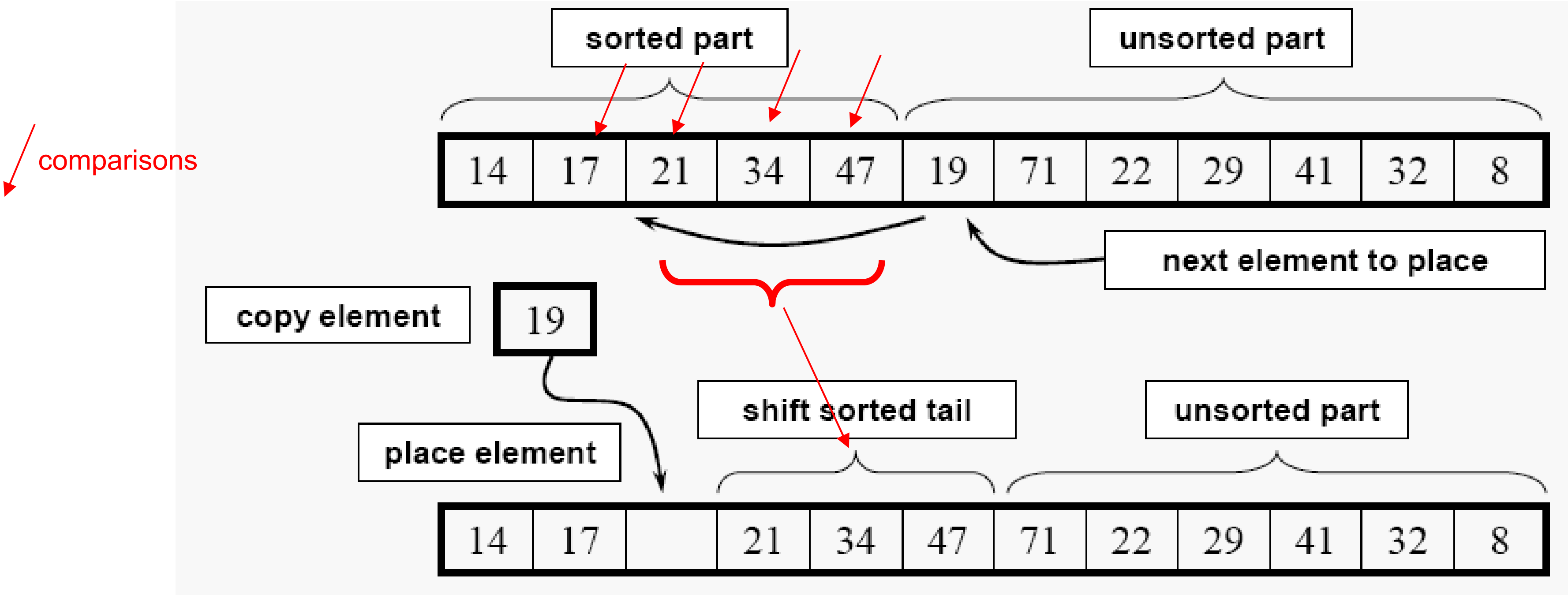
Sorting Algorithms

Sorting methods

- *Comparison based sorting*
  - O(n^2) methods
    - E.g., Insertion, bubble
  - Average time O(n log n) methods
    - E.g., quick sort
  - O(n logn) methods
    - E.g., Merge sort, heap sort
- *Non-comparison based sorting*
  - Integer sorting: linear time
    - E.g., Counting sort, bin sort
  - Radix sort, bucket sort
- *Stable vs. non-stable sorting*

Insertion sort: snapshot at a given iteration



14	17	21	34	47	19	71	22	29	41	32	8

14	17		21	34	47	71	22	29	41	32	8

Worst-case run-time complexity: $\Theta(n^2)$ — When?

Best-case run-time complexity: $\Theta(n)$ — When?

Image courtesy: McQuain WD, VA Tech, 2004

The Divide and Conquer Technique

- Input: A problem of size n
- Recursive
- At each level of recursion:
  - (Divide)
    - Split the problem of size n into a fixed number of sub-problems of smaller sizes, and solve each sub-problem recursively
  - (Conquer)
    - Merge the answers to the sub-problems

Two Divide & Conquer sorts

- Merge sort
  - Divide is trivial
  - Merge (i.e, conquer) does all the work

- Quick sort
  - Partition (i.e, Divide) does all the work
  - Merge (i.e, conquer) is trivial

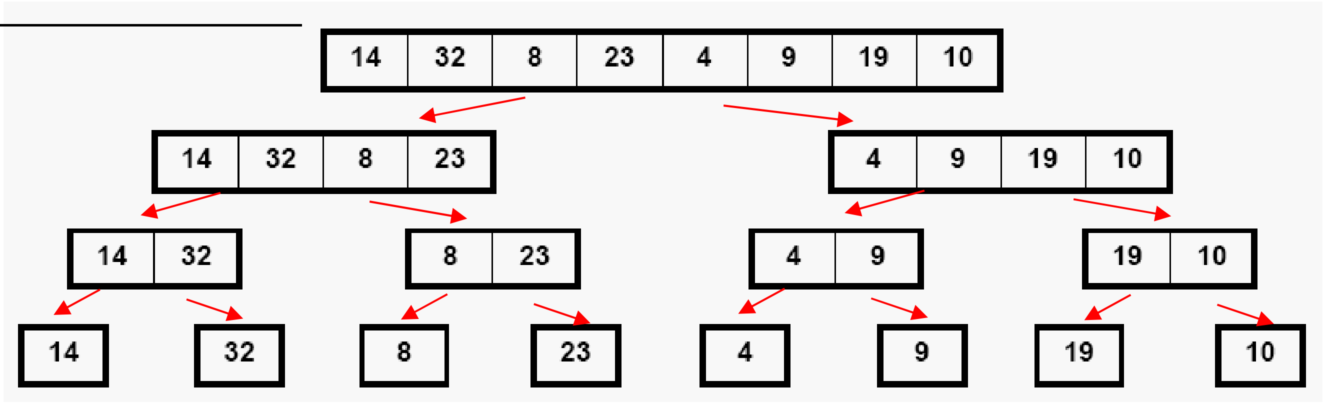
Merge Sort

Main idea:
- Dividing is trivial
- Merging is non-trivial

Input

(divide)

How much work at every step?

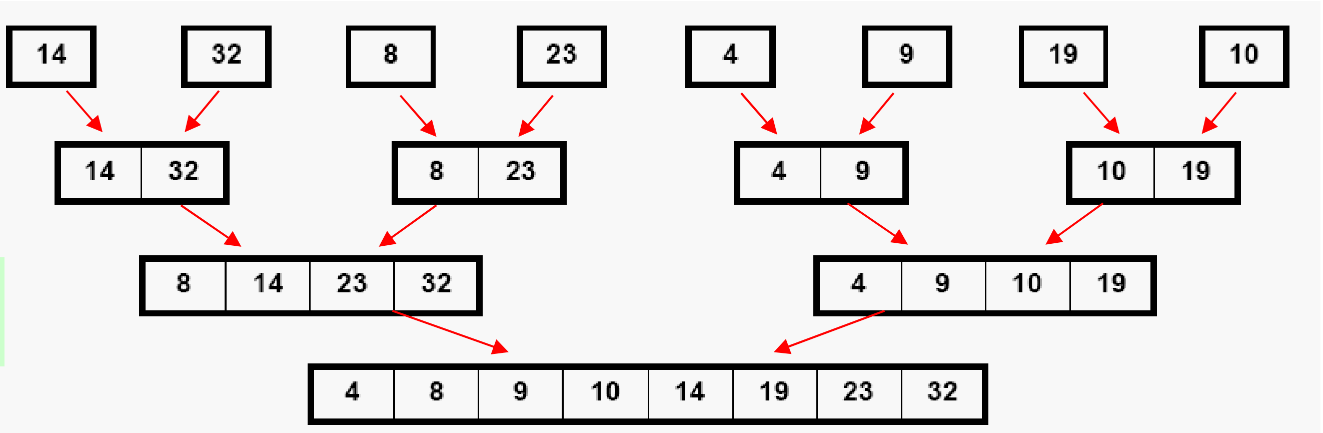


O(lg n) steps

O(n) sub-problems

(conquer)

