynomial.

$$Z(s) = \frac{a_{n+1}s^{n+1} + a_n s^n + \ldots + a_1 s + a_0}{b_n s^n + b_{n-1}s^{n-1} + \ldots + b_1 s + b_0} = Hs + \frac{c_n s^n + c_{n-1}s^{n-1} + \ldots + c_1 s + c_0}{b_n s^n + b_{n-1}s^{n-1} + \ldots + b_1 s + b_0}$$

where $\quad H = \dfrac{a_{n+1}}{b_n}$

Let $\quad Z_1(s) = Hs$

and $\quad Z_2(s) = \dfrac{c_n s^n + c_{n-1}s^{n-1} + \ldots + c_1 s + c_0}{b_n s^n + b_{n-1}s^{n-1} + \ldots + b_1 s + b_0} = Z(s) - Hs$

$Z_1(s) = Hs$ represents impedance of an inductor of value H. Hence, the removal of a pole at infinity corresponds to the removal of an inductor from the network of Fig. 16.6(a).

If the given function is an admittance function $Y(s)$, then $Y_1(s) = Hs$ represents the admittance of a capacitor $Y_C(s) = Cs$. The network for $Y_1(s)$ is a capacitor of value $C = H$ as shown in Fig. 16.6(b).

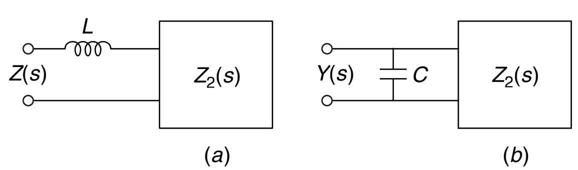


Fig. 16.6 *Network interpretation of the removal of a pole at infinity*

16.4.2 Removal of a Pole at Origin

If $Z(s)$ has a pole at the origin then it may be written as

$$Z(s) = \frac{a_0 + a_1 s + \ldots + a_{n-1}s^{n-1} + a_n s^n}{b_1 s + b_2 s^2 + \ldots + b_m s^m} = \frac{K_0}{s} + \frac{d_1 + d_2 s + \ldots + d_n s^{n-1}}{b_1 + b_2 s + \ldots + b_m s^{m-1}} = Z_1(s) + Z_2(s)$$

where $K_0 = \frac{a_0}{b_1}$

$Z_1(s) = \frac{K_0}{s}$ represents the impedance of a capacitor of value $\frac{1}{K_0}$.

If the given function is an admittance function $Y(s)$ then removal of $Y_1(s) = \frac{K_0}{s}$ corresponds to an inductor of value $\frac{1}{K_0}$.

Thus, removal of a pole from the impedance function $Z(s)$ at the origin corresponds to the removal of a capacitor, and from admittance function $Y(s)$ corresponds to removal of an inductor as shown in Fig. 16.7.

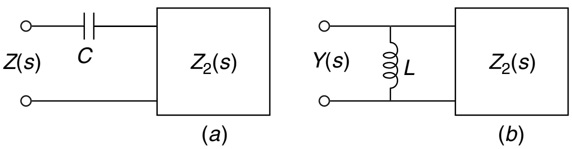


Fig. 16.7 *Network interpretation of the removal of a pole at origin*

16.4.3 Removal of Conjugate Imaginary Poles

If $Z(s)$ contains poles on the imaginary axis, i.e., at $s = \pm j\omega_1$ then $Z(s)$ will have factors $(s + j\omega_1)(s - j\omega_1) = s^2 + \omega_1^2$ in the denominator polynomial

$$Z(s) = \frac{p(s)}{(s^2 + \omega_1^2)\, q_1(s)}$$

By partial-fraction expansion,

$$Z(s) = \frac{K_1}{s + j\omega_1} + \frac{K_1^*}{s - j\omega_1} + Z_2(s)$$

For a positive real function, $j\omega$ axis poles must themselves be conjugate and must have equal, positive and real residues.

$$K_1 = K_1^*$$

Hence,
$$Z(s) = \frac{2K_1 s}{s^2 + \omega_1^2} + Z_2(s)$$

Thus,
$$Z_1(s) = \frac{2K_1 s}{s^2 + \omega_1^2} = \frac{1}{\frac{s}{2K_1} + \frac{\omega_1^2}{2K_1 s}} = \frac{1}{Y_a + Y_b}$$

where $Y_a = \frac{s}{2K_1}$ is the admittance of a capacitor of value $C = \frac{1}{2K_1}$

and $Y_b = \frac{\omega_1^2}{2K_1 s}$ is the admittance of an inductor of value $L = \frac{2K_1}{\omega_1^2}$

If the given function is an admittance function $Y(s)$ then

$$Y_1(s) = \frac{2K_1 s}{s^2+\omega_1^2} = \frac{1}{Z_a + Z_b} = \frac{1}{\dfrac{s}{2K_1} + \dfrac{\omega_1^2}{2K_1 s}}$$

where $Z_a = \dfrac{s}{2K_1}$ is the impedance of an inductor of value $L = \dfrac{1}{2K_1}$ and $Z_b = \dfrac{\omega_1^2}{2K_1 s}$ is the impedance of a capacitor of value $C = \dfrac{2K_1}{\omega_1^2}$.

Thus, removal of conjugate imaginary poles from impedance function $Z(s)$ corresponds to the removal of the parallel combination of $L - C$ and from admittance function $Y(s)$ corresponds to removal of series combination of $L - C$ as shown in Fig. 16.8.

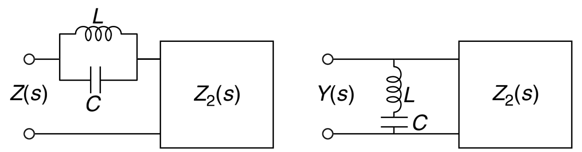


Fig. 16.8 *Network interpretation of the removal of conjugate imaginary poles*

16.4.4 Removal of a Constant

If a real number R_1 is subtracted from $Z(s)$ such that

$$Z_2(s) = Z(s) - R_1$$
$$Z(s) = R_1 + Z_2(s)$$

then R_1 represents a resistor.

If the given function is an admittance function $Y(s)$, then removal of $Y_1(s) = R_1$ represents a conductance of value R_1.

Thus, removal of a constant from impedance function $Z(s)$ corresponds to the removal of a resistance, and from admittance function $Y(s)$ corresponds to removal of a conductance.

Example 16.33 *Synthesize the impedance function* $Z(s) = \dfrac{s^3+4s}{s^2+2}$.

Solution

By long division of $Z(s)$,

$$s^2+2\overline{)\,s^3+4s\,(}\;s$$
$$\underline{s^3+2s}$$
$$2s$$

$$Z(s) = s + \frac{2s}{s^2+2} = Z_1(s) + Z_2(s)$$

$Z_1(s) = s$ represents impedance of an inductor of value 1 H.

$$Y_2(s) = \frac{1}{Z_2(s)} = \frac{s^2+2}{2s} = \frac{s^2}{2s} + \frac{2}{2s} = \frac{1}{2}s + \frac{1}{s} = Y_3(s) + Y_4(s)$$

$Y_3(s) = \dfrac{1}{2}s$ represents the admittance of a capacitor of value $\dfrac{1}{2}$ F.

$Y_4(s) = \dfrac{1}{s}$ represents the admittance of an inductor of value 1 H.

The impedances are connected in the series branches whereas the admittances are connected in the parallel branches. The network is shown in Fig. 16.9.

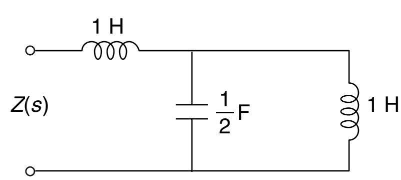


Fig. 16.9

Example 16.34 *Realise the network having impedance function*

$$Z(s) = \frac{s^2 + 2s + 10}{s(s+5)}$$

Solution

By long division of $Z(s)$,

$$s^2 + 5s \overline{\big)\, s^2 + 2s + 10 \,\big(} \frac{2}{s}$$
$$\underline{2s + 10}$$
$$s^2$$

$$Z(s) = \frac{2}{s} + \frac{s^2}{s^2 + 5s} = \frac{2}{s} + \frac{s}{s+5} = Z_1(s) + Z_2(s)$$

$Z_1(s) = \frac{2}{s}$ represents the impedance of capacitor of value $\frac{1}{2}$ F.

$$Y_2(s) = \frac{1}{Z_2(s)} = \frac{s+5}{s} = 1 + \frac{5}{s} = Y_3(s) + Y_4(s)$$

$Y_3(s) = 1$ represents the admittance of a resistor of value 1 Ω.
$Y_4(s) = \frac{5}{s}$ represents the admittance of an inductor of value $\frac{1}{5}$ H.
The impedances are connected in the series branches whereas the admittances are connected in the parallel branches. The network is shown in Fig. 16.10.

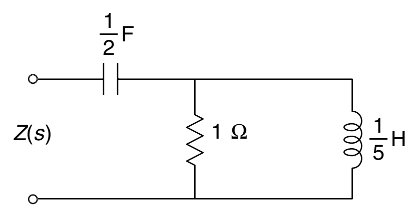


Fig. 16.10

Example 16.35 *Realise the network having impedance function* $Z(s) = \frac{6s^3 + 5s^2 + 6s + 4}{2s^3 + 2s}$.

Solution By long division of $Z(s)$,

$$2s^3 + 2s \overline{\big)\, 6s^3 + 5s^2 + 6s + 4 \,\big(} 3$$
$$\underline{6s^3 \quad + 6s}$$
$$5s^2 \quad + 4$$

$$Z(s) = 3 + \frac{5s^2 + 4}{2s^3 + 2s} = Z_1(s) + Z_2(s)$$

$Z_1(s) = 3$ represents the impedance of a resistor of value 3 Ω.

$$Y_2(s) = \frac{1}{Z_2(s)} = \frac{2s^3 + 2s}{5s^2 + 4}$$

By long division of $Y_2(s)$,

$$5s^2+4\overline{\big)\,2s^3+2s\,}\Big(\frac{2}{5}s$$
$$\underline{2s^3+\frac{8}{5}s}$$
$$\frac{2}{5}s$$

$$Y_2(s)=\frac{2}{5}s+\frac{\frac{2}{5}s}{5s^2+4}=Y_3(s)+Y_4(s)$$

$Y_3(s)=\frac{2}{5}s$ represents the admittance of a capacitor of value $\frac{2}{5}$ F.

$$Z_4(s)=\frac{1}{Y_4(s)}=\frac{5s^2+4}{\frac{2}{5}s}=\frac{25s^2+20}{2s}=\frac{25}{2}s+\frac{10}{s}=Z_5(s)+Z_6(s)$$

$Z_5(s)=\frac{25}{2}s$ represents the impedance of an inductor of value $\frac{25}{2}$ H.

$Z_6(s)=\frac{10}{s}$ represents the impedance of a capacitor of value $\frac{1}{10}$ F.

The impedances are connected in the series branches, whereas the admittances are connected in the parallel branches. The network is shown in Fig. 16.11.

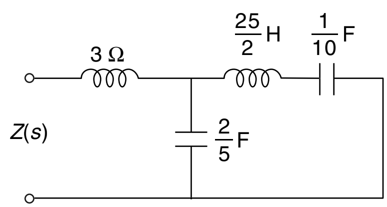


Fig. 16.11

Example 16.36

Realise the network having impedance function

$$Z(s)=\frac{s^4+10s^2+7}{s^3+2s}$$

Solution By long division of $Z(s)$,

$$s^3+2s\overline{\big)\,s^4+10s^2+7\,}\Big(s$$
$$\underline{s^4+2s^2}$$
$$8s^2+7$$

$$Z(s)=s+\frac{8s^2+7}{s^3+2s}=Z_1(s)+Z_2(s)$$

$Z_1(s)=s$ represents the impedance of an inductor of value 1 H.

$$Y_2(s)=\frac{1}{Z_2(s)}=\frac{s^3+2s}{8s^2+7}$$

By long division of $Y_2(s)$,

$$8s^2+7\overline{\smash{)}\,s^3+2s\,}\left(\frac{1}{8}s\right.$$
$$\underline{s^3+\frac{7}{8}s}$$
$$\frac{9}{8}s$$

$$Y_2(s)=\frac{1}{8}s+\frac{\frac{9}{8}s}{8s^2+7}=Y_3(s)+Y_4(s)$$

$Y_3(s)=\frac{1}{8}s$ represents the admittance of a capacitor of value $\frac{1}{8}$ F.

$$Z_4(s)=\frac{1}{Y_4(s)}=\frac{8s^2+7}{\frac{9}{8}s}=\frac{64}{9}s+\frac{56}{9s}=Z_5(s)+Z_6(s)$$

$Z_5(s)=\frac{64}{9}s$ represents the impedance of an inductor of value $\frac{64}{9}$ H.

$Z_6(s)=\frac{56}{9s}$ represents the impedance of a capacitor of value $\frac{9}{56}$ F.

The impedances are connected in the series branches, whereas the admittances are connected in the parallel branches. The network is shown in Fig. 16.12.

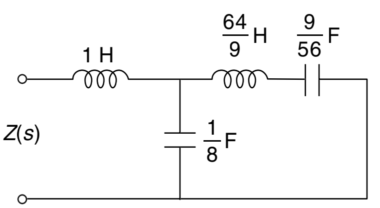


Fig. 16.12

Example 16.37

Realise the network having admittance function $Y(s)=\frac{4s^2+6s}{s+1}$.

Solution By long division of $Y(s)$,

$$s+1\overline{\smash{)}\,4s^2+6s\,}\left(4s\right.$$
$$\underline{4s^2+4s}$$
$$2s$$

$$Y(s)=4s+\frac{2s}{s+1}=Y_1(s)+Y_2(s)$$

$Y_1(s)=4s$ represents the admittance of a capacitor of value 4 F.

$$Z_2(s)=\frac{1}{Y_2(s)}=\frac{s+1}{2s}=\frac{1}{2}+\frac{1}{2s}=Z_3(s)+Z_4(s)$$

$Z_3(s) = \frac{1}{2}$ represents the impedance of a resistor of value $\frac{1}{2}\,\Omega$.

$Z_4(s) = \frac{1}{2s}$ represents the impedance of a capacitor of value 2 F.

The impedances are connected in the series branches, whereas the admittances are connected in the parallel branches. The network is shown in Fig. 16.13.

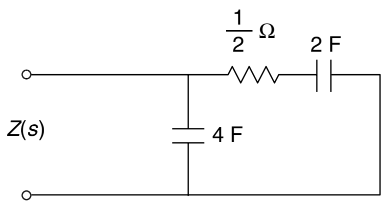


Fig. 16.13

Example 16.38 *Realise the admittance function* $Y(s) = \dfrac{3+5s}{4+2s}$.

Solution By long division,

$$4+2s\overline{)\,3+5s\,}\left(\frac{3}{4}\right.$$
$$\frac{3+\frac{3}{2}s}{\frac{7}{2}s}$$

$$Y(s) = \frac{3}{4} + \frac{\frac{7}{2}s}{4+2s} = Y_1(s) + Y_2(s)$$

$Y_1(s) = \frac{3}{4}$ represents the admittance of a resistor of value $\frac{4}{3}\,\Omega$.

$$Z_2(s) = \frac{1}{Y_2(s)} = \frac{4+2s}{\frac{7}{2}s} = \frac{8+4s}{7s} = \frac{8}{7s} + \frac{4}{7} = Z_3(s) + Z_4(s)$$

$Z_3(s) = \frac{8}{7s}$ represents the impedance of a capacitor of value $\frac{7}{8}$ F.

$Z_4(s) = \frac{4}{7}$ represents the impedance of a resistor of value $\frac{4}{7}\,\Omega$.

The impedances are connected in the series branches, whereas the admittances are connected in the parallel branches. The network is shown in Fig. 16.14.

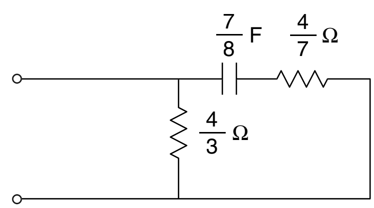


Fig. 16.14

16.5 REALISATION OF *LC* FUNCTIONS

LC driving point immitance functions have the following properties.

1. It is the ratio of odd to even or even to odd polynomials.
2. The poles and zeros are simple and lie on the $j\omega$-axis.
3. The poles and zeros interlace on the $j\omega$-axis.
4. There must be either a zero or a pole at the origin and infinity.
5. The difference between any two successive powers of numerator and denominator polynomials is at most two. There cannot be any missing terms.

6. The highest powers of numerator and denominator polynomials must differ by unity; the lowest powers also differ by unity.

There are a number of methods of realising an *LC* function. But we will study only four basic forms—Foster I, Foster II, Cauer I and Cauer II forms. The Foster forms are obtained by partial-fraction expansion of $F(s)$, and the Cauer forms are obtained by continued fraction expansion of $F(s)$.

16.5.1 Foster Realisation

Consider a general *LC* function $F(s)$ given by

$$F(s) = \frac{H(s^2+\omega_1^2)(s^2+\omega_3^2)...}{s(s^2+\omega_2^2)(s^2+\omega_4^2)...}$$

where $0 \le \omega_1^2 < \omega_2^2 < \omega_3^2 \ldots$ and H is positive.

By partial-fraction expansion of $F(s)$,

$$F(s) = \frac{K_0}{s} + \frac{K_2}{s+j\omega_2} + \frac{K_2}{s-j\omega_2} + ... + K_\infty s$$

Combining terms with conjugate poles,

$$F(s) = \frac{K_0}{s} + \frac{2K_2 s}{s+\omega_2^2} ... + K_\infty s$$

where K_0, K_i and K_∞ are the residues of $F(s)$ at the origin, at $j\omega_i$ and at infinity respectively.
These residues are given by

$$K_0 = sF(s)\big|_{s=0}$$

$$K_i = \left.\frac{\left(s^2+\omega_i^2\right)F(s)}{2s}\right|_{s^2=-\omega_i^2}$$

$$K_\infty = \left.\frac{F(s)}{s}\right|_{s\to\infty}$$

Foster I Form If $F(s)$ represents an impedance function, it gives a series connection of impedances.

$$F(s) = Z(s) = \frac{K_0}{s} + \frac{2K_2 s}{s^2+\omega_2^2} \ldots + K_\infty s = Z_1(s) + Z_2(s) + \ldots + Z_n(s)$$

The first term $\frac{K_0}{s}$ represents the impedance of a capacitor of $\frac{1}{K_0}$ farad.

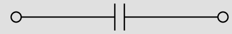
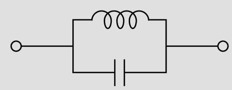
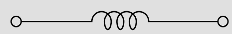
The last term $K_\infty s$ represents the impedance of an inductor of K_∞ henry.

The remaining terms, i.e., $\frac{2K_i s}{s^2+\omega_i^2}$ represent the impedance of a parallel combination of capacitor C_i and inductor L_i. For parallel combination of L_i and C_i,

$$Z(s) = \frac{1}{C_i s + \frac{1}{L_i s}} = \frac{\left(\frac{1}{C_i}\right)s}{s^2 + \frac{1}{L_i C_i}} = \frac{2K_i s}{s^2+\omega_i^2}$$

$$C_i = \frac{1}{2K_i} \quad \text{and} \quad L_i = \frac{2K_i}{\omega_i^2}$$

Table 16.1 *Realisation of Foster-I form of LC network*

Impedance function	Element
$\frac{K_0}{s} = \frac{1}{C_0 s}$	 $C_0 = \frac{1}{K_0}$
$\frac{2K_i s}{s^2+\omega_i^2} = \frac{\left(\frac{1}{C_i}\right)s}{s^2 + \frac{1}{L_i C_i}}$	$L_i = \frac{2K_i}{\omega_i^2}$  $C_i = \frac{1}{2K_i}$
$K_\infty s = Ls$	 $L_\infty = K_\infty$

The network corresponding to Foster I form is shown in Fig. 16.15.

