Thoroughly Revised and Updated

Engineering Mathematics

For

GATE, ESE Prelims, ISRO & Other Technical Examinations

Comprehensive Theory with Solved Examples

Including **Previous Solved Questions of**

GATE (2003-2018) and **ESE-Prelims** (2017-2018)

Note: *Syllabus of ESE Mains Electrical Engineering also covered*

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Why this Book....

Over the period of time, the National Level Examination like ESE, GATE, ISRO, etc. have become more challenging & highly competitive due to the increasing number of aspirants and changing employment scenario. Engineering Mathematics is a key to success in all these competitive exams.

To excel in such National Level Examinations, students require a book that covers the entire syllabus, depth concepts of subject and previous years questions with solution to gain the confidence. So, there was a need for a comprehensive book in the market which can serve the needs of the students who are preparing for the same.

Experts at ICE GATE Institute have been doing a deep study and research on questions asked previously in such Examinations. It is found that a book containing rich theory and question bank will help the students in securing excellent marks in the Exam. Our team has come with a solution for all the students in the form of Engineering Mathematics. The highlighting features of this book are:

- It covers complete Engineering Mathematics Syllabus for GATE/ESE/ISRO etc.
- Previous Years GATE questions with solutions.
- Previous Years ESE questions with solutions.

Our experts have put a large amount of efforts to minimize the errors and give a perfect and reliable product for preparation. Any suggestions and feedback from the readers to improve our work will be acknowledged and appreciated. You can connect with us and send your valuable suggestions on content@icegateinstitute.com

SYLLABUS

GATE and ESE Prelims: Civil Engineering

Linear Algebra : Matrix algebra; Systems of linear equations; Eigen values and Eigen vectors.

Calculus : Functions of single variable; Limit, continuity and differentiability; Mean value theorems, local maxima and minima, Taylor and Maclaurin series; Evaluation of definite and indefinite integrals, application of definite integral to obtain area and volume; Partial derivatives; Total derivative; Gradient, Divergence and Curl, Vector identities, Directional derivatives, Line, Surface & Volume integrals, Stokes, Gauss and Green's theorems.

Ordinary Differential Equation (ODE) : First order (linear and non-linear) equations; higher order linear equations with constant coefficients; Euler-Cauchy equations; Laplace transform and its application in solving linear ODEs; initial and boundary value problems.

Partial Differential Equation (PDE): Fourier series; separation of variables; solutions of one-dimensional diffusion equation; first and second order onedimensional wave equation and two-dimensional Laplace equation.

Probability and Statistics : Definitions of probability and sampling theorems; Conditional probability; Discrete Random variables: Poisson and Binomial distributions; Continuous random variables: normal and exponential distributions; Descriptive statistics - Mean, median, mode and standard deviation; Hypothesis testing.

Numerical Methods : Accuracy and precision; error analysis. Numerical solutions of linear and non-linear algebraic equations; Least square approximation, Newton's & Lagrange polynomials, numerical differentiation, Integration by trapezoidal and Simpson's rule, single and multi-step methods for first order differential equations.

GATE and ESE Prelims: Mechanical Engineering

Linear Algebra : Matrix algebra, systems of linear equations, eigenvalues and eigenvectors.

Calculus : Functions of single variable, limit, continuity and differentiability, mean value theorems, indeterminate forms; evaluation of definite and improper integrals; double and triple integrals; partial derivatives, total derivative, Taylor series (in one and two variables), maxima and minima, Fourier series; gradient, divergence and curl, vector identities, directional derivatives, line, surface and volume integrals, applications of Gauss, Stokes and Green's theorems.

Differential equations : First order equations (linear and nonlinear); higher order linear differential equations with constant coefficients; Euler-Cauchy equation; initial and boundary value problems; Laplace transforms; solutions of heat, wave and Laplace's equations.

Complex Variables : Analytic functions; Cauchy-Riemann equations; Cauchy's integral theorem and integral formula; Taylor and Laurent series.

Probability and Statistics : Definitions of probability, sampling theorems, conditional probability; mean, median, mode and standard deviation; random variables, binomial, Poisson and normal distributions.

Numerical Methods : Numerical solutions of linear and non-linear algebraic equations; integration by trapezoidal and Simpson's rules; single and multi-step methods for differential equations.

GATE and ESE Prelims: Electrical Engineering

Linear Algebra : Matrix Algebra, Systems of linear equations, Eigenvalues, Eigenvectors.

Calculus : Mean value theorems, Theorems of integral calculus, Evaluation of definite and improper integrals, Partial Derivatives, Maxima and minima, Multiple integrals, Fourier series, Vector identities, Directional derivatives, Line integral, Surface integral, Volume integral, Stokes's theorem, Gauss's theorem, Green's theorem.

Differential equations : First order equations (linear and nonlinear), Higher order linear differential equations with constant coefficients, Method of variation of parameters, Cauchy's equation, Euler's equation, Initial and boundary value problems, Partial Differential Equations, Method of separation of variables. Complex Variables: Analytic functions, Cauchy's integral theorem, Cauchy's integral formula, Taylor series, Laurent series, Residue theorem, Solution integrals.

Probability and Statistics : Sampling theorems, Conditional probability, Mean, Median, Mode, Standard Deviation, Random variables, Discrete and Continuous distributions, Poisson distribution, Normal

distribution, Binomial distribution, Correlation analysis, Regression analysis. Numerical Methods: Solutions of nonlinear algebraic equations, Single and Multi-step methods for differential equations.
Transform Theory : Fourier Transform, Laplace Transform, z-Transform.

Electrical Engineering ESE Mains

Matrix theory, Eigen values & Eigen vectors, system of linear equations, Numerical methods for solution of non-linear algebraic equations and differential equations, integral calculus, partial derivatives, maxima and minima, Line, Surface and Volume Integrals. Fourier series, linear, nonlinear and partial differential equations, initial and boundary value problems, complex variables, Taylor's and Laurent's series, residue theorem, probability and statistics fundamentals, Sampling theorem, random variables, Normal & Poisson distributions, correlation and regression analysis.

GATE and ESE Prelims: Electronics Engineering

Linear Algebra : Vector space, basis, linear dependence and independence, matrix algebra, eigen values and eigen vectors, rank, solution of linear equations – existence and uniqueness.

Calculus : Mean value theorems, theorems of integral calculus, evaluation of definite and improper integrals, partial derivatives, maxima and minima, multiple integrals, line, surface and volume integrals, Taylor series.

Differential equations : First order equations (linear and nonlinear), higher order linear differential equations, Cauchy's and Euler's equations, methods of solution using variation of parameters, complementary function and particular integral, partial differential equations, variable separable method, initial and boundary value problems.

Vector Analysis : Vectors in plane and space, vector operations, gradient, divergence and curl, Gauss's, Green's and Stoke's theorems.

Complex Analysis : Analytic functions, Cauchy's integral theorem, Cauchy's integral formula; Taylor's and Laurent's series, residue theorem.

Numerical Methods : Solution of nonlinear equations, single and multi-step methods for differential equations, convergence criteria.

Probability and Statistics : Mean, median, mode and standard deviation; combinatorial probability, probability distribution functions - binomial, Poisson, exponential and normal; Joint and conditional probability; Correlation and regression analysis.

GATE : Instrumentation Engineering

Linear Algebra : Matrix algebra, systems of linear equations, Eigen values and Eigen vectors.

Calculus : Mean value theorems, theorems of integral calculus, partial derivatives, maxima and minima, multiple integrals, Fourier series, vector identities, line, surface and volume integrals, Stokes, Gauss and Green's theorems.

Differential Equations : First order equation (linear and nonlinear), higher order linear differential equations with constant coefficients, method of variation of parameters, Cauchy's and Euler's equations, initial and boundary value problems, solution of partial differential equations: variable separable method.

Analysis of complex variables : Analytic functions, Cauchy's integral theorem and integral formula, Taylor's and Laurent's series, residue theorem, solution of integrals.

Complex Variables : Analytic functions, Cauchy's integral theorem and integral formula, Taylor's and Laurent' series, Residue theorem, solution integrals.

Probability and Statistics : Sampling theorems, conditional probability, mean, median, mode and standard deviation, random variables, discrete and continuous distributions : normal, Poisson and binomial distributions.

Numerical Methods : Matrix inversion, solutions of non-linear algebraic equations, iterative methods for solving differential equations, numerical integration, regression and correlation analysis.

GATE : Computer Science & IT Engineering

Linear Algebra : Matrices, determinants, system of linear equations, eigenvalues and eigenvectors, LU decomposition.

Calculus : Limits, continuity and differentiability. Maxima and minima. Mean value theorem. Integration.

Probability : Random variables. Uniform, normal, exponential, poisson and binomial distributions. Mean, median, mode and standard deviation. Conditional probability and Bayes theorem.

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1 BASIC CONCEPTS

1.1 Powers & Roots

(i) $a^0 = 1,\ a \neq 0$ (ii) $a^m a^n = a^{m+n}$ (iii) $\dfrac{a^m}{a^n} = a^{m-n}$

(iv) $(ab)^m = a^m b^m$ (v) $\left(\dfrac{a}{b}\right)^m = \dfrac{a^m}{b^m}$ (vi) $\left(a^m\right)^n = a^{mn}$

(vii) $a^{-m} = \dfrac{1}{a^m}$ (viii) $\sqrt[n]{ab} = \sqrt[n]{a}\sqrt[n]{b}$ (ix) $\sqrt[n]{\dfrac{a}{b}} = \dfrac{\sqrt[n]{a}}{\sqrt[n]{b}}$

(x) $a^{\frac{m}{n}} = \sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m$

1.2 Logarithms

Definition : $y = \log_a(x)$, If and only if $a^y = x$ where a , $x > 0$ and $a \neq 1$.

Natural logarithm : $y = \log_e(x) = \ln(x)$, If and only if $e^y = x$

Where, $e = \lim\limits_{x\to\infty}\left(1+\dfrac{1}{x}\right)^x = 2.71828182846...$

(i) $\log_a 1 = 0$

(ii) $\log_a a = 1$

(iii) $\log_a(mn) = \log_a m + \log_a n$

(iv) $\log_a\left(\dfrac{m}{n}\right) = \log_a m - \log_a n$

(v) $\log_a\left(m^n\right) = n\log_a m$

(vi) $\log_b a = \dfrac{1}{\log_a b}$

(vii) $\log_{(a^k)}(m) = \dfrac{1}{k}\log_a m$

(viii) $\log_a m = \log_a b \log_b m$ where $b > 0$ and $b \neq 1$

(ix) $\log_a m = \dfrac{\log_b m}{\log_b a}$

(x) $x^{\log_a y} = y^{\log_a x}$

(xi) $x = x^{\log_a a} = a^{\log_a x}$

(xii) $x = e^{\ln x} = \ln\left(e^x\right)$

1.3 Binomial Theorem

(i) Factorials

(a) $n! = 1 \times 2 \times 3 \times ... \times (n-1) \times n$ (b) $0! = 1! = 1$

(ii) Binomial Coefficient $^nC_r = \dfrac{n!}{r!(n-r)!}$

(iii) Binomial Theorem

$$(x+y)^n = {}^nC_0x^n + {}^nC_1x^{n-1}y + {}^nC_2x^{n-2}y^2 + ... + {}^nC_rx^{n-r}y^r + ... + {}^nC_ny^n$$

(iv) Product Formulas

(a) $(a+b)^2 = a^2 + 2ab + b^2$

(b) $(a-b)^2 = a^2 - 2ab + b^2$

(c) $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

(d) $(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$

(v) Factoring Formulas

(a) $a^2 - b^2 = (a-b)(a+b)$

(b) $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$

(c) $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$

(d) $a^{2n} - b^{2n} = (a^n - b^n)(a^n + b^n)$

(e) $a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + a^{n-3}b^2 + ... + ab^{n-2} + b^{n-1})$

Ex : $(1-x^n) = (1-x)(1 + x + x^2 + x^3 + ... + x^{n-1})$

(f) If n is odd then,

$a^n + b^n = (a+b)(a^{n-1} - a^{n-2}b + a^{n-3}b^2 - ... - ab^{n-2} + b^{n-1})$

(g) $a^5 - b^5 = (a-b)(a^4 + a^3b + a^2b^2 + ab^3 + b^4)$

(h) $a^5 + b^5 = (a+b)(a^4 - a^3b + a^2b^2 - ab^3 + b^4)$

1.4 Sequences

(i) Arithmetic sequence

a, a + d, a + 2d, a + 3d, ...

n^{th} term $= t_n = a + (n-1)d$

Sum of First n-terms $= S_n = \dfrac{n}{2}[2a + (n-1)d] = \dfrac{n}{2}(\text{First Term} + \text{Last Term})$

(ii) Geometric Sequence :

$a, ar, ar^2, ar^3, ...$

n^{th} term $= t_n = ar^{n-1}$

$$\text{Sum of First n-terms} = S_n = \begin{cases} \dfrac{a(r^n-1)}{(r-1)} & |r|>1 \\ na & r=1 \\ \dfrac{a(1-r^n)}{(1-r)} & |r|<1 \end{cases}$$

(iii) Sum to infinite terms of geometric sequence

$$S_\infty = a + ar + ar^2 + ar^3 .... = \frac{a}{1-r}, \qquad -1<r<1$$

1.5 Mean Values of n real numbers

$a_1, a_2, a_3, ..., a_n$

(i) Arithmetic mean : $\dfrac{a_1+a_2+a_3+....+a_n}{n}$

(ii) Geometric mean : $(a_1 \cdot a_2 \cdot a_3 \cdot .... \cdot a_n)^{\frac{1}{n}}$

(iii) Harmonic mean : $\dfrac{n}{\left(\dfrac{1}{a_1}+\dfrac{1}{a_2}+\dfrac{1}{a_3}+...+\dfrac{1}{a_n}\right)}$

1.6 Formulas for Summation

(i) $1+2+3+....+n = \dfrac{n(n+1)}{2}$

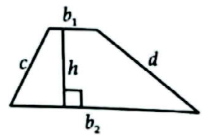
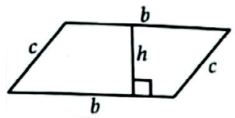
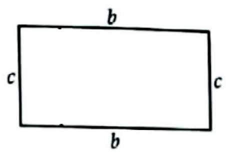
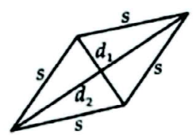
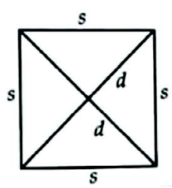
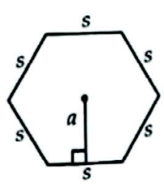
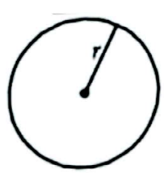
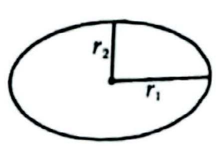
(ii) $1+3+5+....+(2n-1) = n^2$

(iii) $2+4+6+....+(2n) = n(n+1)$

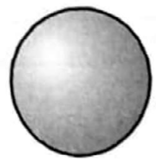
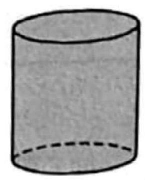
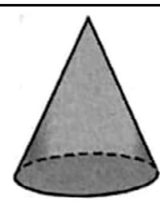
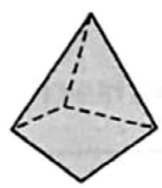
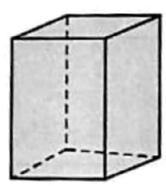
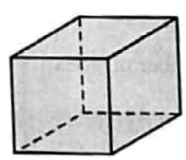
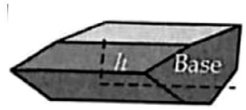
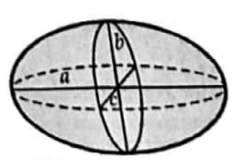
(iv) $1^2+2^2+3^2+....+n^2 = \dfrac{n(n+1)(2n+1)}{6}$

(v) $1^3+2^3+3^3+....+n^3 = \dfrac{n^2(n+1)^2}{4}$

1.7 Two Dimensional Geometry

Shape	Figure	Perimeter	Area
Trapezoid		$P = b_1 + b_2 + c + d$ b_1, b_2 = bases c, d = sides	$A = \frac{1}{2}(b_1 + b_2)h$ b_1, b_2 = bases h = height
Parallelogram		$P = 2b + 2c$ b, c = sides	$A = bh$ b = base h = height
Rectangle		$P = 2b + 2c$ b, c = sides	$A = bc$
Rhombus		$P = 4s$ s = side	$A = \frac{1}{2}(d_1 d_2)$ d_1, d_2 = diagonals
Square		$P = 4s$ s = side	$A = s^2 = \frac{1}{2}d^2$ d = diagonals
Regular Polygon		$P = ns$ n = number of sides s = side	$A = \frac{1}{2} a \cdot P$ a = apothem P = perimeter
Circle		$C = 2\pi r = \pi d$ $r = radius$ $d = diameter$	$A = \pi r^2$ $r = radius$
Ellipse		$P \approx 2\pi\sqrt{\frac{1}{2}(r_1^2 + r_2^2)}$ r_1 = major axis radius r_2 = minor axis radius	$A = \pi r_1 r_2$ r_1 = major axis radius r_2 = minor axis radius

8. Three Dimensional Geometry

Shape	Figure	Perimeter	Area
Sphere		$SA = 4\pi r^2$ r = radius	$V = \frac{4}{3}\pi r^3$ r = radius
Right Cylinder		$SA = 2\pi rh + 2\pi r^2$ h = *height* r = radius of base	$V = \pi r^2 h$ h = *height* r = radius of base
Cone		$SA = \pi rl + \pi r^2$ l = *slant height* r = radius of base	$V = \frac{1}{3}\pi r^2 h$ h = *height* r = radius of base
Square Pyramid		$SA = 2sl + s^2$ s = base side length r = slant height	$V = \frac{1}{3}s^2 h$ s = base side length h = height
Rectangular Prism		$SA = 2(lw + lh + wh)$ l = *length* w = *width* h = *height*	V = lwh l = *length* w = *width* h = height
Cube		$SA = 6s^2$ s = side length (all sides)	$V = s^3$ s = side length (all sides)
General Right Prism		SA = Ph + 2B P = Perimeter of base h = heigth (or length) B = area of base	V = Bh B = area of base h = height
Ellipsoid		–	$\frac{4}{3}\pi abc$

1.9 Theory of Equations

1.9.1 Linear Equation in One Variable

$$ax + b = 0$$

$$x = -\frac{b}{a}$$

1.9.2 Quadratic Equation

$ax^2 + bx + c = 0$ & $a, b, c \in R$

Roots: $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Discriminant : $D = b^2 - 4ac$

If $D > 0$ then Roots are real and distinct
If $D = 0$ then Roots are real and equal
If $D < 0$ then Roots are complex conjugates

Algebraic equation : An equation $f(x) = 0$, where $f(x)$ contains algebraic functions (Polynomial , Rational function etc).

Transcendental Equation : An equation $f(x) = 0$, where $f(x)$ contains non-algebraic functions(Trigonometric , Exponential , Logarithmic Function etc).

1.9.3 Polynomial Equations

A polynomial equation has the form

$f(x) = a_n x^n + a_{n-1} x^{n-1} + ... + a_1 x + a_0 = 0$, Where all a_i are real

Results

(i) a is root of f(x) = 0 if and only if f(a) = 0

(ii) **Factor Theorem :** a is root of f(x) = 0, If and only If (x-a) is factor of f(x).

(iii) Every polynomial equation of *n*th degree has exactly *n* roots (real or complex).

(iv) Every polynomial equation of odd degree has at least one real root.

(v) **Descartes' rule of sign :** A polynomial equation f(x) = 0 cannot have more number of positive roots than the number of sign changes in coefficients of f(x) and cannot have more number of negative roots than the changes of sign in coefficients of $f(-x)$.

(a) Number of positive real roots $\leq$ Number of sign changes in coefficient of f(x).

(b) Number of negative real roots $\leq$ Number of sign changes in coefficient of f(-x).

Remember :

(1) If all the coefficients are positive then the equation has no positive real root.

(2) If the coefficients of even powers of x are all of one sign and the coefficient of the odd powers are all of opposite sign, then the Polynomial equation f(x) = 0 has no negative real root.

(3) If the equation contains only even powers of x and the coefficients are all of the same sign, the equation has no real root.

(4) If the equation contains only odd powers of x, and the coefficients are all of the same sign, the equation has no real root. Except x=0.

1.9.4 Roots of Polynomial Equation

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + ... + a_{n-1}x + a_n = 0$$

Sum of roots $= -\frac{a_1}{a_0}$

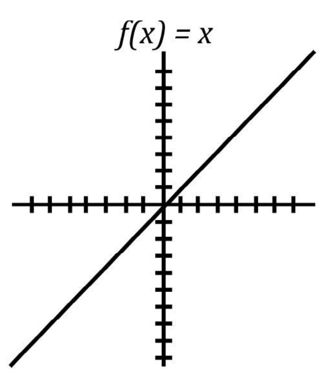
Sum of the products of the roots taken two at a time $= (-1)^2 \frac{a_2}{a_0}$

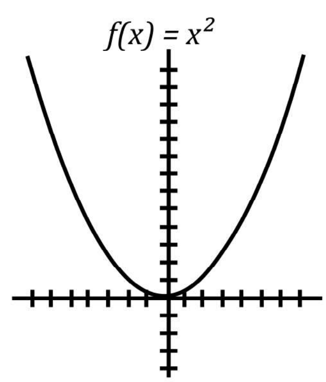
Sum of the products of the roots taken three at a time $= (-1)^3 \frac{a_3}{a_0}$

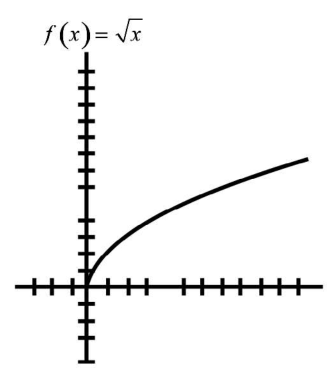
Product of roots $= (-1)^n \frac{a_n}{a_0}$

1.10. Basic Function

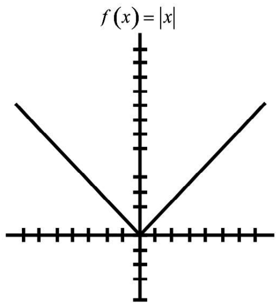
(i) Identity Function (ii) Squaring Function (iii) Square Root Function

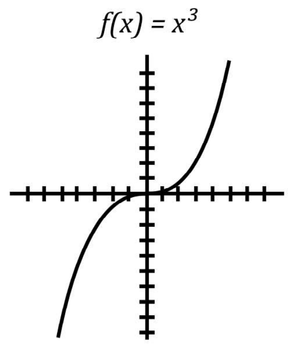


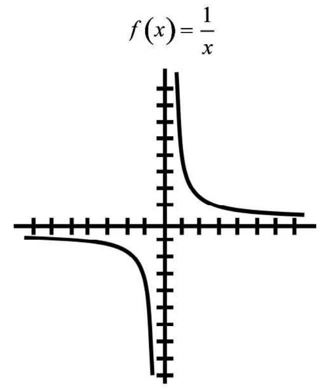




(iv) Absolute Function (v) Cubing Function (vi) Reciprocal Function

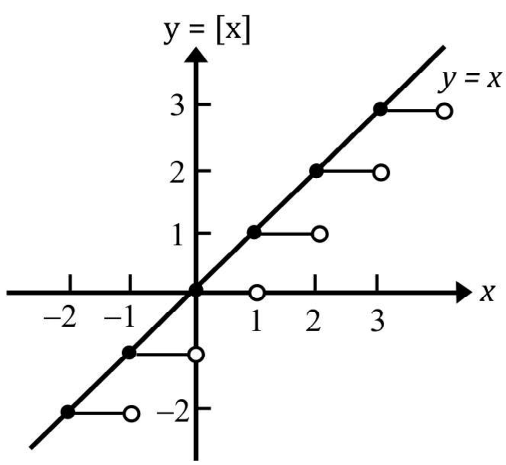






(vii) Greatest Integer Function :

$\Rightarrow [x]$: Greatest integer less than or equal to x

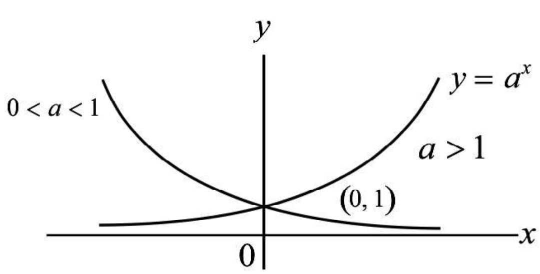


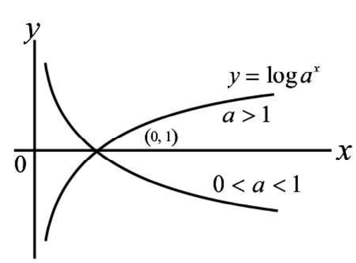
(viii) Exponential Function

$y = a^x, \ a > 0, (a \neq 1)$

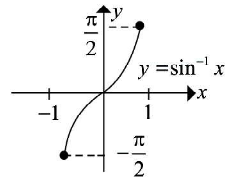
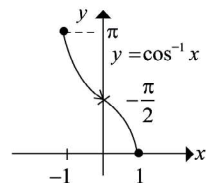
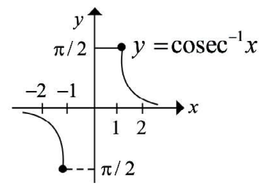
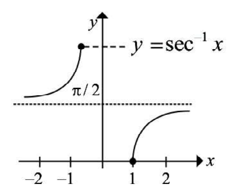
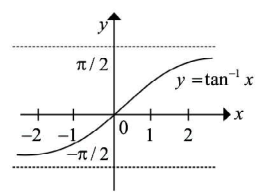
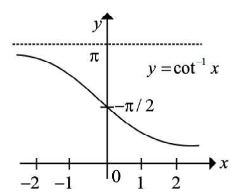
(ix) Logarithmic Function

$y = \log_a x, \ a > 0, (a \neq 1) \Rightarrow y = \log_e x = \ln x$





(ix) Inverse Trigonometric Function

Shape	Domain	Range	Graph
1. $y = \sin^{-1} x$ iff $\sin y = x$	$-1 \le x \le 1$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$	
2. $y = \cos^{-1} x$ iff $\cos y = x$	$-1 \le x \le 1$	$0 \le y \le \pi$	
3. $y = \operatorname{cosec}^{-1} x$ iff $\operatorname{cosec} y = x$	$x \le -1$ or $x \ge 1$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2},\ y \ne 0$	
4. $y = \sec^{-1} x$ iff $\sec y = x$	$x \le -1$ or $x \ge 1$	$0 \le y \le \pi,\ y \ne \frac{\pi}{2}$	
5. $y = \tan^{-1} x$ iff $\tan y = x$	$-\infty < x < \infty$	$-\frac{\pi}{2} < y < \frac{\pi}{2}$	
6. $y = \cot^{-1} x$ iff $\cot y = x$	$-\infty < x < \infty$	$0 < y < \pi$	

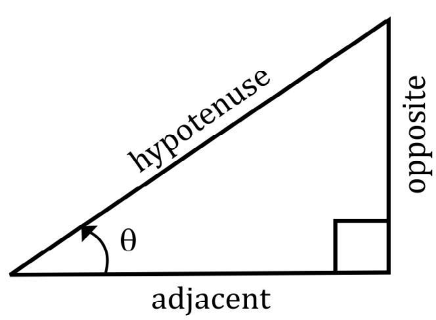
11.1. Trigonometry

1. Trigonometric Functions

$$\sin\theta = \frac{\text{opp}}{\text{hyp}} \qquad \operatorname{cosec}\theta = \frac{\text{hyp}}{\text{opp}}$$

$$\cos\theta = \frac{\text{adj}}{\text{hyp}} \qquad \sec\theta = \frac{\text{hyp}}{\text{adj}}$$

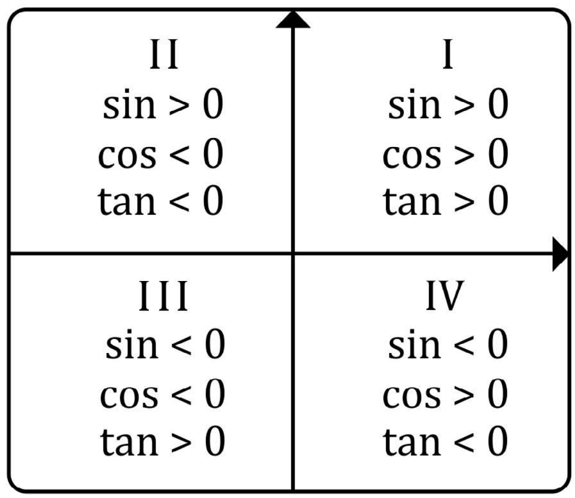
$$\tan\theta = \frac{\text{opp}}{\text{adj}} \qquad \cot\theta = \frac{\text{adj}}{\text{opp}}$$



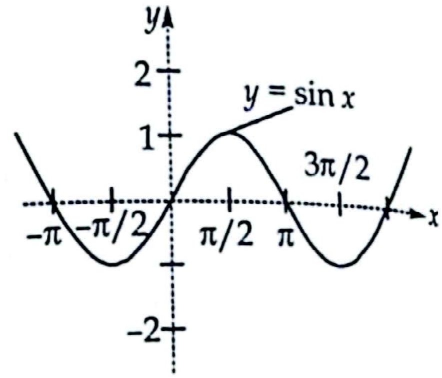
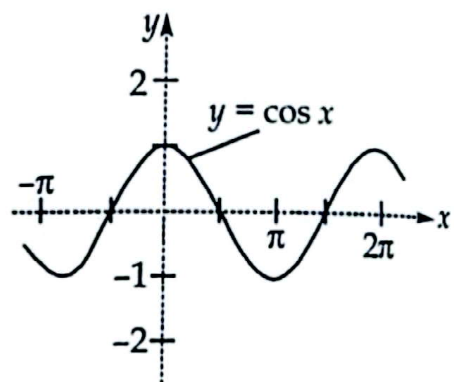
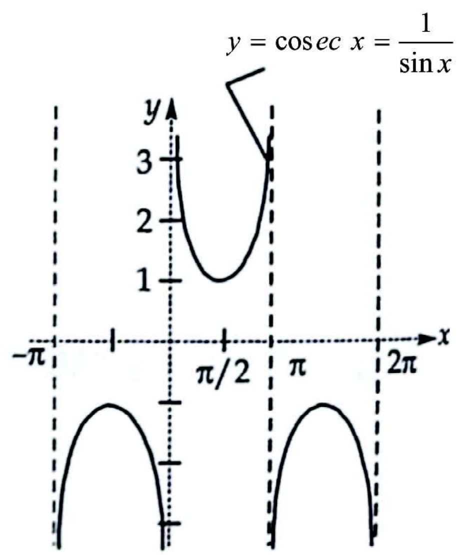
2. Trigonometric Functions of common angles

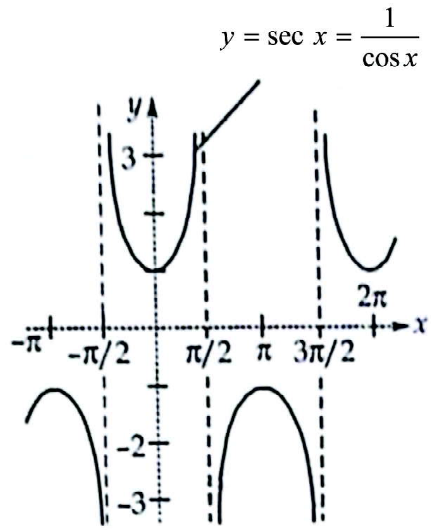
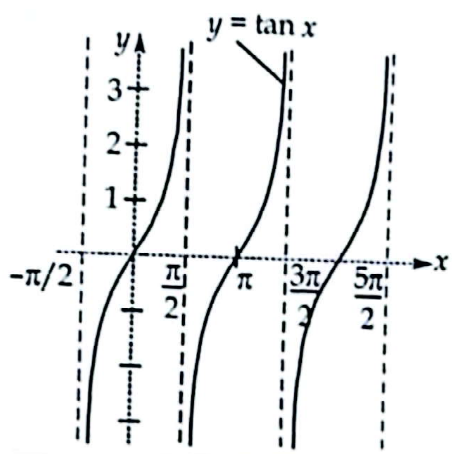
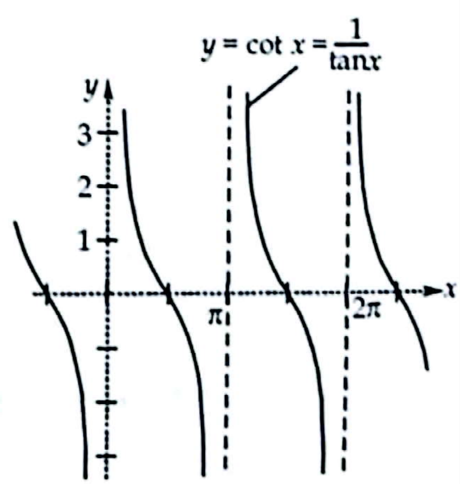
θ	0	30	45	60	90
$\sin\theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
$\tan\theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	Not define