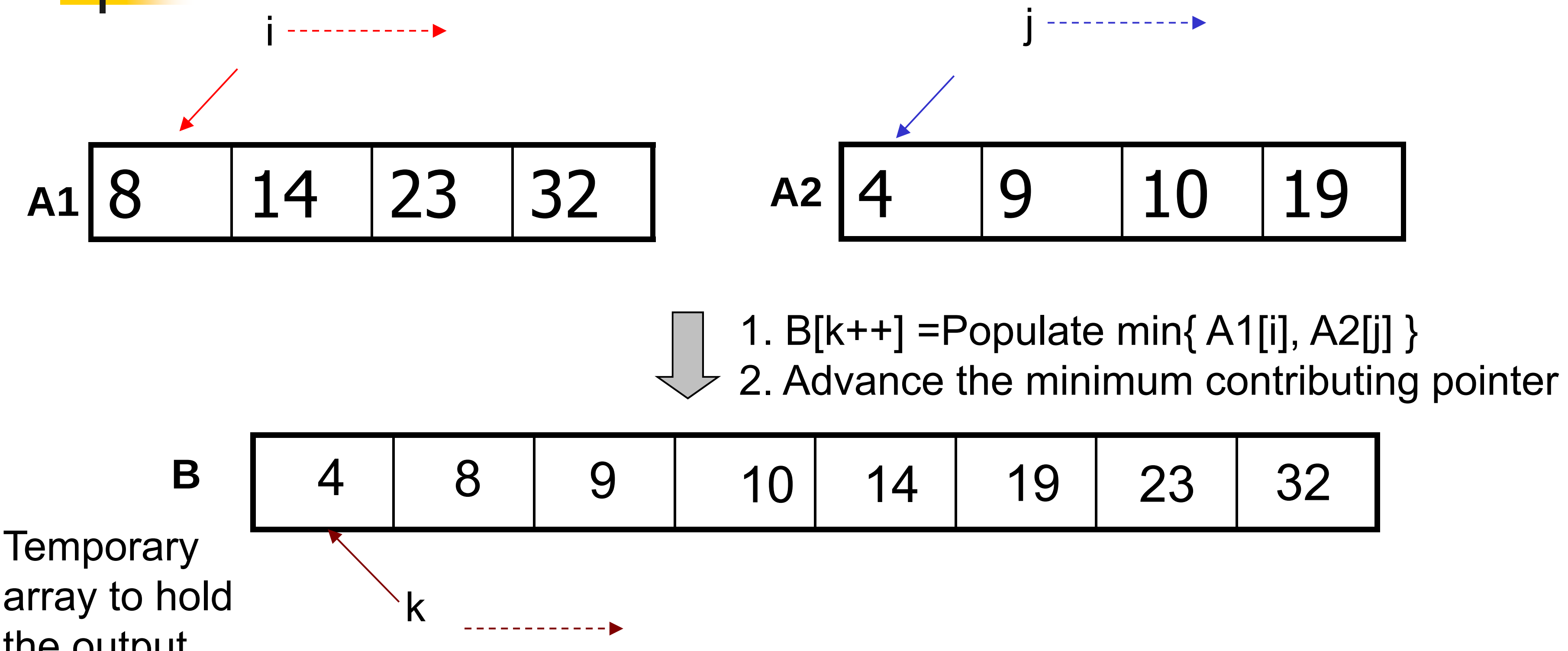
How much work at every step?



O(lg n) steps

Image courtesy: McQuain WD, VA Tech, 2004

How to merge two sorted arrays?



A1			
8	14	23	32

A2			
4	9	10	19

1. B[k++] =Populate min{ A1[i], A2[j] }
2. Advance the minimum contributing pointer

B							
4	8	9	10	14	19	23	32

Temporary array to hold the output

$\Theta(n)$ time

Do you always need the temporary array B to store the output, or can you do this inplace?

Merge Sort : Analysis

Merge Sort takes Θ(n lg n) time

Proof:

- Let T(n) be the time taken to merge sort n elements
- Time for each comparison operation=O(1)

Main observation: To merge two *sorted* arrays of size n/2, it takes n comparisons at most.

Therefore:

- ***T(n) = 2 T(n/2) + n***
- Solving the above recurrence:
  - $T(n) = 2\,[\,2\,T(n/2^2) + n/2\,] + n$
    $= 2^2\,T(n/2^2) + 2n$
    … (k times)
    $= 2^k\,T(n/2^k) + kn$
  - At $k = \lg n$, $T(n/2^k) = T(1) = 1$ (termination case)
  - ==> $T(n) = \Theta(n \lg n)$

<u>Main idea:</u>
•Dividing ("partitioning") is non-trivial
•Merging is trivial

QuickSort

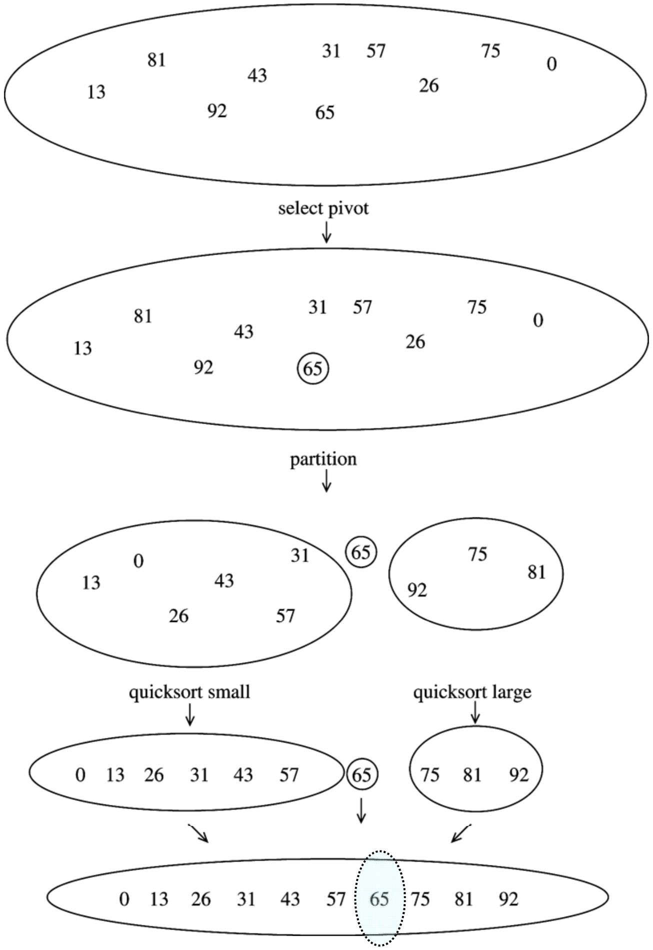
- Divide-and-conquer approach to sorting
- Like MergeSort, except
  - Don't divide the array in half
  - Partition the array based elements being less than or greater than some element of the array (the pivot)
  - i.e., divide phase does all the work; merge phase is trivial.
- Worst case running time $O(N^2)$
- Average case running time O(N log N)
- Fastest generic sorting algorithm in practice
- Even faster if use simple sort (e.g., InsertionSort) when array becomes small

QuickSort Algorithm

```
QuickSort( Array:  S)
1.   If size of S is 0 or 1, return

2.   Pivot = Pick an element v in S

1.   Partition S – {v} into two disjoint groups
       ▪ S1 = {x ∈ (S – {v}) | x < v}
       ▪ S2 = {x ∈ (S – {v}) | x > v}

2.   Return QuickSort(S1), followed by v, followed by
     QuickSort(S2)
```

Q) What's the best way to pick this element? (arbitrary? Median? etc)

QuickSort Example



QuickSort vs. MergeSort

- Main problem with quicksort:
  - QuickSort may end up dividing the input array into subproblems of size 1 and N-1 in the worst case, at every recursive step (unlike merge sort which always divides into two halves)
    - When can this happen?
    - Leading to O(N^2) performance

  =>Need to choose pivot wisely (but efficiently)

- MergeSort is typically implemented using a temporary array (for merge step)
  - QuickSort can partition the array “in place”

Goal: A “good” pivot is one that creates two even sized partitions

=> Median will be best, but finding median could be as tough as sorting itself

Picking the Pivot

How about choosing the first element?

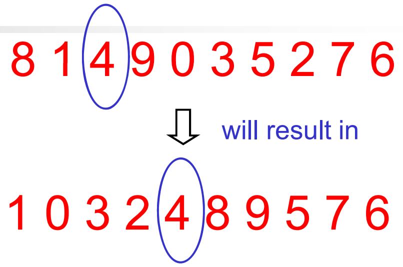
- What if array already or nearly sorted?
- Good for a randomly populated array

How about choosing a random element?

- Good in practice if “truly random”
- Still possible to get some bad choices
- Requires execution of random number generator

Picking the Pivot

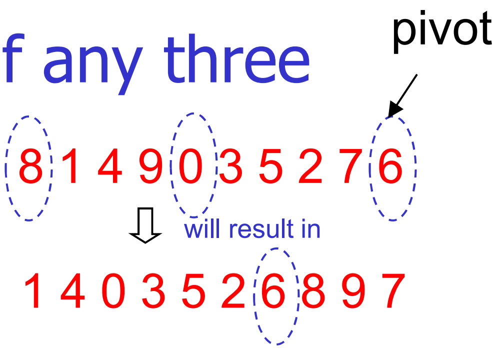
- Best choice of pivot
  - Median of array
  - But median is expensive to calculate



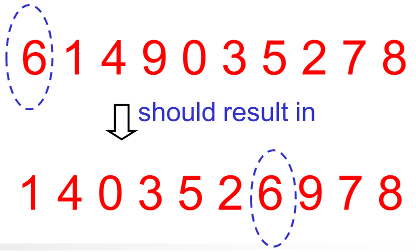
- *Next strategy: Approximate the median*
  - *Estimate* median as the median of any three elements

    Median = median {first, middle, last}

    Has been shown to reduce running time (comparisons) by 14%



How to write the partitioning code?