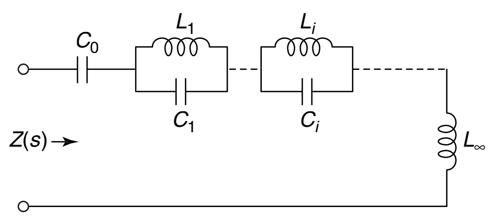


Fig. 16.15 *Foster-I form of LC network*

If $Z(s)$ has no pole at the origin then capacitor C_0 is not present in the network. Similarly, if there is no pole at ∞, inductor L_∞ is not present in the network.

Foster II Form If $F(s)$ represents an admittance function, it gives the parallel combination of admittances.

$$F(s) = Y(s) = \frac{K_0}{s} + \frac{2K_2 s}{s^2+\omega_2^2} + \ldots + K_\infty s = Y_1(s) + Y_2(s) + \ldots Y_n(s)$$

The first term $\frac{K_0}{s}$ represents the admittance of an inductor of $\frac{1}{K_0}$ henry.

The last term $K_\infty s$ represents the admittance of a capacitor of K_∞ farad.

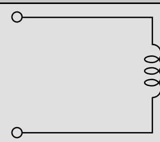
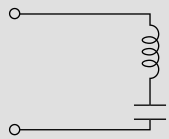
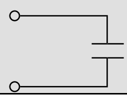
The remaining terms, i.e., $\frac{2K_i s}{s^2+\omega_i^2}$ represent the admittance of a series combination of an inductor L_i and a capacitor C_i.

For series combination of L_i and C_i,

$$Y(s) = \frac{1}{L_i s + \dfrac{1}{C_i s}} = \frac{\left(\dfrac{1}{L_i}\right)s}{s^2 + \dfrac{1}{L_i C_i}} = \frac{2K_i s}{s^2 + \omega_i^2}$$

$$L_i = \frac{1}{2K_i} \quad \text{and} \quad C_i = \frac{2K_i}{\omega_i^2}$$

Table 16.2 *Realisation of Foster-II form of LC network*

Admittance function	Element
$\dfrac{K_0}{s} = \dfrac{1}{L_0 s}$	 $L_0 = \dfrac{1}{K_0}$
$\dfrac{2K_i s}{s^2 + \omega_i^2} = \dfrac{\left(\dfrac{1}{L_i}\right)s}{s^2 + \dfrac{1}{L_i C_i}}$	 $L_i = \dfrac{1}{2K_i}$, $C_i = \dfrac{2K_i}{\omega_i^2}$
$K_\infty s = Cs$	 $C_\infty = K_\infty$

The network corresponding to the Foster II form is shown in Fig. 16.16.

If $Y(s)$ has no pole at the origin then inductor L_0 is not present. Similarly, if there is no pole at infinity, capacitor C_∞ is not present.

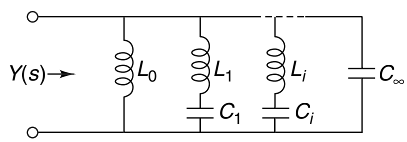


Fig. 16.16 *Foster-II form of LC network*

16.5.2 Cauer Realisation or Ladder Realisation

Cauer I Form Since the numerator and denominator polynomials of an *LC* function always differ in degrees by unity, there is always a zero or a pole at $s = \infty$. The Cauer I Form is obtained by successive removal of a pole or a zero at infinity from the function.

Consider an impedance function $Z(s)$ having a pole at infinity.

By removing the pole at infinity, we get

$$Z_2(s) = Z(s) - L_1 s$$

Now, $Z_2(s)$ has a zero at $s = \infty$. If we invert $Z_2(s)$, $Y_2(s)$ will have a pole at $s = \infty$.

By removing this pole,

$$Y_3(s) = Y_2(s) - C_2 s$$

Now $Y_3(s)$ has a zero at $s = \infty$, which we can invert and remove. This process continues until the remainder is zero. Each time we remove a pole, we remove an inductor or a capacitor depending on whether the function is an impedance or an admittance. The impedance $Z(s)$ can be written as a continued fraction expansion.

$$Z(s) = L_1 s + \cfrac{1}{C_2 s + \cfrac{1}{L_3 s + \cfrac{1}{C_4 s + \dots}}}$$

Thus, the final structure is a ladder network whose series arms are inductors and shunt arms are capacitors. The Cauer I network is shown in Fig. 16.17.

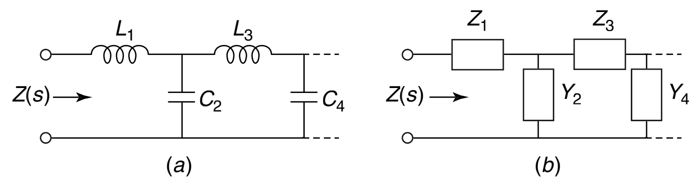


Fig. 16.17 *Cauer I form of LC network*

If the impedance function has zero at infinity, i.e., if degree of numerator is less than that of its denominator by unity, the function is first inverted and continued fraction expansion proceeds as usual. In this case, the first element is a capacitor as shown in Fig. 16.18.

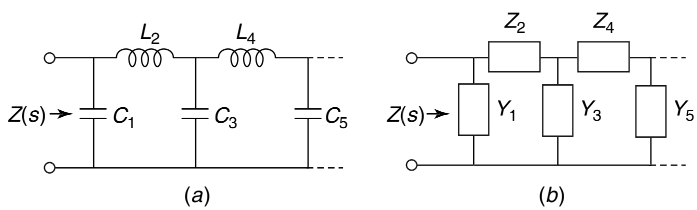


Fig. 16.18 *Cauer-I form of LC network*

Cauer II Form Since the lowest degrees of numerator and denominator polynomials of *LC* function must differ by unity, there is always a zero or a pole at $s = 0$. The Cauer II form is obtained by successive removal of a pole or a zero at $s = 0$ from the function.

In this method, continued fraction expansion of $Z(s)$ is carried out in terms of poles at the origin by removal of the pole at the origin, inverting the resultant function to create a pole at the origin which is removed and this process is continued until the remainder is zero. To do this, we arrange both numerator and denominator polynomials in ascending order and divide the lowest power of the denominator into the lowest power of the numerator. Then we invert the remainder and divide again. The impedance $Z(s)$ can be written as a continued fraction expansion.

$$Z(s) = \frac{1}{C_1 s} + \cfrac{1}{\cfrac{1}{L_2 s} + \cfrac{1}{\cfrac{1}{C_3 s} + \cfrac{1}{\cfrac{1}{L_4 s} + \dots}}}$$

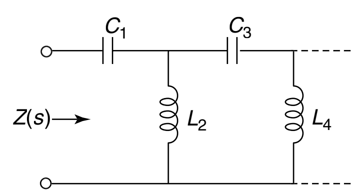


Fig. 16.19 *Cauer II form of LC network*

Thus, the final structure is a ladder network whose first element is a series capacitor and second element is a shunt inductor as shown in Fig. 16.19.

If the impedance function has a zero at the origin then the first element is a shunt inductor and the second element is a series capacitor as shown in Fig. 16.20.

Thus, the *LC* function $F(s)$ can be realised in four different forms. All these forms have the same number of elements and the number is equal to the number of poles and zeros of $F(s)$ includng any at infinity.

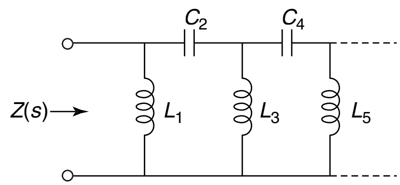


Fig. 16.20 *Cauer–II form of LC network*

Example 16.39 *State whether the following functions are driving point immittance of LC networks are not:*

(a) $Z(s) = \dfrac{5s(s^2+4)}{(s^2+1)(s^2+3)}$ (b) $Z(s) = \dfrac{2(s^2+1)(s^2+9)}{s(s^2+2)}$

Solution

(a) $Z(s) = \dfrac{5s(s^2+4)}{(s^2+1)(s^2+3)}$

The function $Z(s)$ has poles at $s = \pm j1$ and $s = \pm j\sqrt{3}$ and zeros at $s = 0$ and $s = \pm j2$ as shown in Fig. 16.21. Since the poles and zeros do not interlace on the $j\omega$ axis, the function $Z(s)$ is not an *LC* impedance function.

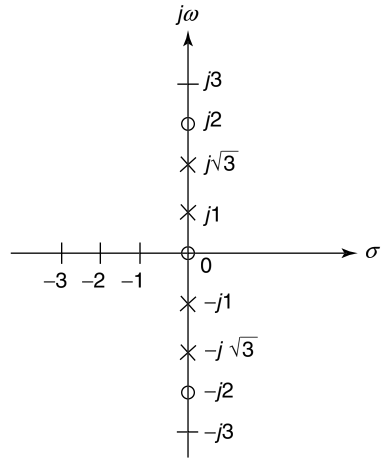


Fig. 16.21

(b) $Z(s) = \dfrac{2(s^2+1)(s^2+9)}{s(s^2+2)}$

The function $Z(s)$ has poles at $s = 0$ and $s = \pm j\sqrt{2}$ and zeros at $s = \pm j1$ and $s = \pm j3$ as shown in Fig. 16.22. The poles and zeros are simple and lie on the $j\omega$-axis. The poles and zeros interlace on the $j\omega$ axis. Hence, the function $Z(s)$ is an *LC* impedance function.

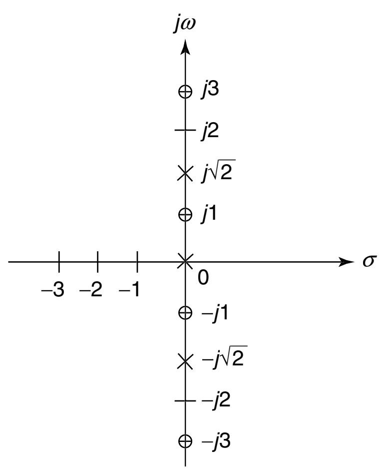


Fig. 16.22

Example 16.40 *Realise the Foster and Cauer forms of the following impedance function*

$$Z(s)=\frac{4(s^2+1)(s^2+9)}{s(s^2+4)}$$

Solution The function $Z(s)$ has poles at $s=0$ and $s=\pm j2$ and zeros at $s=\pm j1$ and $s=\pm j3$ as shown in Fig. 16.23.

From the pole-zero diagram, it is clear that poles and zeros are simple and lie on the $j\omega$ axis. Poles and zeros are interlaced. Hence, the given function is an LC function.

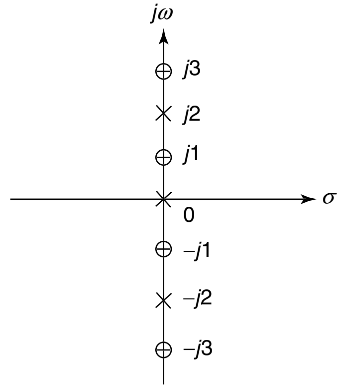


Fig. 16.23

Foster I Form The Foster I form is obtained by partial-fraction expansion of the impedance function $Z(s)$. But degree of numerator is greater than degree of denominator. Hence, division is first carried out.

$$Z(s)=\frac{4(s^2+1)(s^2+9)}{s(s^2+4)}=\frac{4s^4+40s^2+36}{s^3+4s}$$

$$s^3+4s\,)\,4s^4+40s^2+36\,(4s$$
$$\underline{4s^4+16s^2}$$
$$24s^2+36$$

$$Z(s)=4s+\frac{24s^2+36}{s^3+4s}=4s+\frac{24s^2+36}{s(s^2+4)}$$

By partial-fraction expansion,

$$Z(s)=4s+\frac{K_0}{s}+\frac{K_1}{s+j2}+\frac{K_1^*}{s-j2}=4s+\frac{K_0}{s}+\frac{2K_1s}{s^2+4}$$

where
$$K_0=sZ(s)\big|_{s=0}=\frac{4(1)(9)}{4}=9$$

$$K_1=\frac{(s^2+4)Z(s)}{2s}\bigg|_{s^2=-4}=\frac{4(-4+1)(-4+9)}{2(-4)}=\frac{15}{2}$$

$$Z(s)=4s+\frac{9}{s}+\frac{15s}{s^2+4}$$

The first term represents the impedance of an inductor of 4 H. The second term represents the impedance of a capacitor of $\frac{1}{9}$ F. The third term represents the impedance of a parallel LC network.

For a parallel LC network,

$$Z_{LC}(s)=\frac{\left(\frac{1}{C}\right)s}{s^2+\frac{1}{LC}}$$

By direct comparison,

$$C=\frac{1}{15}\text{F}$$

$$L = \frac{15}{4}\text{H}$$

The network is shown in Fig. 16.24.

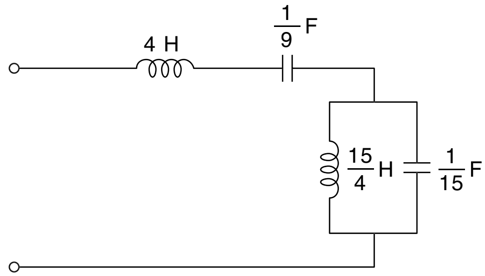


Fig. 16.24

Foster II Form The Foster II form is obtained by partial-fraction expansion of the admittance function $Y(s)$.

$$Y(s) = \frac{s(s^2+4)}{4(s^2+1)(s^2+9)}$$

By partial-fraction expansion,

$$Y(s) = \frac{K_1}{s+j1} + \frac{K_1^*}{s-j1} + \frac{K_2}{s+j3} + \frac{K_2^*}{s-j3} = \frac{2K_1 s}{s^2+1} + \frac{2K_2 s}{s^2+9}$$

where
$$K_1 = \frac{(s^2+1)}{2s}Y(s)\bigg|_{s^2=-1} = \frac{(-1+4)}{8(-1+9)} = \frac{3}{64}$$

$$K_2 = \frac{(s^2+9)}{2s}Y(s)\bigg|_{s^2=-9} = \frac{(-9+4)}{8(-9+1)} = \frac{5}{64}$$

$$Y(s) = \frac{\left(\frac{3}{32}\right)s}{s^2+1} + \frac{\left(\frac{5}{32}\right)s}{s^2+9}$$

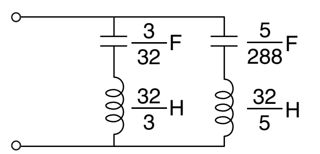
These two terms represent admittance of a series LC network. For a series LC network,

$$Y_{LC}(s) = \frac{\left(\frac{1}{L}\right)s}{s^2 + \frac{1}{LC}}$$

By direct comparison,

$$L_1 = \frac{32}{3}\text{H} \qquad C_1 = \frac{3}{32}\text{F}$$

$$L_2 = \frac{32}{5}\text{H} \qquad C_2 = \frac{5}{288}\text{F}$$



The network is shown in Fig. 16.25.

Fig. 16.25

Cauer I Form The Cauer I form is obtained from continued fraction expansion about the pole at infinity.

$$Z(s) = \frac{4s^4 + 40s^2 + 36}{s^3 + 4s}$$

Since the degree of the numerator is greater than the degree of the denominator by one, it indicates the presence of a pole at infinity.

By continued fraction expansion,

$$s^3+4s\overline{)\,4s^4+40s^2+36\,}\left(4s \leftarrow Z\right.$$
$$\underline{4s^4+16s^2}$$
$$24s^2+36\overline{)\,s^3+4s\,}\left(\frac{1}{24}s \leftarrow Y\right.$$
$$\underline{s^3+\frac{3}{2}s}$$
$$\frac{5}{2}s\overline{)\,24s^2+36\,}\left(\frac{48}{5}s \leftarrow Z\right.$$
$$\underline{24s^2}$$
$$36\overline{)\,\frac{5}{2}s\,}\left(\frac{5}{72}s \leftarrow Y\right.$$
$$\underline{\frac{5}{2}s}$$
$$0$$

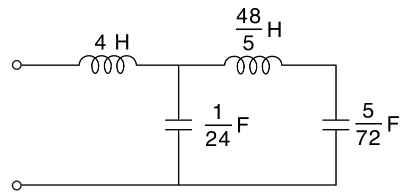


Fig. 16.26

The impedances are connected in the series branches whereas the admittances are connected in the parallel branches in a Cauer or ladder realisation. The network is shown in Fig. 16.26.

Cauer II Form The Cauer II form is obtained from partial-fraction expansion about pole at origin.

$$Z(s)=\frac{4(s^2+1)(s^2+9)}{s(s^2+4)}=\frac{4s^4+40s^2+36}{s^3+4s}$$

The function $Z(s)$ has a pole at origin. Arranging the numerator and denominator polynomials in ascending order of s,

$$Z(s)=\frac{36+40s^2+4s^4}{4s+s^3}$$

By continued fraction expansion,

$$4s+s^3\overline{)\,36+40s^2+4s^4\,}\left(\frac{9}{s} \leftarrow Z\right.$$
$$\underline{36+\ \ 9s^2}$$
$$31s^2+4s^4\overline{)\,4s+s^3\,}\left(\frac{4}{31s} \leftarrow Y\right.$$
$$\underline{4s+\frac{16}{31}s^3}$$
$$\frac{15}{31}s^3\overline{)\,31s^2+4s^4\,}\left(\frac{961}{15s} \leftarrow Z\right.$$
$$\underline{31s^2}$$

$$4s^4\,\Big)\,\frac{15}{31}s^3\,\Big(\,\frac{15}{124s} \leftarrow Y$$
$$\frac{\frac{15}{31}s^3}{0}$$

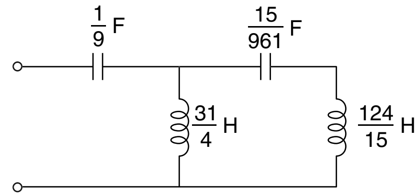


Fig. 16.27

The impedances are connected in the series branches whereas the admittances are connected in the parallel branches in a Cauer or ladder realisation. The network is shown in Fig. 16.27.

Example 16.41 *Realise Foster forms of the LC impedance function*

$$Z(s) = \frac{(s^2 + 1)(s^2 + 3)}{s(s^2 + 2)}$$

Solution

Foster I Form The Foster I form is obtained by partial-fraction expansion of the impedance function $Z(s)$. Since the degree of the numerator is greater than the degree of the denominator, division is first carried out.

$$Z(s) = \frac{(s^2+1)(s^2+3)}{s(s^2+2)} = \frac{s^4+4s^2+3}{s^3+2s}$$

$$s^3+2s\,\Big)\,s^4+4s^2+3\,\Big(\,s$$
$$\frac{s^4+2s^2}{2s^2+3}$$

$$Z(s) = s + \frac{2s^2+3}{s^3+2s} = s + \frac{2s^2+3}{s(s^2+2)}$$

By partial-fraction expansion,

$$Z(s) = s + \frac{K_0}{s} + \frac{K_1}{s+j2} + \frac{K_1^*}{s-j2} = s + \frac{K_0}{s} + \frac{2K_1 s}{s^2+2}$$

where

$$K_0 = sZ(s)|_{s=0} = \frac{(1)(3)}{2} = \frac{3}{2}$$

$$K_1 = \frac{(s^2+2)}{2s}Z(s)\bigg|_{s^2=-2} = \frac{(-2+1)(-2+3)}{2(-2)} = \frac{1}{4}$$

$$Z(s) = s + \frac{\left(\frac{3}{2}\right)}{s} + \frac{\left(\frac{1}{2}\right)s}{s^2+2}$$

The first term represents the impedance of an inductor of 1 H. The second term represents the impedance of a capacitor of $\frac{2}{3}$ F. The third term represents the impedance of a parallel LC network.

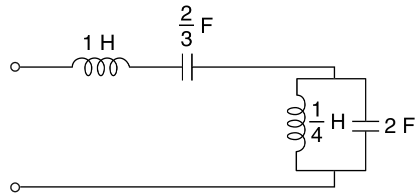
For a parallel *LC* network,

$$Z_{LC}(s) = \frac{\left(\frac{1}{C}\right)s}{s^2 + \frac{1}{LC}}$$

By direct comparison,

$$C = 2\text{F}$$

$$L = \frac{1}{4}\text{H}$$



The network is shown in Fig. 16.28.