3. Signs of Trigonometry Functions by Quadrant.



4. Properties

Function	Domain	Range	Period	Zeros	Graph
1. $\sin x$	$-\infty < x < \infty$	$-1 \le y \le 1$	2π	$\sin x = 0$ *iff* $x = n\pi$ n is integer	
2. $\cos x$	$-\infty < x < \infty$	$-1 \le y \le 1$	2π	$\cos x = 0$ *iff* $x = (2n+1)\frac{\pi}{2}$ n is integer	
3. $\operatorname{cosec} x$	$x \ne n\pi$	$y \le -1$ or $y \ge 1$	2π	No zeros	

Function	Domain	Range	Period	Zeros	Graph
4. sec x	$x \neq (2n+1)\frac{\pi}{2}$	$y \leq -1$ or $y \geq 1$	2π	No zeros	
5. $\tan x$	$x \neq (2n+1)\frac{\pi}{2}$	$-\infty < y < \infty$	π	$\tan x = 0$ iff $x = n\pi$ n is integer	
6. $\cot x$	$x \neq n\pi$	$-\infty < y < \infty$	π	$\cot x = 0$ iff $x = (2n+1)\frac{\pi}{2}$ n is integer	

5. Important Identities

(i) Pythagorean Identities

$$\sin^2 x + \cos^2 x = 1$$
$$1 + \tan^2 x = \sec^2 x$$
$$1 + \cot^2 x = \operatorname{cos}ec^2 x$$

(ii) Even-Odd Identities

$$\sin(-x) = -\sin x$$
$$\cos(-x) = \cos x$$
$$\tan(-x) = -\tan x$$

(iii) Sum-Difference Formulas

$$\sin(x+y) = \sin x \cos y + \cos x \sin y$$
$$\sin(x-y) = \sin x \cos y - \cos x \sin y$$
$$\cos(x+y) = \cos x \cos y - \sin x \sin y$$
$$\cos(x-y) = \cos x \cos y + \sin x \sin y$$
$$\tan(x+y) = \frac{\tan(x) + \tan(y)}{1 - \tan(x)\tan(y)}$$
$$\tan(x-y) = \frac{\tan(x) - \tan(y)}{1 + \tan(x)\tan(y)}$$

(iv) Double Angle Formulas

$$\sin 2x = 2 \sin x \cos x$$
$$\cos 2x = \cos^2 x - \sin^2 x$$
$$= 2\cos^2 x - 1$$
$$= 1 - 2\sin^2 x$$
$$\tan(2x) = \frac{2\tan(x)}{1 - \tan^2(x)}$$
$$\sin(2x) = \frac{2\tan(x)}{1 + \tan^2(x)}$$
$$\cos(2x) = \frac{1 - \tan^2(x)}{1 + \tan^2(x)}$$

(v) Half Angle Formulas

$$\sin^2(x) = \frac{1 - \cos(2x)}{2}$$
$$\cos^2(x) = \frac{1 + \cos(2x)}{2}$$
$$\tan^2(x) = \frac{1 - \cos(2x)}{1 + \cos(2x)}$$

(vi) Product-to-Sum Formulas

$$\sin(x)\sin(y) = \frac{1}{2}\left[\cos(x-y) - \cos(x+y)\right]$$
$$\cos(x)\cos(y) = \frac{1}{2}\left[\cos(x-y) + \cos(x+y)\right]$$
$$\sin(x)\cos(y) = \frac{1}{2}\left[\sin(x+y) + \sin(x-y)\right]$$
$$\cos(x)\sin(y) = \frac{1}{2}\left[\sin(x+y) - \sin(x-y)\right]$$

(vii) Sum to Product Formulas

$$\sin(x) + \sin(y) = 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$
$$\sin(x) - \sin(y) = 2\sin\left(\frac{x-y}{2}\right)\cos\left(\frac{x+y}{2}\right)$$
$$\cos(x) + \cos(y) = 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$
$$\cos(x) - \cos(y) = -2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$$

(viii) Multiple Angle Formulas

$$\sin 3x = 3\sin x - 4\sin^3 x$$
$$\cos 3x = 4\cos^3 x - 3\cos x$$
$$\tan 3x = \frac{3\tan x - \tan^3 x}{1 - 3\tan^2 x}$$

(ix) Relations to Hyperbolic function

$$\sin(ix) = i \sinh x$$
$$\cos(ix) = \cosh x$$
$$\sec(ix) = \operatorname{sech} x$$
$$\operatorname{cosec}(ix) - i \operatorname{cosech} x$$
$$\tan(ix) = i \tanh x$$
$$\cot(ix) = -i \coth x$$

2 LINEAR ALGEBRA

2.1 INTRODUCTION

Linear algebra is the branch of mathematics concerning vector spaces, often finite or countably infinite dimensional, as well as linear mappings between such spaces. Such an investigation is initially motivated by a system of linear equations containing several unknowns. Such equations are naturally represented using the formalism of matrices and vectors.

2.2 MATRICES

2.2.1 DEFINITION

A matrix is defined as a rectangular array of numbers or functions arranged in 'm' rows and 'n' columns such that each row has same number of elements (n) and each column has same number of elements(m).

If A = $[a_{ij}]_{mxn}$ be any matrix of order m x n then it is written as

$$A = \left[a_{ij}\right]_{mxn} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}_{mxn}$$

Where, a_{ij} represent the element of i^{th} row and j^{th} column.

2.2.2 TYPES OF MATRICES

(1) SQUARE MATRIX

A matrix with m=n i.e. with no of rows= no of columns is called a square matrix.

Example : $A = \begin{bmatrix} 1 & 2 & -1 \\ 4 & 8 & 6 \\ 1 & 3 & 7 \end{bmatrix}_{3x3}$ is a 3×3 square matrix

Note : **(1)** The elements a_{ij} with i=j i.e. a_{11}, a_{22}, a_{33} are called **diagonal elements** and all the elements together called as **principle diagonal** of a Square matrix.

(2) Trace of matrix is the addition of all the diagonal elements.

Example : $A = \begin{bmatrix} 1 & 2 & -1 \\ 4 & 8 & 6 \\ 1 & 3 & 7 \end{bmatrix}_{3x3}$

Trace (A) = 1+8+7=16

(2) DIAGONAL MATRIX

A diagonal matrix is a square matrix with all the non-diagonal elements 0. The diagonal matrix is completely defined by the diagonal elements. The diagonal elements may or may not be zero.

Example: $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 7 \end{bmatrix}_{3x3}$ is a diagonal matrix.

A diagonal Matrix can also be written as A=diag[1,8,7]

Properties of Diagonal Matrix:

(1) diag[a,b,c] + diag[d , e , f] = diag[a+d, b+e, c+f]

(2) diag[a, b, c] diag[d, e, f]=diag[ad, be, cf]

(3) $(\text{diag}[a, b, c])^{-1} = \text{diag}\left[\frac{1}{a}, \frac{1}{b}, \frac{1}{c}\right]$

(4) $(\text{diag}[a, b, c])^{T} = \text{diag}[a, b, c]$

(5) $(\text{diag}[a, b, c])^{n} = \text{diag}[a^n, b^n, c^n]$

(6) Eigen Values of diag[a , b , c]= a, b and c

(7) Determinant of diag[a , b , c] = abc

(3) SCALAR MATRIX

A scalar matrix S is a diagonal matrix whose all diagonal elements contain the same scalar value.

$$S = \begin{cases} a_{ij} = 0 \text{ If } i \neq j \\ a_{ij} = k \text{ If } i = j \end{cases}$$

Example : $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}_{3x3}$ is a scalar matrix.

(4) UNIT MATRIX or IDENTITY MATRIX

A square matrix each of whose diagonal elements is 1 and each of whose non-diagonal elements are zero is called unit matrix or an identity matrix which is denoted by I. Identity matrix is always square.

$$I = \begin{cases} a_{ij} = 0 \text{ If } i \neq j \\ a_{ij} = 1 \text{ If } i = j \end{cases}$$

Example : $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3x3}$ is an Identity matrix.

Properties of Identity Matrix :

(1) AI=IA =A

(2) I is identity element for multiplication ,so it is called multiplicative identity.

(3) $I^n = I$

(4) $I^{-1} = I$

(5) $|I| = 1$

(5) NULL MATRIX

The m x n matrix whose all elements are zero is called null matrix.

Null matrix is denoted by O. Null matrix need not be square .$a_{ij}=0 \ \forall$ i , j.

Example : $O_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \ O_{1x2} = \begin{bmatrix} 0 & 0 \end{bmatrix}$

Properties of Null Matrix :

(1) A + O=O + A =A

So, O is additive identity.

(2) A + (-A)=O

(6) UPPER TRIANGULAR MATRIX

A square matrix is called upper triangular if all the entries below the principal diagonal are zero.(denoted by U)

$$[U] = \begin{cases} 0 & \text{If } \ i > j \\ a_{ij} & \text{If } \ i \leq j \end{cases}$$

Example : $A=\begin{bmatrix}1&3&9\\0&2&6\\0&0&2\end{bmatrix}_{3\times3}$ is an upper triangular matrix.

(7) LOWER TRIANGULAR MATRIX

A square matrix is called lower triangular if all the entries above the principal diagonal are zero.(denoted by L)

$$[L]=\begin{cases}a_{ij} & \text{If } i\ge j\\ 0 & \text{If } i<j\end{cases}$$

Example : $A=\begin{bmatrix}1&0&0\\6&2&0\\5&8&2\end{bmatrix}_{3\times3}$ is an lower triangular matrix.

(8) ROW MATRIX

A row vector or row matrix is a 1 × m matrix, i.e. a matrix consisting of a single row of m elements.

Example : $A=\begin{bmatrix}4&6&8\end{bmatrix}_{1\times3}$ is a row matrix.

(9) COLUMN MATRIX

A column vector or column matrix is an m × 1 matrix, i.e. a matrix consisting of a single column of m elements.

Example : $A=\begin{bmatrix}4\\5\\9\end{bmatrix}_{3\times1}$ is a column matrix.

(10) IDEMPOTENT MATRIX

An idempotent matrix is a matrix which, when multiplied by itself, yields itself. i.e. $\mathbf{A^2=A}$

Examples : $\begin{bmatrix}0&0\\0&0\end{bmatrix},\begin{bmatrix}1&0\\0&1\end{bmatrix},\begin{bmatrix}2&-2&-4\\-1&3&4\\1&-2&-3\end{bmatrix}$

(11) INVOLUTARY MATRIX

A matrix can be called as an involuntary if $\mathbf{A^2=I}$.

Example : $A=\begin{bmatrix}4&-1\\15&-4\end{bmatrix}_{2\times2}$ is an involuntary matrix.

Properties of Involutary Matrix

(1) The determinant of an involutory matrix over any field is +1 or -1

(2) If **A** is an $n \times n$ matrix, then **A** is involutory if and only if $\frac{1}{2}(A+I)$ is Idempotent.

(3) If **A** and **B** are two Involutory matrices which commute with each other then **AB** is also Involutory.

(4) If **A** is Involutory matrix then every natural power of **A** is also Involutory. In fact, $\mathbf{A}^n$ will be equal to **A** if n is odd and **I** if n is even.

(12) NILPOTENT MATRIX

A Nilpotent matrix is a square matrix A such that $A^k=0$ and $A^{k-1} \neq 0$ for some positive integer k. The such smallest value of k is sometimes called the degree of A. In other words, all the elements below or above principle diagonal are zero ,then it is termed as Nilpotent Matrix.

Example : $A = \begin{bmatrix} 0 & 2 & 1 & 6 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}_{4\times 4}$ is a Nilpotent matrix of class 3(k=3) since

$A \neq 0$ and $A^2 \neq 0$, but $A^3=0$

(13) SINGULAR MATRIX

A square matrix whose determinant is equal to zero called as a Singular matrix.

i.e. If A is a singular matrix then **|A| = 0**

Example : $A = \begin{bmatrix} 2 & 6 \\ 1 & 3 \end{bmatrix}$

Note: If determinant of a matrix is not zero, then matrix is called as non-singular matrix. For a singular matrix, Inverse doesn't exist.

2.3 OPERATIONS ON MATRICES

2.3.1 ADDITION OF MATRICES

The matrix addition is defined for two matrices of the same dimensions. The sum of two m × n matrices **A** and **B**, denoted by **A + B**, is again an m × n matrix computed by adding corresponding elements

Example : $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$, $A+B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 6 & 8 \\ 10 & 12 \end{bmatrix}$

Properties of Matrix Addition:

(1) A+B=B+A i.e. matrix addition is cumulative.

(2) (A+B)+C=A+(B+C) i.e. matrix addition is associative.

(3) The equation A + X=0 has a unique solution in the set of all m x n matrices.

2.3.2 SUBTRACTION OF MATRICES

The matrix substraction is defined for two matrices of the same dimensions. The substraction of two m × n matrices **A** and **B**, denoted by **A-B=A+(-B)**, is again an m × n matrix computed by subtracting corresponding elements

Example : $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$

$$A - B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} - \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} -4 & -4 \\ -4 & -4 \end{bmatrix}$$

Note : Subtraction of matrices is neither cummulative nor associative.

2.3.3 MULTIPLICATION OF A MATRIX WITH A SCALAR

Let A be any mxn matrix and k be any scalar. The multiplication of a matrix A with the scalar k is denoted by kA.

Example : $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$,then $3A = \begin{bmatrix} 3 & 6 \\ 9 & 12 \end{bmatrix}$

2.3.4 MULTIPLICATION OF 2 MATRICES

Let $A = \left[a_{ij}\right]_{mxn}$ and $B = \left[b_{jk}\right]_{nxp}$ be two matrices such that the number of columns in A is equal to the number of rows in B.

Then the multiplication of A and B is matrix C such that $C = \sum_{j=1}^{n} a_{ij} b_{jk}$

Example : $A = \begin{bmatrix} 1 & 3 & 4 \\ -1 & 2 & -2 \end{bmatrix}_{2\times3}$ and $B = \begin{bmatrix} -6 & 5 \\ 1 & -1 \\ 3 & 0 \end{bmatrix}_{3\times2}$

$$AB = \begin{bmatrix} 1\times(-6) + 3\times1 + 4\times3 & 1\times5 + 3\times(-1) + 4\times0 \\ (-1)\times(-6) + 2\times1 + (-2)\times3 & (-1)\times5 + 2\times(-1) + (-2)\times0 \end{bmatrix} = \begin{bmatrix} 9 & 2 \\ 2 & -7 \end{bmatrix}_{2\times2}$$

Properties of Matrix Multiplication

Let *A*, *B* and *C* be matrices of dimensions such that the following are defined. Then

(1) Associative : A(BC) = (AB)C

(2) Distributive : A(B + C) = AB + AC

Distributive : (A + B)C = AC + BC

(3) c(AB) = (cA)B = A(cB), where c is a constant, note that in general $AB \neq BA$

NOTE : If A_{mxn} and B_{nxp} are two matrices ,then for AB

Total number of multiplications required = mpn

Total number of additions required = $mp(n-1)$

2.3.5 TRANSPOSE OF A MATRIX

A matrix which is formed by turning all the rows of a given matrix into columns is called transpose of a matrix. The transpose of matrix A is written as A^T.

Properties of Transpose Matrix

(1) $(A^T)^T = A$

(2) $(A + B)^T = A^T + B^T$

(3) $(AB)^T = B^T A^T$

(4) $(ABC)^T = C^T B^T A^T$

(5) $(kA)^T = kA^T$, k is scalar element

2.3.6 CONJUGATE OF A MATRIX

The matrix obtained from given matrix A on replacing its elements by the corresponding complex conjugate is called conjugate of matrix A. The conjugate of A is denoted by $\bar{A}$.

Example : $A = \begin{bmatrix} 5 & 3-2j & 2+j \\ 1-j & 3+6j & 6 \end{bmatrix}$ Then $\bar{A} = \begin{bmatrix} 5 & 3+2j & 2-j \\ 1+j & 3-6j & 6 \end{bmatrix}$

Properties of Conjugate Matrices

If $\overline{A}$ and $\overline{B}$ be the conjugate of A and B respectively .Then

(1) $\left(\overline{\overline{A}}\right) = A$

(2) $\left(\overline{A+B}\right) = \overline{A} + \overline{B}$

(3) $\overline{A} = A$ iff A is purely real matrix

(4) $\overline{A} = -A$ iff A is zero or purely imaginary matrix

(5) $\left(\overline{kA}\right) = \bar{k}\,\bar{A}$, k being any complex number

2.3.7 CONJUGATE TRANSPOSE OF A MATRIX

The transpose of the conjugate of a matrix is called conjugate transpose of a matrix. Conjugate transpose of a matrix A is denoted by $(\overline{A})^T$ or A^θ.

Example : $A = \begin{bmatrix} j & 2 \\ 1 & 1-j \end{bmatrix}$ Then $A^\theta = \begin{bmatrix} -j & 1 \\ 2 & 1+j \end{bmatrix}$

Properties : **(1)** $(A^{\theta})^{\theta} = A$

(2) $(A + B)^{\theta} = A^{\theta} + B^{\theta}$

(3) $(AB)^{\theta} = B^{\theta} A^{\theta}$

(4) $(kA)^{\theta} = \bar{k} A^{\theta}$, k being a complex number

2.4 CLASSIFICATION OF REAL MATRICES

Based on the relation between A and A^T real matrices are classified into 3 types

2.4.1 SYMMETRIC MATRIX

A square matrix $A = [a_{ij}]_{mxm}$ is said to be a symmetric matrix if $a_{ij} = a_{ji}$ for all values of i & j. i.e. If $A^T = A$ then A is a symmetric matrix.

Example : $A = \begin{bmatrix} 5 & 2 & 3 \\ 2 & 1 & 4 \\ 3 & 4 & 0 \end{bmatrix}$ is a symmetric matrix as $A^T = A$

2.4.2 SKEW-SYMMETRIC MATRIX

A square matrix $A = [a_{ij}]_{nxn}$ is said to be a Skew-Symmetric matrix if $a_{ij} = -a_{ji}$ for all values of i and j i.e. $A^T = -A$ for a symmetric matrix.

In Skew-Symmetric matrix, all the elements of principle diagonal are zero.

Example : $A = \begin{bmatrix} 0 & 2 & 3 \\ -2 & 0 & -4 \\ -3 & 4 & 0 \end{bmatrix}$ is a symmetric matrix as $A^T = -A$

Properties : **(1)** AA^T is always a symmetric matrix.

(2) $\frac{A + A^T}{2}$ is always symmetric matrix.

(3) $\frac{A - A^T}{2}$ is always a Skew symmetric matrix.

If A and B are symmetric,then

(4) A + B and A - B are also symmetric.

(5) AB, BA may or may not be symmetric.

2.4.3 ORTHOGONAL MATRIX

A square matrix is said to be an orthogonal matrix if $A^T = A^{-1}$.

Example : $A = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$

Properties : **(1)** For an orthogonal matrix $AA^T = AA^{-1} = I$

(2) Determinant of an orthogonal matrix is either $+1$ or -1.

(3) If A is an orthogonal matrix, A^T and A^{-1} are also orthogonal.

2.5 CLASSIFICATION OF COMPLEX MATRICES

Based on the relation between A and A^θ complex matrices are classified into 3 types

2.5.1 HERMITIAN MATRIX

A complex square matrix $A = \left[a_{ij}\right]_{m\times n}$ is said to be a Hermitian matrix if $A^\theta = A$

Example : $A=\begin{bmatrix} 4 & 2+3j \\ 2-3j & 5 \end{bmatrix}$ is a Hermitian matrix as $A^\theta = A$

Where, A^θ is conjugate transpose of A

Properties : **(1)** The principle diagonal elements of Hermitian matrix are always real.

(2) The determinant of any Hermitian matrix is a real number.

2.5.2 SKEW HERMITIAN MATRIX

A complex square matrix A is said to be a skew Hermitian matrix if $A = -A^\theta$

Example : $\begin{bmatrix} 0 & 2+i \\ -2+i & i \end{bmatrix}$

Properties of Skew Hermitian Matrix

(1) For a skew-Hermitian matrix P, P^k would be a Hermitian matrix for every even integer k and it would be skew-Hermitian for every odd integer k.

Example : Which of the following is **not** a skew-Hermitian matrix?

(a) $\begin{bmatrix} 0 & 2+i \\ -2+i & i \end{bmatrix}$ (b) $\begin{bmatrix} 0 & 2i & -3-i \\ 2i & 0 & -2-i \\ 3-i & 2-i & 0 \end{bmatrix}$

(c) $\begin{bmatrix} -i & -2-3i & 8-i \\ 2-3i & i & -7+2i \\ -8-i & 7+2i & -i \end{bmatrix}$ (d) $\begin{bmatrix} 3 & 2+3i & -i \\ -2+3i & 6 & 4i \\ -i & 4i & 2i \end{bmatrix}$

Solution : (d) In a skew-Hermitian matrix $a_{ij} = -\bar{a}_{ji}$ & the diagonal elements are zero or purely imaginary.

2.5.3 UNITARY MATRIX

A Complex square matrix A is said to be an Unitary matrix if $AA^{\theta} = I$

Example : $A = \begin{bmatrix} 2^{(-1/2)} & 2^{(-1/2)} & 0 \\ -i2^{(-1/2)} & i2^{(-1/2)} & 0 \\ 0 & 0 & i \end{bmatrix}$

Properties : (1) For an Unitary matrix $A^{-1} = A^{\theta}$

(2) Magnitude of determinant of unitary matrix is always 1. $||A|| = 1$, If for an Unitray matrix A, $|A| = a + ib$, then $a^2 + b^2 = 1$.

2.6 DETERMINANT OF A MATRIX

Determinant of a matrix is defined as sum of product of elements of any row or column with corresponding co-factor. In linear algebra, the determinant is a value associated with a square matrix only.

For **1x1 matrix** the number itself is the determinant.

For instance let A=[1]

Then |A|=1

For a **2x2 matrix** $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, its determinant value is defined as

$$\Delta = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

For a **3x3 matrix** $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$,its determinant value is defined as

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

2.6.1 MINORS

For the Matrix $\left[a_{ij}\right]_{n\times n}$ the minor of element a_{ij} is defined by the determinant of the matrix obtained by deleting the elements in the i-th row and j-th column and it is denoted by M_{ij}.

Example : $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

Minor of element $a_{11} = M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$

Minor of element $a_{22} = M_{22} = \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}$

Minor of element $a_{23} = M_{23} = \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$

2.6.2 COFACTORS

The cofactor of a matrix $[a_{ij}]_{n\times n}$ is given by $C_{ij} = (-1)^{(i+j)} M_{ij}$.

Cofactor of element $a_{11} = (-1)^{1+1} M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$

Cofactor of element $a_{22} = (-1)^{2+2} M_{22} = \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}$

Cofactor of element $a_{23} = (-1)^{2+3} M_{23} = -\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$

Cofactor matrix : If the elements of the matrix A is replace by its cofactors then the matrix obtained is called as cofactor matrix of A.

Example : Cofactor Matrix of $A = \begin{bmatrix} 2 & -1 \\ 3 & 6 \end{bmatrix}$ is $\begin{bmatrix} 6 & -3 \\ 1 & 2 \end{bmatrix}$

2.6.3 ADJOINT

The adjoint of a matrix A is the transpose of the cofactor matrix of A. It is denoted by adj A.

$$\text{adj } A = [\text{Cofactor } A]^T$$

Example : Find the adjoint of the matrix $A = \begin{bmatrix} 1 & -1 & 2 \\ 4 & 0 & 6 \\ 0 & 1 & -1 \end{bmatrix}$

Solution : $(c_{11}) = +\begin{vmatrix} 4 & 0 \\ 0 & 1 \end{vmatrix} = -6$, $(c_{12}) = -\begin{vmatrix} 4 & 6 \\ 0 & -1 \end{vmatrix} = 4$, $(c_{21}) = -\begin{vmatrix} -1 & 2 \\ 1 & -1 \end{vmatrix} = 1$

$(c_{22}) = +\begin{vmatrix} 1 & 2 \\ 0 & -1 \end{vmatrix} = -1$, $(c_{31}) = +\begin{vmatrix} -1 & 2 \\ 0 & 6 \end{vmatrix} = -6$, $(c_{33}) = +\begin{vmatrix} 1 & -1 \\ 4 & 0 \end{vmatrix} = 4$

$(c_{13}) = +\begin{vmatrix} 4 & 0 \\ 0 & 1 \end{vmatrix} = 4$, $(c_{23}) = -\begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} = -1$, $(c_{33}) = +\begin{vmatrix} 1 & -1 \\ 4 & 0 \end{vmatrix} = 4$

$$\text{adj}A = [(c_{ij})]^T = \begin{bmatrix} -6 & 4 & 4 \\ 1 & -1 & -1 \\ -6 & 2 & 4 \end{bmatrix}^T = \begin{bmatrix} -6 & 1 & -6 \\ 4 & -1 & 2 \\ 4 & -1 & 4 \end{bmatrix}$$

2.6.3 PROPERTIES OF DETERMINANTS

(1) The determinant value for A^T is same as that for A i.e. $|A^T| = |A|$.

(2) If all the elements of any row or column of matrix A are zero then |A|=0.

(3) If any two rows or columns are interchanged, the determinant get multiplied by -1.

(4) If all the elements of any one row or column of matrix A are multiplied by any scalar k, then determinant will be **k|A|**.

(5) If all the elements of matrix A are multiplied by any scalar k, then **$|kA|=k^n|A|$**, where n is order of square matrix.

(6) $|AB| = |A||B|$ where both A and B are square matrix of same size.

(7) $|A^n| = |A|^n$

(8) $|A^{-1}| = |A|^{-1} = \dfrac{1}{|A|}$

(9) If |A|=0,then A is called singular matrix.

(10) If $|A| \neq 0$,then A is called as non-singular matrix.

(11) If A is an upper or lower triangular matrix ,then |A| is equal to the product of principle diagonal elements.

(12) |cofactor matrix of A|=$|A|^{n-1}$, where n=order.

(13) A(adj A)=|A|I

(14) |adj A|=$|A|^{n-1}$, where n is order of matrix.

(15) $\text{adj}(\text{adj}(A)) = |A|^{n-2} A$

Example : If the determinant of a matrix A= $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 0 \\ 1 & 3 & 2 \end{bmatrix}$ is 3, then the determinant of the transposed matrix i.e.$|A^T|$ is equal to ,

(a) 6 (b) 9

(c) 2 (d) none of these

Solution : (d) $|A^T| = |A| = 3$

Example : If A be a 2×2 matrix and $|A|=3$ then $|3A|$ is

(a) 9 (b) 27

(c) 81 (d) 243

Solution : (b) $|A|=3$, $|3A| = 3^2|A| = 9 \times 3 = 27$

2.7 INVERSE OF A MATRIX

For every non singular matrix A, there exist a another matrix B such that AB=I. This matrix B is called as inverse of A.

$$AA^{-1}=A^{-1}A=I$$

$$A^{-1} = \frac{Adj(A)}{|A|}$$

Example : Adjoint of a 2×2 matrix can be calculated directly by exchanging diagonal elements and reversing sign of other elements.

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad adj\,(A) = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Example : The inverse of the Matrix $A = \begin{bmatrix} -1 & 8 \\ 4 & 6 \end{bmatrix}$ is _________.

Solution : $A^{-1} = \frac{Adj(A)}{|A|}$, $Adj(A) = \begin{bmatrix} 6 & -8 \\ -4 & -1 \end{bmatrix}$, $|A| = 6(-1) - (8 \times 4) = -38$

$$A^{-1} = -\frac{1}{8}\begin{bmatrix} 6 & -8 \\ -4 & -1 \end{bmatrix}$$

Properties of Inverse

(1) $(A^{-1})^{-1} = A$

(2) $AA^{-1} = A^{-1}A = I$

(3) $(AB)^{-1} = B^{-1}A^{-1}$

(4) $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$

(5) If AB = BA = I, then A and B are inverse of each other.

Note : Shortcut formula for inverse of 2 by 2 matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{(ad-bc)}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

2.8 RANK OF A MATRIX

2.8.1 DEFINITION

Rank of a matrix A is defined as the largest positive integer r such that, A has at least one non zero minor of order r. Rank of a matrix A is also defined as the **number of linearly independent rows or columns** of matrix A.

Properties of Rank of matrix

(1) For a matrix of size (m,n) : rank $r \le \min\{m, n\}$.

(2) $r(AB) \le \min\{r(A), r(B)\}$

(3) $r(A + B) \le r(A) + r(B)$

(4) $r(A) - r(B) \le r(A - B)$

(5) For a **non-singular** matrix, its rank is equal to its size.

(6) For a null matrix its rank is zero.

(7) $\text{Rank}(A^T) = \text{Rank}(A)$

2.8. 2 LINEARLY DEPENDENT VECTORS

A subset S of a vector space V is called linearly dependent if there exist a finite number of distinct vectors $v_1, v_2, \ldots, v_n$ in S and scalars a1, a2,..., an, not all zero, such that

$$a_1V_1 + a_2V_2 + \ldots\ldots\ldots\ldots\ldots + a_nV_n = 0$$

If no such set of scalar exists, the vectors are called as **linearly independent**.

Example : $A = \begin{bmatrix}1 & 2 & 3\end{bmatrix}$ $B = \begin{bmatrix}2 & 4 & 6\end{bmatrix}$

A & B are linearly dependent vectors as $2A - B = 0$

1.8.3 ORTHOGONAL VECTORS

Two vector u and v are said to be **orthogonal** if $u \cdot v = 0$

Example : $u = \begin{bmatrix}-2 \\ 3\end{bmatrix}$ and $v = \begin{bmatrix}1 \\ 2/3\end{bmatrix}$ Then $u.v = -2\times1 + 3\times\frac{2}{3} = 0$

2.8.4 ORTHONORMAL VECTORS

Two vectors are orthonormal if

1. Their dot product is zero.

2. The two vectors are unit vectors.

2.9 SOLUTION OF LINEAR SYSTEM OF EQUATIONS

In mathematics, a system of linear equations (or linear system) is a collection of linear equations involving the same set of variables.

Example : $3x + 2y - z = 1,\quad 2x - 2y + 4z = -2,\quad x + 5y - z = 0$

is a system of three equations in the three variables x, y, z. A solution to a linear system is an assignment of numbers to the variables such that all the equations are simultaneously satisfied.

2.9.1 METHOD OF DETERMINANTS-CRAMER'S RULE

Consider the equations

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

If the determinant of the coefficients be

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix},\ \Delta_1 = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix},\ \Delta_2 = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix},\ \Delta_3 = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

Then $x = \dfrac{\Delta_1}{\Delta},\ y = \dfrac{\Delta_2}{\Delta}\ \&\ z = \dfrac{\Delta_3}{\Delta}$

Above result constitutes the **Cramer's Rule,** which gives the solution of the linear equations.

2.9.2 MATRIX INVERSE METHOD

If AX=D be a system of linear equation, where $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $D = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$

Then the solution of system of equation is given by $X = A^{-1}D = \dfrac{1}{|A|}\text{adj}(A)D$

Example : Solve the equations $3x + y + 2z = 3,\ 2x - 3y - z = -3,\ ax + 2y + z = 4$

by (i)determinants (ii)matrices.