6 1 4 9 0 3 5 2 7 8

should result in

1 4 0 3 5 2 6 9 7 8

- Goal of partitioning:
  - i) Move all elements < pivot to the left of pivot
  - ii) Move all elements > pivot to the right of pivot

- Partitioning is conceptually straightforward, but easy to do inefficiently

- One bad way:
  - Do one pass to figure out how many elements should be on either side of pivot
  - Then create a temp array to copy elements relative to pivot

Partitioning strategy

6 1 4 9 0 3 5 2 7 8

⇩ should result in

1 4 0 3 5 2 6 9 7 8

- A good strategy to do partition : ***do it <u>in place</u>***

```
// Swap pivot with last element S[right]
i = left
j = (right – 1)
While (i < j) {
        // advance i until first element > pivot

        // decrement j until first element < pivot

        // swap A[i] & A[j]  (only if i<j)

}
Swap ( pivot , S[i] )
```

OK to also swap with S[left] but then the rest of the code should change accordingly

This is called “*in place*” because all operations are done in place of the input array (i.e., without creating temp array)

Partitioning Strategy

- An in place partitioning algorithm

```
Swap pivot with last element S[right]
i = left
j = (right – 1)
while (i < j)
    { i++; } until S[i] > pivot
    { j--; } until S[j] < pivot
    If (i < j), then swap( S[i] , S[j] )
Swap ( pivot , S[i] )
```

Needs a few
boundary case
handling

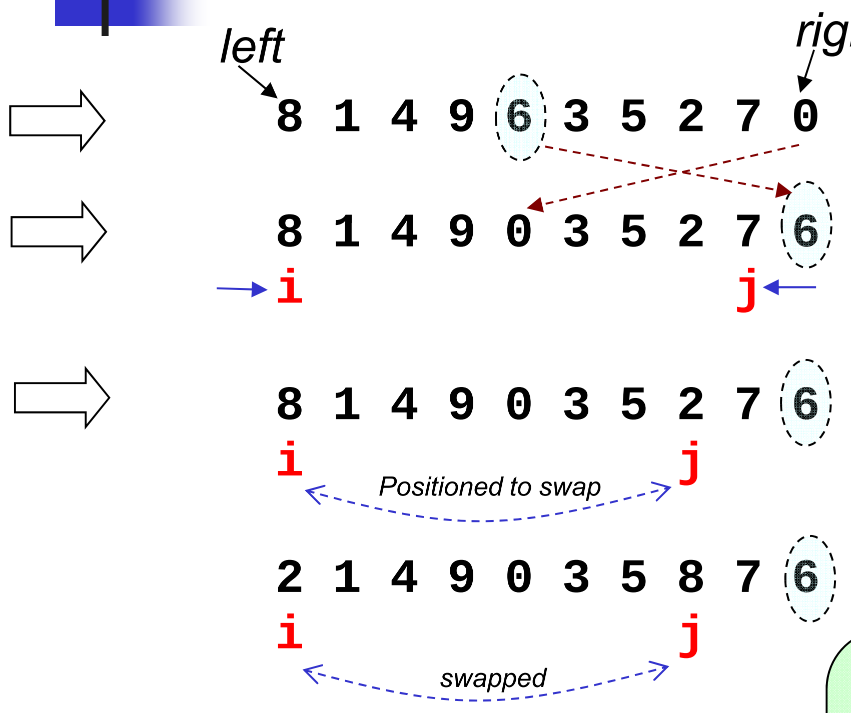
"Median of three" approach to picking the pivot:

=> compares the first, last and middle elements and pick the median of those three

pivot = min{8,6,0} = 6

Partitioning Example

Swap pivot with last element S[right]
i = left
j = (right – 1)



8 1 4 9 6 3 5 2 7 0 — Initial array

8 1 4 9 0 3 5 2 7 6 — Swap pivot; initialize i and j

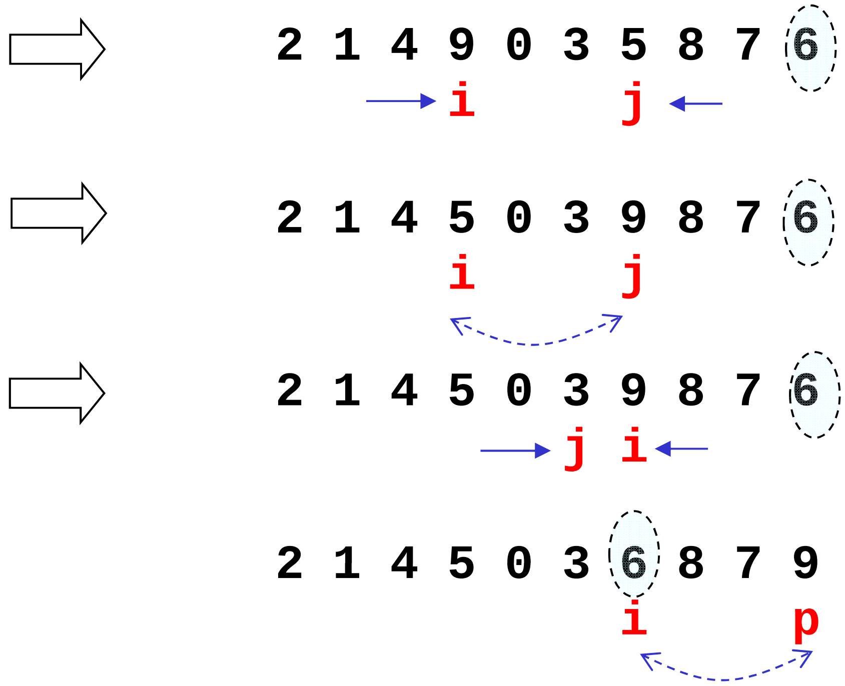
8 1 4 9 0 3 5 2 7 6 — Move i and j inwards until conditions violated

2 1 4 9 0 3 5 8 7 6 — After first swap

```
While (i < j) {
        { i++; } until S[i] > pivot
        { j--; } until S[j] < pivot
        If (i < j), then swap( S[i] , S[j] )
    }
```

Partitioning Example (cont.)

After a few steps …



```
2 1 4 9 0 3 5 8 7 6
      i       j
```

Before second swap

```
2 1 4 5 0 3 9 8 7 6
      i       j
```

After second swap

```
2 1 4 5 0 3 9 8 7 6
          j i
```

i has crossed *j*

```
2 1 4 5 0 3 6 8 7 9
            i     p
```

After final swap with pivot

Swap (*pivot* , *S[i]*)

Handling Duplicates

What happens if all input elements are equal?

Special case: 6 6 6 6 6 6 6 6 6 6 6 6 6 6 6 6

- Current approach:
  - *{ i++; }* ***until*** *S[i] > pivot*
  - *{ j--; }* ***until*** *S[j] < pivot*

- What will happen?
  - i will advance all the way to the right end
  - j will advance all the way to the left end
  - => pivot will remain in the right position, creating the left partition to contain N-1 elements and empty right partition
    - Worst case $O(N^2)$ performance

Handling Duplicates

- A better code
  - Don’t skip elements equal to pivot
    - *{ i++; }* ***until*** *S[i] ≥ pivot*
    - *{ j--; }* ***until*** *S[j] ≤ pivot*

Special case: 6 6 6 6 6 6 6 6 6 6 6 6 6 6 6 6

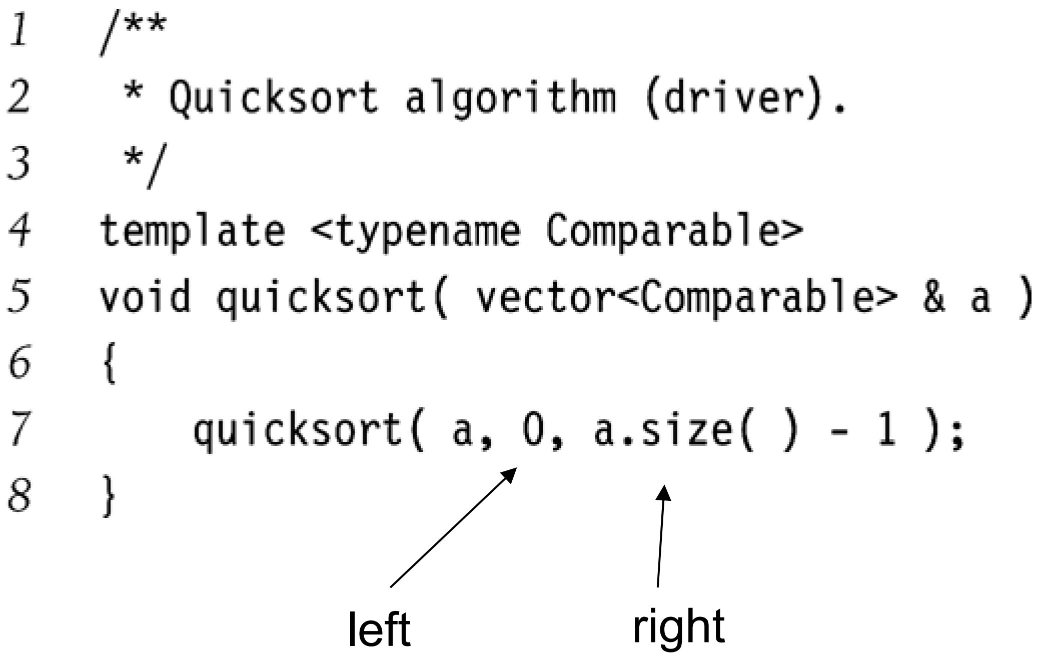
What will happen now?