Fig. 16.28

Foster II Form The Foster II form is obtained by partial-fraction expansion of the admittance function $Y(s)$.

$$Y(s) = \frac{s(s^2+2)}{(s^2+1)(s^2+3)}$$

By partial-fraction expansion,

$$Y(s) = \frac{K_1}{s+j1} + \frac{K_1^*}{s-j1} + \frac{K_2}{s+j\sqrt{3}} + \frac{K_2^*}{s-j\sqrt{3}} = \frac{2K_1 s}{s^2+1} + \frac{2K_2 s}{s^2+3}$$

where

$$K_1 = \frac{(s^2+1)}{2s}Y(s)\bigg|_{s^2=-1} = \frac{(-1+2)}{2(-1+3)} = \frac{1}{4}$$

$$K_2 = \frac{(s^2+3)}{2s}Y(s)\bigg|_{s^2=-3} = \frac{-3+2}{2(-3+1)} = \frac{1}{4}$$

$$Y(s) = \frac{\left(\frac{1}{2}\right)s}{s^2+1} + \frac{\left(\frac{1}{2}\right)s}{s^2+3}$$

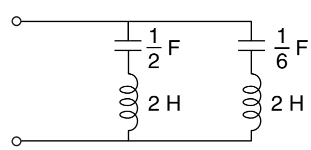
These two terms represent admittance of a series *LC* network. For a series *LC* network,

$$Y_{LC}(s) = \frac{\left(\frac{1}{L}\right)s}{s^2 + \frac{1}{LC}}$$

By direct comparison,

$$L_1 = 2\text{H}, \quad C_1 = \frac{1}{2}\text{F}$$

$$L_2 = 2\text{H}, \quad C_2 = \frac{1}{6}\text{F}$$



The network is shown in Fig. 16.29.

Fig. 16.29

Example 16.42 *Realise Foster forms of the following LC impedance function:*

$$Z(s) = \frac{(s^2+1)(s^2+3)}{s(s^2+2)(s^2+4)}$$

Solution

Foster I Form The Foster I form is obtained by partial-fraction expansion of the impedance function $Z(s)$.

By partial-fraction expansion,

$$Z(s) = \frac{K_0}{s} + \frac{K_1}{s + j\sqrt{2}} + \frac{K_1^*}{s - j\sqrt{2}} + \frac{K_2}{s + j2} + \frac{K_2^*}{s - j2} = \frac{K_0}{s} + \frac{2K_1 s}{s^2 + 2} + \frac{2K_2 s}{s^2 + 4}$$

where

$$K_0 = sZ(s)\Big|_{s=0} = \frac{(1)(3)}{(2)(4)} = \frac{3}{8}$$

$$K_1 = \frac{(s^2 + 2)}{2s} Z(s)\Bigg|_{s^2=-2} = \frac{(-2+1)(-2+3)}{2(-2)(-2+4)} = \frac{1}{8}$$

$$K_2 = \frac{(s^2 + 4)}{2s} Z(s)\Bigg|_{s^2=-4} = \frac{(-4+1)(-4+3)}{2(-4)(-4+2)} = \frac{3}{16}$$

$$Z(s) = \frac{\frac{3}{8}}{s} + \frac{\left(\frac{1}{4}\right)s}{s^2 + 2} + \frac{\left(\frac{3}{8}\right)s}{s^2 + 4}$$

The first term represents the impedance of a capacitor of $\frac{8}{3}$ F. The other two terms represent the impedance of a parallel LC network.

For a parallel LC network,

$$Z_{LC}(s) = \frac{\left(\frac{1}{C}\right)s}{s^2 + \frac{1}{LC}}$$

By direct comparison,

$$C_1 = 4\text{F}, \quad L_1 = \frac{1}{8}\text{H}$$

$$C_2 = \frac{8}{3}\text{F}, \quad L_2 = \frac{3}{32}\text{H}$$

The network is shown in Fig. 16.30.

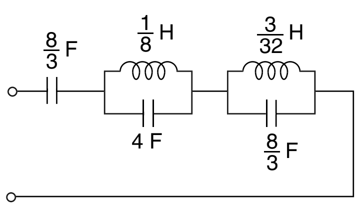


Fig. 16.30

Foster II Form The Foster II form is obtained by partial-fraction expansion of the admittance function $Y(s)$.

$$Y(s) = \frac{s(s^2 + 2)(s^2 + 4)}{(s^2 + 1)(s^2 + 3)} = \frac{s^5 + 6s^3 + 8s}{s^4 + 4s^2 + 3}$$

Since the degree of the numerator is greater than the degree of the denominator, division is first carried out.

$$\begin{array}{r} s^4 + 4s^2 + 3)\, s^5 + 6s^3 + 8s\,(s \\ \underline{s^5 + 4s^3 + 3s} \\ 2s^3 + 5s \end{array}$$

$$Y(s) = s + \frac{2s^3 + 5s}{s^4 + 4s^2 + 3} = s + \frac{2s^3 + 5s}{(s^2 + 1)(s^2 + 3)}$$

By partial-fraction expansion,

$$Y(s) = s + \frac{K_1}{s + j1} + \frac{K_1^*}{s - j1} + \frac{K_2}{s + j\sqrt{3}} + \frac{K_2^*}{s - j\sqrt{3}} = s + \frac{2K_1 s}{s^2 + 1} + \frac{2K_2 s}{s^2 + 3}$$

where

$$K_1 = \frac{(s^2+1)}{2s}Y(s)\bigg|_{s^2=-1} = \frac{(-1+2)(-1+4)}{2(-1+3)} = \frac{3}{4}$$

$$K_2 = \frac{(s^2+3)}{2s}Y(s)\bigg|_{s^2=-3} = \frac{(-3+2)(-3+4)}{2(-3+1)} = \frac{1}{4}$$

$$Y(s) = s + \frac{\left(\frac{3}{2}\right)s}{s^2+1} + \frac{\left(\frac{1}{2}\right)s}{s^2+3}$$

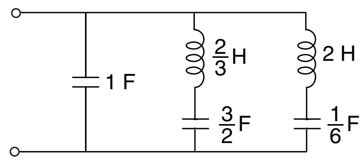
The first term represents the admittance of capacitor of 1 F. The other two terms represent admittance of a series *LC* network. For a series *LC* network,

$$Y_{LC}(s) = \frac{\left(\frac{1}{L}\right)s}{s^2 + \frac{1}{LC}}$$

By direct comparison,

$$L_1 = \frac{2}{3}\,\text{H}, \quad C_1 = \frac{3}{2}\,\text{F}$$

$$L_2 = 2\,\text{H}, \quad C_2 = \frac{1}{6}\,\text{F}$$



The network is shown in Fig. 16.31.

Fig. 16.31

Example 16.43 *Realise Cauer forms of the following LC impedance function:*

$$Z(s) = \frac{10s^4 + 12s^2 + 1}{2s^3 + 2s}$$

Solution

Cauer I Form The Cauer I form is obtained from continued fraction expansion about the pole at infinity.

$$Z(s) = \frac{10s^4 + 12s^2 + 1}{2s^3 + 2s}$$

Since the degree of the numerator is greater than the degree of the denominator by one, it indicates the presence of a pole at infinity.

By continued fraction expansion,

$$2s^3 + 2s)\,10s^4 + 12s^2 + 1\,(5s \leftarrow Z$$
$$\underline{10s^4 + 10s^2}$$
$$2s^2 + 1)\,2s^3 + 2s\,(s \leftarrow Y$$
$$\underline{2s^3 + s}$$
$$s)\,2s^2 + 1\,(2s \leftarrow Z$$
$$\underline{2s^2}$$
$$1)\,s\,(s \leftarrow Y$$
$$\underline{s}$$
$$0$$

The impedances are connected in the series branches whereas the admittances are connected in the parallel branches in a Cauer or ladder realisation. The network is shown in Fig. 16.32.

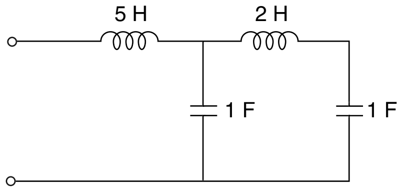


Fig. 16.32

Cauer II Form The Cauer II form is obtained from continued fraction expansion about the pole at the origin.

$$Z(s) = \frac{10s^4 + 12s^2 + 1}{2s^3 + 2s}$$

The function $Z(s)$ has a pole at the origin. Arranging the numerator and denominator polynomials in ascending order of s,

$$Z(s) = \frac{1 + 12s^2 + 10s^4}{2s + 2s^3}$$

By continued fraction expansion of $Z(s)$,

$$2s + 2s^3 \overline{)1 + 12s^2 + 10s^4} \left(\frac{1}{2s} \leftarrow Z \right.$$

$$\underline{1 + \quad s^2}$$

$$11s^2 + 10s^4 \overline{)2s + 2s^3} \left(\frac{2}{11s} \leftarrow Y \right.$$

$$\underline{2s + \frac{20}{11}s^3}$$

$$\frac{2}{11}s^3 \overline{)11s^2 + 10s^4} \left(\frac{121}{2s} \leftarrow Z \right.$$

$$\underline{11s^2}$$

$$10s^4 \overline{)\frac{2}{11}s^3} \left(\frac{2}{110s} \leftarrow Y \right.$$

$$\underline{\frac{2}{11}s^3}$$

$$0$$

The impedances are connected in the series branches whereas the admittances are connected in the parallel branches in Cauer or ladder realisation. The network is shown in Fig. 16.33.

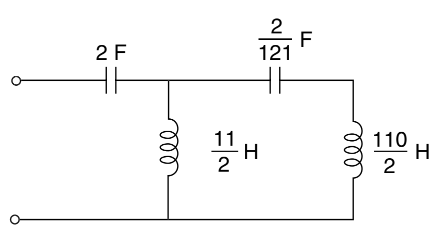


Fig. 16.33

Example 16.44 *Realise the following network function in Cauer I form:*

$$Z(s) = \frac{6s^4 + 42s^2 + 48}{s^5 + 18s^3 + 48s}$$

Solution The Cauer I form is obtained by continued fraction expansion of $Z(s)$ about the pole at infinity. In the above function, the degree of the numerator is less than the degree of the denominator which indicates the presence of a zero at infinity. The admittance function $Y(s)$ has a pole at infinity. Hence, the continued fraction expansion of $Y(s)$ is carried out.

$$Y(s) = \frac{s^5 + 18s^3 + 48s}{6s^4 + 42s^2 + 48}$$

By continued fraction expansion

$$6s^4 + 42s^2 + s^2 \,\big)\, s^5 + 18s^3 + 48s \,\Big(\, \frac{1}{6}s \leftarrow Y$$

$$\underline{s^5 + 7s^3 + 8s}$$

$$11s^3 + 40s \,\big)\, 6s^4 + 42s^2 + 48 \,\Big(\, \frac{6}{11}s \leftarrow Z$$

$$\underline{6s^4 + \frac{240}{11}s^2}$$

$$\frac{222}{11}s^2 + 30 \,\big)\, 11s^3 + 40s \,\Big(\, \frac{121}{222}s \leftarrow Y$$

$$\underline{11s^3 + \frac{5808}{222}s}$$

$$\frac{3072}{222}s \,\big)\, \frac{222}{11}s^2 + 48 \,\Big(\, \frac{49284}{33792}s \leftarrow Z$$

$$\underline{\frac{222}{11}s^2}$$

$$48 \,\big)\, \frac{3072}{222}s \,\Big(\, \frac{128}{444}s \leftarrow Y$$

$$\underline{\frac{3072}{222}s}$$

$$0$$

The impedances are connected in the series branches whereas the admittances are connected in the parallel branches in a Cauer or ladder realisation. The network is shown in Fig. 16.34.

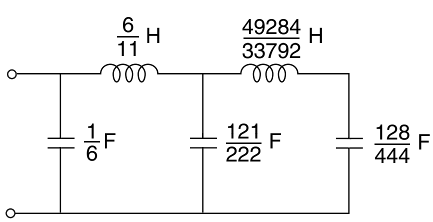


Fig. 16.34

Example 16.45 *Realise Cauer II form of the function:*

$$Z_{LC}(s) = \frac{s(s^4 + 3s^2 + 1)}{3s^4 + 4s^2 + 1}$$

Solution The Cauer II form is obtained by continued fraction expansion about the pole at the origin. The given function has a zero at the origin. The admittance function $Y(s)$ has a pole at origin. Hence, the continued fraction expansion of $Y(s)$ is carried out. Arranging the polynomials in ascending order of s,

$$Y_{LC}(s) = \frac{3s^4 + 4s^2 + 1}{s^5 + 3s^3 + s} = \frac{1 + 4s^2 + 3s^4}{s + 3s^3 + s^5}$$

By continued fraction expansion of $Y(s)$, we have

$$s + 3s^3 + s^5 \overline{\big)\, 1 + 4s^2 + 3s^4} \left(\frac{1}{s} \leftarrow Y \right.$$

$$\underline{1 + 3s^2 + \; s^4}$$

$$s^2 + 2s^4 \overline{\big)\, s + 3s^3 + s^5} \left(\frac{1}{s} \leftarrow Z \right.$$

$$\underline{s + 2s^3}$$

$$s^3 + s^5 \overline{\big)\, s^2 + 2s^4} \left(\frac{1}{s} \leftarrow Y \right.$$

$$\underline{s^2 + \; s^4}$$

$$s^4 \overline{\big)\, s^3 + s^5} \left(\frac{1}{s} \leftarrow Z \right.$$

$$\underline{s^3}$$

$$s^5 \overline{\big)\, s^4} \left(\frac{1}{s} \leftarrow Y \right.$$

$$\underline{s^4}$$

$$0$$

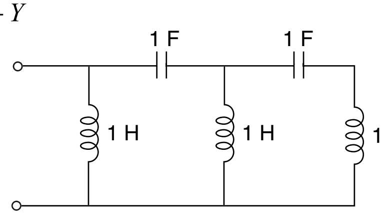


Fig. 16.35

The impedances are connected in the series branches whereas the admittances are connected in the parallel branches in a Cauer or ladder realisation. The network is shown in Fig. 16.35.

Example 16.46 *Obtain the Cauer I form of realisation for the function*

$$Z_{LC}(s) = \frac{s^5 + 7s^3 + 10s}{s^4 + 5s^2 + 4}$$

Solution The Cauer I form is obtained by continued fraction expansion of $Z_{LC}(s)$ about pole at infinity.

$$
\begin{array}{l}
s^4+5s^2+4\,\big)\,s^5+7s^3+10s\,\big(\,s \leftarrow Z \\
\qquad\qquad\quad \underline{s^5+5s^3+\;\;4s} \\
\qquad\qquad\qquad 2s^3+6s\,\big)\,s^4+5s^2+4\,\big(\,\tfrac{1}{2}s \leftarrow Y \\
\qquad\qquad\qquad\qquad\quad \underline{s^4+3s^2} \\
\qquad\qquad\qquad\qquad\qquad 2s^2+4\,\big)\,2s^3+6s\,\big(\,s \leftarrow Z \\
\qquad\qquad\qquad\qquad\qquad\qquad\quad \underline{2s^3+4s} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad 2s\,\big)\,2s^2+4\,\big(\,s \leftarrow Y \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \underline{2s^2} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 4\,\big)\,2s\,\big(\,\tfrac{1}{2}s \leftarrow Z \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \underline{2s} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\;\; 0
\end{array}
$$

The impedances are connected in the series branches, whereas the admittances are connected in the parallel branches in a Cauer or ladder realisation. The network is shown in Fig. 16.36.

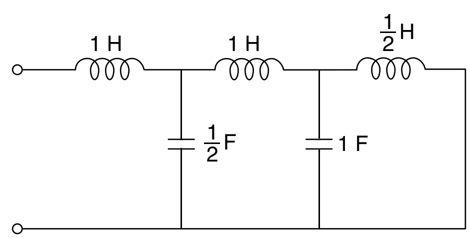


Fig. 16.36

Example 16.47 *Synthesize the following LC impedance function in Cauer II form:*

$$Z(s) = \frac{s^3+2s}{s^4+4s^2+3}$$

Solution The Cauer II form is obtained by continued fraction expansion about the pole at the origin.

$$Z(s) = \frac{s(s^2+2)}{(s^2+3)(s^2+1)}$$

But $Z(s)$ has a zero at the origin. Hence, the continued fraction expansion of $Y(s)$ is carried out. Arranging the polynomials in ascending order of s,

$$Y(s) = \frac{3+4s^2+s^4}{2s+s^3}$$

By continued fraction expansion of $Y(s)$,

$$2s+s^3 \overline{\big)\, 3+4s^2+s^4 \,\Big(\, \frac{3}{2s} \leftarrow Y}$$

$$\underline{3+\frac{3}{2}s^2}$$

$$\frac{5}{2}s^2+s^4 \overline{\big)\, 2s+s^3 \,\Big(\, \frac{4}{5s} \leftarrow Z}$$

$$\underline{2s+\frac{4}{5}s^3}$$

$$\frac{s^3}{5} \overline{\big)\, \frac{5}{2}s^2+s^4 \,\Big(\, \frac{25}{2s} \leftarrow Y}$$

$$\underline{\frac{5}{2}s^2}$$

$$s^4 \overline{\big)\, \frac{1}{5}s^3 \,\Big(\, \frac{1}{5s} \leftarrow Z}$$

$$\underline{\frac{1}{5}s^3}$$

$$0$$

The impedances are connected in the series branches whereas the admittances are connected in the parallel branches in a Cauer or ladder realisation. The network is shown in Fig. 16.37.

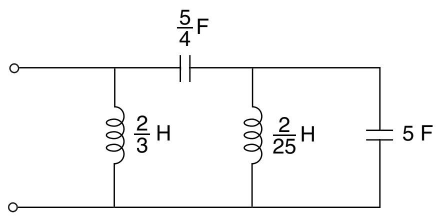


Fig. 16.37

16.6 REALISATION OF *RC* FUNCTIONS

RC driving point immittance functions have following properties:

1. The poles and zeros are simple and are located on the negative real axis of the *s* plane.
2. The poles and zeros are interlaced.
3. The lowest critical frequency nearest to the origin is a pole.
4. The highest critical frequency farthest to the origin is a zero.
5. Residues evaluated at the poles of $Z_{RC}(s)$ are real and positive.
6. The slope $\frac{d}{d\sigma}Z_{RC}$ is negative.
7. $Z_{RC}(\infty) < Z_{RC}(0)$.

RC functions can also be realised in four different ways. The impedance function of RC networks is given by,

$$Z(s) = \frac{H(s+\sigma_1)(s+\sigma_3)...}{s(s+\sigma_2)...}$$

16.6.1 Foster Realisation

Foster I Form The Foster I form is obtained by partial-fraction expansion of $Z(s)$.

$$Z(s) = \frac{K_0}{s} + \frac{K_1}{s+\sigma_1} + \frac{K_2}{s+\sigma_2} + ... + K_\infty$$

where $K_0, K_1, K_2, \dots K_\infty$ are residues of $Z(s)$.

$$K_o = sZ(s)\big|_{s=0}$$
$$K_i = (s+\sigma_i)Z(s)\big| s = -\sigma_i$$
$$K_\infty = \frac{Z(s)}{s}\bigg|_{s\to\infty}$$

The first term $\frac{K_0}{s}$ represents the impedance of a capacitor of $\frac{1}{K_o}$ farads.

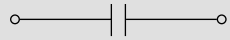
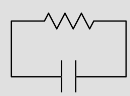
The last term K_∞ represents the impedance of a resistor of K_∞ ohms.