Solution : By determinant Method :

$$\Delta = \begin{vmatrix} 3 & 1 & 2 \\ 2 & -3 & -1 \\ 1 & 2 & 1 \end{vmatrix} = 3(-3+2) - 2(1-4) + (-1+6) = 8 \qquad [\text{Expanding by } C_1]$$

$$\therefore \; x = \frac{1}{\Delta}\begin{vmatrix} 3 & 1 & 2 \\ -3 & -3 & -1 \\ 4 & 2 & 1 \end{vmatrix} = 1, \; y = \frac{1}{\Delta}\begin{vmatrix} 3 & 3 & 2 \\ 2 & -3 & 1 \\ 1 & 4 & 4 \end{vmatrix} = 2 \text{ and } z = \frac{1}{\Delta}\begin{vmatrix} 3 & 1 & 3 \\ 2 & -3 & -3 \\ 1 & 2 & 4 \end{vmatrix} = -1$$

Hence $x = 1, y = 2, z = -1$

By matrices :

$$|A| = \begin{vmatrix} 3 & 1 & 2 \\ 2 & -3 & -1 \\ 1 & 2 & 1 \end{vmatrix} = 8, \; \text{adj}A = \begin{bmatrix} -1 & 3 & 5 \\ -3 & 1 & 7 \\ 7 & -5 & -11 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{|A|}\text{adj}A \times \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} = \frac{1}{8}\begin{bmatrix} -1 & 3 & 5 \\ -3 & 1 & 7 \\ 7 & -5 & -11 \end{bmatrix}\begin{bmatrix} 3 \\ -3 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

Hence, $x = 1, \; y = 2, \; z = -1$

2.9.3 ROW ECHELON AND REDUCED ROW ECHELON FORM

Row Echelon Form:

A matrix is in row echelon form when it satisfies the following conditions:

(i) The first non-zero element in each row, called the leading entry, is 1.

(ii) Each leading entry is in a column to the right of the leading entry in the previous row.

(iii) Rows with all zero elements, if any, are below rows having a non-zero element.

Reduced Row Echelon Form:

A matrix A is said to be in Reduced row echelon form if

(i) The matrix is in echelon form

(ii) The leading non-zero entries (if any) must be 1(leading 1)

(iii) The leading 1 is the only non-zero entry in its column .

Example : $\begin{bmatrix} 0 & a & b & c & 0 \\ 0 & 0 & d & e & 0 \\ 0 & 0 & 0 & f & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

Example 6 : Solve $x + 2y - z = 3$, $3x - y + 2z = 1$, $2x - 2y + 3z = 2$, $x - y + z = 1$

Solution :

$$\text{Augmented matrix } [A:B] = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 3 & -1 & 2 & 1 \\ 2 & -2 & 3 & 2 \\ 1 & -1 & 1 & -1 \end{bmatrix} \begin{array}{l} R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 - 2R_1 \\ R_4 \to R_4 - R_1 \end{array}$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -7 & 5 & -8 \\ 0 & -6 & 5 & -4 \\ 0 & -3 & 2 & -4 \end{bmatrix} \begin{array}{l} R_3 \to 7R_3 - 6R_2 \\ R_4 \to 7R_4 - 3R_2 \end{array} \Rightarrow \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -7 & 6 & -8 \\ 0 & 0 & 5 & 20 \\ 0 & 0 & -1 & -4 \end{bmatrix}$$

$$R_4 \to 5R_4 + R_3 \Rightarrow \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -7 & 6 & -8 \\ 0 & 0 & 5 & 20 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad \begin{array}{l} \Rightarrow 5z = 20 \\ -7y + 6z = -8 \\ x + 2y - z = 3 \\ \Rightarrow z = 4, y = 4, x = -1 \end{array}$$

2.10 CONSISTENCYAND INCONSISTENCY OF A SYSTEM OF EQUATION

Consider the system of m linear equations

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots\cdots + a_{1n}x_n = k_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots\cdots + a_{2n}x_n = k_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots\cdots + a_{mn}x_n = k_m \end{array}\right\} \cdots \text{(i)}$$

Containing the n unknowns $x_1, x_2, x_3, \ldots\ldots x_n$ To determine whether the equations (i) are consistence (i.e. possess a solution) or not, we consider

$$A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ldots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix} \text{ and } [A:B] = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} & k_1 \\ a_{21} & a_{22} & \ldots & a_{2n} & k_2 \\ \vdots & \vdots & \ldots & \vdots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} & k_m \end{bmatrix}$$

A is the coefficient matrix and **[A : B]** is called the augmented matrix of the equation (i)

2.10.1 ROUCHE'S THEOREM

The system of equations (i) is consistent if and only if the coefficient matrix A and the augmented matrix **[A : B]** are of the same rank otherwise the system is inconsistent.

2.10.2 PROCEDURE TO TEST CONSISTENCY OF LINEAR EQUATIONS

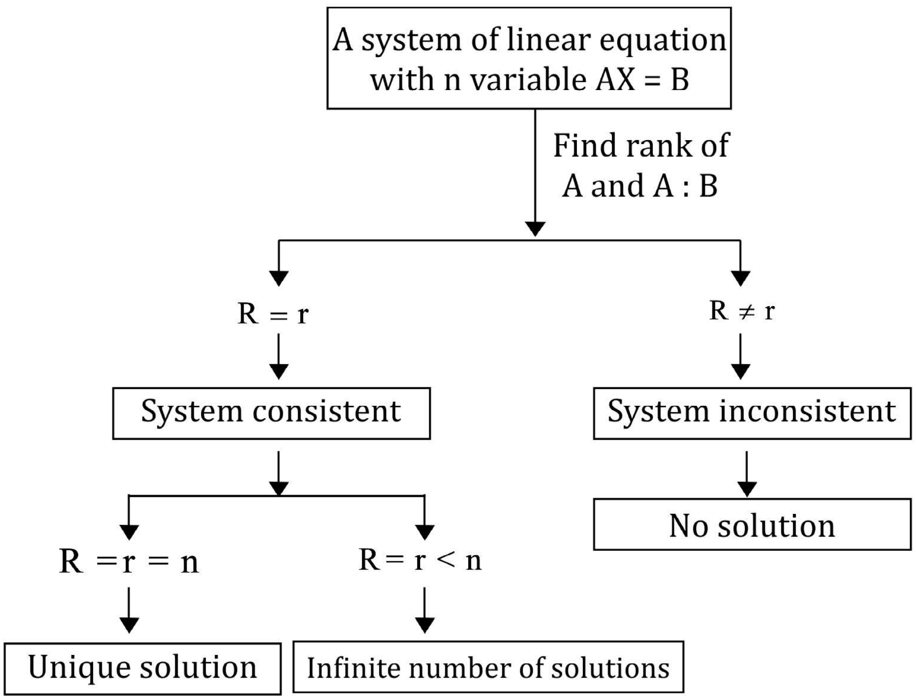
Find the rank of the **coefficient matrix A** and the **augmented matrix [A : B]**, by reducing A to the triangular from by elementary row operation.

Let the **rank of A** be **R , rank of [A:B]** be **r. n** be the **number of unknowns**.

(1) If $R \neq r$ then the equation are **inconsistent**, i.e. there is **no solution**.

(2) If $R = r = n$, the equation are **consistent** and there is a **unique solution**.

(3) If $R = r < n$, the equation are **consistent** and there are **infinite number of Solutions.**



Example : If A is a 4×5 matrix and AX = B is an inconsistent system .Then the highest possible rank of A is ?

Solution : Rank(A) $\leq$ min(4,5), Rank $\leq$ 4

But System is inconsistent ,so rank=4 is not possible.

Therefore highest possible rank of A is 3.

Note : Consider the system of equations ax + by = p and cx + dy = q

The above system of equations have

(i) No solution if $\frac{a}{c} = \frac{b}{d} \neq \frac{e}{f}$

(ii) Unique solution if $\frac{a}{c} \neq \frac{b}{d}$

(iii) Infinitely many Solutions if $\frac{a}{c} = \frac{b}{d} = \frac{e}{f}$

Example : The system of equations 8x + 2y = 7 and 4x + y = 3 has

(i) No solution **(ii)** Unique solution

(iii) Infinitely many Solutions **(iv)** None of the above

Solution : (i) Taking the coefficient $\frac{8}{4} = \frac{2}{1} \neq \frac{7}{3}$

Therfore No solution.

2.10.3 SYSTEM OF LINEAR HOMOGENEOUS EQUATION

consider the homogenous linear equations

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n = 0 \\ a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n = 0 \\ a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n = 0 \end{array}\right\} \quad \ldots(i)$$

Find the rank R of the coefficient matrix A by reducing it to the triangular form by elementary row operation.

(1) If $R = n$, the equation (i) have only a **trivial zero solutions**.

$$x_1 = x_2 = \ldots = x_n = 0$$

(2) When $R < n$, the equation (i) have **infinite number of solutions.**

Example : Solve the equations $x + 2y + 3z = 0,\ 3x + 4y + 4z = 0,\ 7x + 10y + 12z = 0$

Solution : Rank of the coefficient matrix

$$\begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 4 \\ 7 & 10 & 12 \end{bmatrix} \xrightarrow[R_3-3R_1]{} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -5 \\ 7 & 10 & 12 \end{bmatrix} \xrightarrow[R_3-7R_1-2R_2]{} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -5 \\ 0 & 0 & 1 \end{bmatrix} \text{is } 3 = n$$

$\therefore$ the equation have only a trivial solution: x = y = z = 0

Example : The following system of linear equations, $x + 2y = 4$, $2x + y = 3$, $x + y = 1$

(a) has unique solution (b) has no solution

(c)has many solutions (d) has two solutions

Solution : (b)

$$(A:B) = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 1 & 3 \\ 1 & 1 & 1 \end{bmatrix} \xrightarrow[R_3 - R_1]{R_2 - 2R_1} \begin{bmatrix} 1 & 2 & 4 \\ 0 & -3 & -5 \\ 0 & -1 & -3 \end{bmatrix} \xrightarrow[-\frac{1}{3}R_2]{} \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & \frac{5}{3} \\ 0 & -1 & -3 \end{bmatrix} \xrightarrow[R_3 + R_2]{R_1 - 2R_2} \begin{bmatrix} 1 & 0 & \frac{2}{3} \\ 0 & 1 & \frac{5}{3} \\ 0 & 0 & \frac{-4}{3} \end{bmatrix}$$

Here, Rank (A) = 2, Rank (A : B) = 3, System is inconsistent.

Example : The value of α for which the system of equations x + y + z = 0, y+2z = 0, ax + z = 0 has more than one solution, is

(a) -1 (b) 0

(c) $\frac{1}{2}$ (d) 1

Solution : (a)

For more than one solution

$$\begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ \alpha & 0 & 1 \end{vmatrix} = 0$$

$$1+\alpha(2-1)=0$$

$$\alpha=-1$$

2.11 EIGEN VALUES

If A is any square matrix of order n, we can form the matrix $A-\lambda I$, where I is the nth order unit matrix.

The determinant of this matrix $A-\lambda I$ equated to zero,

i.e.

$$|A-\lambda I| = \begin{vmatrix} a_{11}-\lambda & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22}-\lambda & \dots & a_{2n} \\ . & . & . & . \\ a_{n1} & a_{n2} & \dots & a_{nn}-\lambda \end{vmatrix} = 0$$

is called the characteristic equation of A. On expanding the determinant, the characteristic equation takes the form

$$(-1)^n \lambda^n + k_1\lambda^{n-1} + k_2\lambda^{n-2} + \ldots + k_n = 0$$

Where k_i are constants in terms of a_{ij}. The roots of these equation are called the eigen values or latent roots or characteristic roots of the matrix A.

2.11.1 EIGEN VECTORS

$$\text{If } X = \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{bmatrix} \text{ and } A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \dots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}, \ \dots \ (1)$$

then the linear transformation Y = AX carries the column vector X into the column vector Y by means of square-matrix A. In practice, it is often required to find such vectors which transforms into themselves or to a scalar multiple of themselves.

Let, X be the such vector which transformed into λX by means of transformation

Then $AX = \lambda X \Rightarrow AX - \lambda X = 0 \Rightarrow (A - \lambda I)X = 0$

This matrix equation represents n homogenous linear equations

$$\left.\begin{array}{l} (a_{11}-\lambda)x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = 0 \\ a_{21}x_1 + (a_{22}-\lambda)x_2 + \ldots a_{2n}x_n = 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{n1}x_1 + a_{n2}x_2 + \ldots + (a_{nn}-\lambda)x_n = 0 \end{array}\right\}$$

Which will have a non-trivial solution only if the coefficient matrix is singular, i.e. if $|A - \lambda I| = 0$

This is called the characteristic equation of the transformation and is same as the characteristic equation of the matrix A. It has n roots and corresponding to each root, the equation will have a non-zero Solutions.

$X = [x_1, x_2 \ldots\ldots x_n]'$ which is known as eigen vector or latent vector.

2.11.2 PROPERTIES OF EIGEN VALUES

(1) Any square matrix **A** and its transpose $\mathbf{A^T}$ have the **same Eigen values**.

(2) The Eigen values of a **triangular matrix** are just the **diagonal elements** of the matrix.

(3) Eigen Values of diag[a , b , c]= a, b and c

(4) The Eigen values of an **idempotent matrix** are **either zero or unity**.

(5) The **sum** of the Eigen values of a matrix is the **sum** of the elements of the **principal diagonal**.

(6) The **product** of the Eigen values of a matrix A is equal to its **determinant.**

(7) If λ is an eigen values of the non singular matrix **A**, then $\frac{1}{\lambda}$ is the eigen values of A^{-1}.

(8) If λ is an Eigen value of an **orthogonal** matrix, then $\frac{1}{\lambda}$ is also its eigen values.

(9) If $\lambda_1, \lambda_2 \ldots\ldots \lambda_n$ are the Eigen values of a matrix **A**, then A^m has the eigen values $\lambda_1^m, \lambda_2^m \ldots \lambda_n^m$ (m being the positive integer).

2.12 CAYLEY HAMILTON THEOREM

Statement : Every square matrix satisfies its own characteristic equation; i.e. if the characteristic equation for the nth order square matrix A is

$$|A - \lambda I| = (-1)^n \lambda^n + k_1\lambda^{n-1} + k_2\lambda^{n-2} + \ldots + k_n = 0$$

then $(-1)^n A^n + k_1A^{n-1} + k_2A^{n-2} + \ldots + k_nI = 0$ ...(1)

Inverse of a matrix using Caley Hamilton theorem

Multiplying (1) by A^{-1}, we get

$$(-1)^n A^{n-1} + k_1 A^{n-2} + \ldots + k_{n-1} I + k_n A^{-1} = 0$$

Whence, $A^{-1} = -\dfrac{1}{k_n}\left[(-1)^n A^{n-1} + k_1 A^{n-2} + \ldots + k_{n-1} I\right]$

This result gives the inverse of A in the term of powers of A and is considered as a practical method for the computation of the inverse of the large matrices. As a by-product of the computation, the characteristic equation and the determinant of the matrix are also obtained.

Example : The eigen values of the matrix:

$$A = \begin{bmatrix} 0 & 3 & 6 \\ -3 & 0 & -4 \\ -6 & 4 & 0 \end{bmatrix} \text{ are}$$

(a)(0,0,1) (b)(1,2,3)

(c)(-1,2,3) (d)none of these

Solution : **(d)** The given matrix is a real skew symmetric matrix, the eigen values of which are either purely imaginary or zero.

Example : The characteristic roots of the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 & 5 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & -1 \end{bmatrix} \text{ are,}$$

(a) $1, -1, 1, -1$ (b) $1,\ 2, -3,\ 4$

(c) $0, 1, 2, -1$ (d) None of these

Solution : **(a)** Triangular matrix: Eigen value are the elements of principle diagonal. Characteristic roots are $1,\ -1,\ 1, -1$

GATE QUESTIONS

Q.1 Consider the equation of simultaneous equations

$x + 2y + z = 6; 2x + y + 2z = 6; x + y + z = 5$ This system has

(a)unique solution (b)infinite number of solution

(c) no solution (d)exactly two solution **[ME-GATE-2003]**

Q.2 Let A,B,C,D be $n \times n$ matrices, each with non-zero determinant, If ABCD=I, Then B^{-1} is

(a) $D^{-1}C^{-1}A^{-1}$ (b) CDA

(c) ADC (d) does not necessarily exist **[CS-GATE-2004]**

Q.3 Given matrix $[A] = \begin{bmatrix} 4 & 2 & 1 & 3 \\ 6 & 3 & 4 & 7 \\ 2 & 1 & 0 & 1 \end{bmatrix}$, The rank of matrix is

(a) 4 (b) 3

(c) 2 (d) 1 **[CS-GATE-2004]**

Q.4 $x = [x_1, x_2 \dots x_n]^T$ is an n-tuple non-zero vector. The n×nmatrix $V = xx^T$

(a) has rank zero (b) has rank 1

(c) is orthogonal (d) has rank n **[CS-GATE-2004]**

Q.5 Given matrix $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}$, $[AA^T]^{-1}$ is

(a) $\begin{bmatrix} 1/4 & 0 & 0 & 0 \\ 0 & 1/4 & 0 & 0 \\ 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 1/2 \end{bmatrix}$ (b) $\begin{bmatrix} 1/2 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 1/2 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} 1/4 & 0 & 0 & 0 \\ 0 & 1/4 & 0 & 0 \\ 0 & 0 & 1/4 & 0 \\ 0 & 0 & 0 & 1/4 \end{bmatrix}$ **[EC-GATE-2005]**

Q.6 Let, $\begin{bmatrix} 2 & -0.1 \\ 0 & 3 \end{bmatrix}$ and $A^{-1} = \begin{bmatrix} \frac{1}{2} & a \\ 0 & b \end{bmatrix}$ Then $(a+b)=$

(a) $\frac{7}{20}$ (b) $\frac{3}{20}$

(c) $\frac{19}{60}$ (d) $\frac{11}{20}$ **[EC-GATE-2005]**

Q.7 Consider the following system of equations in three real variables

x_1, x_2 and x_3, $2x_1 - x_2 + 3x_3 = 1$, $3x_1 - 2x_2 + 5x_3 = 2$, $-x_1 - 4x_2 + x_3 = 3$

This system of equations has

(a) No solution

(b) A unique solution

(c) More than one but a finite number of solutions

(d) An infinite number of solutions **[CS-GATE-2005]**

Q.8 Given the matrix $\begin{bmatrix} -4 & 2 \\ 4 & 3 \end{bmatrix}$ the Eigen vector is

(a) $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ (b) $\begin{bmatrix} 4 \\ 3 \end{bmatrix}$

(c) $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ (d) $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$ **[EC-GATE-2005]**

Q.9 For the matrix $\begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$ the Eigen value corresponding to the Eigenvector $\begin{bmatrix} 101 \\ 101 \end{bmatrix}$ is

(a) 2 (b) 4

(c) 6 (d) 8 **[EC-GATE-2006]**

Q.10 How many of the following matrices have an Eigen value 1?

$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ and $\begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix}$

(a) One (b) Two

(c) Three (d) Four **[EC-GATE-2006]**

Q.11 Consider the matrices $X_{[4\times 3]}\, Y_{[4\times 3]}$ and $P_{[2\times 3]}$. The order of of $\left(P\left(X^{T}Y\right)^{-1}P^{T}\right)^{T}$ will be

(a) (2×2) (b) (2×3)

(c) (4×3) (d) (3×4) **[ME-GATE-2006]**

Q.12 Multiplication of matrices E and F is G. Matrices E and G are

$$E=\begin{bmatrix}\cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{bmatrix} \text{ and } G=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$ What is the matrix F ?

(a) $\begin{bmatrix}\cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{bmatrix}$ (b) $\begin{bmatrix}\cos\theta & \cos\theta & 0\\ -\cos\theta & \sin\theta & 0\\ 0 & 0 & 1\end{bmatrix}$

(c) $\begin{bmatrix}\cos\theta & \sin\theta & 0\\ -\sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{bmatrix}$ (d) $\begin{bmatrix}\sin\theta & -\cos\theta & 0\\ \cos\theta & \sin\theta & 0\\ 0 & 0 & 1\end{bmatrix}$ **[ME-GATE-2006]**

Q.13 The rank of matrix $\begin{bmatrix}1 & 1 & 1\\ 1 & -1 & 0\\ 1 & 1 & 1\end{bmatrix}$ is

(a) 0 (b) 1
(c) 2 (d) 3 **[EC-GATE-2006]**

Q.14 Solution for the system defined by the set of equations

$4y+3z=8;\ 2x-z=2$ and $3x+2y=5$ is

(a) $x=0; y=1; z=\frac{4}{3}$ (b) $x=0; y=\frac{1}{2}; z=2$

(c) $x=0; y=\frac{1}{2}; z=2$ (d) non-existent **[CE-GATE-2006]**

Statement for linked answer Q.15 & Q.16

Cayley-hamilton Theorem sate that a square of matrix satisfied its own characteristic equation. Consider a matrix $A=\begin{bmatrix}-3 & 2\\ -1 & 0\end{bmatrix}$

Q.15 A satisfied the relation

(a) $A+2I+2A^{-1}=0$ (b) $A^{2}+2A+2I=0$

(c) $(A+I)(A+2I)=0$ (d) $\exp(A)=0$ **[EE-GATE-2007]**

Q.16 A^9 equals

(a) 511A + 510I (b) 309A + 104I

(c) 154A + 155I (d) EXP (9A) **[EE-GATE-2007]**

Q.17 The inverse of the 2×2 matrix $\begin{bmatrix}1 & 2\\5 & 7\end{bmatrix}$ is

(a) $\frac{1}{3}\begin{bmatrix}-7 & 2\\5 & -1\end{bmatrix}$ b) $\frac{1}{3}\begin{bmatrix}7 & 2\\5 & 1\end{bmatrix}$

(c) $\frac{1}{3}\begin{bmatrix}7 & -2\\-5 & 1\end{bmatrix}$ (d) $\frac{1}{3}\begin{bmatrix}-7 & -2\\-5 & -1\end{bmatrix}$ **[CE-GATE-2007]**

Q.18 For what values of α and β, the following simultaneous equations have an infinite number of solution ? $x+y+z=5; x+3y+3z=9; x+2y+\alpha z=\beta$

(a) 2,7 (b) 3,8

(c) 8,3 (d) 7,2 **[CE-GATE-2007]**

Q.19 The system of linear equations 4x + 2y = 7, 2x + y = 6 has

(a) a unique solution (b) no solution

(c) an infinite number of solutions (d) exactly two distinct solutions

[EC-GATE-2008]

Q.20 The following system of equations $x_1+x_2+2x_3=1,\ x_1+2x_2+3x_3=2,\ x_1+4x_2+ax_3=4$ has a unique solution. The only possible values for a is/ are

(a) 0 (b) either 0 or 1

(c) One of 0, 1 or -1 (d) any real number other than 5

[CS-GATE-2008]

Q.21 The Eigen values of the matrix $\begin{bmatrix}4 & -2\\-2 & 1\end{bmatrix}$

(a) are 1 and 4 (b) are -1 and 2

(c) are 0 and 5 (d) cannot be determined **[CS-GATE-2008]**

Q.22 For the matrix the $\begin{bmatrix}4 & 1\\1 & 4\end{bmatrix}$ Eigen value are

(a)3 and -3 (b) -3 and -5

(c)3 and 5 (d) 5 and 0 **[CS-GATE-2008]**

Q.23 Which of the following is an Eigen vector of the matrix $\begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 5 & 5 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 3 & 1 \end{bmatrix}$?

(a) $\begin{bmatrix} 1 \\ -2 \\ 0 \\ 0 \end{bmatrix}$ (b) $\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$

(c) $\begin{bmatrix} 1 \\ 0 \\ 0 \\ -2 \end{bmatrix}$ (d) $\begin{bmatrix} 0 \\ 0 \\ 1 \\ 2 \end{bmatrix}$

Q.24 The number of linearly independent Eigen vector of $\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$ is

(a)0 (b)1
(c)2 (d)infinite **[CS-GATE-2008]**

Q.25 The Eigen vector of the matrix $\begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$ are written in the form of $\begin{bmatrix} 1 \\ a \end{bmatrix}$ & $\begin{bmatrix} 1 \\ b \end{bmatrix}$. What is a + b?

(a) 0 (b) $\frac{1}{2}$
(c) 1 (d) 2 **[CS-GATE-2008]**

Q.26 One of the Eigen vectors of the matrix $A = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}$ is

(a) $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ (b) $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

(c) $\begin{bmatrix} 4 \\ 1 \end{bmatrix}$ (d) $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ **[CS-GATE-2008]**

Q.27 The product of the matrices $(PQ)^{-1}P$ is

(a) P^{-1} (b) Q^{-1}
(c) $P^{-1}Q^{-1}P$ (d) PQP^{-1} **[CE-GATE-2008]**

Q.28 The inverse of the matrix $\begin{bmatrix} 3+2i & i \\ -i & 3-2i \end{bmatrix}$ is

(a) $\frac{1}{12}\begin{bmatrix} 3+2i & -i \\ i & 3-2i \end{bmatrix}$ (b) $\frac{1}{12}\begin{bmatrix} 3-2i & -i \\ i & 3+2i \end{bmatrix}$

(c) $\frac{1}{14}\begin{bmatrix} 3+2i & -i \\ i & 3-2i \end{bmatrix}$ (d) $\frac{1}{14}\begin{bmatrix} 3-2i & -i \\ i & 3+2i \end{bmatrix}$ **[CE-GATE-2008]**

Q. 29 For which value of x will the matrix given below become singular?

$$\begin{bmatrix} 8 & x & 0 \\ 4 & 0 & 2 \\ 12 & 6 & 0 \end{bmatrix}$$

(a) 4 (b) 6

(c) 8 (d) 12 **[CE-GATE-2008]**

Q. 30 For a matrix $[M] = \begin{bmatrix} \frac{3}{5} & \frac{4}{5} \\ x & \frac{3}{5} \end{bmatrix}$, the transpose of the matrix is equal to the inverse of the matrix, $[M]^T = [M]^{-1}$. The value of x is given by

(a) $-\frac{4}{5}$ (b) $-\frac{3}{5}$

(c) $\frac{3}{5}$ (d) $\frac{4}{5}$ **[CE-GATE-2008]**

Q.31 If R= $\begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & -1 \\ 2 & 3 & 2 \end{bmatrix}$, then top row of R^{-1} is

(a) [5 6 4] (b) [5 -3 1]

(c) $[2\ 0\ -1]$ (d) $\left[2\ -1\frac{1}{2}\right]$ **[CE-GATE-2008]**

Q.32 A is m × n full rank matrix with m > n and I is an identity matrix. Let matrix $A' = \left(A^T A\right)^{-1} A^T$, then which one of the following is FALSE ?

(a) AA'A=A (b) $(AA')^2 = AA'$

(c) A'A=I (d) AA'A=A' **[CE-GATE-2008]**

Q.33 The following simultaneous equations $x+y+z=3;\ x+2y+3z=4;\ x+4y+kz=6$ will not have a unique solution for k equal to

(a) 0 (b) 5

(c) 6 (d) 7 **[CE-GATE-2008]**

Q.34 For what value of a, if any, will the following system of equation in x, y and z have a solution?

$2x+3y=4;\ x+y+z=4;\ x+2y-z=a$

(a) Any real number (b) 0

(c) 1 (d) There is no such value **[ME-GATE-2008]**

Q.35 For the matrix $P=\begin{bmatrix}3 & -2 & 2\\ 0 & -2 & 1\\ 0 & 0 & 1\end{bmatrix}$ one of the Eigen values is equal to -2. Which of the following is an Eigen vector?

(a) $\begin{bmatrix}3\\ -2\\ 1\end{bmatrix}$ (b) $\begin{bmatrix}-3\\ 2\\ -1\end{bmatrix}$

(c) $\begin{bmatrix}1\\ -2\\ 3\end{bmatrix}$ (d) $\begin{bmatrix}2\\ 5\\ 0\end{bmatrix}$ **[CS-GATE-2008]**

Q.36 The characteristic equation of a (3 × 3) matrix P is define as $a(\lambda)=|\lambda-PI|=\lambda^3+\lambda^2+2\lambda+1=0$. If I denote identity matrix, then the inverse of matrix P will be

(a) (P^2+P+2I) (b) (P^2+P+1)

(c) $-(P^2+P+1)$ (d) $-(P^2+P+2I)$ **[EE-GATE-2008]**

Q.37 For the set of equation $x_1+2x_2+x_3+4x_4=2,\quad 3x_1+6x_2+3x_3+12x_4=6$

The following statement is true:

(a) Only the trivial solution $x_1=x_2=x_3=x_4=0$ exist.

(b) There are no solutions.

(c) A unique non-trivial solution exists

(d) Multiple non-trivial solutions exist. **[EE-GATE-2010]**

Q.38 Consider the matrix $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 7 \\ 0 & 0 & 3 \end{bmatrix}$. Which one of the following provides the CORRECT values of Eigen values of the matrix ?

(a) 1, 4, 3 (b) 3, 7, 3
(c) 7, 3, 2 (d) 1, 2, 3

[CS-GATE-2011]

Q.39 The system of algebraic equations $x + 2y + z = 4$, $2x + y + 2z = 5$, $x - y + z = 1$ has

(a) A unique solution of x = 1, y = 1 and z = 1.

(b) Only the two solutions of (x = 1, y = 1 and z = 1) and (x = 2, y = 1 and z = 0)

(c) Infinite number of solutions

(d) No feasible solution **[ME-GATE-2012]**

Q.40 For the matrix $A = \begin{bmatrix} 5 & 3 \\ 1 & 3 \end{bmatrix}$, one of the normalized Eigen vectors is given as

(a) $\begin{pmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix}$ (b) $\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{pmatrix}$

(c) $\begin{pmatrix} \frac{3}{\sqrt{10}} \\ \frac{-1}{\sqrt{10}} \end{pmatrix}$ (d) $\begin{pmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{pmatrix}$ **[ME-GATE-2012]**

Q.41 The eigen values of matrix $\begin{bmatrix} 9 & 5 \\ 5 & 8 \end{bmatrix}$ are

(a)-2.42 and 6.86 (b) 3.48 and 13.53
(c) 4.70 and 6.86 (d) 6.86 and 9.50 **[CE-GATE-2012]**

Q.42 There are three matrixes P (4 × 2), Q(2 × 4) and R (4 ×1) . The minimum of multiplication required to compute the matrix PQR is **[CE-GATE-2012]**

Q.43 Let A be the 2×2 matrix with elements $a_{11} = a_{12} = a_{21} = +1$ and $a_{22} = -1$. Then the Eigen values of the matrix A^{19} are

(a) 1024 and -1024 (b) $1024\sqrt{2}$ and $-1024\sqrt{2}$

(c) $4\sqrt{2}$ and $-4\sqrt{2}$ (d) $512\sqrt{2}$ and $-512\sqrt{2}$ **[CS-GATE-2012]**

Q.44 Which one of the following does NOT equal $\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$?