  - Adds some unnecessary swaps
  - But results in perfect partitioning for array of identical elements
    - Unlikely for input array, but more likely for recursive calls to QuickSort

Small Arrays

- When S is small, recursive calls become expensive (*overheads*)
- General strategy
  - When size < threshold, use a sort more efficient for small arrays (e.g., InsertionSort)
  - Good thresholds range from 5 to 20
  - Also avoids issue with finding median-of-three pivot for array of size 2 or less
  - Has been shown to reduce running time by 15%

QuickSort Implementation

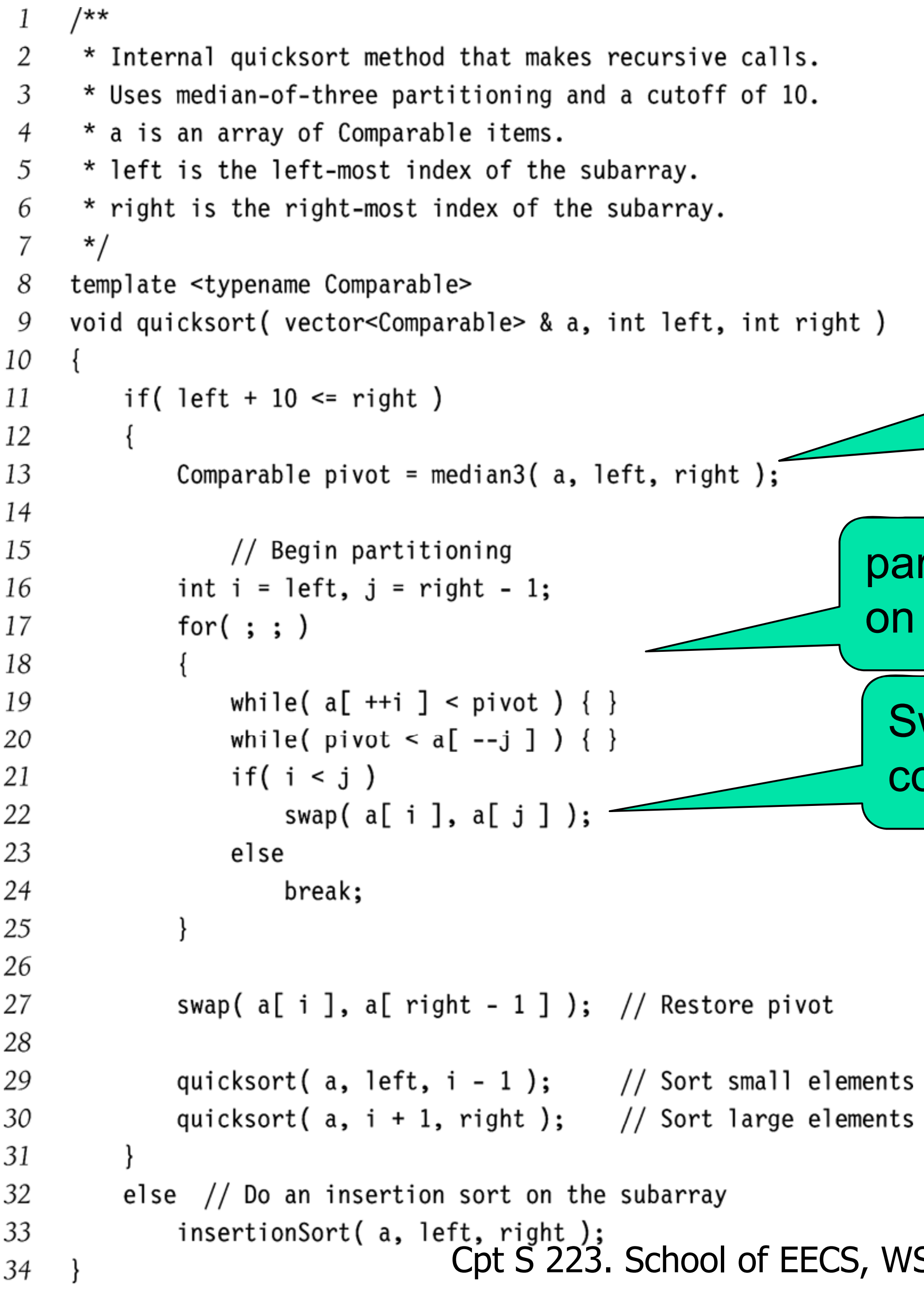


```
/**
 * Quicksort algorithm (driver).
 */
template <typename Comparable>
void quicksort( vector<Comparable> & a )
{
    quicksort( a, 0, a.size( ) - 1 );
}
```

QuickSort Implementation

```
/**
 * Return median of left, center, and right.
 * Order these and hide the pivot.
 */
template <typename Comparable>
const Comparable & median3( vector<Comparable> & a, int left, int right )
{
    int center = ( left + right ) / 2;
    if( a[ center ] < a[ left ] )
        swap( a[ left ], a[ center ] );
    if( a[ right ] < a[ left ] )
        swap( a[ left ], a[ right ] );
    if( a[ right ] < a[ center ] )
        swap( a[ center ], a[ right ] );

        // Place pivot at position right - 1
    swap( a[ center ], a[ right - 1 ] );
    return a[ right - 1 ];
}
```

```
8 1 4 9 6 3 5 2 7 0
L       C         R

6 1 4 9 8 3 5 2 7 0
L       C         R

0 1 4 9 8 3 5 2 7 6
L       C         R

0 1 4 9 6 3 5 2 7 8
L       C         R

0 1 4 9 7 3 5 2 6 8
L       C       P R
```



```
/**
 * Internal quicksort method that makes recursive calls.
 * Uses median-of-three partitioning and a cutoff of 10.
 * a is an array of Comparable items.
 * left is the left-most index of the subarray.
 * right is the right-most index of the subarray.
 */
template <typename Comparable>
void quicksort( vector<Comparable> & a, int left, int right )
{
    if( left + 10 <= right )
    {
        Comparable pivot = median3( a, left, right );

            // Begin partitioning
        int i = left, j = right - 1;
        for( ; ; )
        {
            while( a[ ++i ] < pivot ) { }
            while( pivot < a[ --j ] ) { }
            if( i < j )
                swap( a[ i ], a[ j ] );
            else
                break;
        }

        swap( a[ i ], a[ right - 1 ] );  // Restore pivot

        quicksort( a, left, i - 1 );     // Sort small elements
        quicksort( a, i + 1, right );    // Sort large elements
    }
    else  // Do an insertion sort on the subarray
        insertionSort( a, left, right );
}
```

Assign pivot as median of 3

partition based on pivot

Swap should be compiled inline.

Recursively sort partitions

Analysis of QuickSort

- Let T(N) = time to quicksort N elements
- Let L = #elements in left partition
  => #elements in right partition = N-L-1

- Base: T(0) = T(1) = O(1)
- $$T(N) = \underbrace{T(L)}_{\text{Time to sort left partition}} + \underbrace{T(N - L - 1)}_{\text{Time to sort right partition}} + \underbrace{O(N)}_{\text{Time for partitioning at current recursive step}}$$

Analysis of QuickSort

- <u>Worst-case analysis</u>
  - Pivot is the smallest element (L = 0)

$$
\begin{aligned}
T(N) &= T(0) + T(N-1) + O(N) \\
&= O(1) + T(N-1) + O(N) \\
&= T(N-1) + O(N) \\
&= T(N-2) + O(N-1) + O(N) \\
&= T(N-3) + O(N-2) + O(N-1) + O(N) \\
&= \sum_{i=1}^{N} O(i) = O(N^2)
\end{aligned}
$$

Analysis of QuickSort

- Best-case analysis
  - Pivot is the median (sorted rank = N/2)

$$\begin{aligned} T(N) &= T(N/2) + T(N/2) + O(N) \\ &= 2T(N/2) + O(N) \\ &= O(N \log N) \end{aligned}$$

- Average-case analysis
  - Assuming each partition equally likely
  - $T(N) = O(N \log N)$ HOW?

QuickSort: Avg Case Analysis

- $T(N) = T(L) + T(N\text{-}L\text{-}1) + O(N)$

All partition sizes are equally likely

$\Rightarrow$ Avg $T(L) =$ Avg $T(N\text{-}L\text{-}1) = 1/N \sum_{j=0}^{N-1} T(j)$

$\Rightarrow$ Avg $T(N) = 2/N \left[\sum_{j=0}^{N-1} T(j) \right] + cN$

$\Rightarrow N\,T(N) = 2 \left[\sum_{j=0}^{N-1} T(j) \right] + cN^2$ => (1)

Substituting N by N-1 …

$\Rightarrow (N-1)\,T(N-1) = 2 \left[\sum_{j=0}^{N-2} T(j) \right] + c(N-1)^2$ => (2)

(1)-(2)

$\Rightarrow NT(N) - (N-1)T(N-1)$

$= 2\,T(N-1) + c\,(2N-1)$

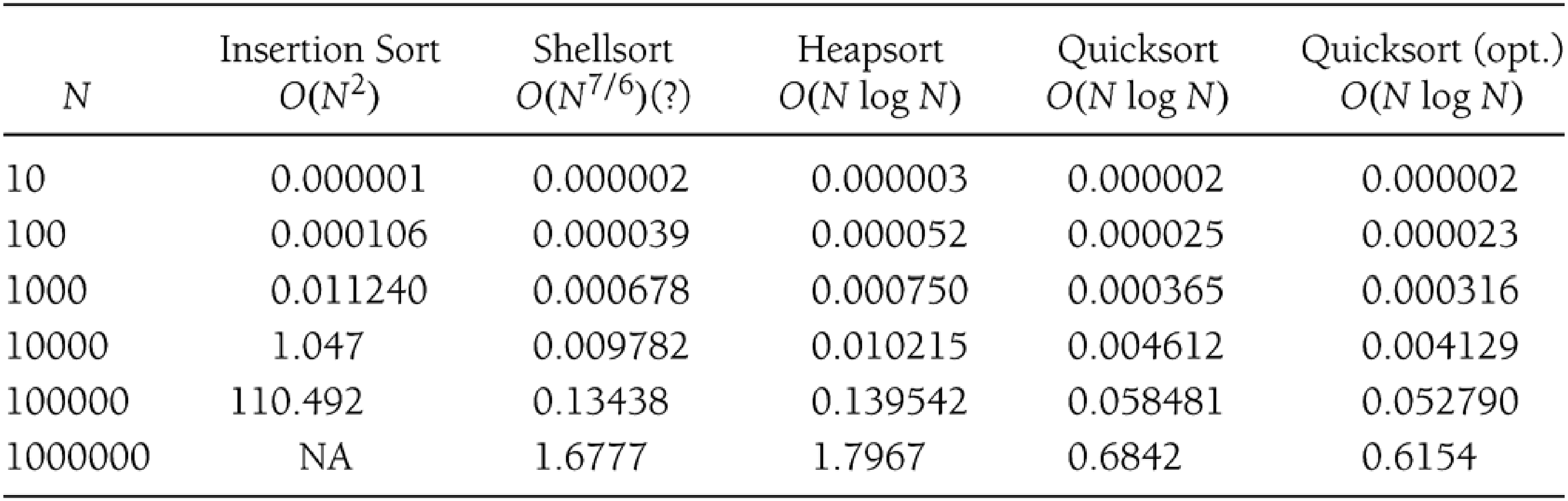
Avg case analysis …

- $NT(N) = (N+1)T(N-1) + c\,(2N-1)$
- $T(N)/(N+1) \approx T(N-1)/N + c2/(N+1)$
- Telescope, by substituting N with N-1, N-2, N-3, .. 2
- …
- $T(N) = O(N \log N)$