The remaining terms, i.e., $\frac{K_i}{s+\sigma_i}$ represent the impedance of the parallel combination of resistor R_i and capacitor C_i. For parallel combinaton of R_i and C_i,

$$Z(s) = \frac{R_i\left(\frac{1}{C_i s}\right)}{R_i + \frac{1}{C_i s}} = \frac{K_i}{s+\sigma_i}$$

$$R_i = \frac{K_i}{\sigma_i} \quad \text{and} \quad C_i = \frac{1}{K_i}$$

Table 16.3 *Realisation of Foster-I form of RC network*

Impedance function	Element
$\frac{K_0}{s} = \frac{1}{C_0 s}$	 $C_0 = \frac{1}{K_0}$
$\frac{K_i}{s+\sigma_i} = \frac{(R_i)\left(\frac{1}{C_i s}\right)}{R_i + \frac{1}{C_i s}}$	$Ri = \frac{K_i}{\sigma_i}$  $C_i = \frac{1}{K_i}$
$K_\infty = R_\infty$	 $R_\infty = K_\infty$

The network corresponding to the Foster-I form is shown in Fig. 16.38.

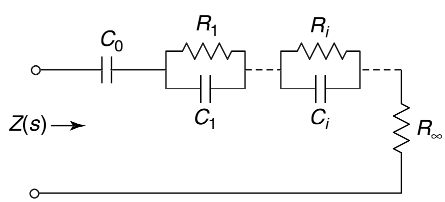


Fig. 16.38 *Foster-I form of RC network*

Foster II Form The Foster-II form is obtained by partial fraction expansion of $Y(s)$. Since $Y(s) = \frac{1}{Z(s)}$ has negative residue at its pole, Foster II form is obtained by expanding $\frac{Y(s)}{s}$.

$$\frac{Y(s)}{s} = \frac{K_o}{s} + \sum_{i=1}^{n} \frac{K_i}{(s+\sigma_i)} + K_\infty$$

Multiplying this equation by s,

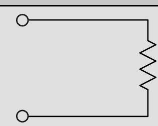
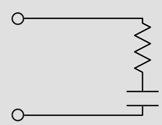
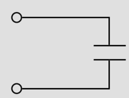
$$Y(s) = K_o + \sum_{i=1}^{n} \frac{K_i s}{(s+\sigma_i)} + K_\infty s$$

The first term K_o represents the conductance of a resistor of $\frac{1}{K_o}$ ohms.

The last term $K_\infty s$ represents the admittance of a capacitor of K_∞ farads.

The remaining terms, i.e., $\frac{K_i s}{s+\sigma_i}$ represent the admittance of series combination of resistor R_i and capacitor C_i with $R_i = \frac{1}{K_i}$ ohms and $C_i = \frac{K_i}{\sigma_i}$ farads.

Table 16.4 *Realisation of Foster II form of RC network*

Admittance function	Element
$K_0 = \frac{1}{R_0}$	 $R_o = \frac{1}{K_o}$
$\frac{K_i}{s+\omega_i} = \frac{\left(\frac{1}{R_i}\right)s}{s+\frac{1}{R_i C_i}}$	 $R_i = \frac{1}{K_i}$ $C_i = \frac{K_i}{\sigma_i}$
$K_\infty s = C_\infty s$	 $C_\infty = K_\infty$

The network corresponding to the Foster II form is shown in Fig. 16.39.

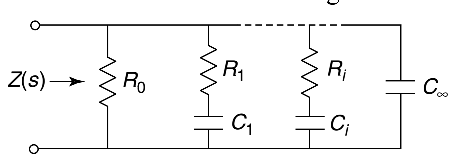


Fig. 16.39 *Foster II form of RC network*

16.6.2 Cauer Realisation

Cauer I Form The Cauer I form is obtained by removal of the pole from the impedance function $Z(s)$ at $s = \infty$. This is the same as a continued fraction expansion of the impedance function about infinity. The impedance $Z(s)$ can be written as a continued fraction expansion.

$$Z(s) = R_1 + \cfrac{1}{C_2 s + \cfrac{1}{R_3 + \cfrac{1}{C_4 s + \ldots}}}$$

The network is shown in Fig. 16.40.

In the network shown in Fig. 16.40, if $Z(s)$ has a zero at $s = \infty$, the first element is the capacitor C_1. If $Z(s)$ is a constant at $s = \infty$, the first element is R_1. If $Z(s)$ has a pole at $s = 0$, the last element is C_n. If $Z(s)$ is a constant at $s = 0$, the last element is R_n.

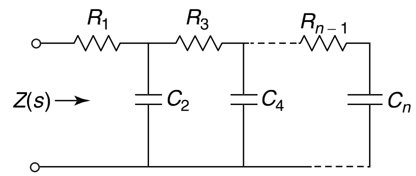


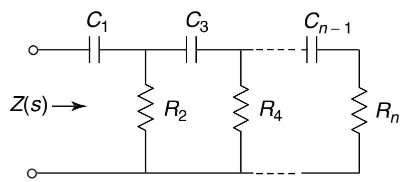
Fig. 16.40 *Cauer-I form of RC network*

Cauer II Form The Cauer II form is obtained by removal of the pole from the impedance function at the origin. This is the same as a continued fraction expansion of an impedance function about the origin.

If the given impedance function has a pole at the origin, it is removed as a capacitor C_1. The reciprocal of the remainder function has a minimum value at $s = 0$ which is removed as a constant of resistor R_2. If the original impedance has no pole at the origin, then the first capacitor is absent and the process is repeated with the removal of the constant corresponding to the resistor R_2.

The impedance $Z(s)$ can be written as a continued fraction expansion.

$$Z(s) = \frac{1}{C_1 s} + \cfrac{1}{\cfrac{1}{R_2} + \cfrac{1}{\cfrac{1}{C_3 s} + \cfrac{1}{\cfrac{1}{R_4} + \dots}}}$$



The network is shown in Fig. 16.41.

Fig. 16.41 *Caver-II form of RC network*

In the network shown in Fig. 16.41, if $Z(s)$ has a pole at $s = 0$, the first element is C_1. If $Z(s)$ is a constant at $s = 0$, the first element is R_2. If $Z(s)$ has a zero at $s = \infty$, the last element is C_n. If $Z(s)$ is constant at $s = \infty$, the last element is R_n.

Example 16.48 *Determine whether following functions are RC impedance function or not.*

(a) $Z(s) = \dfrac{3(s+2)(s+4)}{s(s+3)}$ (b) $\dfrac{2(s+1)(s+3)}{(s+2)(s+6)}$

Solution

(a) $Z(s) = \dfrac{3(s+2)(s+4)}{s(s+3)}$

The function Z(s) has poles at $s = 0$ and $s = -3$ and zeros at $s = -2$ and $s = -4$ as shown in Fig. 16.42. The poles and zeros are simple and located on the negative real axis of the s plane. The poles and zeros are interlaced. The lowest critical frequency nearest to the origin is a pole. Hence, the function $Z(s)$ is an RC impedance function.

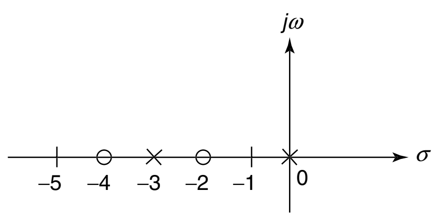


Fig. 16.42

(b) $Z(s) = \dfrac{2(s+1)(s+3)}{(s+2)(s+6)}$

The function $Z(s)$ has poles at $s = -2$ and $s = -6$ and zeros at $s = -1$ and $s = -3$ as shown in Fig. 16.43. The poles and zeros are simple and located on the negative real axis of the s-plane. The poles and zeros are interlaced. But the lowest critical frequency nearest to the origin is not a pole, but zero. Hence, the function $Z(s)$ is not an RC impedance function.

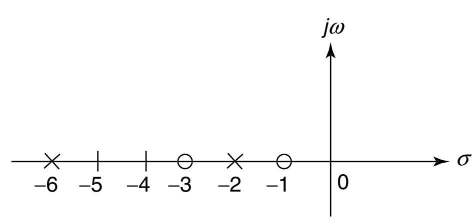


Fig. 16.43

Example 16.49 *Realise the Foster and Cauer forms of the impedance function*

$$Z(s)=\frac{(s+1)(s+3)}{s(s+2)}$$

Solution The function $Z(s)$ has poles at $s = 0$ and $s = -2$ and zeros at $s = -1$ and $s = -3$ as shown in Fig. 16.44.

From the pole-zero diagram, it is clear that poles and zeros are simple and lie on the negative real axis. The poles and zeros are interlaced and the lowest critical frequency nearest to the origin is a pole. Hence, the function $Z(s)$ is an RC function.

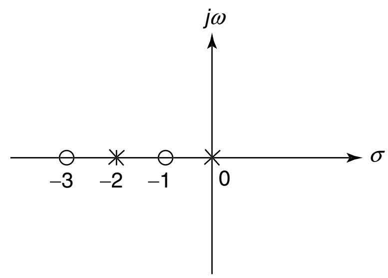


Fig. 16.44

Foster I Form The Foster I form is obtained by partial fraction expansion of impedance function $Z(s)$. Since the degree of the numerator is greater than the degree of the denominator, division is first carried out.

$$Z(s)=\frac{s^2+4s+3}{s^2+2s}$$

$$s^2+2s\overline{\big)\,s^2+4s+3\,\big(}\,1$$
$$\frac{s^2+2s}{\quad 2s+3}$$

$$Z(s)=1+\frac{2s+3}{s^2+2s}=1+\frac{2s+3}{s(s+2)}$$

By partial-fraction expansion,

$$Z(s)=1+\frac{K_1}{s}+\frac{K_2}{s+2}$$

where

$$K_1=sZ(s)\big|_{s=0}=\frac{(1)(3)}{2}=\frac{3}{2}$$

$$K_2=(s+2)Z(s)\big|_{s=-2}=\frac{(-2+1)(-2+3)}{-2}=\frac{1}{2}$$

$$Z(s)=1+\frac{\frac{3}{2}}{s}+\frac{\frac{1}{2}}{s+2}$$

The first term represents the impedance of a resistor of 1 Ω. The second term represents the impedance of a capacitor of $\frac{2}{3}$ F. The third term represents the impedance of parallel RC circuit for which

$$Z_{RC}(s)=\frac{\frac{1}{C_i}}{s+\frac{1}{R_iC_i}}$$

By direct comparison,

$$R=\frac{1}{4}\,\Omega$$
$$C=2\text{ F}$$

The network is shown in Fig. 16.45.

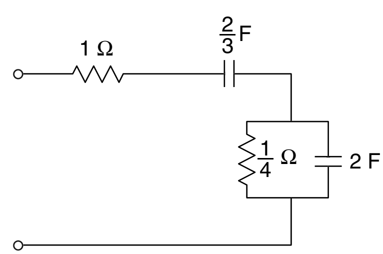


Fig. 16.45

Foster II Form The Foster II form is obtained by the partial-fraction expansion of admittance function $\frac{Y(s)}{s}$.

$$Y(s)=\frac{1}{Z(s)}=\frac{s(s+2)}{(s+1)(s+3)}$$

$$\frac{Y(s)}{s}=\frac{s+2}{(s+1)(s+3)}$$

By partial-fraction expansion,

$$\frac{Y(s)}{s}=\frac{K_1}{s+1}+\frac{K_2}{s+3}$$

where

$$K_1=(s+1)\frac{Y(s)}{s}\bigg|_{s=-1}=\frac{(-1+2)}{(-1+3)}=\frac{1}{2}$$

$$K_2=(s+3)\frac{Y(s)}{s}\bigg|_{s=-3}=\frac{(-3+2)}{(-3+1)}=\frac{1}{2}$$

$$\frac{Y(s)}{s}=\frac{\frac{1}{2}}{s+1}+\frac{\frac{1}{2}}{s+3}$$

$$Y(s)=\frac{\frac{1}{2}s}{s+1}+\frac{\frac{1}{2}s}{s+3}$$

These two terms represent the admittance of a series RC circuit. For a series RC circuit.

$$Y_{RC}(s)=\frac{\left(\frac{1}{R_i}\right)s}{s+\frac{1}{R_iC_i}}$$

By direct comparison,

$$R_1=2\ \Omega,\quad C_1=\frac{1}{2}\text{ F}$$

$$R_2=2\ \Omega,\quad C_2=\frac{1}{6}\text{ F}$$

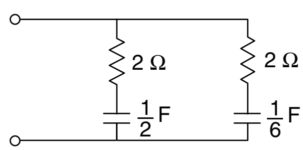


Fig. 16.46

The network is shown in Fig. 16.46.

Cauer I Form The Cauer I form is obtained by continued fraction expansion about the pole at infinity.

$$Z(s)=\frac{s^2+4s+3}{s^2+2s}$$

By continued fraction expansion,

$$s^2+2s\overline{)\,s^2+4s+3\,}\left(1\leftarrow Z\right.$$

$$\underline{s^2+2s}$$

$$2s+3\overline{)\,s^2+2s\,}\left(\frac{1}{2}s\leftarrow Y\right.$$

$$\begin{array}{l}
\quad s^2 + \frac{3}{2}s \\
\hline
\quad\quad \frac{1}{2}s\,\Big)\,2s + 3\,\Big(\,4 \leftarrow Z \\
\quad\quad\quad 2s \\
\hline
\quad\quad\quad\quad 3\,\Big)\,\frac{1}{2}s\,\Big(\,\frac{1}{6}s \leftarrow Y \\
\quad\quad\quad\quad\quad \frac{1}{2}s \\
\hline
\quad\quad\quad\quad\quad 0
\end{array}$$

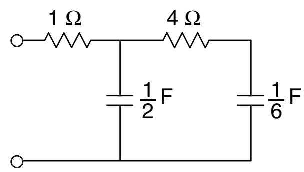


Fig. 16.47

The impedances are connected in the series branches whereas admittances are connected in the parallel branches. The network is shown in Fig. 16.47.

Cauer II Form The Cauer II form is obtained from continued fraction expansion about the pole at the origin. Arranging the numerator and denominator polynomials of $Z(s)$ in ascending order of s,

$$Z(s) = \frac{3 + 4s + s^2}{2s + s^2}$$

By continued fraction expansion,

$$\begin{array}{l}
2s + s^2\,\Big)\,3 + 4s + s^2\,\Big(\,\frac{3}{2s} \leftarrow Z \\
\quad 3 + \frac{3}{2}s \\
\hline
\quad\quad \frac{5}{2}s + s^2\,\Big)\,2s + s^2\,\Big(\,\frac{4}{5} \leftarrow Y \\
\quad\quad\quad 2s + \frac{4}{5}s^2 \\
\hline
\quad\quad\quad\quad \frac{1}{5}s^2\,\Big)\,\frac{5}{2}s + s^2\,\Big(\,\frac{25}{2s} \leftarrow Z \\
\quad\quad\quad\quad\quad \frac{5}{2}s \\
\hline
\quad\quad\quad\quad\quad\quad s^2\,\Big)\,\frac{1}{5}s^2\,\Big(\,\frac{1}{5} \leftarrow Y \\
\quad\quad\quad\quad\quad\quad\quad \frac{1}{5}s^2 \\
\hline
\quad\quad\quad\quad\quad\quad\quad 0
\end{array}$$

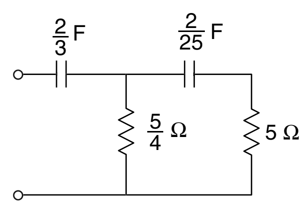


Fig. 16.48

The impedances are connected in the series branches whereas admittances are connected in the parallel branches. The network is shown in Fig. 16.48.

Example 16.50 *Determine the Foster form of realisation of the RC impedance function.*

$$Z(s)=\frac{(s+1)(s+3)}{s(s+2)(s+4)}$$

Solution

Foster I Form The Foster I form is obtained by the partial-fraction expansion of the impedance function $Z(s)$.

By partial-fraction expansion,

$$Z(s)=\frac{K_0}{s}+\frac{K_1}{s+2}+\frac{K_2}{s+4}$$

where

$$K_0=sZ(s)\big|_{s=0}=\frac{(1)(3)}{(2)(4)}=\frac{3}{8}$$

$$K_1=(s+2)Z(s)\big|_{s=-2}=\frac{(-2+1)(-2+3)}{(-2)(-2+4)}=\frac{(-1)(1)}{(-2)(2)}=\frac{1}{4}$$

$$K_2=(s+4)Z(s)\big|_{s=-4}=\frac{(-4+1)(-4+3)}{(-4)(-4+2)}=\frac{(-3)(-1)}{(-4)(-2)}=\frac{3}{8}$$

$$Z(s)=\frac{\frac{3}{8}}{s}+\frac{\frac{1}{4}}{s+2}+\frac{\frac{3}{8}}{s+4}$$

The first term represents the impedance of a capacitor of $\frac{8}{3}$ F. The remaining terms represent the impedance of a parallel *RC* circuit for which

$$Z_{RC}(s)=\frac{\frac{1}{C_i}}{s+\frac{1}{R_iC_i}}$$

By direct comparison,

$$R_1=\frac{1}{8}\,\Omega,\qquad C_1=4\text{ F}$$

$$R_2=\frac{3}{32}\,\Omega,\quad C_2=\frac{8}{3}\text{ F}$$

The network is shown in Fig. 16.49.

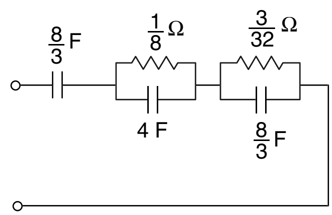


Fig. 16.49

Foster II Form The Foster II form is obtained by partial-fraction expansion of admittance function $\frac{Y(s)}{s}$.

$$Y(s)=\frac{s(s+2)(s+4)}{(s+1)(s+3)}$$

$$\frac{Y(s)}{s}=\frac{(s+2)(s+4)}{(s+1)(s+3)}=\frac{s^2+6s+8}{s^2+4s+3}$$

Since the degree of the numerator is equal to the degree of the denominator, division is carried out first.

$$s^2+4s+3 \overline{\smash{)}\, s^2+6s+8} \left(1\right.$$
$$\underline{s^2+4s+3}$$
$$2s+5$$

$$\frac{Y(s)}{s} = 1 + \frac{2s+5}{s^2+4s+3} = 1 + \frac{2s+5}{(s+1)(s+3)}$$

By partial-fraction expansion,

$$\frac{Y(s)}{s} = 1 + \frac{K_1}{s+1} + \frac{K_2}{s+3}$$

where

$$K_1 = (s+1)\frac{Y(s)}{s}\bigg|_{s=-1} = \frac{(-1+2)(-1+4)}{(-1+3)} = \frac{(1)(3)}{2} = \frac{3}{2}$$

$$K_2 = (s+3)\frac{Y(s)}{s}\bigg|_{s=-3} = \frac{(-3+2)(-3+4)}{(-3+1)} = \frac{(-1)(1)}{(-2)} = \frac{1}{2}$$

$$\frac{Y(s)}{s} = 1 + \frac{\frac{3}{2}}{s+1} + \frac{\frac{1}{2}}{s+3}$$

$$Y(s) = s + \frac{\frac{3}{2}s}{s+1} + \frac{\frac{1}{2}s}{s+3}$$

The first term represents the admittance of a capacitor of 1 F. The other two terms represent the admittance of a series *RC* circuit. For a series *RC* circuit,

$$Y_{RC}(s) = \frac{\left(\frac{1}{R_i}\right)s}{s + \frac{1}{R_iC_i}}$$

By direct comparison,

$$R_1 = \frac{2}{3}\,\Omega, \quad C_1 = \frac{3}{2}\text{ F}$$

$$R_2 = 2\,\Omega, \quad C_2 = \frac{1}{6}\text{ F}$$

The network is shown in Fig. 16.50.