(a) $\begin{vmatrix} 1 & x(x+1) & x+1 \\ 1 & y(y+1) & y+1 \\ 1 & z(z+1) & z+1 \end{vmatrix}$ (b) $\begin{vmatrix} 1 & x+1 & x^2+1 \\ 1 & y+1 & y^2+1 \\ 1 & z+1 & z^2+1 \end{vmatrix}$

(c) $\begin{vmatrix} 0 & x-y & x^2-y^2 \\ 0 & y-z & y^2-z^2 \\ 1 & z & z^2 \end{vmatrix}$ (d) $\begin{vmatrix} 2 & x+y & x^2+y^2 \\ 2 & y+z & y^2+z^2 \\ 1 & z & z^2 \end{vmatrix}$ **[CS-GATE-2013]**

Q.45 The Eigen values of symmetric matrix are all

(a) Complex with non-zero positive imaginary part

(b) Complex with non-zero negative imaginary part

(c) Real

(d) Pure imaginary **[ME-GATE-2013]**

Q.46 Let A be an m×n matrix & B an n×mmatrix. It is given that determinant (I_m+AB)= determinant (I_n+BA), where I_k is the k×k identity matrix. Using the above property, the determinant of the matrix given below is

$$\begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix}$$

(a) 2 (b) 5

(c) 8 (d) 16 **[EC-GATE-2013]**

Q.47 The equation $\begin{bmatrix} 2 & -2 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ has

(a) no solution

(b) only one solution $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

(c) non-zero unique solution

(d) multiple solutions **[EE-GATE-2013]**

Q.48 A Matrix has Eigenvalues -1 & -2.The correspondingEigenvectors are $\begin{bmatrix}1\\-1\end{bmatrix}$ & $\begin{bmatrix}1\\-2\end{bmatrix}$ respectively. The matrix is

(a) $\begin{bmatrix}1 & 1\\-1 & -2\end{bmatrix}$ (b) $\begin{bmatrix}1 & 2\\-2 & -4\end{bmatrix}$

(c) $\begin{bmatrix}-1 & 0\\0 & -2\end{bmatrix}$ (d) $\begin{bmatrix}0 & 1\\-2 & -3\end{bmatrix}$ **[EE-GATE-2013]**

Q.49 The dimension of the null space of the matrix is $\begin{bmatrix}0 & 1 & 1\\1 & -1 & 0\\-1 & 0 & -1\end{bmatrix}$

(a) 0 (b) 1

(c) 2 (d) 3 **[IN-GATE-2013]**

Q.50 If the A- matrix of the state space model of a SISO linear time invariant system is rank deficient, the transfer function of the system must have

(a) a pole with positive real part

(b) a pole with negative real part

(c) a pole with positive imaginary part

(d) a pole at the origin **[IN-GATE-2013]**

Q.51 Given that the determinant of the matrix $\begin{bmatrix}1 & 3 & 0\\2 & 6 & 4\\-1 & 0 & 2\end{bmatrix}$ is -12, the determinant of the matrix

$\begin{bmatrix}2 & 6 & 0\\4 & 12 & 8\\-2 & 0 & 4\end{bmatrix}$ is

(a) -96 (b)-24

(c)24 (d)96 **[ME-GATE-2014(1)]**

Q.52 The matrix form of the linear system $\frac{dx}{dt} = 3x - 5y$ and $\frac{dy}{dt} = 4x + 8y$ is

(a) $\frac{d}{dt}\begin{Bmatrix}x\\y\end{Bmatrix} = \begin{bmatrix}3 & -5\\4 & 8\end{bmatrix}\begin{Bmatrix}x\\y\end{Bmatrix}$ (b) $\frac{d}{dt}\begin{Bmatrix}x\\y\end{Bmatrix} = \begin{bmatrix}3 & 8\\4 & -5\end{bmatrix}\begin{Bmatrix}x\\y\end{Bmatrix}$

(c) $\frac{d}{dt}\begin{Bmatrix}x\\y\end{Bmatrix} = \begin{bmatrix}4 & -5\\3 & 8\end{bmatrix}\begin{Bmatrix}x\\y\end{Bmatrix}$ (d) $\frac{d}{dt}\begin{Bmatrix}x\\y\end{Bmatrix} = \begin{bmatrix}4 & 8\\3 & -5\end{bmatrix}\begin{Bmatrix}x\\y\end{Bmatrix}$ **[ME-GATE-2014(1)]**

Q.53 One of the Eigenvectors of the matrix $\begin{bmatrix} -5 & 2 \\ -9 & 6 \end{bmatrix}$ is

(a) $\begin{Bmatrix} -1 \\ 1 \end{Bmatrix}$ (b) $\begin{Bmatrix} -2 \\ 9 \end{Bmatrix}$

(c) $\begin{Bmatrix} 2 \\ -1 \end{Bmatrix}$ (d) $\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$ **[ME-GATE-2014(2)]**

Q.54 Consider a 3 × 3 real symmetric matrix S such that two of its Eigen values are $a \neq 0, b \neq 0$ with respective Eigenvectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$. If $a \neq b$ then $x_1y_1 + x_2y_2 + x_3y_3$ equals

(a) a (b) b

(c) ab (d) 0 **[ME-GATE-2014(3)]**

Q.55 Which one of the following equations is a correct identity for arbitrary 3×3 real matrices P, Q and R?

(a) $P(Q+R) = PQ + RP$ (b) $(P-Q)^2 = P^2 - 2PQ + Q^2$

(c) $\det(P+Q) = \det(P) + \det(Q)$ (d) $(P+Q)^2 = P^2 + PQ + QP + Q^2$ **[ME-GATE-2014(4)]**

Q.56 Given the matrices J= $\begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 6 \end{bmatrix}$ and K = $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$, the product $K^T JK$ is _______________.

[CE-GATE-2014(1)]

Q.57 The sum of Eigen values of the matrix, [M] is where

$$M = \begin{bmatrix} 215 & 650 & 795 \\ 655 & 150 & 835 \\ 485 & 355 & 550 \end{bmatrix}$$

(a) 915 (b) 1355

(c) 1640 (d) 2180 **[CE-GATE-2014(1)]**

Q.58 With reference to the conventional Cartesian (x, y) coordinate system, the vertices of a triangle have the following coordinates: $(x_1, y_1) = (1, 0)$; $(x_2, y_2) = (2, 2)$; and $(x_3, y_3) = (4,3)$. The area of the triangle is equal to

(a) $\frac{3}{2}$ (b) $\frac{3}{4}$

(c) $\frac{4}{5}$ (d) $\frac{5}{2}$ **[CE-GATE-2014(1)]**

Q.59 The determinant of matrix $\begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 3 & 0 \\ 2 & 3 & 0 & 1 \\ 3 & 0 & 1 & 2 \end{bmatrix}$ is ________. **[CE-GATE-2014(2)]**

Q.60 The rank of matrix $\begin{bmatrix} 6 & 0 & 4 & 4 \\ -2 & 14 & 8 & 18 \\ 14 & -14 & 0 & -10 \end{bmatrix}$ is ________. **[CE-GATE-2014(2)]**

Q.61 Consider the following system of equations: $3x + 2y = 1$, $4x + 7z = 1$, $x + y + z = 3$, $x - 2y + 7z = 0$. The number of solutions for this system is________. **[CS-GATE-2014(1)]**

Q.62 The value of the dot product of the Eigenvectors corresponding to any pair of different Eigen values of a 4 by 4 symmetric positive definite matrix is________. **[CS-GATE-2014(1)]**

Q.63 If the matrix A is such that $A = \begin{bmatrix} 2 \\ -4 \\ 7 \end{bmatrix} \begin{bmatrix} 1 & 9 & 5 \end{bmatrix}$, then the determinant of A is equal to ____________. **[CS-GATE-2014(2)]**

Q.64 The product of the non-zero Eigen values of the matrix $\begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix}$ is______.

[CS-GATE-2014(2)]

Q.65 Which one of the following statements is TRUE about every n × n matrix with only real Eigen values?

(a) If the trace of the matrix is positive and the determinant of the matrix is negative, at least one of its Eigen values is negative.

(b) If the trace of the matrix is positive, all its Eigen values are positive.

(c) If the determinant of the matrix is positive, all its Eigen values are positive.

(d) If the product of the trace and determinant of the matrix is positive, all its Eigen values are positive. **[CS-GATE-2014(3)]**

Q.66 If V1 and V2 are 4-dimensional subspaces of a 6-dimensional vector space V, then the smallest possible dimension of $V_1 \cap V_2$ is ________. **[CS-GATE-2014(3)]**

Q.67 The minimum number of arithmetic operations required to evaluate the polynomial $P(X) = X^5 + 4X^3 + 6x + 5$ for a given value of X, using only one temporary variable is ______.

[CS-GATE-2014(3)]

Q.68 For matrices of same dimension M, N and scalar c, which one of these properties DOES NOT ALWAYS hold?

(a) $(M^T)^T = M$
(b) $(cM)^T = c(M)^T$
(c) $(M+N)^T = M^T + N^T$
(d) $MN = NM$ **[EC-GATE-2014(1)]**

Q.69 A real (4×4) matrix A satisfies the equation $A^2 = I$, where I is the (4×4) identity matrix. The positive Eigen value of A is ________. **[EC-GATE-2014(1)]**

Q.70 Consider the matrix :

$$J_6 = \begin{vmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{vmatrix}$$

Which is obtained by reversing the order of the columns of the identity matrix I_6. Let $P = I_6 + \alpha J_6$, where α is a non-negative real number. The value of α for which det(P)=0 is ____. **[EC-GATE-2014(1)]**

Q.71 The determinant of matrix A is 5 and the determinant of matrix B is 40.The determinant of matrix AB is _________. **[EC-GATE-2014(2)]**

Q.72 The system of linear equations $\begin{pmatrix} 2 & 1 & 3 \\ 3 & 0 & 1 \\ 1 & 2 & 5 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 5 \\ -4 \\ 14 \end{pmatrix}$ has

(a) a unique solution
(b) infinitely many solutions
(c) no solution
(d) exactly two solutions **[EC-GATE-2014(2)]**

Q.73 The maximum value of the determinant among all 2×2 real symmetric matrices with trace 14 is ________. **[EC-GATE-2014(2)]**

Q.74 Which one of the following statements is NOT true for a square matrix?

(a) If A is upper triangular, the Eigen values of A are the diagonal elements of it
(b) If A is real symmetric, the Eigen values of A are always real and positive
(c) If A is real, the Eigen values of A and A^T are always the same
(d) If all the principal minors of A are positive, all the Eigen values of A are also positive

[EC-GATE-2014(3)]

Q.75 Given a system of equations : $x + 2y + 2z = b_1$, $5x + y + 3z = b_2$

Which of the following is true regarding its solutions ?

(a) The system has a unique solution for any given b_1 and b_2

(b) The system will have infinitely many solutions for any given b_1 and b_2

(c) Whether or not a solution exists depends on the given b_1 and b_2

(d) The system would have no solution for any values of b_1 & b_2 **[EE-GATE-2014(1)]**

Q.76 A system matrix is given as follows $A=\begin{bmatrix} 0 & 1 & -1 \\ -6 & -11 & 6 \\ -6 & -11 & 5 \end{bmatrix}$. The absolute value of the ratio of the maximum Eigen value to the minimum Eigen value is **[EE-GATE-2014(1)]**

Q.77 Which one of the following statements is true for all real symmetric matrices?

(a) All the Eigen values are real.

(b) All the Eigen values are positive

(c) All the Eigen values are distinct

(d) Sum of all the Eigen values is zero. **[EE-GATE-2014-Set-2]**

Q.78 Two matrices A and B are $A=\begin{bmatrix} p & q \\ r & s \end{bmatrix}$; $B=\begin{bmatrix} p^2+q^2 & pr+qs \\ pr+qs & r^2+s^2 \end{bmatrix}$

If the rank of matrix A is N, then the rank of matrix B is

(a) $\frac{N}{2}$ (b) $N-1$

(c) N (d) 2 N **[EE-GATE-2014(3)]**

Q.79 Given $x(t)=3\sin(1000\pi t)$ and $y(t)=5\cos\left(1000\ \pi t+\frac{\pi}{4}\right)$. The x-y plot will be

(a) a circle (b) a multi-loop closed curve

(c) a hyperbola (d) an ellipse **[IN-GATE-2014]**

Q.80 A scalar valued function is defined as $f(x)=x^tAx+b^tx+c$, where A is a symmetric positive definite matrix with dimension n×n; b and x are vectors of dimension n×1. The minimum value of f(x) will occur when x equals

(a) $\left(A^TA\right)^{-1}b$ (b) $-\left(A^TA\right)^{-1}b$

(c) $-\left(\frac{A^{-1}b}{2}\right)$ (d) $\frac{A^{-1}b}{2}$ **[IN-GATE-2014]**

Q.81 For the matrix A satisfying the equation given below, then theEigen values of A are

$$[A]\begin{bmatrix}1 & 2 & 3\\ 7 & 8 & 9\\ 4 & 5 & 6\end{bmatrix}=\begin{bmatrix}1 & 2 & 3\\ 4 & 5 & 6\\ 7 & 8 & 9\end{bmatrix}$$

(a) (1, −j, j) (b) (1, 1, 0)
(c) (1, 1, −1) (d) (1, 0, 0) **[IN-GATE-2014]**

Q.82 If any two columns of determinant $P=\begin{vmatrix}4 & 7 & 8\\ 3 & 1 & 5\\ 9 & 6 & 2\end{vmatrix}$ are interchange, which one of the following statement regarding the value of the determinant is CORRECT?
(a) Absolute value remains unchanged but sign will change
(b) Both absolute value and sign will change
(c) Absolute value will change but sign will not change
(d) Both absolute value and sign will remain unchanged **[ME-GATE-2015(1)]**

Q.83 At least one Eigen value of a singular matrix is
(a) Positive (b) Zero
(c) Negative (d) Imaginary **[ME-GATE-2015(2)]**

Q.84 The lowest Eigen value of the 2×2 matrix $\begin{bmatrix}4 & 2\\ 2 & 3\end{bmatrix}$ is ________. **[ME-GATE -2015(3)]**

Q.85 For a given matrix, $P=\begin{bmatrix}4-3i & i\\ -i & 4+3i\end{bmatrix}$ where is $i=\sqrt{-1}$, the inverse of matrix P is

(a) $\frac{1}{24}\begin{bmatrix}4-3i & i\\ -i & 4+3i\end{bmatrix}$ (b) $\frac{1}{25}\begin{bmatrix}i & 4-3i\\ 4+3i & -i\end{bmatrix}$

(c) $\frac{1}{24}\begin{bmatrix}4+3i & -i\\ i & 4-3i\end{bmatrix}$ (d) $\frac{1}{25}\begin{bmatrix}4+3i & -i\\ i & 4-3i\end{bmatrix}$ **[ME-GATE -2015(3)]**

Q.86 For what value of p the following set of equations will have no solution?
2x + 3y = 5 , 3x + py = 10 **[CE-GATE-2015(1)]**

Q.87 The smallest and largest Eigen values of the following matrix are : $\begin{bmatrix}3 & -2 & 2\\ 4 & -4 & 6\\ 2 & -3 & 5\end{bmatrix}$

(a) 1.5 and 2.5 (b) 0.5 and 2.5
(c) 1.0 and 3.0 (d) 1.0 and 2.0 **[CE-GATE-2015(1)]**

Q.88 Let $A = \left[a_{ij}\right],\ 1 \le i,\ j \le n$ with $n \ge 3$ and $a_{ij} = i.j.$ The rank of A is

(a) 0 (b) 1

(c) $n-1$ (d) n **[CE-GATE-2015(2)]**

Q.89 The two Eigen values of the matrix $\begin{bmatrix} 2 & 1 \\ 1 & p \end{bmatrix}$ have a ratio of 3 : 1 for p = 2. What is another value of p for which the Eigen values have the same ratio of 3 : 1?

(a) -2 (b) 1

(c) $\frac{7}{3}$ (d) $\frac{14}{3}$ **[CE-GATE-2015(2)]**

Q.90 Consider the following 2×2 matrix A where two elements are unknown and are marked by a and b. The Eigen values of this matrix are –1 and 7. What are the values of a & b ?

$$A = \begin{pmatrix} 1 & 4 \\ b & a \end{pmatrix}$$

(a) $a = 6,\ b = 4$ (b) $a = 4,\ b = 6$

(c) $a = 3,\ b = 5$ (d) $a = 5,\ b = 3$ **[CS-GATE-2015(1)]**

Q.91 The larger of the two Eigen values of the matrix $\begin{bmatrix} 4 & 5 \\ 2 & 1 \end{bmatrix}$ is ______. **[CS-GATE-2015(2)]**

Q.92 Perform the following operations on the matrix $\begin{bmatrix} 3 & 4 & 45 \\ 7 & 9 & 105 \\ 13 & 2 & 195 \end{bmatrix}$

i) Add the third row to the second row

ii) Subtract the third column from the first column.

The determinant of the resultant matrix is _________. **[CS-GATE-2015(2)]**

Q.93 In the matrix $\begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix}$ one of the Eigen values is 1. The Eigenvectors corresponding to the Eigen value 1 are

(a) $\left\{\alpha\left(4,2,1\right) \mid \alpha \ne 0, \alpha \in R\right\}$ (b) $\left\{\alpha\left(-4,2,1\right) \mid \alpha \ne 0, \alpha \in R\right\}$

(c) $\left\{\alpha\left(\sqrt{2},0,1\right) \mid \alpha \ne 0, \alpha \in R\right\}$ (d) $\left\{\alpha\left(-\sqrt{2},0,1\right) \mid \alpha \ne 0, \alpha \in R\right\}$ **[CS-GATE-2015(3)]**

Q.94 If the following system has non-trivial solution. px + qy + rz = 0, qx + ry + pz = 0, rx + py + qz = 0, Then which one of the following options is TRUE?

(a) p − q + r = 0 or p = q = −r (b) p + q − r = 0 or p = −q = r

(c) p + q + r = 0 or p = q = r (d) p − q + r = 0 or p = −q = −r **[CS-GATE-2015(3)]**

Q.95 In the LU decomposition of the matrix $\begin{bmatrix} 2 & 2 \\ 4 & 9 \end{bmatrix}$, if the diagonal elements of U are both 1, then the lower diagonal entry l_{22} of L is ________. **[CS-GATE-2015(1)]**

Q.96 Consider a system of linear equations : $x - 2y + 3z = -1$, $x - 3y + 4z = 1$ & $-2x + 4y - 6z = k$

The value of k for which the system has infinitely many solutions is ________.

[EC-GATE-2015(1)]

Q.97 The value of p such that the vector $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ is an Eigenvector of the matrix

$\begin{bmatrix} 4 & 1 & 2 \\ p & 2 & 1 \\ 14 & -4 & 10 \end{bmatrix}$ is ____________ **[EC-GATE-2015(1)]**

Q.98 The value of x for which all the Eigen-values of the matrix given below are real is

$$\begin{bmatrix} 10 & 5+j & 4 \\ x & 20 & 2 \\ 4 & 2 & -10 \end{bmatrix}$$

(a) 5 + j (b) 5 − j

(c) 1 − 5j (d) 1 + 5 j **[EC-GATE-2015(2)]**

Q.99 For A = $\begin{bmatrix} 1 & \tan x \\ -\tan x & 1 \end{bmatrix}$ the determinant of $A^T A^{-1}$ is

(a) $\sec^2 x$ b) cos4x

(c) 1 (d) 0 **[EC-GATE-2015(3)]**

Q.100 If the sum of the diagonal elements of a 2×2 matrix is -6, then the maximum possible value of determinant of the matrix is **[EE-GATE-2015(1)]**

Q.101 We have a set of 3 linear equations in 3 unknowns. '$x \equiv y$' means X and Y are equivalent statements and '$X \not\equiv Y$' means X and Y are not equivalent statements.

P : There is a unique solution.

Q : The equations are linearly independent.

R : All Eigen values of the coefficient matrix are nonzero.

S : The determinant of the coefficient matrix is nonzero.

Which one of the following is TRUE?

(a) $P \equiv Q \equiv R \equiv S$ (b) $P \equiv R \not\equiv Q \equiv S$

(c) $P \equiv Q \not\equiv R \equiv S$ (d) $P \equiv Q \not\equiv R \equiv S$ **[EE-GATE-2015(2)]**

Q.102 Let A be an $n \times n$ matrix with the rank $r(0 < r < n)$, Then Ax=0 has p independent solutions, where p is

(a) r (b) n

(c) n - r (d) n + r **[IN-GATE-2015]**

Q.103 The solution to the system of equations $\begin{bmatrix} 2 & 5 \\ -4 & 3 \end{bmatrix}\begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{Bmatrix} 2 \\ -30 \end{Bmatrix}$ is

(a) 6, 2 (b) -6, 2

(c) -6, -2 (d) 6, -2 **[ME-GATE -2016(1)]**

Q.104 The condition for which the Eigen values of the matrix $A = \begin{bmatrix} 2 & 1 \\ 1 & k \end{bmatrix}$ are positive, is

(a) $k > 1/2$ (b) $k > -2$

(c) $k > 0$ (d) $k < -1/2$ **[ME-GATE -2016(2)]**

Q.105 The number of linearly independent Eigenvectors of $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ is _______.

[ME-GATE -2016(3)]

Q.106 If the entries in each column of a square matrix M add up to 1, then an Eigen value of M is

(a) 4 (b) 3

(c) 2 (d) 1 **[CE-GATE-2016(1)]**

Q.107 Consider the following linear system.

$x + 2y - 3z = a, \ 2x + 3y + 3z = b, 5x + 9y - 6z = c$

This system is consistent if a, b and c satisfy the equation

(a) $7a - b - c = 0$ (b) $3a + b - c = 0$

(c) $3a - b + c = 0$ (d) $7a - b + c = 0$ **[CE-GATE-2016(2)]**

Q.108 Two Eigen values of a 3 × 3 real matrix P are $\left(2+\sqrt{-1}\right)$ and 3. The determinant of P is ________. **[CS-GATE-2016(1)]**

Q.109 Consider the systems, each consisting of m linear equations in n variables.

I. If m < n, then all such systems have a solution

II. If m > n, then none of these systems has a solution

III. If m = n, then there exists a system which has a solution

Which one of the following is correct?

(a) I, II and III are true (b) Only II and III are true

(c) Only III is true (d) None of them is true **[CS-GATE-2016(2)]**

Q.110 Suppose that the Eigen values of matrix A are 1, 2, 4. The determinant of $(A^{-1})^T$ is ____. **[CS-GATE-2016(2)]**

Q.111 Let $M^4 = I$, (where I denotes the identity matrix) & $M \neq I, M^2 \neq I$ & $M^3 \neq I$. Then, for any natural number k, M^{-1} equals:

(a) M^{4k+1} (b) M^{4k+2}

(c) M^{4k+3} (d) M^{4k} **[EC-GATE-2016(1)]**

Q.112 A sequence x[n] is specified as $\begin{bmatrix} x[n] \\ x[n-1] \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, for $n \geq 2$ the initial conditions are x[0] = 1, x[1] = 1, and x[n] = 0 for n < 0. The value of x[12] is ___. **[EC-GATE-2016(1)]**

Q.113 The value of x for which the matrix $A = \begin{bmatrix} 3 & 2 & 4 \\ 9 & 7 & 13 \\ -6 & -4 & -9+x \end{bmatrix}$ has zero as an Eigen value is _________. **[EC-GATE-2016(2)]**

Q.114 The matrix $A = \begin{bmatrix} a & 0 & 3 & 7 \\ 2 & 5 & 1 & 3 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & b \end{bmatrix}$ has det(A) = 100 and trace (A) = 14. The value of $|a-b|$ is ___________. **[EC-GATE-2016(2)]**

Q.115 Consider a 2 x 2 square matrix $A = \begin{bmatrix} \sigma & x \\ \omega & \sigma \end{bmatrix}$ Where x is unknown. If the Eigen values of the matrix A are $(\sigma + j\omega)$ and $(\sigma - j\omega)$, the x is equal to

(a) $+j\omega$ (b) $-j\omega$

(c) $+\omega$ (d) $-\omega$ **[EC-GATE-2016(3)]**

Q.116 Consider a linear time invariant system $\dot{x} = Ax$ with initial condition x(0) at t = 0. Suppose α and β are Eigenvectors of (2×2) matrix A corresponding to distinct Eigen values λ_1 and λ_2 respectively. Then the response x (t) of the system due to initial condition x (0) = α is

(a) $\alpha e^{\lambda_1 t}$ (b) $\beta e^{\lambda_2 t}$

(c) $\alpha e^{\lambda_2 t}$ (d) $\alpha e^{\lambda_1 t} + \beta e^{\lambda_2 t}$ **[EE-GATE-2016(2)]**

Q.117 Consider the matrix $A = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 3 & 4 \\ -1 & -1 & -2 \end{bmatrix}$ whose Eigen values are $1, -1$ & 3. Then Trace of $(A^3 - 3A^2)$ is ______. **[IN-GATE-2016]**

Q.118 The product of Eigen values of the matrix $P = \begin{bmatrix} 2 & 0 & 1 \\ 4 & -3 & 3 \\ 0 & 2 & -1 \end{bmatrix}$ is

(a) -6 (b) 2

(c) 6 (d) -2 **[GATE -2017]**

Q.119 Consider the matrix

$$P = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{-1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

Which one of the following statements about P is INCORRECT?

(a) Determinant of P is equal to 1.

(b) P is orthogonal.

(c) Inverse of P is equal to its transpose.

(d) All Eigen values of P are real numbers. **[GATE -2017]**

Q.120 The determinant of a 2×2 matrix is 50.If one Eigen value of the matrix is 10, the other Eigen value is _____. **[GATE-2017]**

Q.121 Consider the matrix $A = \begin{bmatrix} 50 & 70 \\ 70 & 80 \end{bmatrix}$ whose Eigen vectors corresponding to Eigen value λ_1 & λ_2 are $x_1 \begin{bmatrix} 70 \\ \lambda_1 - 50 \end{bmatrix}$ & $x_2 = \begin{bmatrix} \lambda_1 - 80 \\ 70 \end{bmatrix}$ respectively. The value of $x_1^T x_2$ is **[GATE-2017]**

Q.122 For the given Orthogonal Matrix

$$Q = \begin{bmatrix} 3/7 & 2/7 & 6/7 \\ -6/7 & 3/7 & 2/7 \\ 2/7 & 6/7 & -3/7 \end{bmatrix}$$

The inverse of Q is

(a) $\begin{bmatrix} 3/7 & 2/7 & 6/7 \\ -6/7 & 3/7 & 2/7 \\ 2/7 & 6/7 & -3/7 \end{bmatrix}$ (b) $\begin{bmatrix} -3/7 & -2/7 & -6/7 \\ 6/7 & -3/7 & -2/7 \\ -2/7 & -6/7 & 3/7 \end{bmatrix}$

(c) $\begin{bmatrix} 3/7 & -6/7 & 2/7 \\ 2/7 & 3/7 & 6/7 \\ 6/7 & 2/7 & -3/7 \end{bmatrix}$ (d) $\begin{bmatrix} -3/7 & 6/7 & -2/7 \\ -2/7 & -3/7 & -6/7 \\ -6/7 & -2/7 & 3/7 \end{bmatrix}$ **[CE-GATE-2018(1)]**

Q.123 Which one of the following matrices is singular?

(a) $\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$ (b) $\begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$

(c) $\begin{bmatrix} 2 & 4 \\ 3 & 6 \end{bmatrix}$ (d) $\begin{bmatrix} 4 & 3 \\ 6 & 2 \end{bmatrix}$ **[CE-GATE-2018(1)]**

Q.124 The matrix has $\begin{bmatrix} 2 & -4 \\ 4 & -2 \end{bmatrix}$

(a) real eigenvalues and eigen vectors
(b) real eigenvalues but complex eigen vectors
(c) complex eigenvalues but real eigen vectors
(d) complex eigenvalues and eigen vectors **[CE-GATE-2018(2)]**

Q.125 The rank of the following matrix $\begin{bmatrix} 1 & 1 & 0 & -2 \\ 2 & 0 & 2 & 2 \\ 4 & 1 & 3 & 1 \end{bmatrix}$

(a) 1 (b) 2
(c) 3 (d) 4 **[CE-GATE-2018(2)]**

Q.126 Let N be a 3 by 3 matrix with real number entries. The matrix is such that $N^2 = 0$. The Eigen values of are

(a) 0, 0, 0 (b) 0, 0, 1
(c) 0, 1, 1 (d) 1, 1, 1 **[IN-GATE-2018]**

Q.127 Consider the following system of linear equations : $3x + 2ky = 2,\ kx + 6y = 2$

Here x and y are the unknowns and k is a real constant. The value of k for which there are infinite number of solutions is

(a) 3 (b) 1

(c) – 3 (d) – 6 **[IN-GATE-2018]**

Q.128 Consider a non-singular 2×2 square matrix A. If trace (A) = 4 and trace (A^2) = 5. The determinant of the matrix A is ______ (up to 1 decimal place). **[EE-GATE-2018]**

Q.129 Let $A = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$ and $B = A^3 - A^2 - 4A + 5I$, where I is the 3×3 identity matrix. The determinant of the B is _______. **[EE-GATE-2018]**

Q.130 Let M be a real 4×4 matrix. Consider the following statements :

S1 : M has 4 linearly independent eigenvectors.

S2 : M has 4 distinct eigen values.

S3 : M is non-singular (invertible) matrix.

Which one among the following is TRUE ?

(a)S1 implies S2 (b) S1 implies S3

(c) S2 implies S1 (d) S3 implies S2 **[EC-GATE-2018]**

Q.131 Consider matrix $A = \begin{bmatrix} k & 2k \\ k^2 - k & k^2 \end{bmatrix}$ and vector $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. The number of distinct real values of k for which the equation AX = 0 has infinitely many solutions is__________.

[EC-GATE-2018]

Q.132 The rank of the matrix $\begin{bmatrix} -4 & 1 & -1 \\ -1 & -1 & -1 \\ 7 & -3 & 1 \end{bmatrix}$ is

(a) 1 (b) 2

(c) 3 (d) 4 **[ME-GATE-2018(1)]**

Q.133 If $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 1 \end{bmatrix}$ then det (A^{-1}) is ________ (correct to two decimal places).

[ME-GATE-2018(2)]

ANSWER KEY:

1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**
(c)	(b)	(c)	(b)	(d)	(a)	(b)	(c)	(c)	(a)	(a)	(c)	(c)	(d)
15	**16**	**17**	**18**	**19**	**20**	**21**	**22**	**23**	**24**	**25**	**26**	**27**	**28**
(c)	(a)	(a)	(a)	(b)	(d)	(c)	(c)	(a)	(b)	(b)	(a)	(b)	(b)
29	**30**	**31**	**32**	**33**	**34**	**35**	**36**	**37**	**38**	**39**	**40**	**41**	**42**
(a)	(a)	(b)	(d)	(d)	(b)	(d)	(d)	(d)	(a)	(c)	(b)	(b)	16
43	**44**	**45**	**46**	**47**	**48**	**49**	**50**	**51**	**52**	**53**	**54**	**55**	**56**
(d)	(a)	(c)	(b)	(d)	(d)	(b)	(d)	(a)	(a)	(d)	(d)	(d)	23
57	**58**	**59**	**60**	**61**	**62**	**63**	**64**	**65**	**66**	**67**	**68**	**69**	**70**
(a)	(a)	88	2.0	1	0	0	6	(b)	2	7	(d)	1	1
71	**72**	**73**	**74**	**75**	**76**	**77**	**78**	**79**	**80**	**81**	**82**	**83**	**84**
200	(b)	49	(b)	(b)	(1/3)	(a)	(c)	(d)	(c)	(c)	(a)	(b)	(c)
85	**86**	**87**	**88**	**89**	**90**	**91**	**92**	**93**	**94**	**95**	**96**	**97**	**98**
(a)	4.5	(d)	(b)	(d)	(d)	6	0	(b)	(c)	5	2	17	(b)
99	**100**	**101**	**102**	**103**	**104**	**105**	**106**	**107**	**108**	**109**	**110**	**111**	**112**
(c)	(*)	(a)	(c)	(d)	(a)	2	(d)	(b)	15	(c)	0.125	(c)	233
113	**114**	**115**	**116**	**117**	**118**	**119**	**120**	**121**	**122**	**123**	**124**	**125**	**126**
1	3	(d)	(a)	-6	(b)	(d)	5	0	(c)	(c)	(d)	(b)	(a)
127	**128**	**129**	**130**	**131**	**132**	**133**							
(c)	5.5	1	(c)	2	(b)	0.25							

ESE QUESTIONS

Q.1 Let the Eigenvector of the matrix $\begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$ be written in the form $\begin{bmatrix} 1 \\ a \end{bmatrix}$ and $\begin{bmatrix} 1 \\ b \end{bmatrix}$. What is the value of (a+b)?