Comparison Sorting

Sort	Worst Case	Average Case	Best Case	Comments
InsertionSort	$\Theta(N^2)$	$\Theta(N^2)$	$\Theta(N)$	Fast for small N
MergeSort	$\Theta(N \log N)$	$\Theta(N \log N)$	$\Theta(N \log N)$	Requires memory
HeapSort	$\Theta(N \log N)$	$\Theta(N \log N)$	$\Theta(N \log N)$	Large constants
QuickSort	$\Theta(N^2)$	$\Theta(N \log N)$	$\Theta(N \log N)$	Small constants

Comparison Sorting



N	Insertion Sort $O(N^2)$	Shellsort $O(N^{7/6})$(?)	Heapsort $O(N \log N)$	Quicksort $O(N \log N)$	Quicksort (opt.) $O(N \log N)$
10	0.000001	0.000002	0.000003	0.000002	0.000002
100	0.000106	0.000039	0.000052	0.000025	0.000023
1000	0.011240	0.000678	0.000750	0.000365	0.000316
10000	1.047	0.009782	0.010215	0.004612	0.004129
100000	110.492	0.13438	0.139542	0.058481	0.052790
1000000	NA	1.6777	1.7967	0.6842	0.6154

Good sorting applets

- http://www.cs.ubc.ca/~harrison/Java/sorting-demo.html
- http://math.hws.edu/TMCM/java/xSortLab/

Sorting benchmark: http://sortbenchmark.org/

Lower Bound on Sorting

What is the best we can do on comparison based sorting?

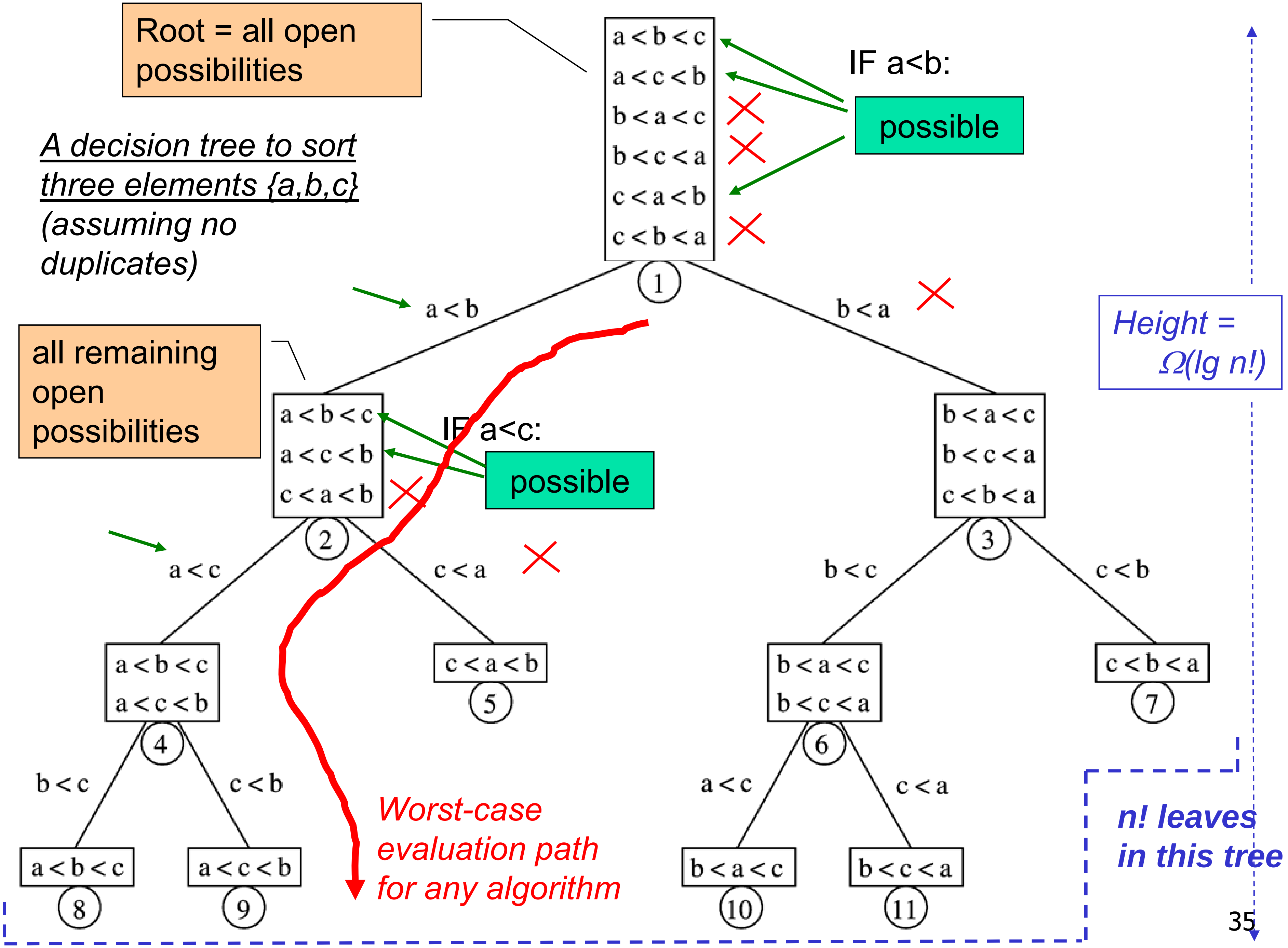
- Best worst-case sorting algorithm (so far) is O(N log N)
  - Can we do better?

- Can we prove a lower bound on the sorting problem, independent of the algorithm?
  - For comparison sorting, no, we can't do better than O(N log N)
  - Can show lower bound of Ω(N log N)

Proving lower bound on sorting using “Decision Trees”

A *decision tree* is a binary tree where:

- Each node
  - lists all left-out open possibilities (for deciding)
- Path of each node
  - represents a decided sorted prefix of elements
- Each branch
  - represents an outcome of a particular comparison
- Each leaf
  - represents a particular ordering of the original array elements



Root = all open possibilities

A decision tree to sort three elements {a,b,c}

(assuming no duplicates)

IF a<b:

possible

all remaining open possibilities

IF a<c:

possible

Height = Ω(lg n!)

Worst-case evaluation path for any algorithm

n! leaves in this tree

Decision Tree for Sorting

- The logic of *any sorting algorithm* that uses comparisons can be represented by a decision tree

- In the worst case, the number of comparisons used by the algorithm equals the HEIGHT OF THE DECISION TREE

- In the average case, the number of comparisons is the average of the depths of all leaves

- There are N! different orderings of N elements

Lower Bound for Comparison Sorting

Lemma: *A binary tree with L leaves must have depth at least ceil(lg L)*

Sorting's decision tree has N! leaves

Theorem: *Any comparison sort may require at least* $\lceil \log(N!) \rceil$ *comparisons in the worst case*

Lower Bound for Comparison Sorting

Theorem: *Any comparison sort requires Ω(N log N) comparisons*

- Proof (uses Stirling's approximation)

$$N! \approx \sqrt{2\pi N}\,(N/e)^N\,(1+\Theta(1/N))$$

$$N! > (N/e)^N$$

$$\log(N!) > N\log N - N\log e = \Theta(N\log N)$$

$$\log(N!) > \Theta(N\log N)$$

$$\therefore \log(N!) = \Omega(N\log N)$$

Implications of the sorting lower bound theorem

- Comparison based sorting cannot be achieved in less than (n lg n) steps

=> Merge sort, Heap sort are optimal

=> Quick sort is not optimal but pretty good as optimal in practice

=> Insertion sort, bubble sort are clearly sub-optimal, even in practice

Non comparison based sorting

Integer sorting

e.g., Counting sort

Bucket sort

Radix sort

Non comparison based

Integer Sorting

- Some input properties allow to eliminate the need for comparison
  - E.g., sorting an employee database by age of employees

- Counting Sort
  - *Given array A[1..N], where $1 \leq A[i] \leq M$*
  - Create array C of size M, where C[i] is the number of i's in A
  - Use C to place elements into new sorted array B
  - Running time $\Theta(N+M) = \Theta(N)$ if $M = \Theta(N)$

Counting Sort: Example

Input A:

1	2	3	4	5	6	7	8	9	10
0	3	2	1	3	2	1	2	2	3

N=10
M=4

(all elements in input between 0 and 3)

Count array C:

0	**1**
1	**2**
2	**4**
3	**3**

Output sorted array:

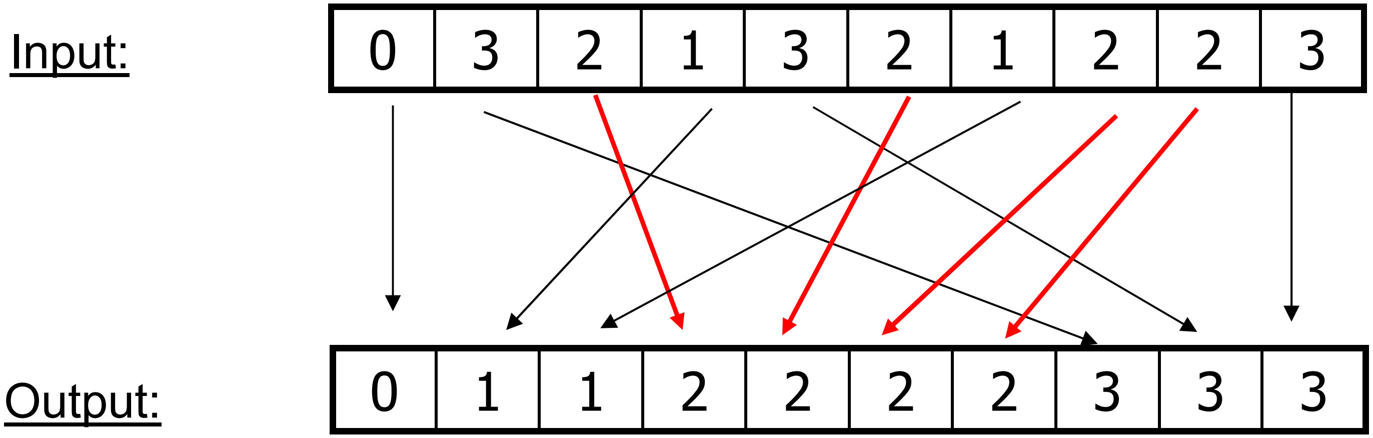
1	2	3	4	5	6	7	8	9	10
0	1	1	2	2	2	2	3	3	3

Time = O(N + M)

If (M < N), Time = O(N)

Stable vs. nonstable sorting

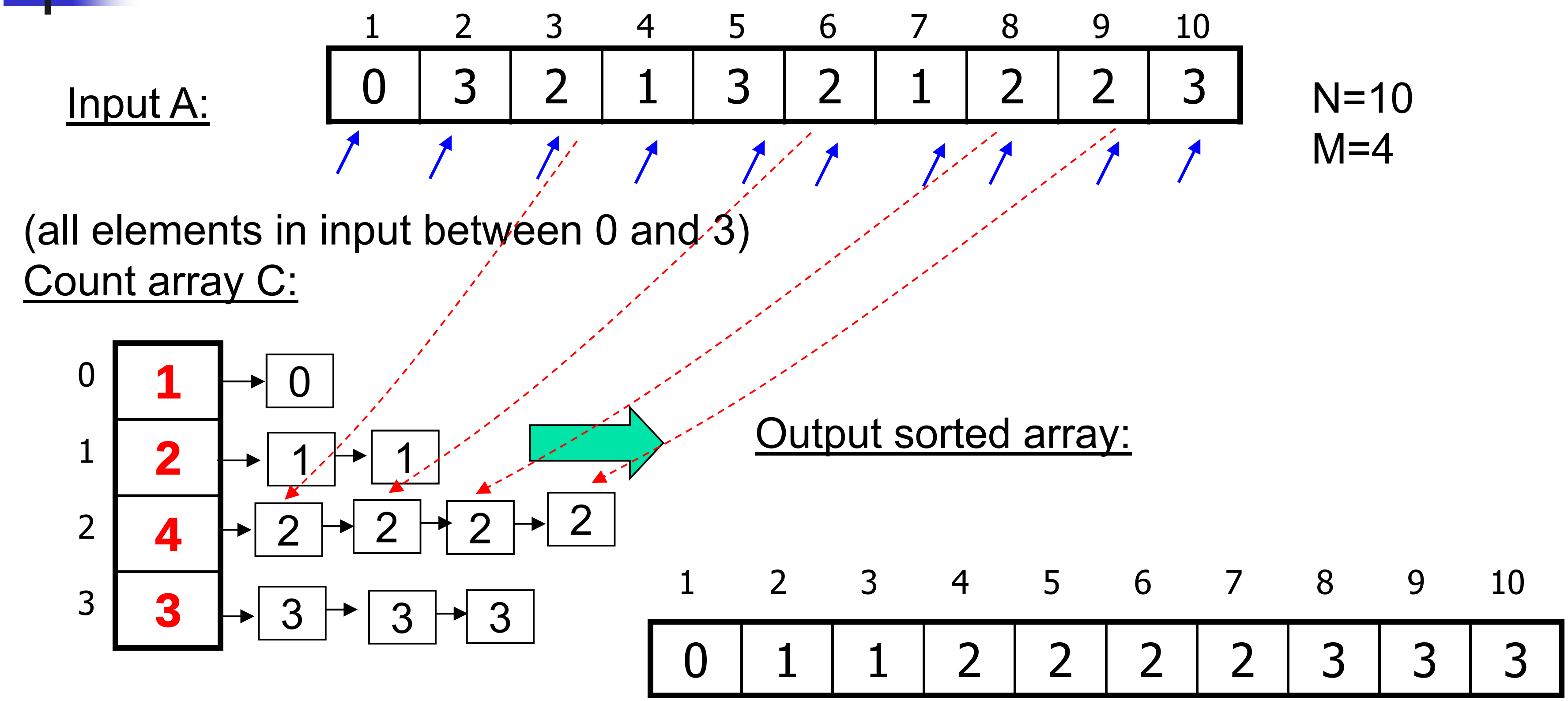
- A "stable" sorting method is one which preserves the original input order among duplicates in the output



Input:	0	3	2	1	3	2	1	2	2	3
Output:	0	1	1	2	2	2	2	3	3	3

Useful when each data is a struct of form { key, value }

How to make counting sort “stable”? (one approach)



Input A:

1	2	3	4	5	6	7	8	9	10
0	3	2	1	3	2	1	2	2	3

N=10
M=4

(all elements in input between 0 and 3)

Count array C:

0	1
1	2
2	4
3	3

Output sorted array:

1	2	3	4	5	6	7	8	9	10
0	1	1	2	2	2	2	3	3	3

But this algorithm is NOT in-place!
Can we make counting sort in place?
(i.e., without using another array or linked list)

How to make counting sort in place?

```
void CountingSort_InPlace(vector a, int n) {
    1. First construct Count array C s.t C[i] stores the last index in the bin
    corresponding to key i, where the next instance of i should be written
    to. Then do the following:

    i=0;
    while(i<n) {
            e=A[i];
            if c[e] has gone below range, then continue after i++;
            if(i==c[e]) i++;
            tmp = A[c[e]];
            A[c[e]--] = e;
            A[i] = tmp;
    }
}
```

Note: This code has to keep track of the valid range for each key

	0	1	2	3	4	5	6	7	8	9
A:	0	3	2	1	3	2	1	2	2	3

C:		End points	
0	1	**0**	-1
1	2	**2**	1 0
2	4	**6**	5 4 3 2
3	3	**9**	8 7 6

0	3	2	1	3	2	1	2	2	3
0	**3**	2	1	3	2	1	2	2	**3**
0	**2**	2	1	3	2	1	2	**3**	3
0	**1**	2	1	3	2	**2**	2	3	3
0	**2**	**1**	1	3	2	2	2	3	3
0	**2**	1	1	3	**2**	2	2	3	3
0	**3**	1	1	**2**	2	2	2	3	3
0	**3**	1	1	2	2	2	2	3	3
0	**2**	1	1	2	2	2	**3**	3	3
0	**1**	1	**2**	2	2	2	3	3	3

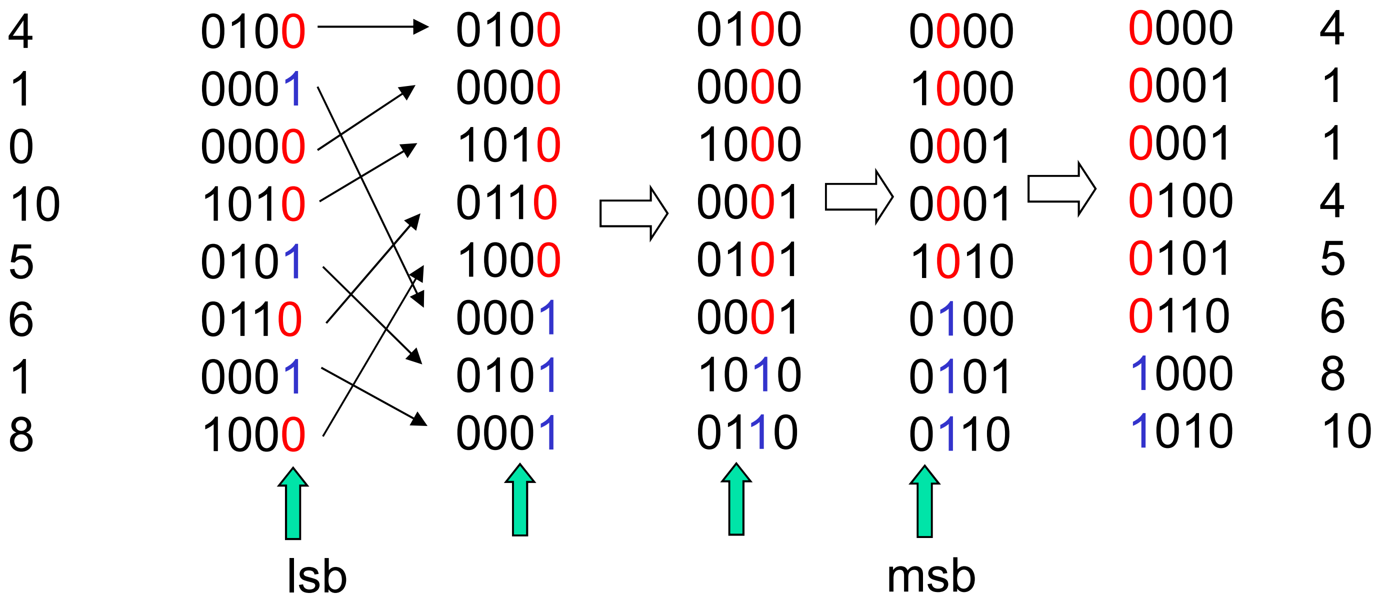
Bucket sort

- Assume N elements of A uniformly distributed over the range [0,1]
- Create M equal-sized buckets over [0,1], s.t., M≤N
- Add each element of A into appropriate bucket
- Sort each bucket internally
  - Can use recursion here, or
  - Can use something like InsertionSort
- Return concatentation of buckets
- Average case running time Θ(N)
  - assuming each bucket will contain Θ(1) elements