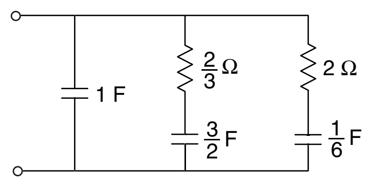


Fig. 16.50

Example 16.51 *Realise Foster forms of the following RC impedance function*

$$Z(s) = \frac{2(s+2)(s+4)}{(s+1)(s+3)}$$

Solution

Foster I Form The Foster I form is obtained by the partial-fraction expansion of the impedance function $Z(s)$. Since the degree of the numerator is equal to the degree of the denominator, division is carried out first.

$$Z(s) = \frac{2s^2 + 12s + 16}{s^2 + 4s + 3}$$

$$s^2 + 4s + 3\overline{)\,2s^2 + 12s + 16\,}\Big(\,2$$
$$\underline{2s^2 + 8s + 6}$$
$$4s + 10$$

$$Z(s) = 2 + \frac{4s + 10}{s^2 + 4s + 3} = 2 + \frac{4s + 10}{(s+1)(s+3)}$$

By partial-fraction expansion,

$$Z(s) = 2 + \frac{K_1}{s+1} + \frac{K_2}{s+3}$$

where
$$K_1 = (s+1)Z(s)\big|_{s=-1} = \frac{2(-1+2)(-1+4)}{(-1+3)} = 3$$

$$K_2 = (s+3)Z(s)\big|_{s=-3} = \frac{2(-3+2)(-3+4)}{(-3+1)} = 1$$

$$Z(s) = 2 + \frac{3}{s+1} + \frac{1}{s+3}$$

The first term represents the impedance of a resistor of 2 Ω. The remaining terms represent the impedance of a parallel RC circuit for which

$$Z_{RC}(s) = \frac{\dfrac{1}{C_i}}{s + \dfrac{1}{R_i C_i}}$$

By direct comparison,

$$R_1 = 3\,\Omega, \quad C_1 = \frac{1}{3}\,\text{F}$$

$$R_2 = \frac{1}{3}\,\Omega, \quad C_2 = 1\,\text{F}$$

The network is shown in Fig. 16.51.

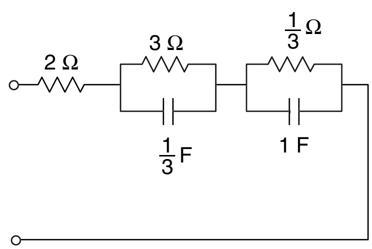


Fig. 16.51

Foster II Form The Foster II form is obtained by partial-fraction expansion of admittance function $\dfrac{Y(s)}{s}$.

$$Y(s) = \frac{(s+1)(s+3)}{2(s+2)(s+4)}$$

$$\frac{Y(s)}{s} = \frac{(s+1)(s+3)}{2s(s+2)(s+4)}$$

By partial-fraction expansion,

$$\frac{Y(s)}{s} = \frac{K_0}{s} + \frac{K_1}{s+2} + \frac{K_2}{s+4}$$

where
$$K_0 = s\frac{Y(s)}{s}\bigg|_{s=0} = \frac{(1)(3)}{(2)(2)(4)} = \frac{3}{16}$$

$$K_1 = (s+2)\frac{Y(s)}{s}\bigg|_{s=-2} = \frac{(-2+1)(-2+3)}{2(-2)(-2+4)} = \frac{(-1)(1)}{2(-2)(2)} = \frac{1}{8}$$

$$K_2 = (s+4)\frac{Y(s)}{s}\bigg|_{s=-4} = \frac{(-4+1)(-4+3)}{2(-4)(-4+2)} = \frac{(-3)(-1)}{2(-4)(-2)} = \frac{3}{16}$$

$$\frac{Y(s)}{s} = \frac{\frac{3}{16}}{s} + \frac{\frac{1}{8}}{s+2} + \frac{\frac{3}{16}}{s+4}$$

$$Y(s) = \frac{3}{16} + \frac{\frac{1}{8}s}{s+2} + \frac{\frac{3}{16}s}{s+4}$$

The first term represents the admittance of a resistor of $\frac{16}{3}$ Ω. The other two terms represent the admittance of a series *RC* circuit. For a series *RC* circuit.

$$Y_{RC}(s) = \frac{\left(\frac{1}{R_i}\right)s}{s + \frac{1}{R_i C_i}}$$

By direct comparison,

$$R_1 = 8\ \Omega, \qquad C_1 = \frac{1}{16}\text{ F}$$

$$R_2 = \frac{16}{3}\ \Omega, \quad C_2 = \frac{3}{64}\text{ F}$$

The network is shown in Fig. 16.52.

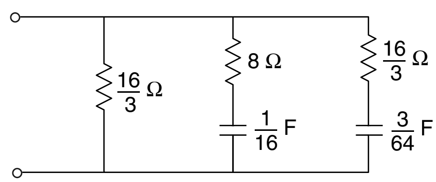


Fig. 16.52

Example 16.52 *Obtain the Cauer forms of the RC impedance function*

$$Z(s) = \frac{(s+2)(s+6)}{2(s+1)(s+3)}$$

Solution

Cauer I Form The Cauer I form is obtained by continued fraction expansion about the pole at infinity.

$$Z(s) = \frac{(s+2)(s+6)}{2(s+1)(s+3)} = \frac{s^2+8s+12}{2s^2+8s+6}$$

By continued fraction expansion,

$$2s^2+8s+6\,\big)\,s^2+8s+12\,\big(\,\frac{1}{2} \leftarrow Z$$
$$\underline{s^2+4s+3}$$
$$4s+9\,\big)\,2s^2+8s+6\,\big(\,\frac{1}{2}s \leftarrow Y$$
$$\underline{2s^2+\frac{9}{2}s}$$
$$\frac{7}{2}s+6\,\big)\,4s+9\,\big(\,\frac{8}{7} \leftarrow Z$$
$$\underline{4s+\frac{48}{7}}$$

$$
\begin{array}{l}
\dfrac{15}{7}\Big)\,\dfrac{7}{2}s+6\,\Big(\dfrac{49}{30}s \leftarrow Y \\
\qquad \underline{\dfrac{7}{2}s\qquad\quad} \\
\qquad\qquad 6\Big)\,\dfrac{15}{7}\,\Big(\dfrac{5}{14} \leftarrow Z \\
\qquad\qquad\quad \underline{\dfrac{15}{7}} \\
\qquad\qquad\quad\;\; 0
\end{array}
$$

The impedances are connected in the series branches whereas the admittances are connected in the parallel branches. The network is shown in Fig. 16.53.

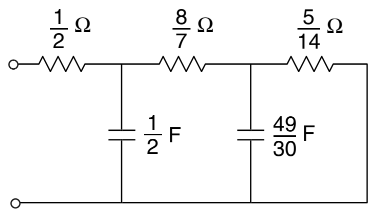


Fig. 16.53

Cauer II Form The Cauer II form is obtained by continued fraction expansion about the pole at the origin. Arranging the polynomials in ascending order of s,

$$Z(s)=\frac{12+8s+s^2}{6+8s+2s^2}$$

By continued fraction expansion,

$$
\begin{array}{l}
6+8s+2s^2\Big)\,12+8s+s^2\,\Big(\,2 \\
\qquad\qquad\quad \underline{12+16s+4s^2} \\
\qquad\qquad\qquad -8s-3s^2
\end{array}
$$

Since negative term results, continued fraction expansion of $Y(s)$ is carried out.

$$Y(s)=\frac{6+8s+2s^2}{12+8s+s^2}$$

By continued fraction expansion,

$$
\begin{array}{l}
12+8s+s^2\Big)\,6+8s+2s^2\,\Big(\dfrac{1}{2} \leftarrow Y \\
\qquad \underline{6+4s+\dfrac{1}{2}s^2} \\
\qquad\quad 4s+\dfrac{3}{2}s^2\Big)\,12+8s+s^2\,\Big(\dfrac{3}{s} \leftarrow Z \\
\qquad\qquad\qquad \underline{12+\dfrac{9}{2}s} \\
\qquad\qquad\qquad\quad \dfrac{7}{2}s+s^2\Big)\,4s+\dfrac{3}{2}+s^2\,\Big(\dfrac{8}{7} \leftarrow Y \\
\qquad\qquad\qquad\qquad\qquad \underline{4s+\dfrac{8}{7}s^2} \\
\qquad\qquad\qquad\qquad\qquad\quad \dfrac{5}{14}s^2\Big)\,\dfrac{7}{2}s+s^2\,\Big(\dfrac{49}{5s} \leftarrow Z
\end{array}
$$

$$\begin{array}{l} \quad\quad \underline{\frac{7}{2}s} \\ \quad\quad\quad s^2\Big)\frac{5}{14}s^2\Big(\frac{5}{14} \leftarrow Y \\ \quad\quad\quad\quad \underline{\frac{5}{14}s} \\ \quad\quad\quad\quad\quad 0 \end{array}$$

The impedances are connected in the series branches, whereas the admittances are connected in the parallel branches. The network is shown in Fig. 16.54.

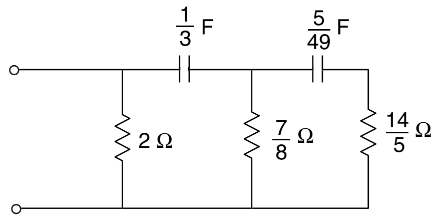


Fig. 16.54

Example 16.53 *Realise the RC impedance in Cauer I and Foster I forms*

$$Z(s) = \frac{s+4}{(s+2)(s+6)}$$

Solution

Cauer I Form The Cauer I form is obtained by continued fraction expansion of $Z(s)$ about the pole at infinity. In the above function, the degree of the numerator is less than the degree of the denominator which indicates presence of a zero at infinity. Hence, the admittance function $Y(s)$ has a pole at infinity.

$$Y(s) = \frac{s^2+8s+12}{s+4}$$

By continued fraction expansion,

$$\begin{array}{l} s+4\Big)s^2+8s+12\Big(s \leftarrow Y \\ \quad\quad \underline{s^2+4s} \\ \quad\quad\quad 4s+12\Big)s+4\Big(\frac{1}{4} \leftarrow Z \\ \quad\quad\quad\quad\quad \underline{s+3} \\ \quad\quad\quad\quad\quad\quad 1\Big)4s+12\Big(4s \leftarrow Y \\ \quad\quad\quad\quad\quad\quad\quad \underline{4s} \\ \quad\quad\quad\quad\quad\quad\quad\quad 12\Big)1\Big(\frac{1}{12} \leftarrow Z \\ \quad\quad\quad\quad\quad\quad\quad\quad\quad \underline{1} \\ \quad\quad\quad\quad\quad\quad\quad\quad\quad 0 \end{array}$$

The impedances are connected in series branches, whereas the admittances are connected in parallel branches. The network is shown in Fig. 16.55.

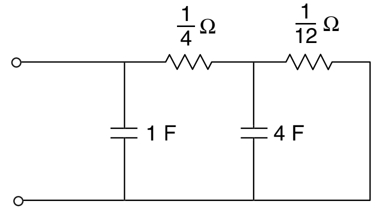


Fig. 16.55

Foster I Form The Foster I form is obtained by partial fraction expansion of $Z(s)$.

$$Z(s) = \frac{s+4}{(s+2)(s+6)}$$

By partial-fraction expansion,

$$Z(s) = \frac{K_1}{s+2} + \frac{K_2}{s+6}$$

where
$$K_1 = (s+2)Z(s)\big|_{s=-2} = \frac{(-2+4)}{(-2+6)} = \frac{1}{2}$$
$$K_2 = (s+6)Z(s)\big|_{s=-6} = \frac{(-6+4)}{(-6+2)} = \frac{1}{2}$$
$$Z(s) = \frac{\frac{1}{2}}{s+2} + \frac{\frac{1}{2}}{s+6}$$

These two terms represent the impedance of a parallel *RC* circuit for which

$$Z_{RC}(s) = \frac{\frac{1}{C_i}}{s + \frac{1}{R_i C_i}}$$

By direct comparison,

$$R_1 = \frac{1}{4}\,\Omega, \qquad C_1 = 2\text{ F}$$
$$R_2 = \frac{1}{12}\,\Omega, \quad C_2 = 2\text{ F}$$

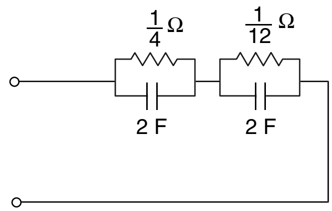


Fig. 16.56

The network is shown in Fig. 16.56.

Example 16.54 *The RC driving-point impedance function is given as* $Z(s) = H\frac{(s+1)(s+4)}{s(s+3)}$ *Realise the impedance function in the ladder form, given* $Z(-2) = 1$.

Solution Putting $s = -2$,

$$Z(-2) = H\frac{(-2+1)(-2+4)}{(-2)(-2+3)} = H$$
$$H = 1$$
$$Z(s) = \frac{(s+1)(s+4)}{s(s+3)}$$

Cauer I Form The Cauer I form is obtained by continued fraction expansion of $Z(s)$ about the pole at infinity.

By continued fraction expansion of $Z(s)$,

$$s^2+3s\,\big)\,s^2+5s+4\,\big(\,1 \leftarrow Z$$
$$\underline{s^2+3s}$$
$$2s+4\,\big)\,s^2+3s\,\big(\,\tfrac{1}{2}s \leftarrow Y$$
$$\underline{s^2+2s}$$
$$s\,\big)\,2s+4\,\big(\,2 \leftarrow Z$$
$$\underline{2s}$$
$$4\,\big)\,s\,\big(\,\tfrac{1}{4}s \leftarrow Y$$
$$\underline{s}$$
$$0$$

The impedances are connected in the series branches whereas admittances are connected in the parallel branches. The network is shown in Fig. 16.57.

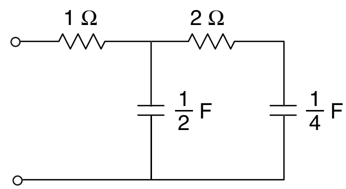


Fig. 16.57

Cauer II Form The Cauer II form is obtained by continued fraction expansion about the pole at the origin. Arranging the polynomials in ascending order of s,

$$Z(s) = \frac{4 + 5s + s^2}{3s + s^2}$$

By continued fraction expansion,

$$3s + s^2 \overline{\big)\, 4 + 5s + s^2 \,\Big(}\; \frac{4}{3s} \leftarrow Z$$

$$\underline{4 + \frac{4}{3}s}$$

$$\frac{11}{3}s + s^2 \overline{\big)\, 3s + s^2 \,\Big(}\; \frac{9}{11} \leftarrow Y$$

$$\underline{3s + \frac{9}{11}s^2}$$

$$\frac{2}{11}s^2 \overline{\big)\, \frac{11}{3}s + s^2 \,\Big(}\; \frac{121}{6s} \leftarrow Z$$

$$\underline{\frac{11}{3}s}$$

$$s^2 \overline{\big)\, \frac{2}{11}s^2 \,\Big(}\; \frac{2}{11} \leftarrow Y$$

$$\underline{\frac{2}{11}s^2}$$

$$0$$

The impedances are connected in the series branches whereas the admittances are connected in the parallel branches. The network is shown in Fig. 16.58.

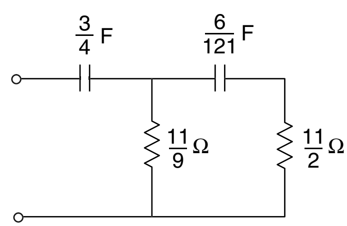


Fig. 16.58

Example 16.55 *An impedance function has the pole-zero diagram as shown in Fig. 16.59. Find the impedance function such that* $Z(-4) = \dfrac{3}{4}$ *and realise in Cauer I and Foster II forms.*

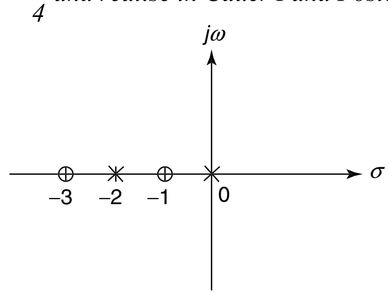


Fig. 16.59

Solution The function $Z(s)$ has poles at $s = 0$ and $s = -2$ and zeros at $s = -1$ and $s = -3$.

$$Z(s) = H\frac{(s+1)(s+3)}{s(s+2)}$$

Putting $s = -4$,

$$Z(-4) = H\frac{(-4+1)(-4+3)}{(-4)(-4+2)} = H\frac{(-3)(-1)}{(-4)(-2)} = \frac{3}{8}\text{H}$$

$$\frac{3}{4} = \frac{3}{8}\text{H}$$

$$H = 2$$

$$Z(s) = \frac{2(s+1)(s+3)}{s(s+2)} = \frac{2s^2+8s+6}{s^2+2s}$$

Cauer I Form The Cauer I form is obtained by continued fraction expansion of $Z(s)$ about the pole at infinity.
By continued fraction expansion of $Z(s)$,

$$s^2+2s\,\big)\,2s^2+8s+6\,\big(\,2 \leftarrow Z$$

$$\underline{2s^2+4s}$$

$$4s+6\,\big)\,s^2+2s\,\big(\,\frac{1}{4}s \leftarrow Y$$

$$\underline{s^2+\frac{3}{2}s}$$

$$\frac{1}{2}s\,\big)\,4s+6\,\big(\,8 \leftarrow Z$$

$$\underline{4s}$$

$$6\,\big)\,\frac{1}{2}s\,\big(\,\frac{1}{12}s \leftarrow Y$$

$$\underline{\frac{1}{2}s}$$

$$0$$

The impedances are connected in the series branches whereas admittances are connected in the parallel branches. The network is shown in Fig. 16.60.

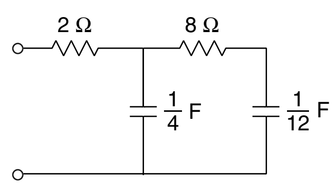


Fig. 16.60

Foster II Form The Foster II form is obtained by partial fraction expansion of $\frac{Y(s)}{s}$.

$$\frac{Y(s)}{s} = \frac{s+2}{2(s+1)(s+3)}$$

By partial fraction expansion,

$$\frac{Y(s)}{s} = \frac{K_1}{s+1} + \frac{K_2}{s+3}$$

where

$$K_1 = (s+1)\left.\frac{Y(s)}{s}\right|_{s=-1} = \frac{(-1+2)}{2(-1+3)} = \frac{1}{4}$$

$$K_2 = (s+3)\left.\frac{Y(s)}{s}\right|_{s=-3} = \frac{(-3+2)}{2(-3+1)} = \frac{-1}{2(-2)} = \frac{1}{4}$$

$$\frac{Y(s)}{s} = \frac{\frac{1}{4}}{s+1} + \frac{\frac{1}{4}}{s+3}$$

$$Y(s) = \frac{\frac{1}{4}s}{s+1} + \frac{\frac{1}{4}s}{s+3}$$

Two terms represent the admittance of a series *RC* circuit. For a series *RC* circuit,

$$Y_{RC}(s) = \frac{\left(\frac{1}{R_i}\right)s}{s + \frac{1}{R_i C_i}}$$

By direct comparison,

$$R_1 = 4\ \Omega, \quad C_1 = \frac{1}{4}\ \text{F}$$

$$R_2 = 4\ \Omega, \quad C_2 = \frac{1}{12}\ \text{F}$$

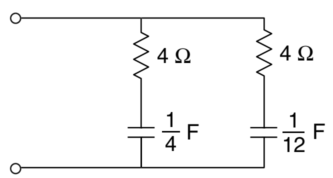


Fig. 16.61

The network is shown in Fig. 16.61.

16.7 REALISATION OF *RL* FUNCTIONS

RL driving point immittance functions have following properties:

1. The poles and zeros are simple and are located on the negative real axis of the s plane.
2. The poles and zeros are interlaced.
3. The lowest critical frequency is a zero which may be at $s = 0$.
4. The highest critical frequency is a pole which may be at infinity.
5. Residues evaluated at the poles of $Z_{RL}(s)$ are real and negative while that of $\frac{Z_{RL}(s)}{s}$ are real and positive.
6. The slope $\frac{d}{d\sigma} Z_{RL}$ is positive.
7. $Z_{RL}(0) < Z_{RL}(\infty)$.

The admittance of an inductor is similar to the impedance of a capacitor. Hence, properties of an *RL* admittance are identical to those of an *RC* impedance and vice-versa, i.e.,

$$Z_{RC}(s) = Y_{RL}(s)$$
$$Z_{RL}(s) = Y_{RC}(s)$$

An *RL* admittance can be considered as the dual of an *RC* impedance and vice-versa.