(a) 0 (b) $\frac{1}{2}$

(c) 1 (d) 2 **[ESE 2018 –GS]**

Q.2 Eigen values of the Matrix $\begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix}$ are

(a)1, 1, 1 (b)1, 1, 2

(c)1, 4, 4 (d)1, 2, 4 **[ESE 2018 –EE]**

Q.3 If the system 2x – y + 3z = 2, x + y + 2z = 2, 5x – y + az = b has infinitely many solutions, then the values of a and b, respectively, are

(a) – 8 and 6 (b) 8 and 6

(c) – 8 and –6 (d) 8 and –6 **[ESE 2018 –EE]**

ANSWER KEY :

1. (b)

2. (c)

3. (b)

GATE SOLUTIONS

Q.1 **(c)** Given equation are

$x+2y+z=6; 2x+y+2z=6; x+y+z=5$ Given system can be written as

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 6 \\ 6 \\ 5 \end{bmatrix} \text{ Augmented matrix is } \left[\begin{array}{ccc|c} 1 & 2 & 1 & 6 \\ 2 & 1 & 2 & 6 \\ 1 & 1 & 1 & 5 \end{array}\right]$$

By Gauss elimination

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 6 \\ 2 & 1 & 2 & 6 \\ 1 & 1 & 1 & 5 \end{array}\right] \xrightarrow[R_3-R_1]{R_2-2R_1} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 6 \\ 0 & -3 & 0 & -6 \\ 0 & -1 & 0 & -1 \end{array}\right] \xrightarrow{R_3-\frac{1}{3}R_2} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 6 \\ 0 & -3 & 0 & -6 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

r(A) = 2, r(A|B) = 3

Since the rank of coefficient matrix is 2 and rank of augmented matrix is 3, which is not equal. Hence system has no solution i.e. system is inconsistent.

Q.2 **(b)** A, B, C, D is an n × n matrix

Given $ABCD = I$, $ABCDD^{-1}C^{-1}=D^{-1}C^{-1}$, $AB = D^{-1}C^{-1}$

$A^{-1}AB = A^{-1}D^{-1}C^{-1}$, $B = A^{-1}D^{-1}C^{-1}$, $B^{-1} = \left(A^{-1}D^{-1}C^{-1}\right)^{-1}$

$B^{-1} = \left(C^{-1}\right)^{-1}\left(D^{-1}\right)^{-1}\left(A^{-1}\right)^{-1}$, $B^{-1} = CDA$

Q.3 **(c)** Consider first 3×3 minors, since maximum possible rank is 3

$$\begin{vmatrix} 4 & 2 & 1 \\ 6 & 3 & 4 \\ 2 & 1 & 0 \end{vmatrix}=0, \quad \begin{vmatrix} 2 & 1 & 3 \\ 3 & 4 & 7 \\ 1 & 0 & 1 \end{vmatrix}=0, \quad \begin{vmatrix} 4 & 1 & 3 \\ 6 & 4 & 7 \\ 2 & 0 & 1 \end{vmatrix}=0, \quad \begin{vmatrix} 4 & 2 & 3 \\ 6 & 3 & 7 \\ 2 & 1 & 1 \end{vmatrix}=0$$

Since all 3×3 minors are zero therefore $\text{rank} \neq 3$, now try 2×2 minors.

$$\begin{bmatrix} 4 & 2 \\ 6 & 3 \end{bmatrix}=0, \quad \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}=8-3=5\neq 0, \quad \text{So, rank=2}$$

Q.4 **(b)** If $x=\left[x_1, x_2 \dots x_n\right]^T$

Then, $\text{Rank}(V)=\text{Rank}(XX^T)=\text{Rank}(X^TX)=1$

Q.5 **(d)** Here, $AA^T = (1/4)I$

Q.6 **(d)**

$$AA^{-1} = I, \quad \begin{bmatrix} 2 & -0.1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & a \\ 0 & b \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 2a - 0.1b \\ 0 & 3b \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$2a - 0.1b = 0, \quad 3b = 1, \quad a + b = \frac{7}{20}$$

Q.7 **(b)** The augmented matrix for given system is $\left[\begin{array}{ccc|c} 2 & -1 & 3 & 1 \\ 3 & -2 & 5 & 2 \\ -1 & -4 & 1 & 3 \end{array}\right]$ Using gauss-elimination method on above matrix we get

$$\left[\begin{array}{ccc|c} 2 & -1 & 3 & 1 \\ 3 & -2 & 5 & 2 \\ -1 & -4 & 1 & 3 \end{array}\right] \xrightarrow[R_3 + \frac{1}{2}R_1]{R_2 - \frac{3}{2}R_1} \left[\begin{array}{ccc|c} 2 & -1 & 3 & 1 \\ 0 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & -\frac{9}{2} & \frac{5}{2} & \frac{7}{2} \end{array}\right] \xrightarrow{R_3 - 9R_2} \left[\begin{array}{ccc|c} 2 & -1 & 3 & 1 \\ 0 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & -2 & -1 \end{array}\right]$$

Rank ([A|B]) = 3, Rank ([A]) = 3, Since Rank ([A|B]) = Rank ([A]) = number of variables, the system has unique solution.

Q.8 **(c)** First, find the Eigen values of $A = \begin{bmatrix} -4 & 2 \\ 4 & 3 \end{bmatrix}$

$$|A - \lambda I| = 0, \begin{bmatrix} -4-\lambda & 2 \\ 4 & 3-\lambda \end{bmatrix} = 0, \; \lambda^2 + \lambda - 20 = 0, \; \lambda_1 = -5 \text{ and } \lambda_2 = 4$$

Corresponding to $\lambda_1 = -5$ we need to find the Eigen vector : The Eigen value problem is

$$|A - \lambda I|X = 0, \begin{bmatrix} -4-\lambda & 2 \\ 4 & 3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

Putting $\lambda_1 = -5$

$$\begin{bmatrix} 1 & 2 \\ 4 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, x_1 + 2x_2 = 0$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

Q.9 **(c)** $M=\begin{bmatrix}4 & 2\\2 & 4\end{bmatrix}$, $[M-\lambda I]=\begin{bmatrix}4-\lambda & 2\\2 & 4-\lambda\end{bmatrix}$, $|M-\lambda I|\widehat{X}=0, \begin{bmatrix}4-\lambda & 2\\2 & 4-\lambda\end{bmatrix}\begin{bmatrix}101\\101\end{bmatrix}=0,\ \lambda=6$

Q.10 **(a)** For Eigen value of $\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}$, $\begin{bmatrix}1-\lambda & 0\\0 & 0-\lambda\end{bmatrix}=0,\ (1-\lambda)(-\lambda)=0,\ \lambda=0\ \&\ \lambda=1$

Eigen value of $\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}$, $\begin{bmatrix}1-\lambda & 0\\0 & 0-\lambda\end{bmatrix}=0\ (1-\lambda)(-\lambda)=0,\ \lambda=0\ \&\ \lambda=1$

Eigen value of $\begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}$, $\begin{bmatrix}-\lambda & 1\\0 & -\lambda\end{bmatrix}=0,\ \lambda^2=0,\ \ \lambda=0,0$

Eigen value of $\begin{bmatrix}1 & -1\\1 & 1\end{bmatrix}$, $\begin{bmatrix}1-\lambda & -1\\1 & 1-\lambda\end{bmatrix}=0,\ (1-\lambda)^2+1=0,\ \lambda=1-i\ \&\ 1+i$

Eigen value of $\begin{bmatrix}-1 & 0\\1 & -1\end{bmatrix}$, $\begin{bmatrix}-1-\lambda & 0\\1 & -1-\lambda\end{bmatrix}=0,\ (1+\lambda)^2=0,\ \lambda=-1,-1$

Only one matrix has an Eigen value of 1 which is $\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}$, Correct choice is (a).

Q.11 **(a)**

Matrix	Order
X^T	3×4
Y	4×3
X^TY	3×3
$(X^TY)^{-1}$	3×3
P	2×3
P^T	3×2
$P\,(X^TY)^{-1}\,P^T$	2×2
$(P\,(X^TY)^{-1}\,P^T)^T$	2×2

Q.12 **(c)** $G=\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$, $E\equiv\begin{bmatrix}\cos\theta & -\sin\theta & 0\\\sin\theta & \cos\theta & 0\\0 & 0 & 1\end{bmatrix}$

Given EF=G and G is an Identity matrix, therefore F is Inverse of E.

$$F=E^{-1}=\frac{\text{Adj}(E)}{|E|}=\begin{bmatrix}\cos\theta & \sin\theta & 0\\-\sin\theta & \cos\theta & 0\\0 & 0 & 1\end{bmatrix}$$

Q.13 **(c)** Perform, $R_3 \to -R_3 + R_1$, $\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, Rank = 2

Q.14 **(d)** The augmented matrix for given system is

$$\left[\begin{array}{ccc|c} 0 & 4 & 3 & 8 \\ 2 & 0 & -1 & 2 \\ 3 & 2 & 0 & 5 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 2 & 0 & -1 & 2 \\ 0 & 4 & 3 & 8 \\ 0 & 2 & 0 & 5 \end{array}\right] \xrightarrow{R_3 - \frac{3}{2}R_1 - \frac{1}{2}R_2} \left[\begin{array}{ccc|c} 2 & 0 & -1 & 8 \\ 0 & 4 & 3 & 8 \\ 0 & 0 & 0 & -2 \end{array}\right]$$

r(A)=2, r(A|B)=3, r(A) $\neq$ r(A|B), means system is inconsistent.

Q.15 **(c)** $|A - \lambda I| = 0$, $\begin{bmatrix} -3-\lambda & 2 \\ -1 & 0-\lambda \end{bmatrix} = 0$, $\lambda^2 + 3\lambda + 2 = 0$

A satisfies this equation according to Cayley-Hamilton theorem

i.e. $A^2 + 3A + 2I = 0$

Q.16 **(a)** $A^2 + 3A + 2I = 0$, $A^2 = -3A - 2I$, $A^4 = \left(A^2\right)^2 = \left(-3A - 2I\right)^2 = 9A^2 + 12A + 4I$

$A^4 = 9(-3A - 2I) + 12A + 4I = -15A - 14I$, $A^8 = \left(A^4\right)^2 = (-15A - 14I)^2$

$A^8 = \left(225A^2 + 420A + 196I\right) = -255A - 254I$,

$A^9 = A \times A^8$, $= A(-255A - 254I) = 511A + 510I$

Q.17 **(a)** Inverse of $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \dfrac{1}{(ad - bc)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Inverse of $\begin{bmatrix} 1 & 2 \\ 5 & 7 \end{bmatrix} = \dfrac{1}{7-10} \begin{bmatrix} 7 & -2 \\ -5 & 1 \end{bmatrix} = \dfrac{1}{3} \begin{bmatrix} -7 & 2 \\ 5 & -1 \end{bmatrix}$

Q.18 **(a)** The augmented matrix for given system is $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 1 & 3 & 3 & 9 \\ 1 & 2 & \alpha & \beta \end{array}\right]$

by Gauss elimination Process

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 1 & 3 & 3 & 9 \\ 1 & 2 & \alpha & \beta \end{array}\right] \xrightarrow[R_3 - R_1]{R_2 - R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 2 & 2 & 4 \\ 0 & 1 & \alpha - 1 & \beta - 5 \end{array}\right] \xrightarrow{R_3 - \frac{1}{2}R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 2 & 2 & 4 \\ 0 & 0 & \alpha - 2 & \beta - 7 \end{array}\right]$$

Now, for infinite solution last row must be completely zero

i.e. $\alpha - 2 = 0$ and $\beta - 7 = 0$, $\alpha = 2$ and $\beta = 7$

Q.19 **(b)** The system can be written in matrix form as $\begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} 7 \\ 6 \end{bmatrix}$

$$[A:B]=\left[\begin{array}{cc|c} 4 & 2 & 7 \\ 2 & 1 & 6 \end{array}\right]=\left[\begin{array}{cc|c} 4 & 2 & 7 \\ 2 & 1 & 6 \end{array}\right]\xrightarrow{R_2-\frac{1}{2}R_1}\left[\begin{array}{cc|c} 4 & 2 & 7 \\ 0 & 0 & \frac{5}{2} \end{array}\right]$$

$\text{Rank}[A|B] \neq \text{Rank}[A]$, The system has no solution.

Q.20 **(d)** $$[A:B]=\left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 2 \\ 1 & 4 & a & 4 \end{array}\right]\xrightarrow[R_3-R_1]{R_2-R_1}\left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 3 & a-2 & 3 \end{array}\right]\xrightarrow{R_3-3R_2}\left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & a-5 & 0 \end{array}\right]$$

Now as long as $a-5\neq 0$, rank (A) = rank $(A\,|\,B)$ = 3

$\therefore$ a can take any real value except 5.

Q.21 **(c)** Characteristic equation is

$$|A-\lambda I| = 0, \begin{vmatrix} 4-\lambda & -2 \\ -2 & 1-\lambda \end{vmatrix} = 0,\ (4-\lambda)(1-\lambda)-4 = 0,\ \lambda^2-5\lambda=0,\ \lambda(\lambda-5) = 0$$

Hence, Eigen values are 0 & 5.

Q.22 **(c)** $A =\begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix},\ |A-\lambda I|=0,\ \begin{bmatrix} 4-\lambda & 1 \\ 1 & 4-\lambda \end{bmatrix}=0,\ (4-\lambda)^2-1 = 0,\ \lambda = 3\ \&= 5$

Q.23 **(a)** $$|A-\lambda I|=\begin{vmatrix} 5-\lambda & 0 & 0 & 0 \\ 0 & 5-\lambda & 5 & 0 \\ 0 & 0 & 2-\lambda & 1 \\ 0 & 0 & 3 & 1-\lambda \end{vmatrix}=0 \Rightarrow [5-\lambda)(5-\lambda)[(2-\lambda)(1-\lambda)-3]=0 \Rightarrow \lambda=5,\ 5,\ \frac{3\pm\sqrt{13}}{2}$$

For $\lambda=5$, $[A-5I][X]=0$, $$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & -3 & 1 \\ 0 & 0 & 3 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & -3 & 1 \\ 0 & 0 & 3 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},\ 5x_3=0,\ -3x_3+x_4=0,\ 3x_3-4x_4=0$$

Solving which we get $x_3= 0, x_4= 0$ and x_1 and x_2 may be anything.

The Eigen vector corresponding to $\lambda = 5$ may be written as $\widehat{X_1} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} k_1 \\ k_2 \\ 0 \\ 0 \end{bmatrix}$

Where k_1, k_2 may be any real number. Since choice (a) is the only matrix in the form of with both $x_3, x_4 = 0$ so it is the correct answer

Q.24 **(b)** $|A - \lambda I| = 0$, $\begin{bmatrix} 2\text{-}\lambda & 1 \\ 0 & 2\text{-}\lambda \end{bmatrix} = 0$, $(2-\lambda)^2 = 0$, $\lambda = 2, 2$

Now, consider the Eigen value problem

$$|A - \lambda I|\widehat{X} = 0, \begin{bmatrix} 2-\lambda & 1 \\ 0 & 2-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Put $\lambda = 2$ we get,

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \; x_1 = t,\; x_2 = 0, \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Q.25 **(b)** $\begin{vmatrix} (1-\lambda) & 2 \\ 0 & (2-\lambda) \end{vmatrix} = 0$, $(1-\lambda)(2-\lambda) = 0$, $\lambda = 1, 2$

Putting the value of $\lambda = 1$, $\begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ a \end{bmatrix} = 0$, $a = 0$

Putting the value of $\lambda = 2$, $\begin{bmatrix} -1 & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ b \end{bmatrix} = 0$, $b = \frac{1}{2}$ $\therefore a + b = \frac{1}{2}$

Q.26 **(a)** $A = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}$, $[A - \lambda I] = \begin{bmatrix} 2-\lambda & 2 \\ 1 & 3-\lambda \end{bmatrix} = 0$, $(2-\lambda)(3-\lambda) - 2 = 0$

$\lambda^2 - 5\lambda + 4 = 0$, $\lambda = 1 \;\&\; 4$

The Eigen value problem is $|A - \lambda I|\widehat{X} = 0$

$$\begin{bmatrix} 2-\lambda & 2 \\ 1 & 3-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Putting $\lambda = 1$

$$\begin{bmatrix}1 & 2\\1 & 2\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix},\ \begin{bmatrix}1 & 2\\0 & 0\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix},\ x_1+2x_2=0,\ \begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}2\\-1\end{bmatrix}$$

Q.27 **(b)** $(PQ)^{-1}P=\left(Q^{-1}P^{-1}\right)P=\left(Q^{-1}\right)\left(P^{-1}P\right)=\left(Q^{-1}\right)(I)=Q^{-1}$

Q.28 **(b)** $\begin{bmatrix}3+2i & i\\-i & 3-2i\end{bmatrix}^{-1}=\dfrac{1}{\left[(3+2i)(3-2i)+i^2\right]}\begin{bmatrix}3-2i & -i\\i & 3+2i\end{bmatrix}=\dfrac{1}{12}\begin{bmatrix}3-2i & -i\\i & 3+2i\end{bmatrix}$

Q.29 **(a)** For singularity of matrix $\begin{vmatrix}8 & x & 0\\4 & 0 & 2\\12 & 6 & 0\end{vmatrix}=0,\ 8(0-12)-x(0-2\times12)=0$, $x=4$

Q.30 **(a)** M is orthogonal matrix , $M^TM=I$

$$\begin{bmatrix}\frac{3}{5} & x\\\frac{4}{5} & \frac{3}{5}\end{bmatrix}\begin{bmatrix}\frac{3}{5} & \frac{4}{5}\\x & \frac{3}{5}\end{bmatrix}=\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix},\ \begin{bmatrix}\left(\frac{3}{5}\right)^2+x^2 & \left(\frac{3}{5}\cdot\frac{4}{5}\right)+\frac{3}{5}x\\\left(\frac{4}{5}\cdot\frac{3}{5}\right)+\frac{3}{5}x & \left(\frac{4}{5}\right)^2+\left(\frac{3}{5}\right)^2\end{bmatrix}=\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$$

Compare both sides ,

$$a_{12}=\left(\frac{3}{5}\right)\left(\frac{4}{5}\right)+\left(\frac{3}{5}\right)x=0,\ x=-\frac{4}{5}$$

Q.31 **(b)** $R=\begin{bmatrix}1 & 0 & -1\\2 & 1 & -1\\2 & 3 & 2\end{bmatrix},\ |R|=\begin{vmatrix}1 & 0 & -1\\2 & 1 & -1\\2 & 3 & 2\end{vmatrix}=1$

Since we need only the top row of R^{-1}, we need to find only first column of (R) which after the transpose will become first row of adj A

$$\text{cof}\,(1,1)=+\begin{vmatrix}1 & -1\\3 & 2\end{vmatrix}=5,\ \text{cof}\,(2,1)=-\begin{vmatrix}0 & -1\\3 & 2\end{vmatrix}=-3,\ \text{cof}\,(3,1)=+\begin{vmatrix}0 & -1\\1 & -1\end{vmatrix}=1$$

$$\text{cof.}\,(A)=\begin{bmatrix}5 & \dots & \dots\\-3 & \dots & \dots\\1 & \dots & \dots\end{bmatrix},\ \text{Adj}\,(A)=[\text{cof.}(A)^T]=\begin{bmatrix}5 & -3 & 1\\\dots & \dots & \dots\\\dots & \dots & \dots\end{bmatrix},\ R^{-1}=\begin{bmatrix}5 & -3 & 1\\\dots & \dots & \dots\\\dots & \dots & \dots\end{bmatrix}$$

Top row of $R^{-1}=[5\ \ -3\ \ \ 1]$

Q.32 **(d)** $A' = (A^TA)^{-1}A^T = A^{-1}(A^T)^{-1}A^T = A^{-1}$

(a), (b), (c) are correct. (d) is incorrect.

Q.33 **(d)** $[A:B] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & 2 & 3 & 4 \\ 1 & 4 & k & 6 \end{array}\right] \xrightarrow[R_3-R_1]{R_2-R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 3 & k-1 & 3 \end{array}\right] \xrightarrow[R_3-3R_2]{} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & k-7 & 0 \end{array}\right]$

Now, if $k \neq 7$, $\text{rank}(A) = \text{rank}(A\,|\,B) = 3$, $\therefore$ Unique solution,
If $k = 7$, $\text{rank}(A) = \text{rank}(A\,|\,B) = 2$, infinite solution is possible.

Q.34 **(b)** $[A:B] = \left[\begin{array}{ccc|c} 2 & 3 & 0 & 4 \\ 1 & 1 & 1 & 4 \\ 1 & 2 & -1 & a \end{array}\right] \xrightarrow[R_3-\frac{1}{2}R_1]{R_2-\frac{1}{2}R_1} \begin{bmatrix} 2 & 3 & 0 & 4 \\ 0 & -\frac{1}{2} & 1 & 2 \\ 0 & \frac{1}{2} & -1 & a-2 \end{bmatrix} \xrightarrow{R_3+R_2} \left[\begin{array}{ccc|c} 2 & 3 & 0 & 4 \\ 0 & -0.5 & 1 & 2 \\ 0 & 0 & 0 & a \end{array}\right]$

If $a \neq 0$, $r(A)=2$ and $r(A|B)=3$ hence system have no solution.

If $a = 0$, $r(A) = r(A|B) = 2$ Then the system will be consistent and will have infinite solution.

Q.35 **(d)** Since, matrix is triangular; the Eigen values are the diagonal elements themselves namely $\lambda = 3, -2$ and 1. Corresponding to Eigen value, $\lambda = -2$ let us find the Eigen vector. $|A - \lambda I|\widehat{X} = 0$

$$\begin{bmatrix} 5 & -2 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, 5x_1 - 2x_2 = 0,\ x_3 = 0, \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 0 \end{bmatrix}$$

Q.36 **(d)** If characteristic equation is $\lambda^3+\lambda^2+2\lambda+1=0$, Then by Cayley-Hamilton theorem

$P^3 + P^2 + 2P + I = 0$, $I = -P^3 - P^2 - 2P$, Multiplying by on both sides $P^{-1} = -(P^2 + P + 2I)$

Q.37 **(d)** $x_1+2x_2+x_3+4x_4=2$, $3x_1+6x_2+3x_3+12x_4=6$

$[A:B] = \left[\begin{array}{cccc|c} 1 & 2 & 1 & 4 & 2 \\ 3 & 6 & 3 & 12 & 6 \end{array}\right] \xrightarrow{R_2-3R_1} \left[\begin{array}{cccc|c} 1 & 2 & 1 & 4 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$, rank (A)=rank (A:B)=1

System is consistent and infinetley many non trivial solution exists.

Q.38 **(a)** Given matrix is upper triangular matrix, its Eigen values are diagonal elements i. e. = 1, 4, 3

Q.39 **(c)** The given equations are

$$[A: B]=\begin{bmatrix}1&2&1&4\\2&1&2&5\\1&1&1&1\end{bmatrix}=\begin{bmatrix}1&2&1&4\\0&-3&0&-3\\0&-1&0&-3\end{bmatrix}=\begin{bmatrix}1&2&1&4\\0&-3&0&-3\\0&0&0&0\end{bmatrix}$$

$R(A)=R(A:B)=2<$ no of unknown

System is consistent and have infinetly many solutions.

Q.40 **(b)** $|A-\lambda I|=0 \Rightarrow \begin{vmatrix}5-\lambda & 3\\ 1 & 3-\lambda\end{vmatrix}=0 \Rightarrow (5-\lambda)(3-\lambda)=3=0, \Rightarrow \lambda^2-8\lambda+12=0$

$\Rightarrow \lambda=2, \lambda=6,$ for eigen vector $(A-\lambda I)\,X=0,$

for $\lambda=2, \begin{pmatrix}3&3\\1&1\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}0&0\\1&1\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix},$

$x_1+x_2=0, \begin{pmatrix}x_1\\x_2\end{pmatrix}=\begin{pmatrix}\frac{1}{\sqrt{2}}\\ \frac{-1}{\sqrt{2}}\end{pmatrix}$

Q.41 **(b)** $\begin{vmatrix}9-\lambda & 5\\ 5 & 8-\lambda\end{vmatrix}=0 \Rightarrow (9-\lambda)(8-\lambda)=25 \Rightarrow$ Eigen values are 3.48 and 13.53

Q.42 **(16)** If we multiply QR first then $Q_{2\times4}\,(R)_{4\times1}$ having multiplication number 8.

Therefore $P_{(4\times2)}QR_{2\times1}$ will have minimum number of multiplication = (8 + 8) = 16

Q.43 **(d)** Characteristic equation of A is $|A-\lambda I|=0$ where λ is the Eigenvalue

$$\begin{vmatrix}1-\lambda & 1\\ 1 & -1-\lambda\end{vmatrix}=0 \Rightarrow \lambda^2-2=0$$

Every matrix satisfies its characteristic equation. Therefore $A^2-2I=0 \Rightarrow A^2=2I$

$A^{19}=A^{18}A=\left(A^2\right)^9 A=(2I)^9 A=512A,$ Hence Eigen values of A19 are $\pm\, 512\sqrt{2}$

Q.44 **(a)** If matrix B is obtained from matrix A by replacing the 1th row by itself plus k times the mth row, for $1\neq m$ then det(B)=det(A). With this property given matrix is equal to the matrices given in options (b),(c) and (d).

Q.45 **(c)** The Eigen values of Real symmetric matrices are always purely Real.

Q.46 **(b)** This is a 4×4 matrix, we can use Gauss -Elimination process to reduce the matrix and to find the determinant

Q.47 **(d)**

Q.48 **(d)**

Q.49 **(b)** $\begin{bmatrix} 0 & 1 & 1 \\ 1 & -1 & 0 \\ -1 & 0 & -1 \end{bmatrix}$, Order of matrix = 3, Rank = 2, Dimension of the null space of A=3-2=1

Q.50 **(d)** Rank deficient matrix have nullity ≥ 1, the transfer function of the system must have a pole at origin.

Q.51 **(a)** $\begin{vmatrix} 2 & 6 & 0 \\ 4 & 12 & 8 \\ -2 & 0 & 4 \end{vmatrix} = (2)^3 \begin{vmatrix} 1 & 3 & 0 \\ 2 & 6 & 2 \\ -1 & 0 & 2 \end{vmatrix} = 8\times(-12) = -96$

Q.52 **(a)** $\frac{dx}{dt} = 3x - 5y,\ \frac{dy}{dt} = 4x + 8y,$ can be written in Matrix form as $\frac{d}{dt}\begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{bmatrix} 3 & -5 \\ 4 & 8 \end{bmatrix}\begin{Bmatrix} x \\ y \end{Bmatrix}$

Q.53 **(d)** Eigen values of the matrix $\begin{bmatrix} -5 & 2 \\ -9 & 6 \end{bmatrix}$ are $4, -3$. The Eigen vector corresponding to Eigen value λ is given by $Ax = \lambda x$. $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is Eigen vector corresponding to Eigen value $\lambda = 3$

Q.54 **(d)** We known that the Eigen vectors corresponding to distinct Eigen values of real symmetric

matrix are orthogonal. $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}^T \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = x_1y_1 + x_2y_2 + x_3y_3 = 0$

Q.55 **(d)**

Q.56 **(23)** $K^T JK = \begin{bmatrix} 1 & 2 & -1 \end{bmatrix}\begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 6 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 6 & 8 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = 23$

Q.57 **(a)** Sum of the Eigen values = Trace of the matrix= 215 + 150 + 550= 915

Q.58 **(a)** Area of triangle $= \frac{1}{2}\left\| \begin{matrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ 1 & 1 & 1 \end{matrix} \right\|$

Q.59 **(88)** $\begin{vmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 3 & 0 \\ 2 & 3 & 0 & 1 \\ 3 & 0 & 1 & 2 \end{vmatrix} = -1\begin{vmatrix} 1 & 2 & 3 \\ 3 & 0 & 1 \\ 0 & 1 & 2 \end{vmatrix} - 3\begin{vmatrix} 0 & 1 & 3 \\ 2 & 3 & 1 \\ 3 & 0 & 2 \end{vmatrix} = 88$

Q.60 **(2.0)** $\begin{bmatrix} 6 & 0 & 4 & 4 \\ -2 & 14 & 8 & 18 \\ 14 & -14 & 0 & -10 \end{bmatrix} \xrightarrow[R_3 \to 6R_3 - 14R_1]{R_2 \to 3R_2 + R_1} \begin{bmatrix} 6 & 0 & 4 & 4 \\ 0 & 42 & 28 & 58 \\ 0 & -84 & -56 & -116 \end{bmatrix}$

$$\begin{bmatrix} 6 & 0 & 4 & 4 \\ 0 & 42 & 28 & 58 \\ 0 & -84 & -56 & -116 \end{bmatrix} \xrightarrow{R_3 \to R_3 + 2R_2} \begin{bmatrix} 6 & 0 & 4 & 4 \\ 0 & 42 & 28 & 58 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \text{Rank} = 2$$

Q.61 **(1)** $[A:B] = \begin{bmatrix} 3 & 2 & 0 & 1 \\ 4 & 0 & 7 & 1 \\ 1 & 1 & 1 & 3 \\ 1 & -2 & 7 & 0 \end{bmatrix} R_1 \leftrightarrow R_3 \begin{bmatrix} 1 & 1 & 1 & 3 \\ 4 & 0 & 7 & 1 \\ 3 & 2 & 0 & 1 \\ 1 & -2 & 7 & 0 \end{bmatrix} \begin{matrix} R_3 \to 4R_3 - R_2 \\ \to \\ R_4 \to 4R_4 - 3R_3 \end{matrix}$

$$\begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & -4 & 3 & -11 \\ 0 & 0 & -15 & -21 \\ 0 & 0 & 15 & 21 \end{bmatrix} \begin{matrix} R_4 \to R_4 + R_3 \\ \to \end{matrix} \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & -4 & 3 & -11 \\ 0 & 0 & -15 & -21 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\rho(A : B) = \rho(A) = 3 =$ no. of variables, Unique solution exists

Q.62 **(0)** (The Eigen vectors corresponding to distinct Eigen values of real symmetric matrix are orthogonal)

Q . 63 **(0)** $|A| = \begin{vmatrix} 2 & 18 & 10 \\ -4 & -36 & 20 \\ 7 & 63 & 35 \end{vmatrix} = 0$

Q.64(6) A have $(3+2)$ identical rows, hence it must have $(2+1) = 3$ zeros eigen values.

Sum of eigen values $= \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 = 5(\text{Trace}),\ \lambda_1 + \lambda_2 = 5$ ....(1)

$$\begin{vmatrix} 1-\lambda & 0 & 0 & 0 & 1 \\ 0 & 1-\lambda & 1 & 1 & 0 \\ 0 & 1 & 1-\lambda & 1 & 0 \\ 0 & 1 & 1 & 1-\lambda & 0 \\ 1 & 0 & 0 & 0 & 1-\lambda \end{vmatrix} = 0 \xrightarrow{R_1 \to R_1 - R_5} \begin{vmatrix} -\lambda & 0 & 0 & 0 & \lambda \\ 0 & 1-\lambda & 1 & 1 & 0 \\ 0 & 1 & 1-\lambda & 1 & 0 \\ 0 & 1 & 1 & 1-\lambda & 0 \\ 1 & 0 & 0 & 0 & 1-\lambda \end{vmatrix} = 0$$

$$\xrightarrow{R_5 \to R_5 + \frac{1}{\lambda} R_1} \begin{vmatrix} -\lambda & 0 & 0 & 0 & \lambda \\ 0 & 1-\lambda & 1 & 1 & 0 \\ 0 & 1 & 1-\lambda & 1 & 0 \\ 0 & 1 & 1 & 1-\lambda & 0 \\ 0 & 0 & 0 & 0 & 2-\lambda \end{vmatrix} = 0,$$ hence $\lambda = 2$ is a non zero eigen value.

From (1) another non zero eigen value is 3. Hence, product of non-zero eigen value = 6

Q.65 **(a)** Determinant of a matrix is equal to the product of the Eigen values. For a matrix with only real Eigen values, if determinant is negative, at least one of the Eigen value must be negative.

Q.66 **(2)** Let the basis of 6-dimensional vector space be {e1, e2, e3, e4, e5, e6}. In order for $V1 \cap V2$ to have smallest possible dimension V1 and V2 could be, say, {e1, e2, e3,e4} and {e3, e4, e5, e6} respectively. The basis of $V1 \cap V2$ would then be {e3, e4}. = > Smallest possible dimension = 2.