Radix Sort

- Radix sort achieves stable sorting
- To sort each column, use counting sort (O(n))
  => To sort k columns, O(nk) time

- Sort N numbers, each with k bits
- E.g, input {4, 1, 0, 10, 5, 6, 1, 8}



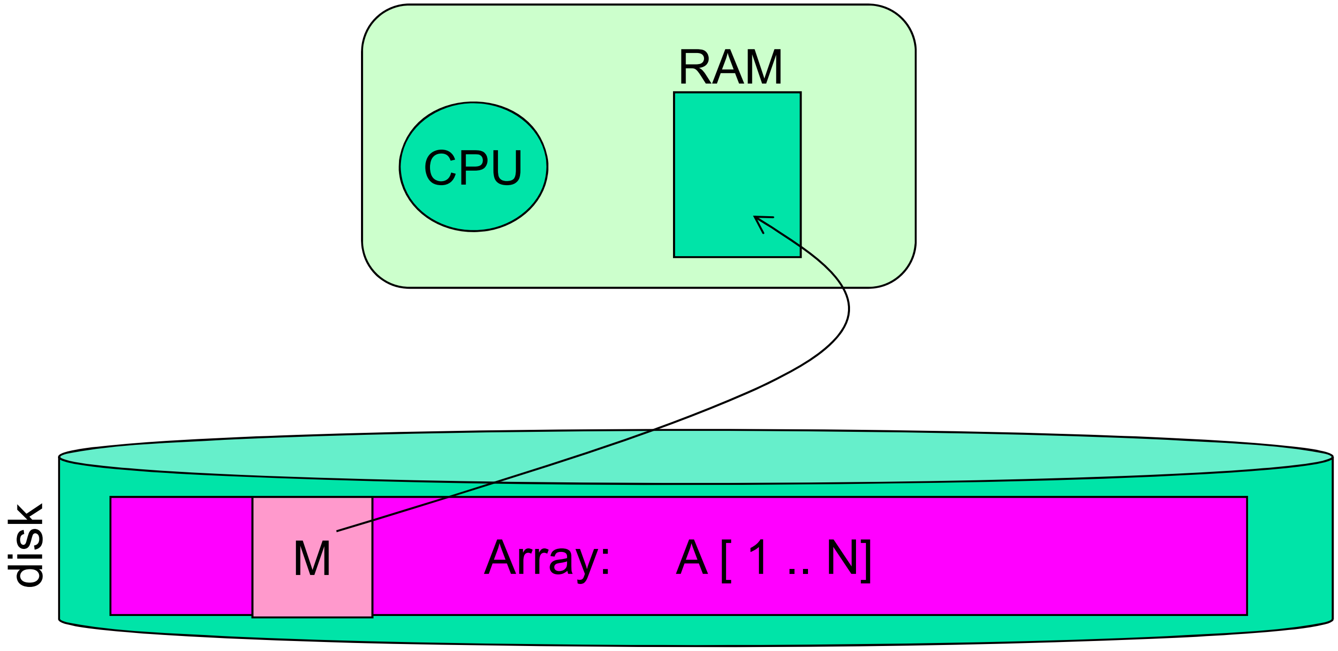
4	0100	0100	0100	0000	0000	4
1	0001	0000	0000	1000	0001	1
0	0000	1010	1000	0001	0001	1
10	1010	0110	0001	0001	0100	4
5	0101	1000	0101	1010	0101	5
6	0110	0001	0001	0100	0110	6
1	0001	0101	1010	0101	1000	8
8	1000	0001	0110	0110	1010	10

External Sorting

- What if the number of elements N we wish to sort do not fit in memory?
- Obviously, our existing sort algorithms are inefficient
  - Each comparison potentially requires a disk access
- Once again, we want to minimize disk accesses

External MergeSort

- N = number of elements in array A[1..N] to be sorted
- M = number of elements that fit in memory at any given time
- K = $\lceil N/M \rceil$



External MergeSort

O(M log M)

- Approach

  1. Read in M amount of A, sort it using local sorting (e.g., quicksort), and write it back to disk

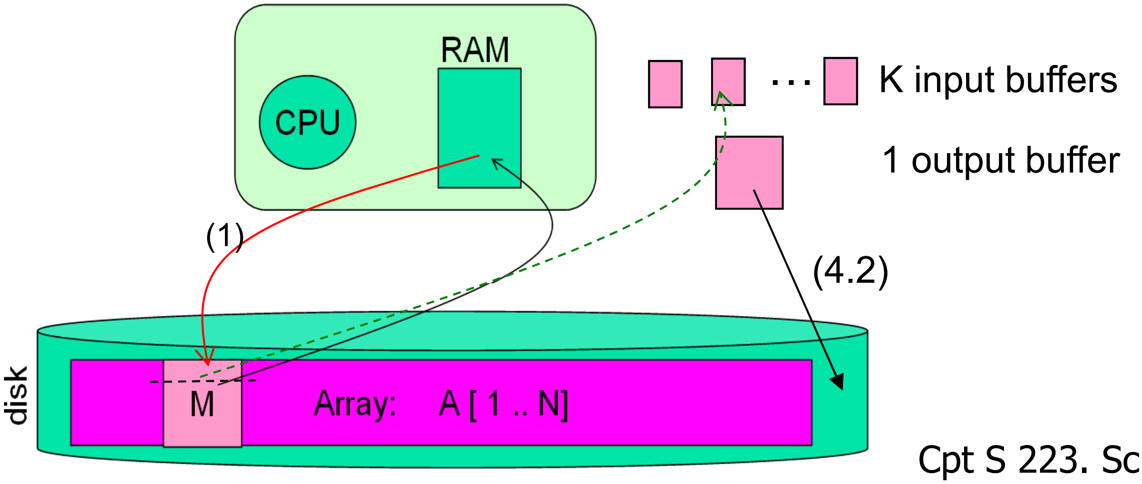
  Repeat above K times until all of A processed

  3. Create K input buffers and 1 output buffer, each of size M/(K+1)
  4. Perform a *K-way merge*:
     1. Update input buffers one disk-page at a time
     2. Write output buffer one disk-page at a time

O(KM log M)

O(N log k)

How?



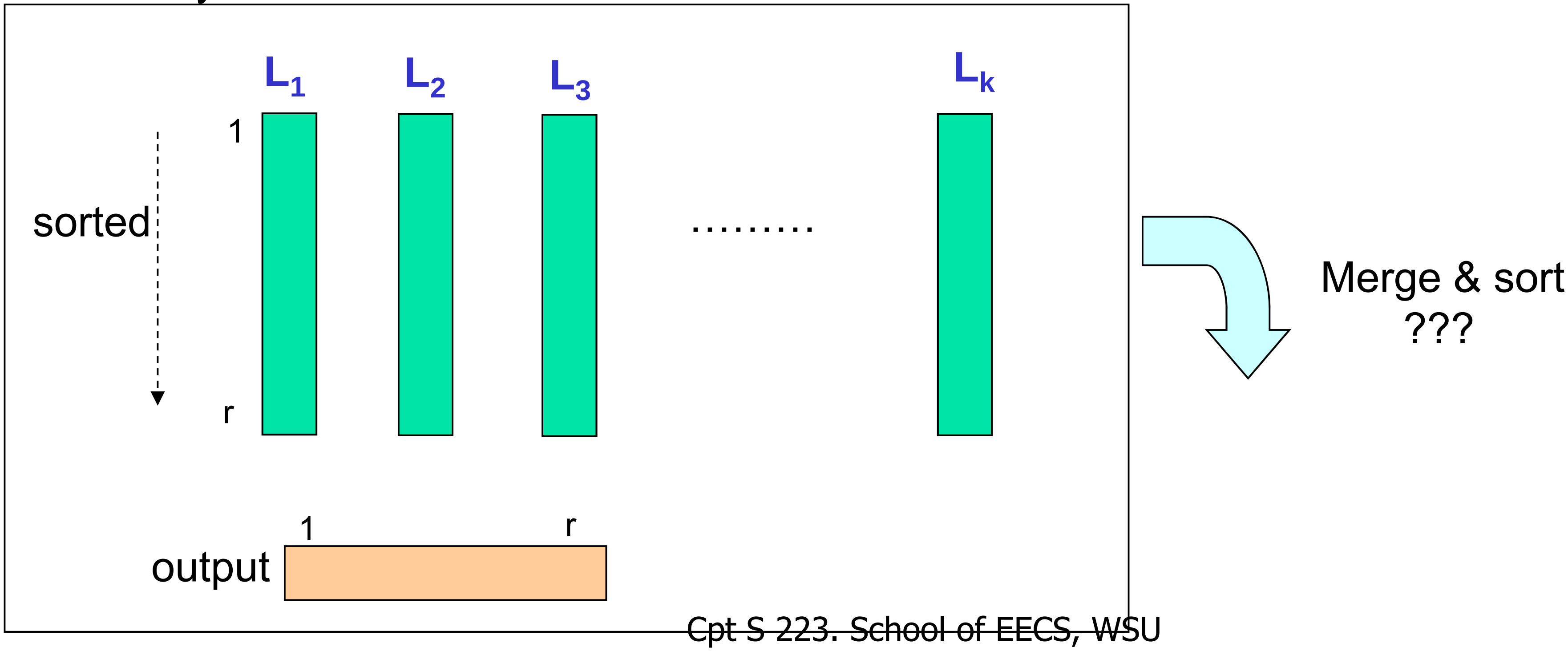
K-way merge

Q) How to merge k sorted arrays of total size N in O(N lg k) time?