Example 16.56 *Indicate which of the following functions are either RL, RC or LC impedance functions.*

(a) $Z(s) = \dfrac{4(s+1)(s+3)}{s(s+2)}$ (b) $Z(s) = \dfrac{s(s+4)(s+8)}{(s+1)(s+6)}$

(c) $Z(s) = \dfrac{(s+1)(s+4)}{s(s+2)}$ (d) $Z(s) = \dfrac{2(s+1)(s+3)}{(s+2)(s+16)}$

Solution (a) $Z(s) = \dfrac{4(s+1)(s+3)}{s(s+2)}$

This is an *RC* impedance function since (i) poles and zeros are on the negative real axis, (ii) they are interlaced, and (iii) critical frequency nearest to the origin is a pole.

(b) $Z(s) = \dfrac{s(s+4)(s+8)}{(s+1)(s+6)}$

This is an *RL* impedance function as (i) poles and zeros are on the negative real axis, (ii) they are interlaced, and (iii) critical frequency nearest to the origin is a zero.

(c) $Z(s) = \dfrac{(s+1)(s+4)}{s(s+2)}$

This is an *RC* impedance function since (i) poles and zeros are on the negative real axis, (ii) they are interlaced, and (iii) critical frequency nearest to the origin is a pole.

(d) $Z(s) = \dfrac{2(s+1)(s+3)}{(s+2)(s+6)}$

This is an *RL* impedance function as (i) poles and zeros are on the negative real axis, (ii) they are interlaced, and (iii) critical frequency nearest to the origin is a zero.

Example 16.57 *Realise following RL impedance function in Foster-I and Foster-II form.*

$$Z(s) = \frac{2(s+1)(s+3)}{(s+2)(s+6)}$$

Solution

Foster I Form The Foster I form is obtained by partial-fraction expansion of the impedance function $Z(s)$. By partial-fraction expansion,

$$Z(s) = \frac{K_1}{s+2} + \frac{K_2}{s+6}$$

where
$$K_1 = (s+2)Z(s)\Big|_{s=-2} = \frac{2(-2+1)(-2+3)}{(-2+6)} = -\frac{1}{2}$$
$$K_2 = (s+6)Z(s)\Big|_{s=-6} = \frac{2(-6+1)(-6+3)}{(-6+2)} = -\frac{15}{2}$$

Since residues of $Z(s)$ are negative, partial fraction expansion of $\frac{Z(s)}{s}$ is carried out.

$$\frac{Z(s)}{s} = \frac{2(s+1)(s+3)}{s(s+2)(s+6)}$$

By partial fraction expansion,

$$\frac{Z(s)}{s} = \frac{K_0}{s} + \frac{K_1}{s+2} + \frac{K_2}{s+6}$$

where
$$K_0 = s\frac{Z(s)}{s}\Big|_{s=0} = \frac{2(1)(3)}{(2)(6)} = \frac{1}{2}$$
$$K_1 = (s+2)\frac{Z(s)}{s}\Big|_{s=-2} = \frac{2(-2+1)(-2+3)}{(2)(-2+6)} = \frac{1}{4}$$
$$K_2 = (s+6)\frac{Z(s)}{s}\Big|_{s=-6} = \frac{2(-6+1)(-6+3)}{(-6)(-6+2)} = \frac{5}{4}$$

$$\frac{Z(s)}{s} = \frac{\frac{1}{2}}{s} + \frac{\frac{1}{4}}{s+2} + \frac{\frac{5}{4}}{s+6}$$
$$Z(s) = \frac{1}{2} + \frac{\frac{1}{4}s}{s+2} + \frac{\frac{5}{4}s}{s+6}$$

The first term represents the impedance of the resistor of $\frac{1}{2}\,\Omega$. The other two terms represent the impedance of the parallel RL circuit for which

$$Z_{RL}(s) = \frac{R_i s}{s + \frac{R_i}{L_i}}$$

By direct comparison,

$$R_1 = \frac{1}{4}\,\Omega, \quad L_1 = \frac{1}{8}\text{ H}$$
$$R_2 = \frac{5}{4}\,\Omega, \quad L_2 = \frac{5}{24}\text{ H}$$

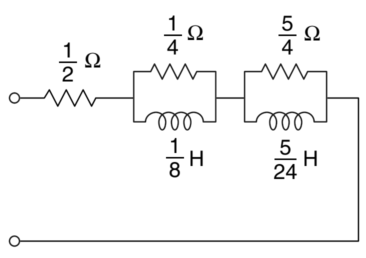


Fig. 16.62

The network is shown in Fig. 16.62.

Foster II Form The Foster II form is obtained by partial fraction expansion of $Y(s)$. Since the degree of the numerator is equal to the degree of the denominator, division is first carried out.

$$Y(s) = \frac{(s+2)(s+6)}{2(s+1)(s+3)} = \frac{s^2+8s+12}{2s^2+8s+6}$$

$$2s^2+8s+6\overline{\big)\, s^2+8s+12\,}\Big(\frac{1}{2}$$
$$\underline{s^2+4s+3}$$
$$4s+9$$

$$Y(s)=\frac{1}{2}+\frac{4s+9}{2s^2+8s+6}=\frac{1}{2}+\frac{4s+9}{2(s+1)(s+3)}$$

By partial-fraction expansion,

$$Y_1(s)=\frac{4s+9}{2(s+1)(s+3)}=\frac{K_0}{s+1}+\frac{K_1}{s+3}$$

where
$$K_0=(s+1)Y_1(s)\Big|_{s=-1}=\frac{(-4+9)}{2(-1+3)}=\frac{5}{4}$$
$$K_1=(s+3)Y_1(s)\Big|_{s=-3}=\frac{(-12+9)}{2(-3+1)}=\frac{3}{4}$$
$$Y(s)=\frac{1}{2}+\frac{\frac{5}{4}}{s+1}+\frac{\frac{3}{4}}{s+3}$$

The first term represents the admittance of a resistor of 2 Ω. The other two terms represent the admittance of a series *RL* circuit. For a series *RL* circuit,

$$Y_{RL}(s)=\frac{\frac{1}{L_i}}{s+\frac{R_i}{L_i}}$$

By direct comparison,

$$R_1=\frac{4}{5}\,\Omega,\quad L_1=\frac{4}{5}\,\text{H}$$
$$R_2=4\,\Omega,\quad L_2=\frac{4}{3}\,\text{H}$$

The network is shown in Fig. 16.63.

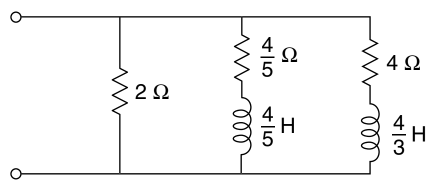


Fig. 16.63

Example 16.58 *Find the Foster forms of the RL impedance function:*

$$Z(s)=\frac{(s+1)(s+4)}{(s+5)(s+3)}$$

Solution

Foster I Form The Foster I form is obtained by partial-fraction expansion of impedance function $\frac{Z(s)}{s}$.

$$\frac{Z(s)}{s}=\frac{(s+1)(s+4)}{s(s+5)(s+3)}$$

By partial-faction expansion,

$$\frac{Z(s)}{s}=\frac{K_0}{s}+\frac{K_1}{s+3}+\frac{K_2}{s+5}$$

where
$$K_0=s\frac{Z(s)}{s}\Big|_{s=0}=\frac{(1)(4)}{(5)(3)}=\frac{4}{15}$$

$$K_1 = (s+3)\frac{Z(s)}{s}\bigg|_{s=-3} = \frac{(-3+1)(-3+4)}{(-3)(-3+5)} = \frac{(-2)(1)}{(-3)(2)} = \frac{1}{3}$$

$$K_2 = (s+5)\frac{Z(s)}{s}\bigg|_{s=-5} = \frac{(-5+1)(-5+4)}{(-5)(-5+3)} = \frac{(-4)(-1)}{(-5)(-2)} = \frac{2}{5}$$

$$\frac{Z(s)}{s} = \frac{\frac{4}{15}}{s} + \frac{\frac{1}{3}}{s+3} + \frac{\frac{2}{5}}{s+5}$$

$$Z(s) = \frac{4}{15} + \frac{\frac{1}{3}s}{s+3} + \frac{\frac{2}{5}s}{s+5}$$

The first term represents the impedance of the resistor of $\frac{4}{15}\,\Omega$. The other two terms represent the impedance of a parallel RL circuit for which

$$Z_{RL}(s) = \frac{R_i s}{s + \frac{R_i}{L_i}}$$

By direct comparison,

$$R_1 = \frac{1}{3}\,\Omega, \quad L_1 = \frac{1}{9}\,\text{H}$$

$$R_2 = \frac{2}{5}\,\Omega, \quad L_2 = \frac{2}{25}\,\text{H}$$

The network is shown in Fig. 16.64.

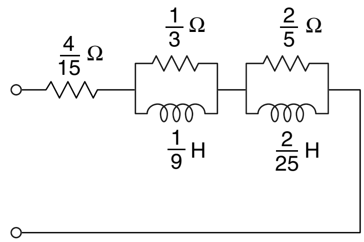


Fig. 16.64

Foster II Form The Foster II form is obtained by partial fraction expansion of $Y(s)$. Since the degree of the numerator is equal to the degree of the denominator, division is first carried out.

$$Y(s) = \frac{(s+5)(s+3)}{(s+1)(s+4)} = \frac{s^2+8s+15}{s^2+5s+4}$$

$$s^2+5s+4)\,s^2+8s+15\,(1$$
$$\underline{s^2+5s+4}$$
$$3s+11$$

$$Y(s) = 1 + \frac{3s+11}{(s+1)(s+4)}$$

By partial-fraction expansion,

$$Y_1(s) = \frac{K_0}{s+1} + \frac{K_1}{s+4}$$

where

$$K_0 = (s+1)Y_1(s)\big|_{s=-1} = \frac{(-3+11)}{(-1+4)} = \frac{8}{3}$$

$$K_1 = (s+4)Y_1(s)\big|_{s=-4} = \frac{(-12+11)}{(-4+1)} = \frac{1}{3}$$

$$Y_1(s) = \frac{\frac{8}{3}}{s+1} + \frac{\frac{1}{3}}{s+4}$$

$$Y(s) = 1 + \frac{\frac{8}{3}}{s+1} + \frac{\frac{1}{3}}{s+4}$$

The first term represents the admittance of a resistor of 1 Ω. The other two terms represent the admittance of a series *RL* circuit.
For a series *RL* circuit,

$$Y_{RL}(s) = \frac{\frac{1}{L_i}}{s + \frac{R_i}{L_i}}$$

By direct comparison,

$$R_1 = \frac{3}{8}\ \Omega, \quad L_1 = \frac{3}{8}\ \text{H}$$
$$R_2 = 12\ \Omega, \quad L_2 = 3\ \text{H}$$

The network is shown in Fig. 16.65.

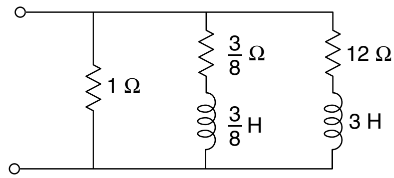


Fig. 16.65

Example 16.59 *Find the Cauer forms of the RL impedance function:*

$$Z(s) = \frac{2(s+1)(s+3)}{(s+2)(s+6)}$$

Solution

Cauer I Form The Cauer I form is obtained by a continued fraction expansion of $Z(s)$ about the pole at infinity.

$$Z(s) = \frac{2(s+1)(s+3)}{(s+2)(s+6)} = \frac{2s^2+8s+6}{s^2+8s+12}$$

By continued fraction expansion,

$$s^2+8s+12)\,2s^2+8s+6\,(2 \leftarrow Z$$
$$\underline{2s^2+16s+24}$$
$$-8s-18$$

Since a negative term results, continued fraction expansion of $Y(s)$ is carried out.

$$Y(s) = \frac{s^2+8s+12}{2s^2+8s+6}$$

By continued fraction expansion,

$$2s^2+8s+6\,\Big)\,s^2+8s+12\,\Big(\frac{1}{2} \leftarrow Y$$
$$\underline{s^2+4s+\ 3}$$
$$4s+9\,\Big)\,2s^2+8s+6\,\Big(\frac{1}{2}s \leftarrow Z$$
$$\underline{2s^2+\frac{9}{2}s}$$
$$\frac{7}{2}s+6\,\Big)\,4s+9\,\Big(\frac{8}{7} \leftarrow Y$$
$$\underline{4s+\frac{48}{7}}$$

$$\frac{15}{7}\Big)\frac{7}{2}s+6\Big(\frac{49}{30}s \leftarrow Z$$
$$\frac{7}{2}s$$
$$\overline{\quad 6\Big)\frac{15}{7}\Big(\frac{15}{42} \leftarrow Y}$$
$$\frac{15}{7}$$
$$\overline{\quad 0 \quad}$$

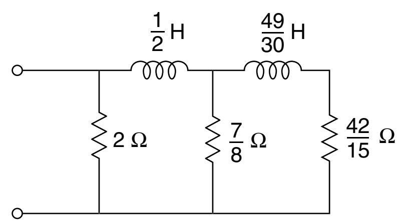


Fig. 16.66

The impedances are connected in the series branches whereas the admittances are connected in the parallel branches. The network is shown in Fig. 16.66.

Cauer II Form The Cauer II form is obtained from a continued fraction expansion about the pole at the origin. Arranging the numerator and denominator polynomials of $Z(s)$ in ascending order of s,

$$Z(s)=\frac{6+8s+2s^2}{12+8s+s^2}$$

By continued fraction expansion,

$$12+8s+s^2\Big)6+8s+2s^2\Big(\frac{1}{2} \leftarrow Z$$
$$6+4s+\frac{1}{2}s^2$$
$$\overline{\quad 4s+\frac{3}{2}s^2\Big)12+8s+s^2\Big(\frac{3}{s} \leftarrow Y}$$
$$12+\frac{9}{2}s$$
$$\overline{\quad \frac{7}{2}s+s^2\Big)4s+\frac{3}{2}s^2\Big(\frac{8}{7} \leftarrow Z}$$
$$4s+\frac{8}{7}s^2$$
$$\overline{\quad \frac{5}{14}s^2\Big)\frac{7}{2}s+s^2\Big(\frac{98}{10s} \leftarrow Y}$$
$$\frac{7}{2}s$$
$$\overline{\quad s^2\Big)\frac{5}{14}s^2\Big(\frac{5}{14} \leftarrow Z}$$
$$\frac{5}{14}s$$
$$\overline{\quad 0 \quad}$$

The impedances are connected in the series branches whereas the admittances are connected in the parallel branches. The network is shown in Fig. 16.67.

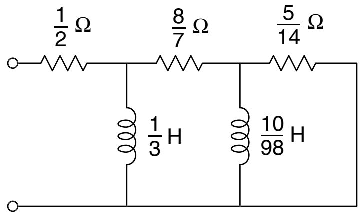


Fig. 16.67

Example 16.60 *Obtain the Foster I and Cauer I forms of the RL impedance function.*

$$Z(s)=\frac{s(s+4)(s+8)}{(s+1)(s+6)}$$

Solution

Foster I Form The Foster I form is obtained by partial fraction expansion of $\frac{Z(s)}{s}$.

$$\frac{Z(s)}{s}=\frac{(s+4)(s+8)}{(s+1)(s+6)}$$

Since the degree of the numerator is equal to the degree of the denominator, division is first carried out.

$$s^2+7s+6)s^2+12s+32(1$$
$$\underline{s^2+7s+6}$$
$$5s+26$$

$$\frac{Z(s)}{s}=1+\frac{5s+26}{s^2+7s+6}=1+\frac{5s+26}{(s+1)(s+6)}$$

By partial-fraction expansion,

$$\frac{Z(s)}{s}=1+\frac{K_0}{s+1}+\frac{K_1}{s+6}$$

where

$$K_0=\left.\frac{5s+26}{s+6}\right|_{s=-1}=\frac{-5+26}{-1+6}=\frac{21}{5}$$

$$K_1=\left.\frac{5s+26}{s+1}\right|_{s=-6}=\frac{-30+26}{-6+1}=\frac{4}{5}$$

$$\frac{Z(s)}{s}=1+\frac{\frac{21}{5}}{s+1}+\frac{\frac{24}{5}}{s+6}$$

$$Z(s)=s+\frac{\frac{21}{5}s}{s+1}+\frac{\frac{4}{5}s}{s+6}$$

The first term represents the impedance of the inductor of 1 H. The other two terms represent the impedance of a parallel *RL* circuit for which

$$Z_{RL}(s) = \frac{R_i s}{s + \frac{R_i}{L_i}}$$

By direct comparison,

$$R_1 = \frac{21}{5}\ \Omega, \quad L_1 = \frac{21}{5}\ \text{H}$$

$$R_2 = \frac{4}{5}\ \Omega, \quad L_2 = \frac{4}{30}\ \text{H}$$

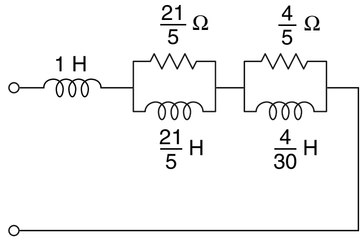


Fig. 16.68

The network is shown in Fig. 16.68.

Cauer I Form The Cauer I form is obtained by continued fraction expansion of $Z(s)$ about the pole at infinity.

$$Z(s) = \frac{s^3 + 12s^2 + 32s}{s^2 + 7s + 6}$$

By continued fraction expansion,

$$s^2 + 7s + 6 \,\big)\, s^3 + 12s^2 + 32s \,\big(\, s \leftarrow Z$$

$$\underline{s^3 + 7s^2 + 6s}$$

$$5s^2 + 26s \,\big)\, s^2 + 7s + 6 \,\big(\, \frac{1}{5} \leftarrow Y$$

$$\underline{s^2 + \frac{26}{5}s}$$

$$\frac{9}{5}s + 6 \,\big)\, 5s^2 + 26s \,\big(\, \frac{25}{9}s \leftarrow Z$$

$$\underline{5s^2 + \frac{50}{3}s}$$

$$\frac{28}{3}s \,\big)\, \frac{9}{5}s + 6 \,\big(\, \frac{27}{140} \leftarrow Y$$

$$\underline{\frac{9}{5}s}$$

$$6 \,\big)\, \frac{28}{3}s \,\big(\, \frac{28}{18}s \leftarrow Z$$

$$\underline{\frac{28}{3}s}$$

$$0$$

The impedances are connected in the series branches, whereas the admittances are connected in the parallel branches. The network is shown in Fig. 16.69.