Q.67 **(7)** $P(X) = X^5 + 4X^3 + 6x + 5$ can be rewritten as follows $P(X) = X^3(X^2 + 4) + 6x + 5$

Now using only one temporary variable t and any number of data transfer as well as memory related operation the polynomial can be evaluated as follows

1. t = x * x [Evaluate x^2 and store in memory]

2. t = t + 4 [Evaluate (x^2+4) & store in memory]

3. t = x^2 [Retinue x^2 from memory]

4. t = t *x [Evaluate x^3 and store in memory]

5. t = t *(x^2 + 4) [Evaluate x^3(x^2+4) & store in memory]

6. t = 6 * x [Evaluate 6x and store in memory]

7. t = t + 5 [Evaluate (6x+5) & store in memory]

8. t = t + x^3 *(x^2 + 4) [Retrieve x^3(x^2 + 4) from memory and evaluate $\left\{x^3\left(x^2+4\right)+6x+5\right\}$

In the above 8 steps of evaluation, the total number of arithmetic operations required ane 7 [4 Multiplications, 3 Additions] So answer is 7 arithmetic operations.

Q.68 **(d)** Matrix multiplication is not commutative in general.

Q.69 **(1)** $A^2 = I \Rightarrow \lambda^2 = 1,\ \lambda = \pm 1,\ \lambda = 1$ is only Positive eigen value.

Q.70 **(1)** (i) Let $P = I_2 + \alpha J_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \alpha \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & \alpha \\ \alpha & 1 \end{bmatrix},\ \Rightarrow |P| = 1 - \alpha^2 = 0,\ \alpha = \pm 1$

(ii) Let $P = I_4 + \alpha J_4 = \begin{bmatrix} 1 & 0 & 0 & \alpha \\ 0 & 1 & \alpha & 0 \\ 0 & \alpha & 1 & 0 \\ \alpha & 0 & 0 & 1 \end{bmatrix},\ |P| = 0 \Rightarrow \left(1-\alpha^2\right)^2 = 0,\ \alpha = \pm 1$

Similarly, if $P = I_6 + \alpha J_6$ then we get $|P| = \left(1 - \alpha^2\right)^3 = 0,\ \alpha = \pm 1$

as α is non negative, $\therefore \alpha = 1$

Q.71 **(200)** $|AB| = |A|\,|B| = (5)(40) = 200$

Q.72 **(b)** $[A\,|B] = \left[\begin{array}{ccc|c} 2 & 1 & 3 & 5 \\ 3 & 0 & 1 & -4 \\ 1 & 2 & 5 & 14 \end{array}\right] \begin{array}{c} R_2 \to 2R_2 - 3R_1 \\ R_3 \to 2R_3 - R_1 \end{array} \left[\begin{array}{ccc|c} 2 & 1 & 3 & 5 \\ 0 & -3 & -7 & -23 \\ 0 & 3 & 7 & 23 \end{array}\right] \xrightarrow{R_3 + R_2} \left[\begin{array}{ccc|c} 2 & 1 & 3 & 5 \\ 0 & -3 & -7 & -23 \\ 0 & 0 & 0 & 0 \end{array}\right]$

Since, rank(A)=rank (A|B) < number of variables

$\therefore$ Equations have infinitely many solutions.

Q.73 **(49)** General 2x 2 real symmetric matrix is $\begin{bmatrix} y & x \\ x & z \end{bmatrix} \Rightarrow \det = yz - x^2$ and trace is

$y + z = 14 \Rightarrow z = 14 - y \ldots\ldots\ldots(1)$

Let $f = yz - x^2 = -x^2 - y^2 + 14y$ (using (1))

Using maxima and minima of a function of two variables, we. have f is maximum at x = 0, y = 7 and therefore, maximum value of the determinant is 49

Q.74 **(b)** Consider, $A\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$ which is real symmetric matrix Characteristic equation is

$|A - \lambda I| = 0 \Rightarrow (1+\lambda)^2 - 1 = 0, \quad \Rightarrow \lambda + 1 = \pm 1 \quad \therefore \lambda = 0, -2$ (not positive)

(b) is not true

(a), (c), (d) are true using properties of Eigen values

Q.75 **(b)** $[A|B] = \left[\begin{array}{ccc|c} 1 & 2 & 2 & b_1 \\ 5 & 1 & 3 & b_2 \end{array}\right] R_2 \to R_2 - 5R_1 \left[\begin{array}{ccc|c} 1 & 2 & 2 & b_1 \\ 0 & -9 & -7 & b_2 - 5b_1 \end{array}\right]$

Rank (A)=Rank(A|B) < number of unknowns, for all values of b_1 & b_2

The equations have infinitely many solutions, for any given b_1 and b_2

Q.76 **(1/3)** Characteristic equation is $|A-\lambda I|=0$ i.e. $\begin{vmatrix} -\lambda & 1 & -1 \\ -6 & -11-\lambda & 6 \\ -6 & -11 & 5-\lambda \end{vmatrix} = 0$

$\Rightarrow \lambda^3 + 6\lambda^2 + 11\lambda + 6 = 0$

$\Rightarrow \lambda = -1, -2\ -3$ are the Eigen values of A

$\lambda_{max} = -1$ and $\lambda_{min} = -3$

$\therefore \left|\dfrac{\lambda_{max}}{\lambda_{min}}\right| = \left|\dfrac{-1}{-3}\right| = \dfrac{1}{3}$

Q.77 **(a)** Eigen values of a real symmetric matrix are all real

Q.78 **(c)** Rank of a matrix is unaltered by the elementary transformations i.e., Row/column operations (Here B is obtained from A by applying row/column operations on A) . Since rank of A is N, rank of B is also N

Q.79 **(d)** $x(t) = 3\sin(1000\pi t)$

$y(t) = 5\cos\left(1000\ \pi t + \dfrac{\pi}{4}\right)$

at $t = 0$

$x(0) = 0,\quad y(0) = 5\cos\dfrac{\pi}{4} = \dfrac{5}{\sqrt{2}}$

at $t = \dfrac{1}{2}$

$x\left(\dfrac{1}{2}\right) = \sin\left(1000\dfrac{\pi}{2}\right) = 0,\ y\left(\dfrac{1}{2}\right) = 5\cos\left(500\pi + \dfrac{\pi}{4}\right) = \dfrac{5}{\sqrt{2}}$

at $t = \dfrac{1}{4000}$

$x\left(\dfrac{1}{4000}\right) = 3\sin\dfrac{\pi}{4} = \dfrac{3}{\sqrt{2}},\ y\left(\dfrac{1}{4000}\right) = 5\cos\left(\dfrac{\pi}{4} + \dfrac{\pi}{4}\right) = 0$

Thus this represents the equation of ellipse.

Q.80 **(c)**

Q.81 **(c)**

Q.82 **(a)**

Q.83 **(b)**

Q.84 **(2)**

Q.85 **(c)** $|P| = (4+3i)(4-3i)-(i)(-i) = 16+9-1 = 24,$

$$\text{Adj}(P) = \begin{bmatrix} 4+3i & -i \\ i & 4-3i \end{bmatrix},\quad P^{-1} = \frac{1}{24}\begin{bmatrix} 4+3i & -i \\ i & 4-3i \end{bmatrix}$$

Q.86 **(4.5)** Given $2x + 3y = 5, \quad 3x + py = 10$

$$[A|B] = \begin{bmatrix} 2 & 3 & 5 \\ 3 & P & 10 \end{bmatrix} R_2 \Rightarrow 2R_2 - 3R_1 \begin{bmatrix} 2 & 3 & 5 \\ 0 & 2P-9 & 5 \end{bmatrix}$$

For No Solution $R(A|B) \neq R(A)$

$\Rightarrow 2P - 9 = 0,\ P = 4.5$

Q.87 **(d)** Characteristic equation is $\Rightarrow \begin{bmatrix} 3-\lambda & -2 & 2 \\ 4 & -4-\lambda & 6 \\ 2 & -3 & 5-\lambda \end{bmatrix} = 0$

$\Rightarrow \lambda^3 - 4 + 5\lambda - 2 = 0$

$\Rightarrow (\lambda-1)(\lambda^2 - 3\lambda + 2) = 0$

$(\lambda-1)(\lambda-1)(\lambda-2) = 0$

$\lambda = 1, 2$

Q.88 **(b)** Given $A = [a_{ij}]\ 1 \le i, j \le n, n \ge 3$ and $a_{ij} = i.j$

$$\Rightarrow = \begin{Bmatrix} 1 & 2 & 3 & --- \\ 2 & 4 & 6 & --- \\ 3 & 6 & 9 & --- \\ - & - & - & --- \end{Bmatrix}$$

If we apply $R_2 - 2R_1,\ R_3 - 3R_1$ ...

Every row will be zero row, except first row in echelon form

$\therefore\ \rho(A) = 1$

Q.89 **(d)** $\lambda_1 + \lambda_2 = 2 + p,\ \lambda_1\lambda_2 = 2p - 1$

as eigen values are having a ratio $3:1$

$3x + x = 2 + p,\ (3x)(x) = 2p - 1$

$4x = 2 + p,\ (3x^2) = 2p - 1$

$x = \dfrac{2+p}{4},\ \left(\dfrac{2+p}{4}\right)^2 = \dfrac{2p-1}{3}$

$p = \dfrac{14}{3}, 2$

Q.90 **(d)** $\lambda_1 = -1$ and $\lambda_2 = 7$ are Eigen values of A

By properties $\lambda_1 + \lambda_2 = \text{tr}\,(A)$ and $\lambda_1 . \lambda_2 = |A|$

Trace of A i.e., sum of theprinciple diagonal elements of A

$\Rightarrow 6 = 1 + a$ and $-7 = a - 4b$
$\Rightarrow a = 5$ $\quad\Rightarrow -7 = 5 - 4b$
$\Rightarrow b = 3$

Q.91 **(6)** Characteristic equation is $\begin{vmatrix} 4-\lambda & 5 \\ 2 & 1-\lambda \end{vmatrix} = 0$

$\Rightarrow \lambda^2 - 5\lambda - 6 = 0 \Rightarrow (\lambda - 6)(\lambda + 1) = 0 \Rightarrow \lambda = 6, -1 \therefore$ Largest Eigen value is 6

Q.92 **(0)** Determinant of the matrices dont changes by elementary operations.

Determinant is unaltered by the operations (i) and (ii)

Determinant of the resultant matrix = Determinant of the given matrix

$$= \begin{vmatrix} 3 & 4 & 45 \\ 7 & 9 & 105 \\ 13 & 2 & 195 \end{vmatrix} = 0$$

Q.93 **(b)** X be an Eigen vector corresponding to Eigen value $\lambda = 1$, then

$$AX = \lambda X \Rightarrow (A - I)X = 0$$

$$\begin{bmatrix} 0 & -1 & 2 \\ 0 & 0 & 0 \\ 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = 0$$

$\Rightarrow -y + 2z = 0$ and $x + 2y = 0$

$\Rightarrow y = 2z$ and $\dfrac{x}{-2} = y$

$$\therefore \frac{x}{-2} = y = 2z \Rightarrow \frac{x}{-4} = \frac{y}{2} = \frac{Z}{1} = \alpha(\text{say}) \Rightarrow X = \begin{pmatrix} -4 \\ 2 \\ 1 \end{pmatrix} \alpha; \alpha \neq 0$$

$\therefore$ Eigen vectors are $\{\alpha(-4, 2, 1) | \alpha \neq 0, \alpha \in R|\}$

Q.94 **(c)** For non-trivial solution, we have $|A| = 0$

$$\begin{vmatrix} p & q & r \\ q & r & p \\ r & p & q \end{vmatrix} = 0 \rightarrow \begin{matrix} C_1 \rightarrow C_1 + C_2 + C_3 \\ R_2 \rightarrow R_2 - R_1 \\ R_3 \rightarrow R_3 - R_1 \end{matrix} \rightarrow (p+q+r)\begin{vmatrix} 1 & q & r \\ 0 & r-q & p-r \\ 0 & p-q & q-r \end{vmatrix} = 0$$

$$\Rightarrow (p+q+r)\left((r-q)^2 - (p-q)(p-r)\right) = 0$$

$$\Rightarrow (p+q+r)\left(p^2 + q^2 + r^2 - pq - qr - pr\right) = 0$$

$$\Rightarrow (p+q+r)\left((p-q)^2 + (q-r)^2 + (r-p)^2\right) = 0$$

$$\Rightarrow p+q+r = 0 \text{ or} \Rightarrow p = q = r$$

Q.95 **(5)** $A = LU \Rightarrow \begin{bmatrix} 2 & 2 \\ 4 & 9 \end{bmatrix} = \begin{bmatrix} \ell_{11} & 0 \\ \ell_{21} & \ell_{22} \end{bmatrix}\begin{bmatrix} 1 & u_{12} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \ell_{11} & \ell_{11}u_{12} \\ \ell_{21} & \ell_{22}u_{12} + \ell_{22} \end{bmatrix}$

$\ell_{11} = 2$; $\ell_{11}u_{12} = 2 \Rightarrow u_{12} = 1$

$\ell_{21} = 4$; $\ell_{21}u_{12} + \ell_{22} = 9 \Rightarrow \ell_{22} = 5$

Q.96 **(2)** $(A|B) = \begin{bmatrix} 1 & -2 & 3 & -1 \\ 1 & -3 & 4 & 1 \\ -2 & 4 & -6 & k \end{bmatrix} \xrightarrow[R_3 \rightarrow R_3 - 2R_1]{R_2 \rightarrow R_2 - R_1,} \begin{bmatrix} 1 & -2 & 3 & -1 \\ 0 & -1 & 1 & 2 \\ 0 & 0 & 0 & k-2 \end{bmatrix}$

The system will have infinitely many solution if r(A|B) = r(A) = r < number of variables

$\Rightarrow k - 2 = 0 \Rightarrow k = 2$

Q.97 **(17)**

$$AX = \lambda X \Rightarrow \begin{bmatrix} 4 & 1 & 2 \\ P & 2 & 1 \\ 14 & -4 & 10 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 12 \\ P+7 \\ 36 \end{bmatrix} = \begin{bmatrix} \lambda \\ 2\lambda \\ 3\lambda \end{bmatrix}$$

$\Rightarrow \lambda = 12$....(1) and $2\lambda = P + 7$ ....(2)

from equations (1) and (2): $P + 7 = 24 \Rightarrow P = 17$

Q.98 **(b)** Let $A = \begin{bmatrix} 10 & 5+j & 4 \\ x & 20 & 2 \\ 4 & 2 & -10 \end{bmatrix}$

Given that all Eigen values of A are real.

$\Rightarrow$ A is Hermitian

$\Rightarrow \left(\overline{A}\right)^T = A$

$$\begin{bmatrix} 10 & \overline{x} & 4 \\ 5-j & 20 & 2 \\ 4 & 2 & -10 \end{bmatrix} = \begin{bmatrix} 10 & 5+j & 4 \\ x & 20 & 2 \\ 4 & 2 & -10 \end{bmatrix} \Rightarrow x = 5-j$$

Q.99 **(c)** $\left|A^T A^{-1}\right| = \left|A^T\right|\left|A^{-1}\right| = 1$

Q.100 **Wrong question**

Q.101 **(a)**

Q.102 **(c)** Given AX = 0, $\rho\left(A_{n\times n}\right) = r\left(0 < r < n\right)$, p = no. of independent solution = nullity

We know that rank + nullity = n

r + p =n, p = n - t

Q.103 **(d)** By verification method ; $\begin{bmatrix} 2 & 5 \\ -4 & 3 \end{bmatrix}\begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{Bmatrix} 2 \\ -30 \end{Bmatrix}$

Q.104 **(a)** As A is a 2×2 matrix, if all the Eigen values are positive, then sum of the Eigen values(Trace(A)) is positive and product of the Eigen values(det(A)) is also positive.

Using the above,

1. $k + 2 > 0, k > -2$
2. $2k - 1 > 0, k > 1/2$

By combining 1. and 2. , $k > \dfrac{1}{2}$

Q.105 **(2)** Here $\lambda = 2, 2, 3$

For $\lambda = 2$,

No. of L.I. Eigen vectors $= 3 - \text{rank of } \left(A - 2I\right) = 3 - 2 = 1$

For $\lambda = 3$,

No. of L.I. Eigen vectors $= 1$

$\therefore$ Total Number of L.I. Eigen vectors = 2

Q.106 **(d)** If sum of the elements in each row/column of a square matrix is equal to 'x' then 'x' is an Eigen value of that matrix A.

Q.107 **(b)** R(A)=R(A|B)

Q.108 **(15)** Given that $2+\sqrt{-1}$ and 3 are two Eigen values of 3 × 3 real matrix.

Product of Eigen values $= (2+i)(2-i)(3) = 15$

Q.109 **(c)** I is not correct

x + y + z = 1

x + y + z = 0

has no solution, when no of equations is less than no of variables.

II is not correct

Example : x − 2y = 2, 2x + 8y = 16, x + y = 5 has a solution (x = 4, y = 1).

III is correct

Example : x + y = 4, x + 2y = 0 has solutions (x = 6, y = −2)

Q.110 **(0.125)** Given that 1,2,4 are Eigen values of A

$$\Rightarrow |A| = 8 \text{ and } |A^{-1}| = \frac{1}{|A|} = \frac{1}{8} \text{ Now } \left|\left(A^{-1}\right)^{T}\right| = |A^{-1}| = \frac{1}{8} = 0.125$$

Q.111 **(c)** $M^4 = I \Rightarrow M^{4k+4} = I \Rightarrow M^{4k+3} = M^{-1}$

Q.112 **(233)** Sequence form a Fibonacci series

Q.113 **(1)** One of the Eigen values zero implies determinant of matrix is also zero. From the matrix A, we can see that for determinant to be zero, Row 1 and Row 3 can be made same.

$$\Rightarrow \frac{-9+x}{2} = -4, \quad x = 1$$

Q.114 **(3)** $|A| = 100 \Rightarrow 10ab = 100 \Rightarrow ab = 10$

$\text{Trace}(A) = 14 \Rightarrow a + 5 + 2 + b = 14,\ a + b = 7$

$(a-b)^2 = (a+b)^2 - 4ab = 49 - 40 = 9$

$|a-b| = 3$

Q.115 **(d)** Product of Eigen values= determinant

$\Rightarrow (\sigma + j\omega)(\sigma - j\omega) = \sigma^2 - x\omega$

$= \sigma^2 + \omega^2 = \sigma^2 - x\omega, \ \Rightarrow x = -\omega$

Q.116 **(a)**

Q.117 **(-6)** Eigen values of given matrix A are 1, -1, 3

eigen values of $A^3 - 3A^2$ are $(1)^3 - 3(1)^2$, $(-1)^3 - 3(-1)^2$, $(3)^3 - 3(3)^2$

eigen values of $A^3 - 3A^2$ are $-2, -4, 0$

$\text{Trace}(A^3 - 3A^2) = (-2) + (-4) + (0) = -6$

Q.118 **(b)** Product of Eigen value = determinant of matrix $=2$

Q.119 **(d)** (i) $|P|=\frac{1}{\sqrt{2}}\left[\frac{1}{\sqrt{2}}-0\right]+\frac{1}{\sqrt{2}}\left[0+\frac{1}{\sqrt{2}}\right]$, $|P|=\frac{1}{2}+\frac{1}{2}=1$

(ii) $P^T=\begin{bmatrix}\frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}}\\ 0 & 1 & 0\\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}}\end{bmatrix}$

$$PP^T=\begin{bmatrix}\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}}\\ 0 & 1 & 0\\ \frac{-1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}}\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}}\\ 0 & 1 & 0\\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}}\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}=I$$

Hence P is orthogonal

(iii) $PP^T=I$

$\therefore P^T=P^{-1}$

Hence (iv) is wrong

Q.120 Product of Eigen values = Determinant of matrix

$10\times\lambda=50,\ \lambda=5$

Q.121 The Dot product of eigen vectors of any two different eigen values of a symmetric matrix is always zero.

Q.122 (c) Given,G is orthogonal Matrix. For orthogonal matrix $Q^{-1}=Q^T$

$$Q^{-1}=\begin{bmatrix}3/7 & 2/7 & 6/7\\ -6/7 & 3/7 & 2/7\\ 2/7 & 6/7 & -3/7\end{bmatrix}^T=\begin{bmatrix}3/7 & -6/7 & 2/7\\ 2/7 & 3/7 & 6/7\\ 6/7 & 2/7 & -3/7\end{bmatrix}$$

Q.123 **(c)** A square matrix A is said to be singular if |A|=0

$\begin{vmatrix}2 & 4\\ 3 & 6\end{vmatrix}=12-12=0\Rightarrow\begin{bmatrix}2 & 4\\ 3 & 6\end{bmatrix}$ is singular matrix.

Q.124 **(d)** Let $A=\begin{bmatrix}2 & -4\\ 4 & -2\end{bmatrix}$

Characteristic equation of A is

$\lambda^2-(\text{trace of A})\lambda+|A|=0$

$\Rightarrow \lambda^2 - 0 + 12 = 0$

$\Rightarrow \lambda = \pm 2\sqrt{3}i$

and coreresponding eigen vectors are

$$\begin{bmatrix} \frac{1}{2} + \frac{\sqrt{3}}{2}i \\ 1 \end{bmatrix} \text{and} \begin{bmatrix} \frac{1}{2} - \frac{\sqrt{3}}{2}i \\ 1 \end{bmatrix}$$

Q.125 **(b)** Reducing into Echelon form

$$\begin{bmatrix} 1 & 1 & 0 & -2 \\ 2 & 0 & 2 & 2 \\ 4 & 1 & 3 & 1 \end{bmatrix} \xrightarrow[R_3 \to R_3 - 4R_1]{R_2 \to R_2 - 2R_1} \begin{bmatrix} 1 & 1 & 0 & -2 \\ 0 & -2 & 2 & 6 \\ 0 & -3 & 3 & 9 \end{bmatrix} \xrightarrow{R_3 \to 2R_3 - 3R_2} \begin{bmatrix} 1 & 1 & 0 & -2 \\ 0 & -2 & 2 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Rank = No. of non-zero rows = 2

Q.126 **(a)** N is a 3×3 Niplotent Matrix of degree 2. All the eigen values of any Nilpotent matrix is always zero.

Q.127 **(c)** Given : $3x + 2ky = -2$ and $kx + 6y = 2$

Condition for infinite solution $\rho(A) = \rho(A:B) <$ No of Unknown

$$(A:B) = \begin{bmatrix} 3 & 2k & -2 \\ k & 6 & 2 \end{bmatrix}$$

$$(A:B) = \begin{bmatrix} 3 & 2k & -2 \\ k+3 & 6+2k & 0 \end{bmatrix}$$

$k + 3 = 0$, $k = -3$

$6 + 2k = 0$, $k = -3$

Q.128 **(5.5)** $\text{Trace}(A) = \lambda_1 + \lambda_2 = 4$

$\text{Trace}(A^2) = \lambda_1^2 + \lambda_2^2 = 5$

$\text{Trace} = \sum \lambda$

Determinant A = Product (λ)

We have to find Determinant (A) that means.

$\det(A) = \lambda_1 . \lambda_2 = ?$

By squaring trace (A), we get

$(\lambda_1 + \lambda_2)^2 = (4)^2$

$\lambda_1^2 + \lambda_2^2 + 2\lambda_1\lambda_2 = 16$

$\therefore 5 + 2\lambda_1\lambda_2 = 16$

$\lambda_1\lambda_2 = \frac{11}{2} = 5.5$

Hence, the determinant of the matrix A is 5.5.

Q.129 **(1)** To find the eigen value of A.

$$|A-\lambda I|=0$$

$$\begin{bmatrix} 1-\lambda & 0 & -1 \\ -1 & 2-\lambda & 0 \\ 0 & 0 & -2-\lambda \end{bmatrix}=0$$

$$(-2-\lambda)(2-\lambda)(1-\lambda)=0$$

$$\lambda=1,2,-2$$

$$B=A^3-A^2-4A+5I$$

The eigen value of matrix B is given by,

$$\lambda_1=1^3-1^2-4+5=1$$

$$\lambda_2=2^3-2^2-4(2)+5=1$$

$$\lambda_3=(-2)^3-(-2)^2-4(-2)+5=1$$

$$|B|=\lambda_1.\lambda_2.\lambda_3=1\times1\times1=1$$

Hence, the determinant of B is 1.

Q.130 **(c)** If the eigen values of a matrix $A_{n\times n}$ are distinct, then it have $n-$ linearly independent eigen vectors.
If the matrix is invertible, that is $|A|\neq 0$. It only implies that eigen values are non zero. For an invertible matrix, eigen values can be repeated.

Q.131 **(2)** $A=\begin{bmatrix} k & 2k \\ k^2-k & k^2 \end{bmatrix}$, $X=\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $AX=0$

is a Homogenous system of linear equations which is always consistent.

If r(A) = Number of variables , Unique solution

If r(A) < Number of variables, infinitely many solution

Here we have given that the system must have infinitely many solutions.

Hence, r(A) < Number of variables

Which is possible only if |A|=0

$$\begin{vmatrix} k & 2k \\ k^2-k & k^2 \end{vmatrix}=0 \Rightarrow k=0,2$$

Hence number of distinct values of K for infinite number of solutions = 2.

Q.132 **(b)** Given that $A=\begin{bmatrix} -4 & 1 & -1 \\ -1 & -1 & -1 \\ 7 & -3 & 1 \end{bmatrix}$

$$R_2\to 4R_2-R_1\ ,R_3\to 4R_3+\ 7\ R_1$$

$$A \sim \begin{bmatrix} -4 & 1 & -1 \\ 0 & -5 & -3 \\ 0 & -5 & -3 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$A \sim \begin{bmatrix} -4 & 1 & -1 \\ 0 & -5 & -3 \\ 0 & 0 & 0 \end{bmatrix}$$

So, The Rank of a matrix is R(A) = 2

Q.133 (0.25)

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 1 \end{bmatrix}$$

$$|\mathrm{A}| = 4$$

$$|\mathrm{A}^{-1}| = \frac{1}{|\mathrm{A}|} = \frac{1}{4} = 0.25$$

ESE SOLUTIONS

Q.1 **(b)** Eigen values are $\lambda=1, 2$

For eigen vectors $(A-\lambda I)X_1=0$

at $\lambda=1$, $(A-I)X_1=0$

$$\begin{bmatrix}1 & 2\\0 & 2\end{bmatrix}\begin{bmatrix}x_1\\y_1\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$$

$2y_1=0 \Rightarrow y_1=0,\ x_1=k$

$$X_1=\begin{pmatrix}k\\0\end{pmatrix}=k\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}1\\a\end{pmatrix}\Rightarrow a=0$$

at $\lambda=2$, $(A-2I)X_2=0$

$$\begin{bmatrix}-1 & 2\\0 & 0\end{bmatrix}\begin{bmatrix}x_2\\y_2\end{bmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$$

$-x_2+2y_2=0 \Rightarrow x_2=2y_2$

$y_2=k \Rightarrow x_2=2k$

$$X_2=\begin{pmatrix}2k\\k\end{pmatrix}=k\begin{pmatrix}2\\1\end{pmatrix}=2k\begin{pmatrix}1\\\frac{1}{2}\end{pmatrix}=\begin{pmatrix}1\\b\end{pmatrix}\Rightarrow b=\frac{1}{2}$$

$$\therefore a+b=0+\frac{1}{2}=\frac{1}{2}$$

Q.2 (c) Using the property: Trace of matrix=Sum of Eigen values

Q.3 **(b)** $$(A:B)=\left(\begin{array}{ccc|c}2 & -1 & 3 & 2\\1 & 1 & 2 & 2\\5 & -1 & a & b\end{array}\right)\begin{matrix}(R_3-5R_2)\\(2R_2-R_1)\end{matrix}\rightarrow\left(\begin{array}{ccc|c}2 & -1 & 3 & 2\\0 & 3 & 1 & 2\\0 & -6 & a-10 & b-10\end{array}\right)$$

$$(R_3+2R_2)\rightarrow\left(\begin{array}{ccc|c}2 & -1 & 3 & 2\\0 & 3 & 1 & 2\\0 & 0 & a-8 & b-6\end{array}\right)$$

The given equations have infinite number of solutions if ρ (A) = ρ (AB) < 3(number of variables). It is possible only if a = 8 & b = 6

3 CALCULUS

3.1 Introduction

Calculus is the broad area of mathematics dealing with such topics as instantaneous rates of change, areas under curves, and sequences and series. Underlying all of these topics is the concept of a limit, which consists of analyzing the behavior of a function at points ever closer to a particular point, but without ever actually, reaching that point. As a typical application of the methods of calculus, consider a moving car. It is possible to create a function describing the displacement of the car (where it is located in relation to a reference point) at any point in time as well as a function describing the velocity (speed and direction of movement) of the car at any point in time. If the car were traveling at a constant velocity, then algebra would be sufficient to determine the position of the car at any time; if the velocity is unknown but still constant, the position of the car could be used (along with the time) to find the velocity.

However, the velocity of a car cannot jump from zero to 35 miles per hour at the beginning of a trip, stay constant throughout, and then jump back to zero at the end. As the accelerator is pressed down, the velocity rises gradually, and usually not at a constant rate (i.e., the driver may push on the gas pedal harder at the beginning, in order to speed up). Describing such motion and finding velocities and distances at particular times cannot be done using methods taught in pre-calculus, whereas it is not only possible but straightforward with calculus.

3.2 Function

$f : A \to B$ such that each element of A is Associated with unique element of B.

A function is an association which from a set A to set B such that each element of A is associated with only one element of B.

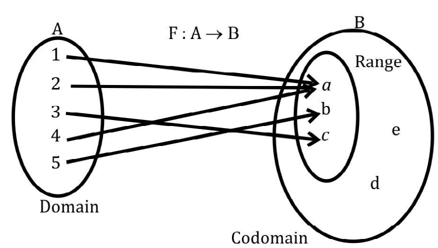
$f = \{(1,a)\ (2,a)\ (3,c)\ (4,a)\ (5,b)\}$

$\Rightarrow \ y = f(x),\ x \in A,\ y \in B$

Range of function, $f(x) = \{f(x) \,/\, x \in A\}$

Range of f $= \{a,\ b,\ c\}$, Range $\subseteq$ Co domain

$\Rightarrow f(x) = x^2 + 4,\ f : R \to R$

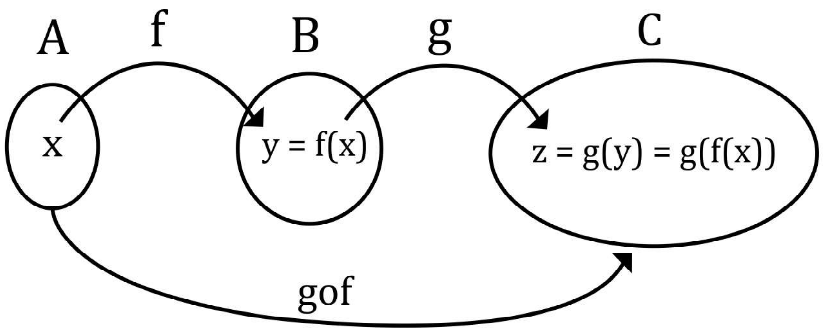


The Codomain and Range are both on the output side, but are subtly different. The Codomain is the set of values that could possibly come out. The Codomain is actually part of the definition of the function and The Range is the set of values that actually do come out.

Explicit Function : $z \to$ Dependent Variable, $x \to$ Independent Variable

Implicit Function : $\phi(z, x_1, x_2, x_3, x_4 \ldots . x_n) = c$

3.2.1 Product of function / Composite Function



$$f : A \to B \, , \; g : B \to C$$
$$gof(x) = g\{f(x)\}$$
$$gof(x) : A \to C$$
$$g(x) = x^2 + 4 \, , \; f(x) = 3x + 9$$
$$fog(x) = f\{g(x)\} = f\{x^2 + 4\} = 3\{x^2 + 4\} + 9 = 3x^2 + 21$$
$$gof(x) = g\{f(x)\} = g\{3x + 9\} = \{3x + 9\}^2 + 4 = 9x^2 + 54x + 85$$
$$\Rightarrow \text{ generally} : fog(x) \neq gof\,(x)$$

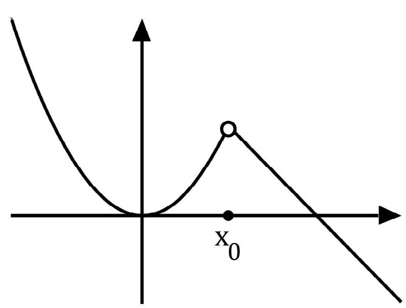
3.3 Limit of Function

Recall that, our aim is to understand af $: A \subseteq R \to R$ function by analyzing various properties of f. Forexample, one would like to analyze :

Does the 'graph' of f have any 'breaks'?