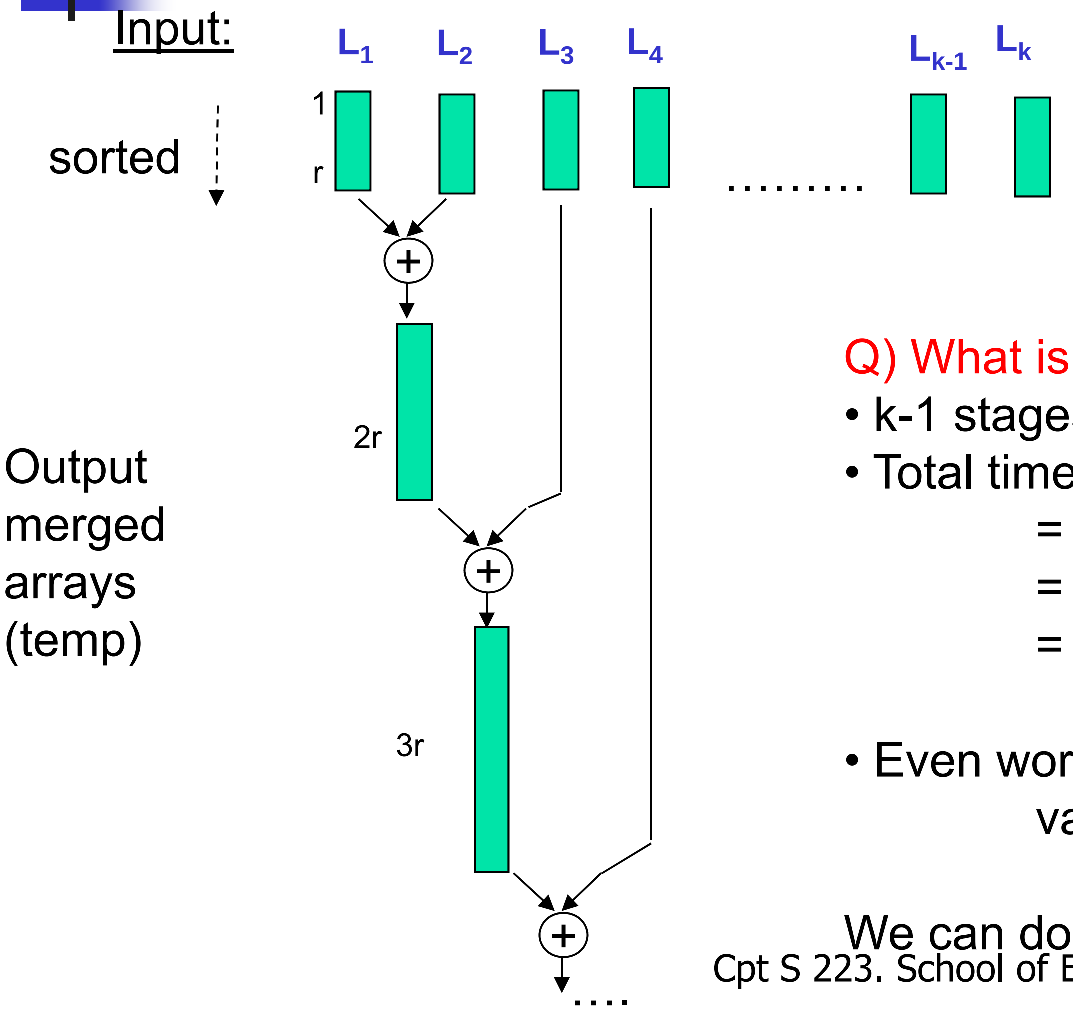
- For external merge sort:

  $r = M/(k+1)$ and $\sum_{i=0}^{k} |L_i| = M$

In memory:



K-way merge – a simple algo



Sum of sorted list lengths
= N (= kr)

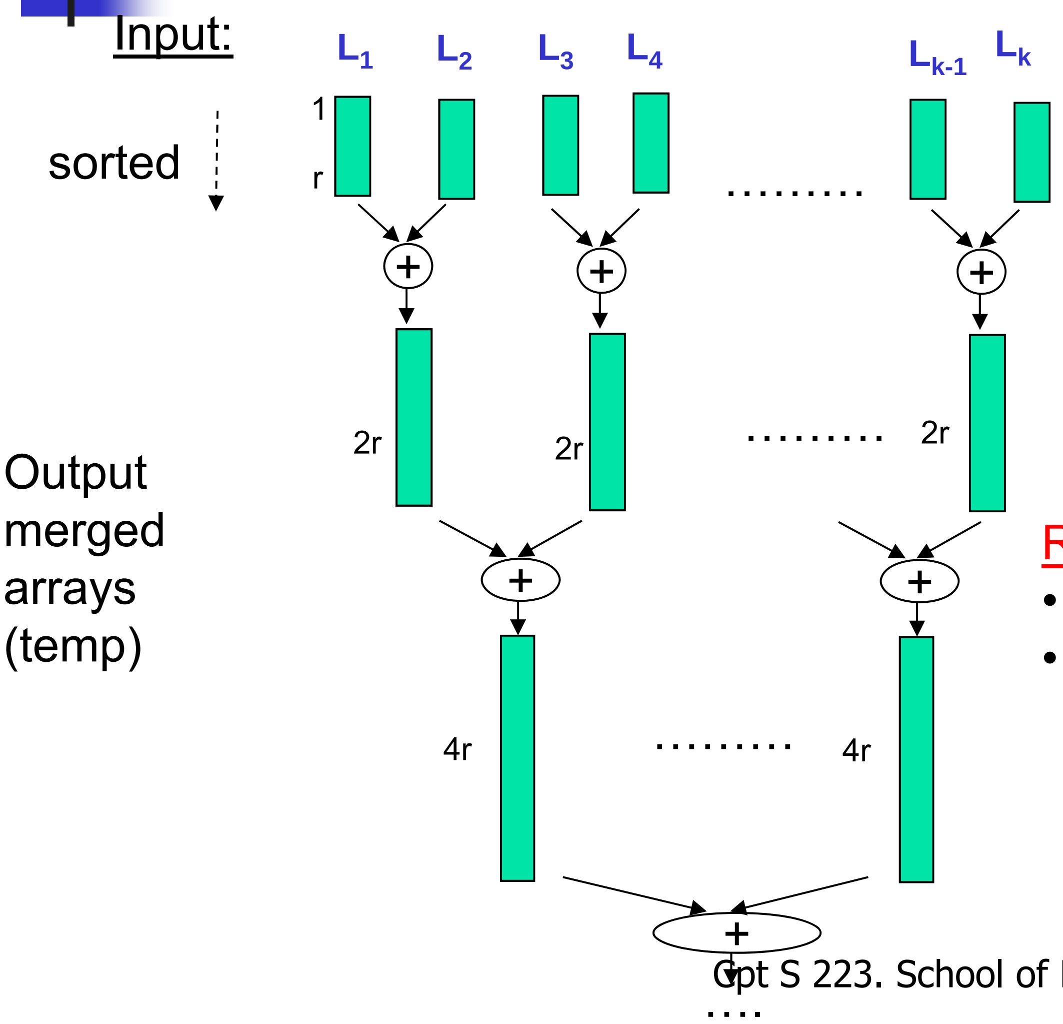
Q) What is the problem with this approach?

- k-1 stages
- Total time
  = 2r + 3r + 4r + 5r + … + kr
  = $O(k^2r)$
  = O(Nk)

- Even worse, if individual input arrays are of variable sizes

We can do better than this!

K-way merge – a better algo



Input:

sorted

Output merged arrays (temp)

Sum of sorted list lengths
= N (= kr)

Run-time Analysis

- lg k stages
- Total time

= (N) + (N) + … : lg k times
= O(N lg k)

External MergeSort

- Computational time T(N,M):

  = O(K*M log M) + O(N log K)

  = O((N/M)*M log M)+O(N log K)

  = O(N log M + N log K)

  = O(N log M)

- Disk accesses (all sequential)
  - P = page size
  - Accesses = O(N/P)

Sorting: Summary

- Need for sorting is ubiquitous in software
- Optimizing the sort algorithm to the domain is essential
- Good general-purpose algorithms available
  - QuickSort
- Optimizations continue…
  - Sort benchmarks

    http://sortbenchmark.org/

    http://research.microsoft.com/barc/sortbenchmark