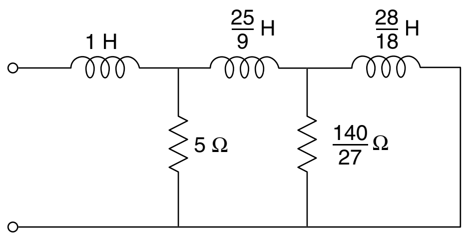


Fig. 16.69

Exercises

16.1 Test the following polynomials for Hurwitz property:

(i) $s^3 + s^2 + 2s + 2$
(ii) $s^4 + s^2 + s + 1$
(iii) $s^3 + 4s^2 + 5s + 2$
(iv) $s^4 + 7s^3 + 6s^2 + 21s + 8$
(v) $s^4 + s^3 + s + 1$
(vi) $s^7 + 3s^6 + 8s^5 + 15s^4 + 17s^3 + 12s^2 + 4s$
(vii) $s^7 + 2s^6 + 2s^5 + s^4 + 4s^3 + 8s^2 + 8s + 4$
(viii) $s^7 + 3s^5 + 2s^3 + s$
(ix) $s^5 + 2s^3 + s$
(x) $s^3 + 2s^2 + 4s + 2$
(xi) $s^4 + s^3 + 4s^2 + 2s + 3$
(xii) $s^5 + 8s^4 + 24s^3 + 28s^2 + 23s + 6$
(xiii) $s^7 - 2s^6 + 2s^5 + 9s^2 + 8s + 4$
(xiv) $s^7 + 3s^5 + 2s^3 + 3$
(xv) $s^5 + s^3 + s$
(xvi) $s^6 + 7s^4 + 5s^3 + s^2 + s$
(xvii) $s^4 + s^3 + 2s^2 + 3s + 2$

16.2 Determine whether the following functions are positive real:

(i) $\dfrac{s^3 + 5s}{s^4 + 2s^2 + 1}$

(ii) $\dfrac{s(s+3)(s+5)}{(s+1)(s+4)}$

(iii) $\dfrac{2s^2 + 2s + 1}{s^3 + 2s^2 + s + 2}$

(iv) $\dfrac{s^4 + 3s^3 + s^2 + s + 2}{s^3 + s^2 + s + 1}$

(v) $\dfrac{2s^3 + 2s^2 + 3s + 2}{s^2 + 1}$

(vi) $\dfrac{s^2 + 2s + 1}{s^2 + 4s + 4}$

(vii) $\dfrac{s^3 + 2s^2 + 3s + 1}{s^3 + 2s^2 + s + 2}$

(viii) $\dfrac{s^3 + 2s^2 + s + 1}{s^2 + s + 1}$

(ix) $\dfrac{s + 4}{s^2 + 2s + 1}$

(x) $\dfrac{s^2 + 4s + 3}{s^2 + 6s + 8}$

(xi) $\dfrac{s^2 + 1}{s^3 + 4s}$

(xii) $\dfrac{s^4 + 2s^3 + 3s^2 + 1}{s^4 + s^3 + 3s^2 + 2s + 1}$

(xiii) $\dfrac{s^2 + 2s + 4}{(s+1)(s+3)}$

(xiv) $\dfrac{2s + 4}{s + 5}$

(xv) $\dfrac{s^2 + 2s}{s^2 + 1}$

(xvi) $\dfrac{s^2 + 4}{s^3 + 3s^2 + 3s + 1}$

16.3 Determine whether the following functions are *LC, RC* or *RL* function:

(i) $F(s) = \dfrac{2(s+1)(s+3)}{(s+2)(s+6)}$

(ii) $Z(s) = \dfrac{3(s+2)(s+4)}{s(s+3)}$

(iii) $Z(s) = \dfrac{(s+1)(s+8)}{(s+2)(s+4)}$

(iv) $Z(s) = \dfrac{Ks(s^2+4)}{(s^2+1)(s^2+3)}$

(v) $Z(s) = \dfrac{2(s^2+1)(s^2+9)}{s(s^2+2)}$

(vi) $Z(s) = \dfrac{4(s+1)(s+3)}{s(s+2)}$

(vii) $Z(s) = \dfrac{(s^2+1)(s^2+3)}{s(s^2+2)}$

(viii) $F(s) = \dfrac{(s+1)(s+2)}{s(s+3)}$

(ix) $Z(s) = \dfrac{(s+1)(s+3)}{(s+2)(s+4)}$

(x) $Z(s) = \dfrac{(s+2)(s+4)}{(s+1)}$

(xi) $Y(s) = \dfrac{4(s+3)}{(s+1)(s+5)}$

(xii) $Y(s) = \dfrac{2(s+1)(s+3)}{(s+2)(s+6)}$

(xiii) $Z(s) = \dfrac{s(s^2+4)(s^2+16)}{(s^2+9)(s^2+25)}$

(xiv) $Z(s) = \dfrac{(s^2+1)(s^2+8)}{s(s^2+4)}$

16.4 Realise the following functions in Foster I form:

(i) $Z(s) = \dfrac{3(s+2)(s+4)}{s(s+3)}$

(ii) $Z(s) = \dfrac{2(s^2+1)(s^2+9)}{s(s^2+4)}$

(iii) $F(s) = \dfrac{4(s+1)(s+3)}{(s+2)(s+6)}$

(iv) $Z(s) = \dfrac{s+4}{(s+2)(s+6)}$

(v) $Z(s) = \dfrac{(s+1)(s+4)}{s(s+2)}$

(vi) $Y(s) = \dfrac{(s+2)(s+5)}{s(s+4)(s+6)}$

(vii) $Z(s) = \dfrac{s^2+2s+2}{s^2+s+1}$

16.5 Realise the following functions in Foster II form:

(i) $Z(s) = \dfrac{3(s+2)(s+4)}{s(s+3)}$

(ii) $Z(s) = \dfrac{2(s^2+1)(s^2+9)}{s(s^2+4)}$

(iii) $Z(s) = \dfrac{4(s+1)(s+3)}{(s+2)(s+6)}$

(iv) $Y(s) = \dfrac{4(s^2+4)(s^2+25)}{s(s^2+16)}$

(v) $Z(s) = \dfrac{(s+2)(s+5)}{s(s+4)(s+6)}$

16.6 Realise the following functions in Cauer I form:

(i) $F(s) = \dfrac{2(s+1)(s+3)}{(s+2)(s+6)}$

(ii) $Z(s) = \dfrac{(s^2+1)(s^2+3)}{s(s^2+2)}$

(iii) $Z(s) = \dfrac{(s+1)(s+3)}{(s+2)(s+4)}$

(iv) $Z(s) = \dfrac{2(s^2+1)(s^2+9)}{s(s^2+4)}$

(v) $F(s) = \dfrac{(s+1)(s+3)}{s(s+2)}$

(vi) $Z(s) = \dfrac{s+4}{(s+2)(s+6)}$

(vii) $Z(s) = \dfrac{6(s+2)(s+4)}{s(s+3)}$

(viii) $Z(s) = \dfrac{s^3+2s}{s^4+4s^2+3}$

(ix) $Z(s) = \dfrac{s(s^2+2)(s^2+5)}{(s^2+1)(s^2+3)}$

(x) $Z(s) = \dfrac{s^2+2s+2}{s^2+s+1}$

16.7 Realise the following function in Cauer II form:

(i) $F(s) = \frac{(s^2+1)(s^2+3)}{s(s^2+2)}$

(ii) $Z(s) = \frac{(s+1)(s+3)}{(s+2)(s+4)}$

(iii) $Z(s) = \frac{2(s^2+1)(s^2+9)}{s(s^2+4)}$

(iv) $Z(s) = \frac{(s+1)(s+3)}{s(s+2)}$

(v) $F(s) = \frac{s^3+12s^2+32s}{s^2+7s+6}$

(vi) $Z(s) = \frac{2(s+1)(s+3)}{(s+2)(s+6)}$

(vii) $Z(s) = \frac{s(s^2+2)(s^2+5)}{(s^2+1)(s^2+3)}$

(viii) $Z(s) = \frac{s^2+2s+2}{s^2+s+1}$

16.8 An impedance function has the pole-zero diagram as shown in Fig. 16.70 below. If $Z(-2) = 3$, synthesise the impedance function in Foster and Cauer forms.

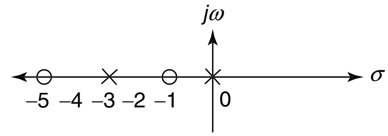


Fig. 16.70

16.9 An impedance function has the pole-zero diagram as shown in Fig. 16.71. Find the impedance function such that $Z(-4) = \frac{8}{3}$ and realise in Cauer forms.

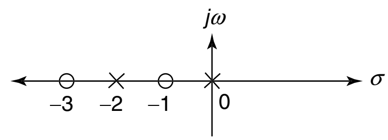


Fig. 16.71

16.10 For the realisation of a given function $F(s)$.

$$F(s) = \frac{K_0}{s} + \sum_{i=1}^{n} \frac{sK_i}{(s^2+\omega_i^2)} + sK_\infty$$

where K_0, K_i (i = 1, 2, 3, … n) and K_∞ are constants.

(i) Mention the type of function (*RC*, *RL* or *LC*)

(ii) Given that $K_0 = 6$, $K_1 = 8$, $\omega_1 = 4$, $K_2 = 10$, $\omega_2 = 8$, $K_\infty = 5$, find the component values of realised network for $F(s) = Z(s)$ and $F(s) = Y(s)$. Draw neat diagrams.

Objective-Type Questions

16.1 The necessary and sufficient condition for a rational function $F(s)$ to be the driving-point impedance of an *RC* network is that all poles and zeros should be

(a) simple and lie on the negative real axis in the *s*-plane
(b) complex and lie in the left half of *s*-plane
(c) complex and lie in the right-half of *s*-plane
(d) simple and lie on the positive real axis of the *s*-plane

16.2 The number of roots of $s^3 + 5s^2 + 7s + 3 = 0$ in the left half of *s*-plane is

(a) zero (b) one
(c) two (d) three

16.3 The first and the last critical frequencies of a driving-point impedance function of a passive network having two kinds of elements, are a pole and a zero respectively. The above property will be satisfied by

(a) *RL* network only
(b) *RC* network only
(c) *LC* network only
(d) *RC* as well as *RL* network

16.4 The pole-zero pattern of a particular network is shown in Fig. 16.72. It is that of an

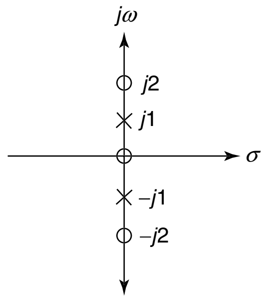


Fig. 16.72

(a) *LC* network (b) *RC* network
(c) *RL* network (d) none of these

16.5 The first critical frequency nearest to the origin of the complex frequency plane for an *RL* driving-point impedance function will be

(a) a zero in the left-half plane
(b) a zero in the right-half plane
(c) a pole in the left-half plane
(d) a pole in the right-half plane

16.6 Consider the following polynomials:
$P_1 = s^8 + 2s^6 + 4s^4$
$P_2 = s^6 - 3s^2 + 2s^2 + 1$
$P_3 = s^4 + 3s^3 + 3s^2 + 2s + 1$
$P_4 = s^7 + 2s^6 + 2s^4 + 4s^3 + 8s^2 + 8s + 4$
which one of these polymials is not Hurwitz?

(a) P_1 (b) P_2
(c) P_3 (d) P_4

16.7 For very high frequencies, the driving-point admittance function, $Y(s) = \dfrac{4(s+1)(s+3)}{s(s+2)(s+4)}$ behaves as

(a) a resistance of $\dfrac{3}{2}\,\Omega$
(b) a capacitance of 4 F
(c) an inductance of $\dfrac{1}{4}$ H
(d) an inductance of 4 H

16.8 The driving-point impedance $Z(s) = \dfrac{s+3}{s+4}$ behaves as

(a) a resistance of 0.75 Ω at low frequencies
(b) a resistance of 1 Ω at high frequencies
(c) both (a) and (b) above
(d) none of the above

16.9 An *RC* driving-point impedance function has zeros at $s = -2$ and $s = -5$. The admissible poles for the functions would be

(a) $s = 0, s = -6$ (b) $s = -1, s = -3$
(c) $s = 0, s = -1$ (d) $s = -3, s = -4$

16.10 Consider the following from the point of view of possible realisation as driving-point impedances using passive elements:

1. $\dfrac{1}{s(s+5)}$ 2. $\dfrac{s+3}{s^2(s+5)}$

3. $\dfrac{s^2+3}{s^2(s^2+5)}$ 4. $\dfrac{s+5}{s(s+5)}$

Of these, the realisable are
(a) 1, 2 and 4 (b) 1, 2 and 3
(c) 3 and 4 (d) none of these

16.11 The poles and zeros of a driving-point function of a network are simple and interlace on the negative real axis with a pole closest to the origin. It can be realised

(a) by an *LC* network
(b) as an *RC* driving point impedance
(c) as an *RC* driving point admittance
(d) only by an *RLC* network

16.12 If $F_1(s)$ and $F_2(s)$ are two positive real functions then the function which is always positive real, is

(a) $F_1(s)\,F_2(s)$ (b) $\dfrac{F_1(s)}{F_2(s)}$

(c) $\dfrac{F_1(s)F_2(s)}{F_1(s)+F_2(s)}$ (d) $F_1(s) - F_2(s)$

16.13 The circuit shown in Fig. 16.73 is

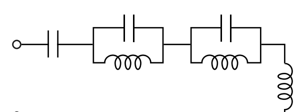


Fig. 16.73

(a) Cauer I form (b) Foster I form
(c) Cauer II form (d) Foster II form

16.14 For an *RC* driving-point impedance function, the poles and zeros

(a) should alternate on the real axis
(b) should alternate only on the negative real axis
(c) should alternate on the imaginary axis
(d) can lie anywhere on the left half-plane

Answers to Objective-Type Questions

16.1 (a)	16.2 (a)	16.3 (b)	16.4 (a)	16.5 (a)	16.6 (b)	16.7 (c)
16.8 (c)	16.9 (b)	16.10 (a)	16.11 (b)	16.12 (c)	16.13 (b)	16.14 (b)

Appendix: Active Filters

A-1 INTRODUCTION

Filters are frequency-selective networks that pass a specified band of frequencies and block or attenuate signals of frequencies outside this band. There are various advantages of active filters.

(i) Gain and frequency adjustment flexibility
(ii) No loading problem
(iii) More economical than passive filters

There are five types of filters.

(i) Low-Pass Filter (LPF)
(ii) High-Pass Filter (HPF)
(iii) Band-Pass Filter (BPF)
(iv) Band-Stop Filter (BSF)
(v) All Pass Filter

A-2 FIRST-ORDER LOW-PASS BUTTERWORTH FILTERS

Figure A.1 shows a first-order low-pass Butterworth filter.

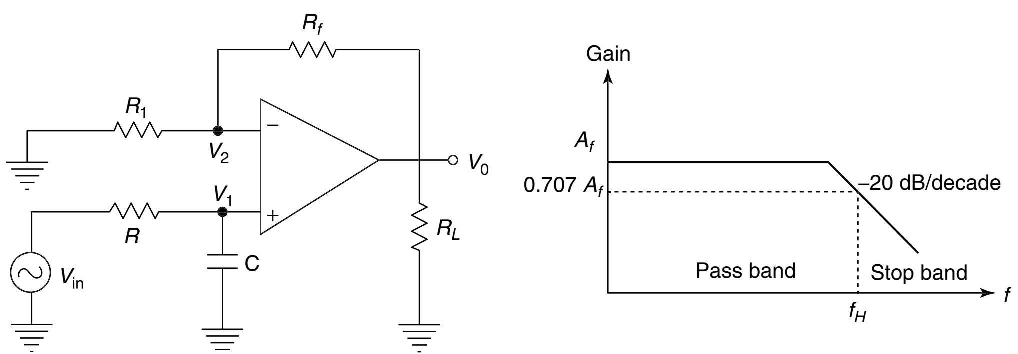


Fig. A.1

Analysis

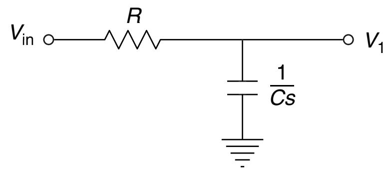


Fig. A.2

$$V_1 = \frac{\frac{1}{Cs}}{R + \frac{1}{Cs}} V_{in}$$

$$= \frac{1}{RCs + 1} V_{in}$$

For low-pass filter

$$V_0 = A_f V_1$$

$$= \frac{A_f}{RCs + 1} V_{in}$$

$$\frac{V_0}{V_{in}} = \frac{A_f}{RCs + 1}$$

Putting $s = j\omega$,

$$\frac{V_0}{V_{in}} = \frac{A_f}{1 + j\omega RC} = \frac{A_f}{1 + j2\pi fRC}$$

$$\text{Let } f_H = \frac{1}{2\pi RC}$$

$$\frac{V_0}{V_{in}} = \frac{A_f}{1 + j\left(\frac{f}{f_H}\right)}$$

where $\frac{V_0}{V_{in}}$ is the transfer gain of the filter as a function of frequency

A_f is the pass-band gain of the filter

f is the frequency of the input signal

$f_H = \frac{1}{2\pi RC}$ is the higher cut-off frequency of the filter.

$$\left|\frac{V_0}{V_{in}}\right| = \frac{A_f}{\sqrt{1 + \left(\frac{f}{f_H}\right)^2}}$$

The phase angle $\phi = -\tan^{-1}\left(\frac{f}{f_H}\right)$

At low frequency, $f < f_H$, $\left|\frac{V_0}{V_{in}}\right| \approx A_f$

At $f = f_H, \left|\frac{V_0}{V_{in}}\right| = \frac{A_f}{\sqrt{2}} = 0.707\ A_f$

For $f > f_H, \left|\frac{V_0}{V_{in}}\right| < A_f$

The low-pass filter has a constant gain A_f from 0 Hz to f_H. At f_H, the gain is 0.707 A_f, thereafter the gain decreases at a constant rate with an increase in frequency. The gain rolls off by 20 dB/decade or 6 dB/octave. The frequency f_H is called the higher cut-off frequency, –3 dB frequency, break frequency, or corner frequency.

Example A.1 *Design an LPF at a cutoff frequency of 1 kHz with a pass-band gain* of *2.*

Solution:

$$f_H = 1\text{ kHz}$$
$$A_f = 2$$
$$f_H = \frac{1}{2\pi RC}$$

Let

$$C = 0.01\ \mu\text{F}$$
$$R = 15.9\text{ k}\Omega$$
$$A_f = 1 + \frac{R_f}{R_1}$$

Let

$$R_1 = 10\text{ k}\Omega$$
$$R_f = 10\text{ k}\Omega$$

A.3 SECOND ORDER LOW-PASS BUTTERWORTH FILTERS

A stop-band response having 40 dB/decade roll-off is obtained with a second-order low-pass filter. A first-order low-pass filter can be converted into a second-order type by adding an additional *RC* network as shown in Fig. A.3.

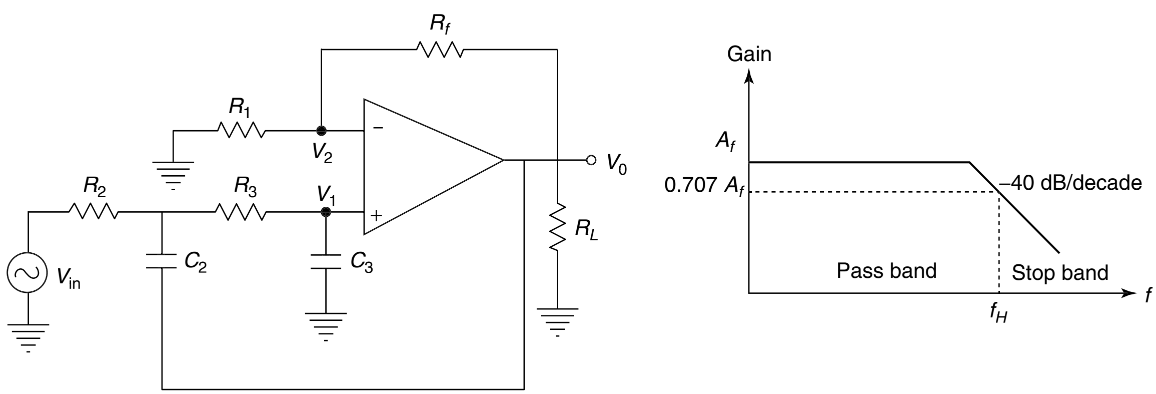


Fig. A.3

The gain of second-order filter is set by R_1 and R_f while the higher cut-off frequency is determined by R_2, C_2, R_3 and C_3.