The **Limit** of a function at a point in its domain (if it exists) is the value that the function approaches as its argument approaches. The concept of a limit is the fundamental concept of calculus and analysis. It is used to define the derivative and the definite integral, and it can also be used to analyze the local behavior of functions near points of interest.

Informally, a function is said to have a limit at if it is possible to make the function arbitrarily close to by choosing values closer and closer to. Note that the actual value at is irrelevant to the value of the limit.



The notation is as follows : $\lim\limits_{x \to x_0} f(x) = L$

Which is read as "the limit of $f(x)$ as x approaches x_0 is L"

Definition of a Function Limit: The limit of $f(x)$ as x approaches x_0 is L, i.e $\lim\limits_{x \to x_0} f(x) = L$

If for every $\in > 0$, there exists $\delta > 0$ such that for all $x, 0 < |x - x_0| < \delta \Rightarrow |f(x) - L| < \in$

It implies that $\lim\limits_{x \to x_0} f(x)$ exists, If $\lim\limits_{x \to x_0^-} f(x) = \lim\limits_{x \to x_0^+} f(x) = L$

In practice, this definition is only used in relatively unusual situations. For many applications, it is easier to use the definition to prove some basic properties of limits, and to use those properties to answer straightforward questions involving limits.

3.3.1 Properties of limit

$$\lim_{x \to \alpha} \left[f(x) + g(x) \right] = \lim_{x \to \alpha} f(x) + \lim_{x \to \alpha} g(x)$$

$$\lim_{x \to \alpha} kf(x) = k \lim_{x \to \alpha} f(x)$$

$$\lim_{x \to \alpha} \left[f(x).g(x) \right] = \lim_{x \to \alpha} f(x) \cdot \lim_{x \to \alpha} g(x)$$

$$\lim_{x \to \alpha} \frac{f(x)}{g(x).} = \frac{\lim\limits_{x \to \alpha} f(x)}{\lim\limits_{x \to \alpha} g(x)}, \; If \lim_{x \to \alpha} g(x) \neq 0$$

$$\lim_{x \to \alpha} \left[f(x) \right]^P = \left[\lim_{x \to \alpha} f(x) \right]^P$$

$$\lim_{x \to \alpha} \sqrt[p]{f(x)} = \sqrt[p]{\lim_{x \to \alpha} f(x)}$$

$$\lim_{x \to \alpha} a^{f(x)} = a^{\lim\limits_{x \to \alpha} f(x)}$$

$$\lim_{x \to \alpha} \left[\log f(x) \right] = \log \left[\lim_{x \to \alpha} f(x) \right]$$

3.3.2 Some Results

$$\Rightarrow \lim_{x \to 0} \frac{\sin x}{x} = 1$$

$$\Rightarrow \lim_{x \to 0} \cos x = 1$$

$$\Rightarrow \lim_{x \to 0} \frac{\tan x}{x} = 1$$

$$\Rightarrow \lim_{x \to 0} (1 + x)^{\frac{1}{x}} = e$$

$$\Rightarrow \lim_{x \to 0} (1 + nx)^{\frac{1}{x}} = e^n$$

$$\Rightarrow \lim_{x \to \infty} \left(1 + \frac{1}{x} \right)^x = e$$

$$\Rightarrow \lim_{x \to \infty} \left(1 + \frac{a}{x} \right)^x = e^a$$

$$\Rightarrow \lim_{x \to \infty} \left(1 - \frac{1}{x} \right)^x = e^{-1}$$

3.3.3 Indeterminate forms

Sometimes when limit is applied, the result is obtained in indeterminate forms like

$$\frac{0}{0}, \frac{\infty}{\infty}, \; 0 \times \infty, \; \infty - \infty, 1^{\infty}, 0^0, \infty^0$$

Note: Limits can be applied to indeterminate forms $\frac{0}{0}$ and $\frac{\infty}{\infty}$ using L'hospitial Rule Other indeterminate forms are first converted into $\frac{0}{0}$ or $\frac{\infty}{\infty}$ & then limit is applied.

3.3.4 L'Hospital Rule

If $\frac{f(x)}{g(x)}$ has the indeterminate form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ and $g'(x) \neq 0$ for $x \neq a$, then

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$

Provide that $\lim_{x\to a}\frac{f'(x)}{g'(x)}$ exists or $\lim_{x\to a}\frac{f'(x)}{g'(x)} = \pm\infty$

Example : Evaluate : $\lim_{x\to 0}\frac{2\sin^4\frac{x}{2}}{\frac{\left(x^4\right)}{2}}$

(a) 16 (b) $\frac{1}{2}$ (c) $\frac{1}{4}$ (d) doesn't exits

Example : $\lim_{x\to 0}\frac{e^x - \left(1 + x + \frac{x^2}{2}\right)}{x^3}$

(a) 0 (b) $\frac{1}{6}$ (c) $\frac{1}{3}$ (d) 1

Solution : (b)

$$\lim_{x\to 0}\frac{e^x - \left(1 + x + \frac{x^2}{2}\right)}{x^3}$$

After applying limit $\frac{0}{0}$ form is obtained, limit can be found using L'Hospitial rule.

$$\lim_{x\to 0}\frac{e^x - \left(1 + x + \frac{x^2}{2}\right)}{x^3}\left(\frac{0}{0}\right)$$

$$\lim_{x\to 0}\frac{e^x - 1}{6x}$$

$$\lim_{x\to 0}\frac{e^x}{6x} = \frac{1}{6}$$

Example : What is the value of $\lim_{n\to\infty}\left(1 - \frac{1}{n}\right)^{2n}$

(a) 0 (b) e^{-2} (c) $e^{-1/2}$ (d) 1

Solution : (b) $\lim_{n\to\infty}\left(1-\frac{1}{n}\right)^{2n} = \lim_{n\to\infty}\left[\left(1-\frac{1}{n}\right)^{n}\right]^2$

$$= \left[\lim_{n\to\infty}\left(1-\frac{1}{n}\right)^{n}\right]^2$$

$$= \left(e^{-1}\right)^2 = e^{-2}$$

3.4 Continuity

A function f(x) is defined for $x = a$ is said to be continuous at $x = a$ if

$\Rightarrow f(a)$ is a finite number

$\Rightarrow \lim_{x\to a} f(x)$ exists and $\lim_{x\to a} f(x) = f(a)$

Let $f(x)$ be a function defined on an open interval I and let a be any point in I. We say that $f(x)$ is continuous to the left of a, if

$$\lim_{x\to a^-} f(x) = f(a)$$

Similarly $f(x)$ is said to be continuous to the right of a, if

$$\lim_{x\to a^+} f(x) = f(a)$$

$\therefore$ A function f(x) is continuous at x = a, if it is continuous from left as well as continuous from right .

3.5 Differentiability

Let I denote an open interval (a, b) in R and let $x_0 \in I$. Then a continuous function $f : I \to R$ is said to be differentiable at x_0, if:

$\lim_{h\to 0}\left[\frac{f(x_0+h)-f(x_0)}{h}\right]$ or $\lim_{x\to x_0}\left[\frac{f(x)-f(x_0)}{x-x_0}\right]$ Exist and is denoted by $f'(x_0)$.

Note : A Function is Differentiable then it must be continuous and if function is continuous than function is may or mayn't be differentiable.

Example : The function y = |2-3x|

(a) Is continuous $\forall x \in R$ and differentiable $\forall x \in R$

(b) Is continuous $\forall x \in R$ and differentiable $\forall x \in R$ except at x = 3/2

(c) Is continuous $\forall x \in R$ and differentiable $\forall x \in R$ except at x = 2/3

(d) Is continuous $\forall x \in R$ except at x = 3 and differentiable $\forall x \in R$

Solution : (c) $y = |2-3x|$

$$f(x) = \begin{matrix} 2-3x, \ 2-3x \geq 0 \\ 3x-2, \ 2-3x \leq 0 \end{matrix}$$

Since 2-3x and 3x-2 are polynomials, these are continuous at all points. The only concern is

at $x = \frac{2}{3}$

Left limit at $\left(x = \frac{2}{3}\right) = 2 - 3 \times \frac{2}{3} = 0$

Right limit at $\left(x = \frac{2}{3}\right) = 3 \times \frac{2}{3} - 2 = 0$

$f\left(\frac{2}{3}\right) = 2 - 3 \times \frac{2}{3} = 0$

As left limit=right limit $= f\left(\frac{2}{3}\right)$

Therefore y is continuous at $x = \frac{2}{3}$

Therefore y is continuous at all $x \in R$

Now, at $x = \frac{2}{3}$ left derivative = - 3 right derivative = 3

As left derivative ≠ right derivative, y is not differentiable at $x = \frac{2}{3}$

Example : $x = a(\theta + \sin\theta)$ and $y = a(1 - \cos\theta)$, then dy/dx will be equal to

(a) $\sin\frac{\theta}{2}$ (b) $\cos\frac{\theta}{2}$ (c) $\tan\frac{\theta}{2}$ (d) $\cot\frac{\theta}{2}$

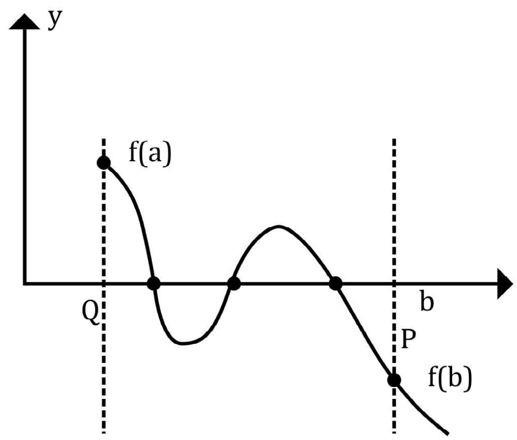
Solution : (c) Given, $x = a(\theta + \sin\theta), y = a(1-\cos\theta), \frac{dx}{d\theta} = a(1+\cos\theta)$

$$\frac{dy}{d\theta} = a\sin\theta$$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{a\sin\theta}{a(1+\cos\theta)} = \frac{2a\sin\left(\frac{\theta}{2}\right).\cos\left(\frac{\theta}{2}\right)}{a \times 2\cos^2\left(\frac{\theta}{2}\right)} = \tan(\theta/2)$$

3.6 Intermediate Mean Value Theorem

If f(x) is continuous on (a,b) such that f(a).f (b) < 0 , then there will be at least one $C \in (a,b)$ such that f(c) = 0. It is also known as Bolzano's theorem.



3.6.1 Rolle's Mean Value Theorem

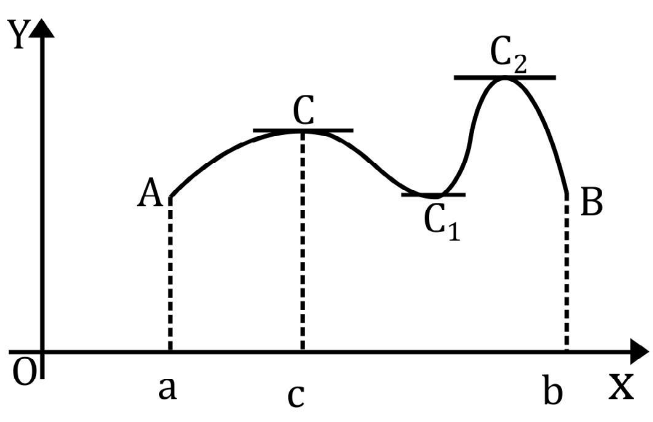
If a function f(x) such that

$\Rightarrow$ $f(x)$ is continuous in the closed interval

$\Rightarrow$ $f'(x)$ exists for every value of x in the open interval (a, b)

$\Rightarrow$ $f(a) = f(b)$,

Then there exists at least one c in (a, b) such that $f'(c) = 0$



Consider the portion AB of the curve y = f (x) , lying between x = a and x=b, such that

(i)It goes continuously from A to B,

(ii) Ordinate of A= ordinate of B.

(iii) It has a tangent at every point between A and B and

From the Fig, it is self evident that there is at least one point C (may be more) of the curve at which the tangent is parallel, to the x-axis.i.e., slope of the tangent at x=c is zero

But the slope of the tangent at C is the value of the differential coefficient of f(x) w.r.t. x, therefore f '(c) =0

Example : Verify Rolle's Theorem for

1) $\frac{\sin x}{e^x}$ in $[0, \pi]$.

2) $(x-a)^m (x-b)^n$

Where m, n are positive integers in [a, b].

Solution : (1) Let $f(x) = \frac{\sin x}{e^x}$

It is continuous in $[0, \pi]$ and differentiable in $(0, \pi)$.

Also $f(0) = f(\pi) = 0$

Hence the conditions of Rolle's Theorem are satisfied

$\therefore\ f'(x) = \frac{e^x \cos x - e^x \sin x}{e^{2x}}$ vanishes

Where $e^x(\cos x - \sin x) = 0$

$\tan x = 1$ or $x = \frac{\pi}{4}$

The value $x = \pi/4$ lies in $(0, \pi)$, so that Rolle's Theorem is verified

2) Let $f(x) = (x-a)^m (x-b)^n$.

Since every polynomial is continuous for all values. $f(x)$ is also continuous in [a,b].

$$f'(x) = m(x-a)^{m-1}(x-b)^n + (x-a)^m n(x-b)^{n-1} = (x-a)^{m-1}(x-b)^{n-1}\left[(m+n)x - (mb+na)\right]$$

Which exists, i.e., $f'(x)$ is derivable in (a,b)

Also $f(a) = 0 = f(b)$.

Thus all the conditions of Rolle's Theorem are satisfied and there exists c in (a, b) such that $f'(c) = 0$

$\therefore (c-a)^{m-1}(c-b)^{n-1}\left[(m+n)c - (mb+na)\right] = 0$ or $c = (mb+na)/(m+n)$.

Hence, Rolle's Theorem is verified.

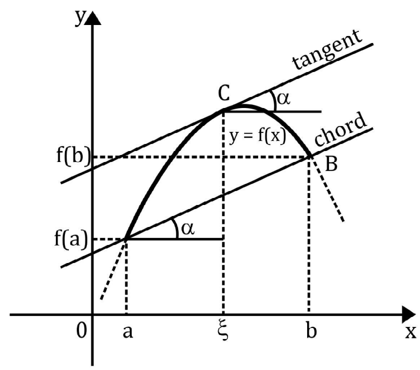
3.6.2 Lagrange's mean value theorem

If a function f(x) is such that

$\Rightarrow$ $f(x)$ is continuous in the closed interval $[a,b]$, and

$\Rightarrow$ $f'(x)$ exists in the open interval (a,b) , then there is at least one value c of x in (a,b),

such that $\frac{f(b)-f(a)}{b-a} = f'(c)$



3.6.3 Cauchy's mean value theorem

Let f(x) and g(x) be defined in [a,b] such that

$\Rightarrow$ $f(x)$ and $g(x)$ are continuous in $[a,b]$

$\Rightarrow$ $f(x)$ and $g(x)$ are differentiable in (a,b)

$\Rightarrow$ $g'(x) \neq 0 \ \forall \ x \in (a,b)$

then their exist atleast one point $c \in (a,b)$ such that $\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$

3.6.4 Taylor's series

$\Rightarrow$ Taylor series expansion of f(x) in powers of (x-a) or Taylor series expansion of f(x) about x = a is

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + ...$$

(1) f(x) and its first (n-1) derivatives be continuous in [a, a+h]

(2) $f^n(x)$ Exists for every value of x in (a, a+h), then there is at least one number $\theta(0 < \theta < 1)$, that

$$f(a+h) = f(a) + hf'(a) + \ldots + \frac{h^n}{n!}f^n(a+\theta h) \ ....(1)$$

Which is called Taylor's theorem with Lagrange's form remainder R_n being $\frac{h^n}{n!}f^n(a+\theta h)$

Note : $\Rightarrow$ Taking n = 1 in (1) , Taylor's theorem reduces to Lagrange's Mean –value theorem.

$\Rightarrow$ Putting a = 0 and h = x in (1) ,we get $f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\ldots+\frac{x^n}{n!}f^n(\theta x)$

Which is known as Maclaurin's theorem with Lagrange's form of reminder.

Example : In the Taylor series expansion of e^x about = 2 .The coefficient of $(x-2)^4$ is

(a) $1/4!$ (b) $2^4/4!$ (c) $e^2/4!$ (d) $e^4/4!$

Solution : (c)

f(x)in the neighborhood of a is,

$$f(x)=\sum_{n=0}^{\infty}b_n(x-a)^n$$

$$\text{Where } b_n=\frac{f^a(a)}{n!}$$

$$f^4(x)=e^x;\ f^4(2)=e^2$$

$$\text{Coefficient of }(x-2)^4=b_4=\frac{f^4(2)}{4!}=\frac{e^2}{4!}$$

3.6.5 MACLAURIN'S SERIES

If f(x) can be expanded as an infinite. Series, then

$$f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{2!}f'''(0)+\ldots\infty$$

Example : Find the Maclaurin's theorem with Lagrange's form of reminder for f(x) = cosx.

Solution : $f^n(x)=\frac{d^n}{dx^n}(\cos x)$

$$=\cos\left(\frac{n\pi}{2}+x\right)\text{so that}f^n_{(0)}$$

$$=\cos\left(n\pi/2\right)$$

$$\text{Thus } f(0)=1$$

$$f^{2n}(0)=\cos\left(n\pi/2\right)=(-1)^n$$

$$f^{2n+1}(0)=\cos\left[(2n+1)n\pi/2\right]=0$$

Substituting these values in the Maclaurin's theorem with Lagrange's form of reminder i.e.

$$f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\ldots+\frac{x^{2n}}{(2n)!}f^{2n}(0)+\frac{x^{2n+1}}{(2n+1)!}f^{2n+1}(\theta x)$$

We get

$$\cos x = 1 + \frac{x^2}{2!}(-1) + \ldots \frac{x^{2n}}{(2n)!}(-1)^n + \frac{x^{2n+1}}{(2n+1)!}(-1)^{n+1}\cos(\theta x)$$

i.e.

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots + (-1)^n \frac{x^{2n}}{(2n)!} + \frac{(-1)^{n+1} x^{2n+1}}{(2n+1)!}\cos(\theta x)$$

Example : Using Maclaurin's series expand tanx up to the term containing x^5.

Solution : $f(x)=\tan x,\ f'(x)=\sec^2 x,\ f''(x) = 2\tan x \sec^2 x,\ f'''(x) = 2\sec^4 x + 4\tan^2 x \sec^2 x$

$$f(0)=0, f'(0) = 1,\ f''(0) = 0,\ f'''(0) = 2$$

$$\tan x = x + \frac{x^3}{3} + \frac{2}{15}x^5 + \ldots$$

3.6.6 EXPANISION BY USE OF KNOWN SERIES

When the expansion of a function is required only up to first few terms, it is often convenient to employ the following well –known series:-

(1) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$

(2) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$

(3) $\tan x = x + \frac{x^3}{3} + \frac{2}{15}x^5 + \ldots$

(4) $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots$

(5) $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{3!} + \ldots$

(6) $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$

(7) $\log(1-x) = -\left(x + \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\right)$

(8) $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots$

3.7 Derivatives

The derivative of a function f(x) with respect to x is defined as

$$f'(x) = \frac{df(x)}{dx} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

3.7.1 RULES

$\Rightarrow\ (f+g)' = f' + g'$

$\Rightarrow\ (f-g)' = f' - g'$

$\Rightarrow\ (f.g)' = fg' + gf'$

$\Rightarrow\ \left(\dfrac{f}{g}\right)' = \dfrac{gf' - fg'}{g^2}$

3.7.2 Important results

$\Rightarrow\ \dfrac{d}{dx} x^n = nx^{n-1}$

$\Rightarrow\ \dfrac{d}{dx} e^x = e^x$

$\Rightarrow\ \dfrac{d}{dx} a^x = a^x \log a$

$\Rightarrow\ \dfrac{d}{dx} \log x = \dfrac{1}{x}$

$\Rightarrow\ \dfrac{d}{dx} \log_a x = \dfrac{1}{x} \log_a e$

$\Rightarrow\ \dfrac{d}{dx} \sin x = \cos x$

$\Rightarrow\ \dfrac{d}{dx} \cos x = -\sin x$

$\Rightarrow\ \dfrac{d}{dx} \tan x = \sec^2 x$

$\Rightarrow\ \dfrac{d}{dx} \cot x = -\operatorname{cosec}^2 x$

$\Rightarrow\ \dfrac{d}{dx} \sec x = \sec x \tan x$

$\Rightarrow\ \dfrac{d}{dx} \operatorname{cosec} x = -\operatorname{cosec} x \cot x$

$\Rightarrow\ \dfrac{d}{dx} \sin^{-1} x = \dfrac{1}{\sqrt{1-x^2}}$

$\Rightarrow\ \dfrac{d}{dx} \cos^{-1} x = -\dfrac{1}{\sqrt{1-x^2}}$

$\Rightarrow\ \dfrac{d}{dx} \tan^{-1} x = \dfrac{1}{1+x^2}$

$\Rightarrow\ \dfrac{d}{dx} \cot^{-1} x = -\dfrac{1}{1+x^2}$

$\Rightarrow\ \dfrac{d}{dx} \sec^{-1} x = \dfrac{1}{x\sqrt{x^2-1}}$

$\Rightarrow\ \dfrac{d}{dx} \operatorname{cosec}^{-1} x = -\dfrac{1}{x\sqrt{x^2-1}}$

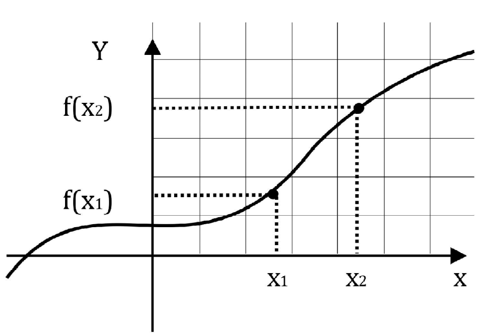
3.7.3 Increasing and Decreasing functions

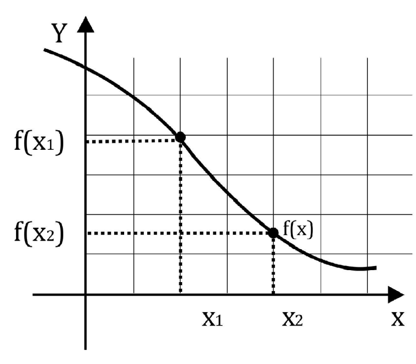
A function is “increasing” if the y-value increases as the x-value increases, like this:

For a function y = f(x):

When $x_2 > x_1$ then $f(x_2) \geq f(x_1)$ Increasing

When $x_2 > x_1$ then $f(x_2) > f(x_1)$ Strictly Increasing





Their value decreases as the x-value increases:

For a function y=f(x):

When $x_2 < x_1$ then $f(x_2) \geq f(x_1)$; Decreasing

When $x_2 < x_1$ then $f(x_2) > f(x_1)$; Strictly Decreasing

Note : $\Rightarrow$ For an **increasing** function dy/dx is always **+ve**

$\Rightarrow$ For an **decreasing** function dy/dx is always **– ve**

$\Rightarrow$ If the derivate is zero, then y is neither increasing nor decreasing .In such cases, we say that the function is **Stationary.**

$\Rightarrow$ The groups of monotonically increasing and monotonically decreasing functions have some special properties. A **monotonically increasing function** is one that increases as x increases for all real x. A **monotonically decreasing function**, on the other hand, is one that decreases as x increases for all real x

Example : As x varies from -1 to +3, which one of the following describes the behavior of the function $f(x) = x^3 - 3x^2 + 1$?

(a) f(x) increase monotonically

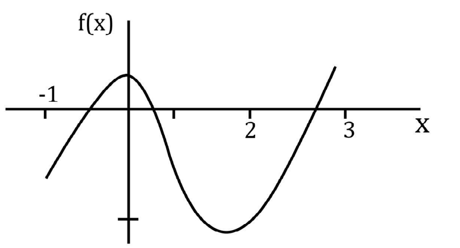
(b) f(x) increases, then decreases and increases again

(c) f(x) decreases, then increases and decreases again

(d) f(x) increases and then decreases

Solution : (b)

We can plot f(x) for various value of x

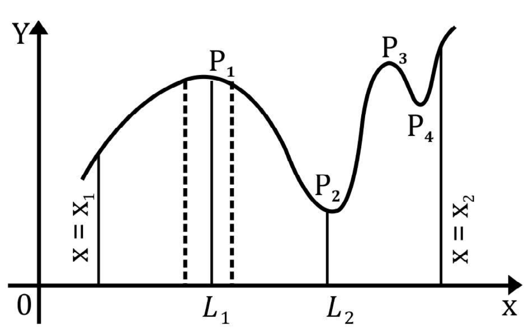


f(x) increase, decrease & again increase

3.7.4 Maxima & Minima of a function

A function f(x) is said to have a **maxima** at x = a, if there exists a small number h, however small, such that f(a) > both f(a-h) and f (a+h).

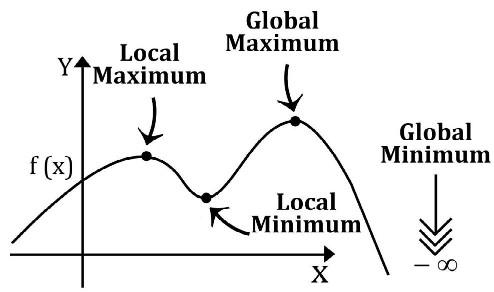
A function f(x) is said to have a **minima** at x = a, if there exists a small number h, however small, such that f(a) < both f(a-h) and f (a+h).



NOTE : • The maxima and minima of a function taken together are called its **extrema** and the points at which the function attains the extreme values are called the **turning points** of the function.

• It is seen from the Fig that maxima and minima occur alternately.

The maximum or minimum over the **entire function** is called an "Absolute" or "Global" maximum or minimum.



3.7.4.1 Conditions for Maxima & Minima

• At each point of extreme value, it is seen from Fig. that the tangent to the curve is parallel to the x-axis, i.e., its slope (dy/dx) is zero. Thus if the function have maxima or minima at x = a then $(dy/dx)_a = 0$.

• f(x) have the point of maxima at x = a if f'(a)=0 and f''(a) is – ve [i.e., f'(a) changes sign from +ve to -ve]

• f(x) have the point of minima at x = a if f'(a)=0 and f''(a) is + ve [i.e., f'(a) changes sign from -ve to +ve]

Example : Show that the function f(x)=sinx(1+cosx) have a point of maxima at $x = \pi/3$

Solution : Let, $f(x) = \sin x\,(1+\cos x)$

$$f'(x) = \cos x(1+\cos x) + \sin x(-\sin x) = \cos x\,(1+\cos x) - (1-\cos^2 x)$$

$$= (1+\cos x)(2+\cos x - 1)$$

$$f'(\pi/3) = 0$$

Now, $f''(x) = \sin x\,(2\cos x - 1) + (1+\cos x)\,(-2\sin x) = -\sin x(4\cos x + 1)$

Since $f''(\pi/3) < 0$

Hence, $x = \pi/3$ is point of maxima

Example : The optimum value of the function f(x) = x^2 - 4x +2 is

(a) 2 (maximum) (b) 2(minimum) (c) -2(maximum) (d)-2(minimum)

Solution : **(d)** $f'(x) = 0 \Rightarrow 2x - 4 = 0 \quad \Rightarrow x = 2$ (stationary point)

$f''(x) = 2 > 0 \Rightarrow f(x)$ is minimum at $x = 2$

and the minimum value is $f(2)$

i.e., $(2)^2 - 4(2) + 2 = -2$

The optimum value of $f(x)$ is -2 (minimum)

Example : The minimum value of function $y = x^2$ in the interval [1, 5] is

(a) 0 (b) 1 (c) 25 (d) Undefined

Solution : (b)

$$y = x^2$$

$$\Rightarrow \frac{dy}{dx} = 2x = 0 \text{ at } x = 0$$

$$\Rightarrow \left[\frac{d^2y}{dx^2}\right]_{x=0} = 2 \text{ which is} + \text{ve}$$

So we have a local minima at x = 0

At x = 0, y = 0

But since x = 0 does not belong to [1, 5]

It is not a candidate for minima or maxima in that range

At the end point x = 1

y = 1

At second end point x = 5

y = 25

So, absolute minimum value of function in [1, 5] is 1.

3.7.5 Partial Derivative

3.7.5.1 Definition of Partial Derivative

If a derivative of a function of several independent variables be found with respect to any one of the independent variables, keeping the others variables as constants. It is said to be a partial derivative. The operation of finding the partial derivative of a function of more than one independent variables is called Partial derivative.

The symbols $\partial / \partial x, \partial / \partial y$ etc., are used to denote such derivative and the expressions $\partial u / \partial x, \partial u / \partial y$ etc. are respectively called partial differential coefficients of u with respect to x and y. If u = f(x, y, z) the partial differential coefficient of u with respect to x i.e., $\partial u / \partial x$ is obtained by differentiating u with respect to x keeping y and z as constants

3.7.5.2 Second order partial differential coefficients

If u = f(x, y) then $\partial u / \partial x$ or f_x and $\partial u / \partial y$ or f_y are themselves function of x and y and can be again differentiated partially.

We call $\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\right)$, $\frac{\partial}{\partial y}\left(\frac{\partial}{\partial y}\right)$, $\frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\right)$, $\frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}\right)$ as second order partial derivatives of u

and these are respectively denoted by $\frac{\partial^2 u}{\partial x^2}$, $\frac{\partial^2 u}{\partial y^2}$, $\frac{\partial^2 u}{\partial x \partial y}$, $\frac{\partial^2 u}{\partial y \partial x}$

3.7.5.3 First & second Partial Derivatives

Let f(x, y) be the function of two variables such that all higher order partial derivative exists

First Partial derivatives

$$p = \frac{\partial f}{\partial x} = f_x \; ; \; q = \frac{\partial f}{\partial y} = f_y$$

Second Partial derivatives

$$r = \frac{\partial^2 f}{\partial x^2} = f_{xx}$$

$$\frac{\partial^2 f}{\partial y \partial x} = f_{xy} \; ; \; \frac{\partial^2 f}{\partial x \partial y} = f_{yx}$$

$$s = f_{xy} = f_{yx}$$

$$t = \frac{\partial^2 f}{\partial y^2} = f_{yy}$$

Example : A particle moves along a curve whose parametric equations are: $x = t^3 + 2t$, $y = -3e^{-2t}$ and z = 2sin(5t), where x, y and z show variations of the distance covered by the particle (in cm) with time t (in s). The magnitude of the acceleration of the particle (in cm/s^2) at t = 0 is _______

Solution : (12 to 12)

$x = t^3 + 2t$, $y = -3^{e-2t}$, z = 2sin (5t)

$$a_x = \frac{\partial^2 x}{\partial t^2} = 6t$$

$$a_y = \frac{\partial^2 y}{\partial t^2} = -12e^{-2t}$$

$$a_z = \frac{\partial^2 z}{\partial t^2} = -50 \sin 5t$$

$$a = a_x \hat{i} + a_y \hat{j} + az\hat{k} = (6t)\hat{i} - \left(12\, e^{-2t}\right) s\, \hat{j} - (50 \sin 5t)\hat{k}$$

$$A\,(t = 0) = -12\hat{j}$$

$$\text{So, } |a|_{t=0}\, 12 \text{ cm/s}^2$$

3.7.5.4 Total Differentiation

If u = f(x, y) where x = f(t) and y = g(t) then the total derivative with

respect to t is $du = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\times\frac{\partial r}{\partial x}+\frac{\partial u}{\partial s}\times\frac{\partial s}{\partial x}$$
$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\times\frac{\partial r}{\partial y}+\frac{\partial u}{\partial s}\times\frac{\partial s}{\partial y}$$

Example : If z = xyln(xy), then

a) $x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=0$ b) $y\frac{\partial z}{\partial x}=x\frac{\partial z}{\partial y}$

c) $x\frac{\partial z}{\partial x}=y\frac{\partial z}{\partial y}$ d) $y\frac{\partial z}{\partial x}+x\frac{\partial z}{\partial y}=0$

Solution : (c) $\frac{\partial z}{\partial x}=y\left[x\times\frac{1}{xy}\times y+\ln xy\right]=y(1+\ln xy)$ and $\frac{\partial z}{\partial x}=x\left(1+\ln xy\right)\Rightarrow x\frac{\partial z}{\partial x}=y\frac{\partial z}{\partial y}$

Example : The contour on the x-y plane, where the partial derivative of x^2+y^2 with respect to y is equal to the partial derivative of 6y+4x with respect to x, is

(a) y = 2 (b) x = 2

(c) x + y = 4 (d) x −y = 0

Solution : (a)

The partial derivative of x^2+y^2 with respect to y is $0 + 2y \Rightarrow 2y$

The partial derivative of 6y + 4x with respect x is 0 + 4 = 4.