A.4 *Network Analysis and Synthesis*

Analysis

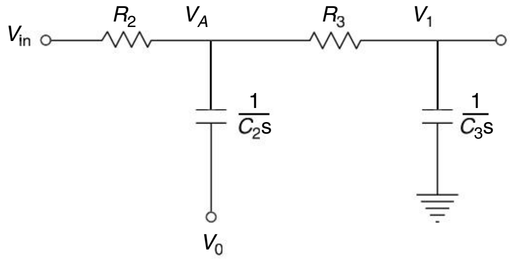


Fig. A.4

$$\frac{V_A - V_{in}}{R_2} + \frac{V_A - V_0}{\frac{1}{C_2 s}} + \frac{V_A - V_1}{R_3} = 0$$

$$\left(\frac{1}{R_2} + \frac{1}{R_3} + C_2 s\right) V_A - \frac{1}{R_3} \mathrm{V}_1 = \frac{V_{in}}{R_2} + C_2 s\, V_0 \qquad \ldots(1)$$

Also,

$$V_1 = \frac{\frac{1}{C_3 s}}{\frac{1}{C_3 s} + R_3} V_A$$

$$= \frac{1}{R_3 C_3 s + 1} V_A$$

$$V_A = (R_3 C_3 s + 1)\ V_1$$

Substituting V_A in Equation (1),

$$\left(\frac{1}{R_2} + \frac{1}{R_3} + C_2 s\right)(R_3 C_3 s + 1)\, V_1 - \frac{1}{R_3} V_1 = \frac{V_{in}}{R_2} + C_2 s\ V_0$$

$$\frac{\left[(R_2 + R_3 R_2 R_3 C_2 s)(R_3 C_3 s H) - R2\right]\ V_1}{R_2 R_3} = \frac{R_3 V_{in} + R_2 R_3 C_2 s V_0}{R_2 R_3}$$

$$V_1 = \frac{R_3 V_{in} + R_3 R_2 C_2 s V_0}{(R_3 C_3 s + 1)(R_2 + R_3 + R_3 R_2 C_2 s) - R_2}$$

$$V0 = \mathrm{A}_f\, V1$$

$$V_0 = \frac{A_f\,[R_3 V_{in} + R_3 R_2 C_2 S V_0]}{(R_3 C_3 S + 1)(R_2 + R_3 + R_3 R_2 C_2 S) - R_2}$$

Simplifying the above equation,

$$\frac{V_0}{V_{in}} = \frac{A_f}{S^2 + \frac{(R_3 C_3 + R_2 C_3 + R_2 C_2 - A_f R_2 C_2)}{R_2 R_3 C_2 C_3} S + \frac{1}{R_2 R_3 C_2 C_3}}$$

For any second-order system,

$$\frac{V_0}{V_{in}} = \frac{A}{s^2 + 2 j \omega_n s + \omega_n^2}$$

For a second-order filter,

$$\omega_n^2 = \omega_H^2 = \frac{1}{R_2 R_3 C_2 C_3}$$

$$\omega_H = \frac{1}{\sqrt{R_2 R_3 C_2 C_3}}$$

$$f_H = \frac{1}{2\pi\sqrt{R_2 R_3 C_2 C_3}}$$

Except for having twice the roll of rate in the stop band, the frequency response of the second-order low-pass filter is identical to that of first-order type and the voltage gain magnitude equation is

$$\left|\frac{V_0}{V_{\text{in}}}\right| = \frac{A_f}{\sqrt{1+\left(\frac{f}{f_H}\right)^4}}$$

A.4 FIRST-ORDER HIGH-PASS BUTTERWORTH FILTERS

High-pass filters are obtained from low-pass filters by interchanging the frequency-determining resistors and capacitors in low-pass filters. The first-order high-pass filter is obtained from a first-order low-pass filter by interchanging *R* and *C* as shown in Fig. A.5.

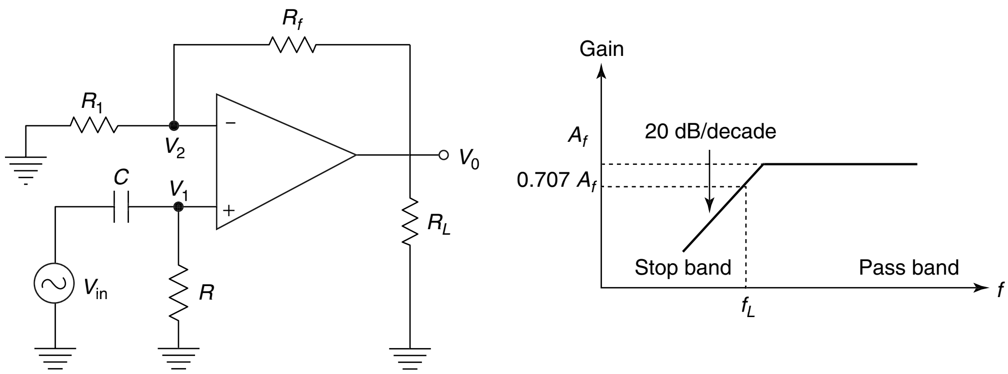


Fig. A.5

Analysis

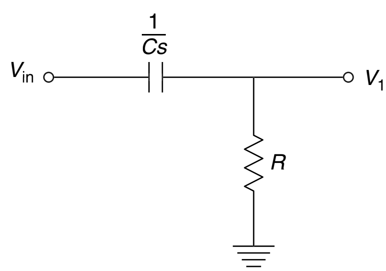


Fig. A.6

$$V_1 = \frac{R}{R + \frac{1}{Cs}} V_{\text{in}}$$

$$= \frac{RCs}{RCs + 1} V_{\text{in}}$$

For a high-pass filter,

$$V_0 = A_f V_1$$

$$= \frac{A_f RCs}{RCs + 1} V_{\text{in}}$$

$$\frac{V_0}{V_{\text{in}}} = \frac{A_f RCs}{RCs + 1}$$

Putting s $= j\omega$,

$$\frac{V_0}{V_{\text{in}}} = \frac{A_f j\omega Rc}{j\omega RC + 1}$$

$$= \frac{A_f j2\pi fRC}{j2\pi fRC + 1} = A_f \left[\frac{j\left(\frac{f}{f_L}\right)}{1 + j\left(\frac{f}{f_L}\right)} \right]$$

where $\frac{V_0}{V_{\text{in}}}$ is the transfer gain of the filter as a function of frequency.

A_f is the pass-band gain of the filter
f is the frequency of the input signal

$f_L = \frac{1}{2\pi RC}$ is the lower cut-off frequency of the filter

$$\left|\frac{V_0}{V_{\text{in}}}\right| = \frac{A_f\left(\frac{f}{f_L}\right)}{\sqrt{1 + \left(\frac{f}{f_L}\right)^2}}$$

The phase angle $\phi = 90° - \tan^{-1}\left(\frac{f}{f_L}\right)$

At low frequency, $f < f_L, \left|\frac{V_0}{V_{\text{in}}}\right| < A_f$

At $f = f_L, \left|\frac{V_0}{V_{\text{in}}}\right| = \frac{A_f}{\sqrt{2}} = 0.707\ A_f$

For $f > f_L, \left|\frac{V_0}{V_{\text{in}}}\right| \approx A_f$

The high-pass filter has a constant gain A_f from f_L to ∞. At f_L, the gain is 0.707 A_f, there after the gain decreases at a constant rate with an decrease in frequency. The gain rolls of by 20 dB/decade or 6 dB/octave. The frequency f_L is called the cut-off frequency, –3 dB frequency, break frequency or corner frequency.

Example A.2 *Design an HPF at a cut-off frequency of 1 kHz with a pass-band gain of 2.*

Solution

$$f_L = 1 \text{ kHz}$$

$$f_L = \frac{1}{2\pi RC}$$

Let $C = 0.01\ \mu\text{F}$

$$R = 15.9\ \text{k}\Omega$$

$$A_f = 1 + \frac{R_f}{R_1}$$

Let $R_1 = 10\ \text{k}\Omega$

$$R_f = 10\ \text{k}\Omega$$

A.5 SECOND-ORDER HIGH-PASS BUTTERWORTH FILTERS

High-pass filters are obtained from low-pass filters by interchanging the frequency determining resistors and capacitors in low-pass filters. The second-order high-pass filter is obtained from second-order low-pass filter by interchanging *R* and *C* as shown in Fig. A.7.

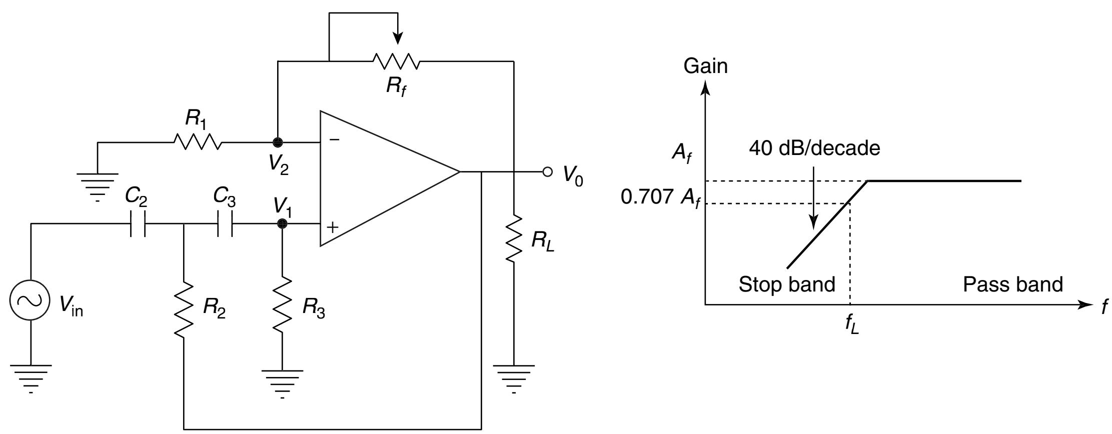


Fig. A.7

The voltage gain magnitude equation of the second-order high-pass filter is given by,

$$\frac{V_0}{V_{\text{in}}} = \frac{A_f}{\sqrt{1 + \left(\frac{f_L}{f}\right)^4}}$$

A.6 BAND-PASS FILTERS

A band-pass filter has a pass band between two cut-off frequencies f_H and f_L such that $f_H > f_L$. Any frequency outside this pass-band is rejected by the filter. There are two types of bandpass filters :

1. Wide band-pass filter
2. Narrow band-pass filter

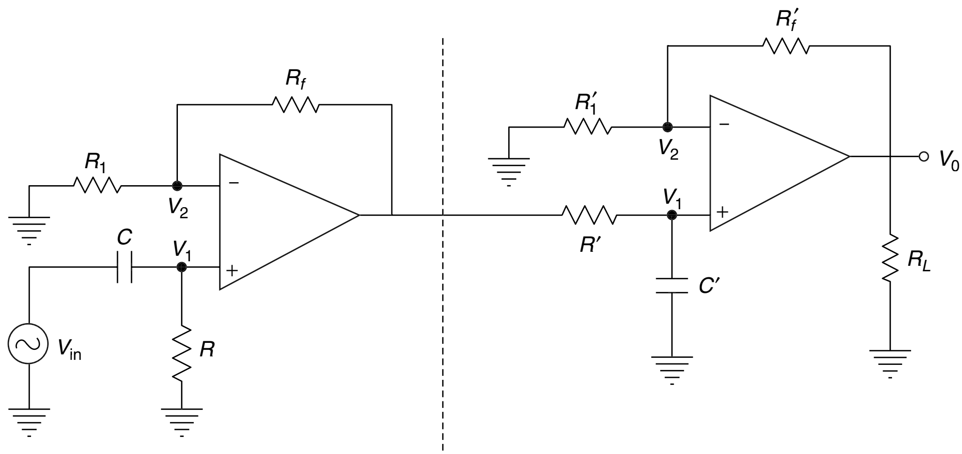
This classification is based on the value of Q, i.e. Fig. A.8 of merit or quality factor. If $Q < 10$ then it is a wide band-pass filter. If $Q > 10$, it is called narrow band-pass filter. Since Q is a measure of selectivity, larger the value of Q, more selective the filter is and the bandwidth decreases. The relationship between Q, the 3 dB bandwidth and the centre frequency f_C is given by

$$Q = \frac{f_c}{BW} = \frac{f_c}{f_H - f_L}$$

For a wide band-pass filter, the centre frequency f_C can be defined as $f_c = \sqrt{f_H\ f_L}$
In a narrow band-pass filter, the output voltage peaks at the centre frequency.

Wide Band-Pass Filters

A wide band-pass filter can be formed by cascading high-pass and low-pass sections as shown. To obtain a ±20 dB/decade band-pass, first order high-pass and first-order low-pass sections are cascaded. To obtain a ±40 dB/decade band-pass, second-order high-pass and second order low-pass section are cascaded. The circuit diagram of first-order wide band-pass filter is shown in Fig. A.8.



(a) First-order high-pass filter (b) First-order low-pass filter

Fig. 6.8

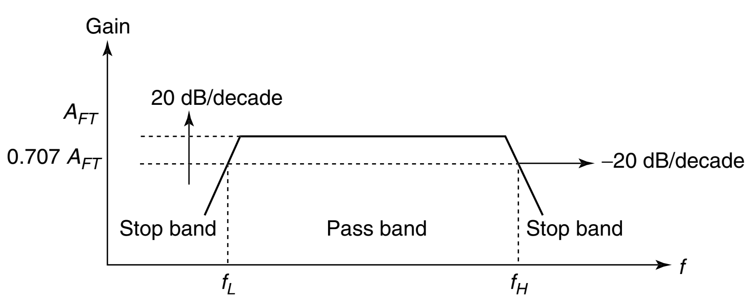


Fig. A.9

The voltage-gain magnitude of band-pass filter is equal to the product of the voltage-gain magnitudes of the high-pass and the low-pass filters.

$$\left|\frac{V_0}{V_{in}}\right| = A_{FT}\frac{\frac{f}{f_L}}{\sqrt{\left[1+\left(\frac{f}{f_L}\right)^2\right]\left[1+\left(\frac{f}{f_H}\right)^2\right]}}$$

where A_{FT} is the total pass-band gain
f is the frequency of the input signal
f_H is higher cut off frequency
f_L is lower cut off frequency

Example A.3 *Design a wide band-pass filter with $f_L = 200$ Hz, $f_H = 1$ kHz and a pass-band gain = 4.*

Solution For a low-pass filter,

$$f_H = 1 \text{ kHz}$$

$$f_H = \frac{1}{2\pi RC}$$

Let $C' = 0.01\ \mu\text{F}$

$$R' = 15.9\ \text{k}\Omega$$

For a high-pass filter,

$$f_L = 200 \text{ kHz}$$

Let $C = 0.05\ \mu\text{F}$

$$R = \frac{1}{2\pi f_L C} = 15.9\ \text{k}\Omega$$

Since the pass-band gain is 4, the gain of the high-pass as well as the low-pass filter can be set at 2.

$$A_f = 2$$

$$A_f = 1 + \frac{R_f}{R_1}$$

Let $R_1 = R_1' = 10\ \text{k}\Omega$

$$R_f = R_f' = 10\ \text{k}\Omega$$

A.7 BAND-STOP FILTERS

In a band-stop filter, frequencies are attenuated in the stop band. Fig. A.10 shows a wideband reject filter. A band-stop filter can be constructed by connecting a high-pass filter and low-pass filter in parallel and then connecting this output to the adder circuit.

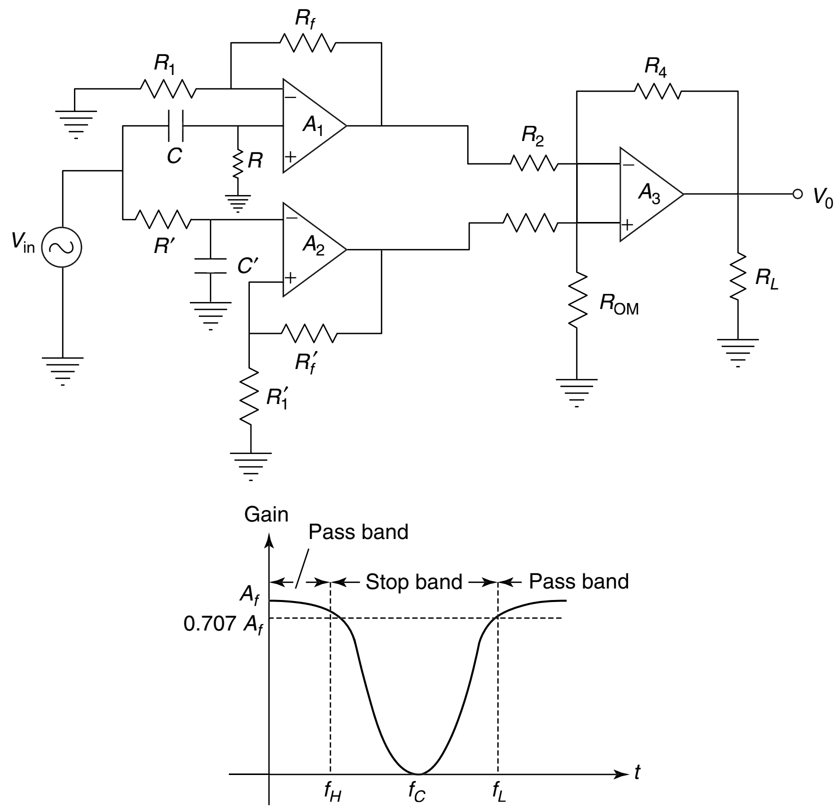


Fig. A.10

To realise the band-stop response, the lower cut-off frequency f_L of high-pass filter must be larger than the high cut-off frequency f_H of the low-pass filter. Also, the pass-band gain of both the filters must be equal. The frequency response of a band-stop filter is shown in Fig. A.10.

A.8 ALL-PASS FILTERS

An all-pass filter passes all signal frequencies without attenuation but with varying phase shifts. Figure. A.11 shows an all-pass filter.

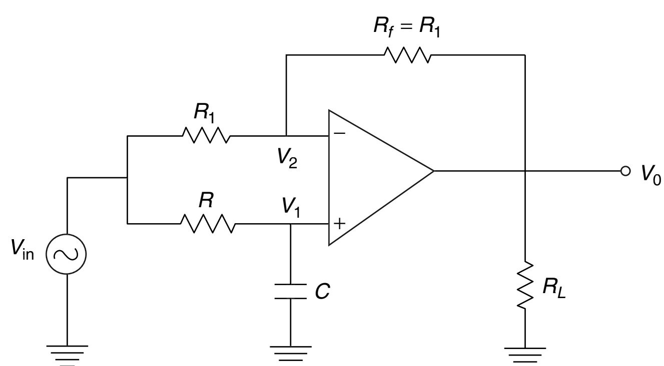


Fig. A.11

With Terminal 1 grounded and voltage V_{in} applied to Terminal 2,

$$V'_0 = -\frac{R_f}{R_1} V_{in}$$

$$= -V_{in} \; (R_f = R_1)$$

With Terminal 2 grounded and voltage V_{in} applied to Terminal 1,

$$V''_0 = \left(1 + \frac{R_f}{R_1}\right) V_1$$

$$= 2V_1 \; (R_f = R_1)$$

$$V_1 = \frac{\frac{1}{Cs}}{R + \frac{1}{Cs}} V_{in}$$

$$= \frac{1}{RCs + 1} V_{in}$$

$$V''_0 = \frac{2}{RCs + 1} V_{in}$$

By superposition principle,

$$V_0 = V'_0 + V''_0$$

$$= -V_{in} + \frac{2}{RCs + 1} V_{in}$$

$$V_0 = V_{in}\left(-1 + \frac{2}{RCs + 1}\right)$$

$$\frac{V_0}{V_{in}} = \frac{1 - RCs}{1 + RCs}$$

Putting $s = j\omega$

$$\frac{V_0}{V_{in}} = \frac{1 - j\omega RC}{1 + j\omega RC}$$

$$\left|\frac{V_0}{V_{in}}\right| = 1$$

The phase angle $\phi = -2\tan^{-1} \omega RC$

Figure A.12 shows a phase shift of 30° between the input V_{in} and output V_o. For fixed values of R and C, the phase angle ϕ changes from 0 to −180° as the frequency f is varied from 0 to ∞.

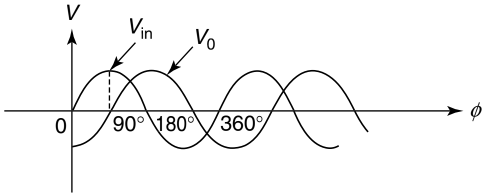


Fig. A.12

When R and C are interchanged, the phase shift between input and output becomes positive. Thus, the output V_o leads the input V_{in}.

Index