Given that both are equal.

$\Rightarrow 2y = 4 \Rightarrow$

Example : Solution of Laplace's equation having continuous second-order partial derivatives are called

(a) Bi-harmonic functions

(b) Harmonic functions

(c) Conjugate harmonic functions

d) Error functions

Solution : (b)

3.7.5.5 Homogeneous Function

Definition

A function f*(x, y)* is said to be homogeneous function of degree n if

$$f(kx, ky) = k^n f(x, y)$$

or $$f(x, y) = \begin{cases} x^n f\left(\frac{y}{x}\right) \\ y^n \psi\left(\frac{x}{y}\right) \end{cases}$$

Example

(i) $x^2yz - 3x^2y^2$ is a HF of degree 4.

(ii) $\dfrac{x^2y + y^3}{2x - y}$ is a HF of degree 2.

(iii) $\cos^{-1}\left(\dfrac{x^2 + y^2}{x - y}\right)$ is non HF

(iv) $x^2 \sin\left(\dfrac{y}{x}\right)$ is HF of degree 2.

Euler's Theorem

If $u = f(x, y)$ is a homogeneous function with degree 'n' then

(a) $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = nu$

(b) $x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x \partial y} + y^2\dfrac{\partial^2 u}{\partial y^2} = n(n-1)u$

Let u(x,y) is non homogeneous function and

(i) If $u(x, y) = f(x, y) + g(x, y)$ where f and g are homogeneous functions with degree m and n respectively then

(a) $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = mf + ng$

(b) $x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x \partial y} + y^2\dfrac{\partial^2 u}{\partial y^2} = m(m-1)f + n(n-1)g$

(ii) If $G(u)$ is a homogeneous function in two variables x and y with degree 'n' then

(a) $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = n\dfrac{G(u)}{G'(u)} = H(u)$

(b) $x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x \partial y} + y^2\dfrac{\partial^2 u}{\partial y^2} = H(u)\left[H'(u) - 1\right]$

3.7.5.6 Maximum and Minima for Two variable Functions

1. Procedure

Let f (x, y) be the function of two variables.

Step 1 : Find $p = \dfrac{\partial f}{\partial x}$, $q = \dfrac{\partial f}{\partial y}$, $r = \dfrac{\partial^2 f}{\partial x^2}$, $s = \dfrac{\partial^2 f}{\partial xy}$, $t = \dfrac{\partial^2 f}{\partial y^2}$

Step 2 : Solve p = 0 and q = 0 to find critical points.

Step 3 : Find *r, s* and *t* at each critical points.

Step 4 : At critical point (*a, b*)

(I) If $rt - s^2 > 0$ and

(a) r > 0 then f(x, y) has a local minimum at (a, b).

(b) r < 0 then f(x, y) has a local maximum at (a, b).

(ii) If $rt - s^2 < 0$ then f(x, y) has no extremum at (a, b) and (a, b) is called saddle

point.**(iii)** If $rt - s^2 = 0$ then the case is undecided.

2. Constrained Maxima and Minima

Sometimes it is required to find the extremum of a function to some other conditions involving the variables. Such problems are called constrained maxima and minima problems and they can be solved by using Lagrange's method of undetermined multipliers.

3. Lagrange's Method of Undetermined Multipliers

Let f(x, y. z) be the given function subject to the condition

$\phi(x, y, z) = 0$ .....(1)

For which extremum is to be found.

F(x, y, z) = f(x, y, z) + λf(x, y, z)

Where λ is the Lagrange's multiplier which is to be determined by the following conditions.

Step 2 : The necessary condition to have extreme value are

$$\frac{\partial F}{\partial x} = 0 \Rightarrow \frac{\partial f}{\partial x} + \lambda\frac{\partial \phi}{\partial x} = 0 \quad ...(2)$$

$$\frac{\partial F}{\partial y} = 0 \Rightarrow \frac{\partial f}{\partial y} + \lambda\frac{\partial \phi}{\partial y} = 0 \quad ...(3)$$

$$\frac{\partial F}{\partial z} = 0 \Rightarrow \frac{\partial f}{\partial z} + \lambda\frac{\partial \phi}{\partial z} = 0 \quad ...(4)$$

Equations (2), (3) and (4) are called Lagrange's equations.

Step 3 : Solving (1), (2), (3) and (4) we obtain the values of x, y, z and λ. Hence the values of x, y, *z* so obtained will give the critical points at which f(x, y, *z)* has extremum.

Example : Given a function $f(x, y) = 4x^2 + 6y^2 - 8x - 4y + 8$, then optimal value of f (x, y)

(a) is a minimum equal to 10/3

(b) is a maximum equal to 10/3

(c) is a minimum equal to 8/3

(d) is a maximum equal to 8/3

Solution : (a)

$$f(x,y)=4x^2+6y^2-8x-4y+8$$

$$\frac{\partial f}{\partial x}=8x-8$$

$$\frac{\partial f}{\partial y}=12y-4y$$

Equating, $\frac{\partial f}{\partial x}=0$ and $\frac{\partial f}{\partial y}=0$

$8x-8=0$ and $12y-4y=0$ $\quad x=1$ and $y=\frac{1}{3}$

$\left(1,\frac{1}{3}\right)$ is the only stationary point

$$r=\left[\frac{\partial^2 f}{\partial x^2}\right]_{\left(1,\frac{1}{3}\right)}=8,\quad s=\left[\frac{\partial^2 f}{\partial x dy}\right]_{\left(1,\frac{1}{3}\right)}=0,\quad t=\left[\frac{\partial^2 f}{\partial y^2}\right]_{\left(1,\frac{1}{3}\right)}=12$$

$rt=8\times12=48,\quad s^2=0$

We have either a maxima or minima at $\left(1,\frac{1}{3}\right)$

Also since $r=\left[\frac{\partial^2 f}{\partial x^2}\right]_{\left(1,\frac{1}{3}\right)}=8>0$ the point $\left(1,\frac{1}{3}\right)$ is a point of minima

The minimum value is

$$f\left(1,\frac{1}{3}\right)=4\times1^2+6\times\frac{1}{3^2}-8\times1-4\times\frac{1}{3}+8=\frac{10}{3}$$

So the optimal value of $f(x,\ y)$ is minimum equal to $\frac{10}{3}$

3.8 Integral calculus

3.8.1 Important results

$$\Rightarrow \quad \int x^{n} dx = \frac{x^{n}+1}{n+1} + c$$

$$\Rightarrow \quad \int \frac{1}{x} dx = \log x + c$$

$$\Rightarrow \quad \int \sin x \, dx = -\cos x + c$$

$$\Rightarrow \quad \int \cos x \, dx = \sin x + c$$

$$\Rightarrow \quad \int \sec^{2} x \, dx = \tan x + c$$

$$\Rightarrow \quad \int \operatorname{cosec}^{2} x \, dx = -\cot x + c$$

$$\Rightarrow \quad \int \sec x \tan x \, dx = \sec x + c$$

$$\Rightarrow \quad \int \operatorname{cosec} x \cot x \, dx = -\operatorname{cosec} x + c$$

$$\Rightarrow \quad \int \frac{1}{\sqrt{1-x^{2}}} dx = \sin^{-1} x + c$$

$$\Rightarrow \quad \int \frac{1}{1+x^{2}} dx = \tan^{-1} x + c$$

$$\Rightarrow \quad \int \frac{1}{x\sqrt{x^{2}-1}} dx = \sec^{-1} x + c$$

$$\Rightarrow \quad \int \cosh(x) dx = \sinh(x) + c$$

$$\Rightarrow \quad \int \sinh(x) \, dx = \cosh(x) + c$$

$$\Rightarrow \quad \int_{0}^{\frac{\pi}{2}} \sin^{n} x \, dx = \int_{0}^{\frac{\pi}{2}} \cos^{n} x \, dx = k \frac{(n-1)(n-3)(n-5)\ldots 1}{n(n-2)(n-4)\ldots 1}$$

$$\text{Where } k = \begin{cases} \dfrac{\pi}{2} & \text{If } n \text{ is even} \\ 1 & \text{otherwise} \end{cases}$$

$$\Rightarrow \quad \int_{0}^{\pi/2} \sin^{m} x \cos^{n} x \, dx = k \frac{(m-1)(m-3)\ldots 1 \times (n-1)(n-3)\ldots 1}{(m+n)(m+n-2)(m+n-4)\ldots 1}$$

$$\text{Where } k = \begin{cases} \dfrac{\pi}{2} & \text{If } m \text{ and } n \text{ is even} \\ 1 & \text{otherwise} \end{cases}$$

3.8.2 Useful trigonometric identities

$\Rightarrow \sin(-x) = -\sin x$

$\Rightarrow \cos(-x) = \cos x$

$\Rightarrow \sin(x+y) = \sin x \cos y + \cos x \sin y$

$\Rightarrow \sin(x-y) = \sin x \cos y - \cos x \sin y$

$\Rightarrow \cos(x+y) = \cos x \cos y - \sin x \sin y$

$\Rightarrow \cos(x-y) = \cos x \cos y + \sin x \sin y$

$$\Rightarrow \tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$$

$$\Rightarrow \tan(x-y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}$$

$$\Rightarrow \tan\left(\frac{\pi}{4} + x\right) = \frac{1 + \tan x}{1 - \tan x}$$

$$\Rightarrow \tan\left(\frac{\pi}{4} - x\right) = \frac{1 - \tan x}{1 + \tan x}$$

$$\Rightarrow \sin 2x = 2\sin x \cos x = \frac{2\tan x}{1 + \tan^2 x}$$

$$\Rightarrow \cos(2x) = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x = \frac{1 - \tan^2 x}{1 + \tan^2 x}$$

3.8.3 Methods to solve integration

3.8.3.1 Integration by substitution

$u = ax + b$

We introduce the technique through some simple examples for which a linear substitution is appropriate.

Example : suppose we want to find the integral

$$\int (x+4)^5 \, dx$$

Solution : You will be familiar already with finding a similar integral $\int u^5 du$ and know that this integral is equal to $\frac{u^6}{6} + c,$ where c is a constant of integration. This is because you know that the rule for integrating powers of a variable tells you to increase the power by 1 and then divide by the new power.

In the integral given by Equation (1) there is still a power 5, but the integrand is more complicated due to the presence of the term x+4. To tackle this problem we make a **substitution.** We let u=x+4. The point of doing this is to change the integrand into the much simpler u^5. However, we must take care to substitute appropriately for the term dx too.

In terms of differentials we have $du = \left(\frac{du}{dx}\right)dx$

Now, in this example, because $u = x + 4$ it follows immediately that $\frac{du}{dx} = 1$ and so du=dx. So, substituting both for x+4 and for dx in Equation (1) we have

$\int (x+4)^5\,dx = \int u^5 du$

The resulting integral can be evaluated immediately to give $\frac{u^6}{6} + c$. We can revert to an expression involving the original variable x by recalling that $u = x + 4$, giving

$$\int (x+4)^5\,dx = \frac{(x+4)^6}{6} + c$$

We have completed the integration by substitution.

3.8.3.2 Integration by parts

Let u and v be two function of x. Then we have from differential calculus.

$$\frac{d}{dx}(uv) = u.\frac{dv}{dx} + v.\frac{du}{dx} \quad \ldots \ (1)$$

Integrating both sides of (1) with respect to x, we have

$$uv = \int u.\frac{dv}{dx}dx + \int v.\frac{du}{dx}dx$$

$$\int u\frac{dv}{dx}dx = uv - \int v.\frac{du}{dx}.dx$$

i.e., $\int udv = uv - \int v.du$

This can also be written as

$$\int uvdx = u\int vdx - \int\left[\frac{du}{dx}\int vdx\right]dx$$

ILATE RULE :

A rule of thumb proposed by Herbert Kasube of Bradly University advises that whichever function comes first in the following list should be taken as u.

I : Inverse trigonometric functions : $\tan^{-1}x$, $\sec^{-1}x$, etc

L : Logarithmic functions: $\ln x, \log_b x$, etc

A : Algebraic functions: $x^2, 3x^{50}$, etc

T : Trigonometric functions: sin x, tan x, etc.

E : Exponential functions: $e^x, 19^x$, etc

3.8.3.3 Integration by partial fraction

Suppose $I=\int \frac{1}{x^2-a^2}dx, (x>a)$

$$\frac{1}{x^2-a^2}=\frac{1}{(x-a)(x+a)}=\frac{1}{2a}\left(\frac{1}{x-a}-\frac{1}{x+a}\right)$$

$$\int \frac{1}{x^2-a^2}dx=\frac{1}{2a}\left\{\int \frac{dx}{x-a}-\int \frac{dx}{x+a}\right\}$$

$$=\frac{1}{2a}\{\log(x-a)-\log(x+a)\}$$

$$=\frac{1}{2a}\log\frac{x-a}{x+a}$$

Thus $\int \frac{1}{x^2-a^2}dx=\frac{1}{2a}\log\frac{x-a}{x+a}+c, x>a$

3.8.4 Definite integrals

If $\int_a^b f(x)dx=[F(x)]_a^b=F(b)-F(a)$ is called the definite integral of f(x) between the limit a and b. where F'(x) = f(x)

3.8.4.1 Properties

$\Rightarrow \int_a^b f(x)dx=\int_a^b f(t)dt$

$\Rightarrow \int_a^b f(x)dx=-\int_b^a f(x)dx$

$\Rightarrow \int_a^b f(x)dx=\int_a^c f(x)dx+\int_c^b f(x)dx$

$\Rightarrow \int_0^a f(x)dx=\int_0^a f(a-x)dx$

$$\Rightarrow \int_{-a}^a f(x)dx=\begin{matrix} 2\int_0^a f(x)dx, & \text{If } f(x) \text{ is an even function} \\ 0, & \text{If } f(x) \text{ is an odd function}\end{matrix}$$

$$\Rightarrow \int_0^{2a} f(x)dx=2\int_0^a f(x)dx, \text{ If } f(2a-x)=f(x)$$
$$=0, \quad \text{If } f(2a-x)=-f(x)$$

Example : Evaluate $\int_0^{\pi/2} \frac{\sqrt{(\sin x)}}{\sqrt{(\sin x)}+\sqrt{(\cos x)}} dx$

$$\text{Let } I = \int_0^{\pi/2} \frac{\sqrt{(\sin x)}}{\sqrt{(\sin x)}+\sqrt{(\cos x)}} dx = \int_0^{\pi/2} \frac{\sqrt{\left[\sin\left(\frac{1}{2}\pi - x\right)\right]}}{\sqrt{\left[\sin\left(\frac{1}{2}\pi - x\right)\right]}+\sqrt{\left[\cos\left(\frac{1}{2}\pi - x\right)\right]}} dx$$

$$I = \int_0^{\pi/2} \frac{\sqrt{(\cos x)}}{\sqrt{(\cos x)}+\sqrt{(\sin x)}} dx,\ 2I = \int_0^{\pi/2} \frac{\sqrt{(\sin x)}+\sqrt{(\cos x)}}{\sqrt{(\sin x)}+\sqrt{(\cos x)}} dx = \int_0^{\pi/2} dx = \frac{\pi}{2} \Rightarrow I = \frac{\pi}{4}$$

Example : Evaluate $\int_0^{\pi} \theta \sin^2\theta \cos^4\theta d\theta$

Solution : Let $I = \int_0^{\pi} \theta \sin^2\theta \cos^4\theta d\theta$

$$\text{Then } I = \int_0^{\pi} (\pi-\theta)\sin^2(\pi-\theta)\cos^4(\pi-\theta)d\theta = \pi\int_0^{\pi}\sin^2\theta\cos^4\theta d\theta - I$$

$$2I = \pi\int_0^{\pi}\sin^2\theta\cos^4\theta d\theta = 2\pi\int_0^{\pi/2}\sin^2\theta\cos^4\theta d\theta = 2\pi\frac{1.3.1}{6.4.2}\left(\frac{\pi}{2}\right) = \frac{\pi^2}{16} \Rightarrow I = \frac{\pi^2}{32}$$

3.8.4.2 Gamma function

For n > 0 , $\Gamma(n) = \int_0^{\infty} e^{-x}x^{n-1}dx$

Results (i) $\Gamma(1) = 1$ (ii) $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

(iii) $\Gamma(n+1) = n!$, When n is positive Integer (iv) $\Gamma(n+1) = n\Gamma(n)$

(v) $\int_0^{\infty} e^{-x^2}dx = \frac{\sqrt{\pi}}{2}$

3.8.4.3 Beta Function

For m , n > 0 , $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx$

Results :

(i) $\beta(m,n) = \beta(n,m)$

(ii) $\beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$

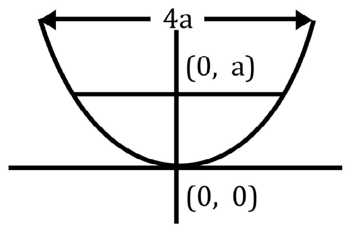
3.8.5 EQUATION OF STANDARD CURVES

(1) Circle

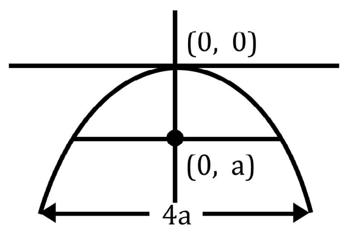
- $x^2 + y^2 = a^2$: Circle with center $(0,\ 0)$ and radius a.
- $(x-h)^2 + (y-k)^2 = a^2$: Circle with center $(h,\ k)$ and radius a.

(2) Parabola

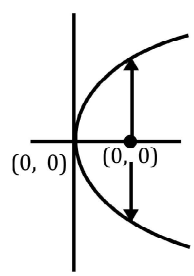
- $x^2 = 4ay$: Parabola with vertex at (0, 0) and focus at (0,a) and latus rectum=4a.



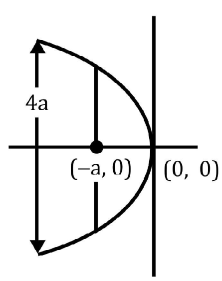
- $x^2 = -4ay$: Parabola with vertex at $(0,\ 0)$ and focus at $(0, -a)$ and latus rectum $= 4a$.



- $y^2 = 4ax$: Parabola with vertex at $(0,\ 0)$ and focus at $(a,\ 0)$ and latus rectum $= 4a$.



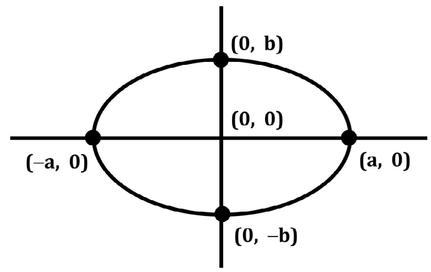
- $y^2 = -4ax$: Parabola with vertex at $(0,0)$ and focus at $(-a,\ 0)$ and latus rectum $= 4a$.



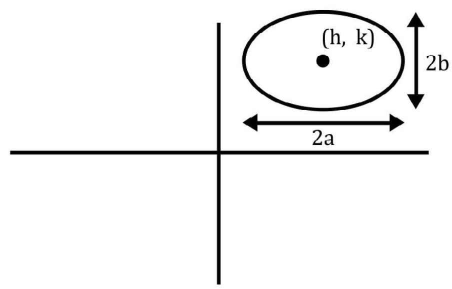
- $(x-h)^2 = 4a(y-k)^2$ Parabola with center at (h, k) focus at (0+h,a+k) and latus rectum=4a.

(3) Ellipse

- $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$**:** Ellipse with center at (0,0) and major axis= 2a**.** and minor axis =2b

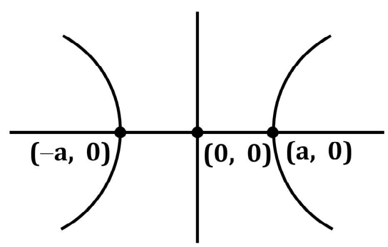


- $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$**:** Ellipse with center at (h, k) and major axis = 2aminor axis = 2b

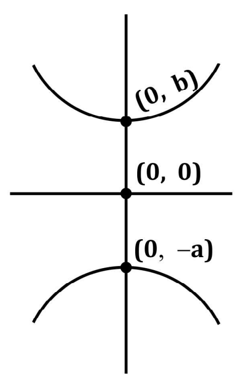


(4) Hyperbola

- $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ **:** Hyperbola with vertex at (a,0) and (-a,0) and center at (0,0)

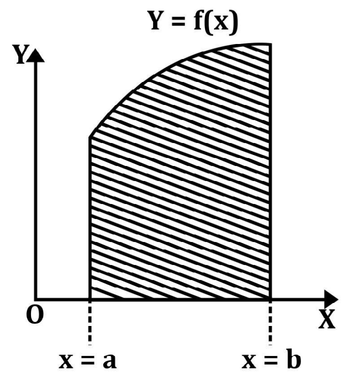


- $\frac{y^2}{b^2}-\frac{x^2}{a^2}=1$**:** Hyperbola with vertex at (0, b)and (0,-b) and center at (0,0)

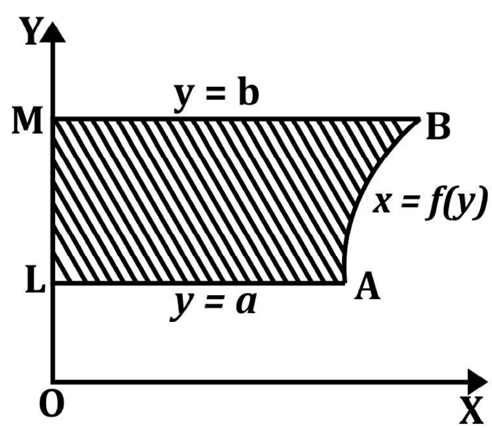


3.8.6 AREAS OF CARTESIAN CURVES

(1) Area bounded by the curve y=f(x) the x-axis and the $x = a$ to b is $\int_a^b y\,dx = \int_a^b f(x)dx$

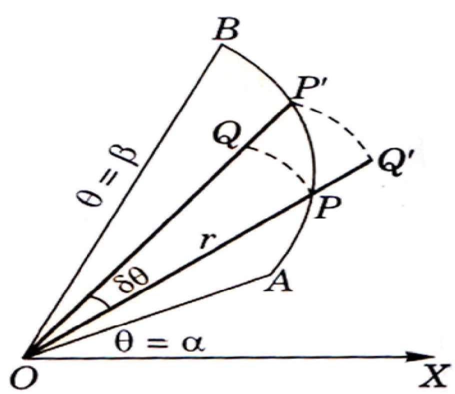


(2) Interchanging x and y in the above formula, we see that the area bounded by the curve x = f(y), the y-axis and curves y = a, y=b is $\int_a^b x\,dy$



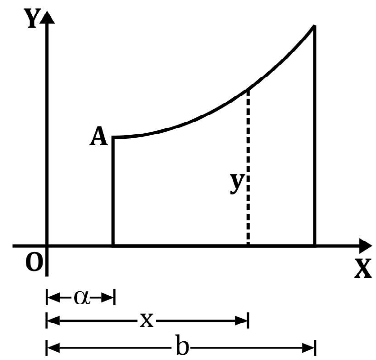
3.8.7 AREAS OF POLAR CURVES

Area bounded by the curve $r = f(\theta)$ and the $\theta = \alpha, \theta = \beta$ is $\frac{1}{2}\int_\alpha^\beta r^2 d\theta$



3.8.8 LENGTH OF CURVES

The length of the curvey $y = f(x)$ between the points a and b are $\int_a^b \sqrt{\left[1+\left(\frac{dy}{dx}\right)^2\right]}dx$



EXAMPLE : A parabolic cable is held between two supports at the same level. The horizontal span between the supports is L. The sag at the mid-span is h. The equation of the parabola is $y = 4h\frac{x^2}{L^2}$ where x is the horizontal coordinate and y is the vertical coordinate with the origin at the center of the cable. The expression for the total length of the cable is

(A) $\int_0^L \sqrt{1+64\frac{h^2x^2}{L^4}}dx$ (B) $2\int_0^{L/2} \sqrt{1+64\frac{h^3x^2}{L^4}}dx$

(C) $\int_0^{L/2} \sqrt{1+64\frac{h^2x^2}{L^4}}dx$ (D) $2\int_0^{L/2} \sqrt{1+64\frac{h^2x^2}{L^4}}dx$

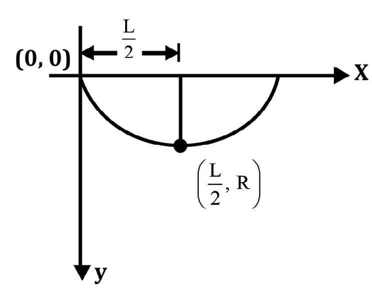
Solution : (D) Length of curve y = f(x) between x = a and x = b is given by $\int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\ dx$

Here, $y = 4h\frac{x^2}{L^2}$ ... (i) $\frac{dy}{dx} = 8h\frac{x}{L^2}$

Since, $y = 0$ at $x = 0$ and $y = h$ at $x = \frac{L}{2}$

(As can be seen from equation (i), by substituting $x = 0$ and)

$\frac{1}{2}(\text{Length of cable}) = \int_a^{L/2} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\ dx = \int_a^{L/2} \sqrt{1+\left(\frac{8hx}{L^2}\right)^2}\ dx$

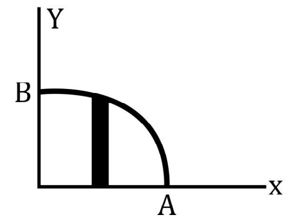


$\text{Length of cable} = 2\int_0^{L/2} \sqrt{1+64\frac{h^2x^2}{L^4}}\ dx$

Example : A path AB in the form of one quarter of a circle of unit radius is shown in the figure. Integration of $(x + y)^2$ on path AB traversed in a counter clockwise sense is

A) $\frac{\pi}{2}-1$ (B) $\frac{\pi}{2}+1$

(C) $\frac{\pi}{2}$ (D) 1



Solution : (b)

Path AB: $x^2+y^2=1$

$x=\cos\theta$

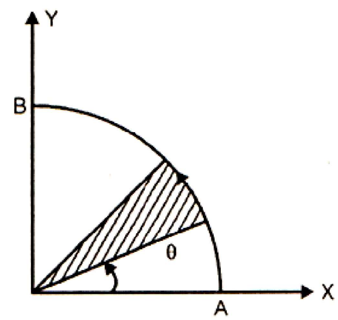
$y=\sin\theta$

Along path AB θ varies from 0° to 90° $[0 \text{ to } \pi/2]$

$$\int_{\text{Path AB}}(x+y)^2(rd\theta)=\int_0^{\pi/2}(\cos\theta+\sin\theta)^2 1.d\theta=\int_0^{\pi/2}(\sin^2\theta+\cos^2\theta+2\sin\theta\cos\theta)d\theta$$

$$=\int_0^{\pi/2}(1+\sin 2\theta)d\theta=\theta+\frac{(-\cos 2\theta)}{2}\Big|_0^{\pi/2}=\frac{\pi}{2}-\frac{1}{2}\left[\cos 2\frac{\pi}{2}-\cos 0\right]$$

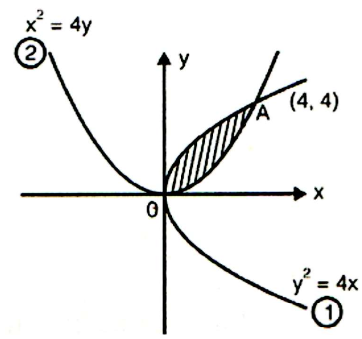
$$=\frac{\pi}{2}-\frac{1}{2}[-1-1]=\frac{\pi}{2}+1$$



Example : The area enclosed between the curves $y^2=4x$ and $x^2=4y$ is

(A) $\frac{16}{3}$ (B) 8 (C) $\frac{32}{3}$ (D) 16

Solution : (a) Curve1: $y^2=4x$, Curve 2: $x^2=4y$



Intersection points of curve 1 and 2

$y^2=4x=4\sqrt{4y}=8\sqrt{y}$, $\quad y^4=8\times 8y \Rightarrow y(y^3-64)=0$

Solution y = 4 and y = 0

Therefore intersection point are $A(4,4)$ and $0(0,0)$

The area enclosed between curves 1 and 2 are given by

$$\text{Area} = \int_{x_1}^{x_2} y_1 dx - \int_{x_1}^{x_2} y_2 dx$$

$$= \int_0^4 2x^{\frac{1}{2}} dx - \int_0^4 \frac{x^2}{4} dx$$

$$= 2\left.\frac{x^{3/2}}{3/2}\right|_0^4 - \left.\frac{x^3}{3\times4}\right|_0^4$$

$$= \frac{4}{3}(4)^{3/2} - \frac{(4)^3}{3\times4} = \frac{16}{3}$$

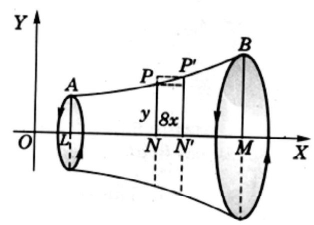
Alternately, the same can been obtained by taking a double integral as follows:
Required Area

$$= \int_0^4 \int_{\frac{x^2}{4}}^{2\sqrt{x}} dx\, dy = \int_0^4 \left(2\sqrt{x} - \frac{x^2}{4}\right) dx = \frac{16}{3}$$

3.8.9 VOLUMES OF REVOLUTION

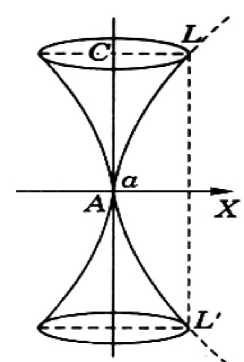
1)Revolution about x-axis : - The volume of the solid generated by the revolution about the x-axis, of the area bounded by the curve y=f(x)

$$\text{Volume} = \int_a^b \pi y^2 dx$$



2) Revolution about the y-axis. :-Interchanging x and y in the above formula, we see that the volume of the solid generated by the revolution about y - axis and bounded by the curve x =f(y),

$$\text{Volume} = \int_a^b \pi\, x^2 dy$$



3.8.10 DOUBLE INTERGRALS

The definite integral $\int_a^b f(x)dx$ is defined as the limit of the sum

$$f(x_1)\delta x_1 + f(x_2)\delta x_2 + \ldots + f(x_n)\delta x_n$$

Where $n \to \infty$ and each of the lengths $\delta x_1, \delta x_2, \ldots$ tends to zero. A double integral is its counterpart in two dimensions. Consider a function f(x,y) of the independent variables x,y defined at each point in the finite region R of the xy-plane, Divide R into n elementary Areas $\delta A_1, \delta A_2, \ldots \delta A_n$. Let (x_r, y_r) be any point within the rth elementary area δA_r.Consider the sum $f(x_1, y_1)\delta A_1 + f(x_2, y_2)\delta A_2 + \ldots f(x_n, y_n)\delta A_n$, i.e., $\sum_{r=1}^{n} f(x_r, y_r)\delta A_r$

The limit of this sum, if it exists , as number of sub-divisions increases indefinitely and area of each sub-division decreases to zero, is defined as the double integral of f(x,y) over the region R and is written as $\iint_R f(x, y)\, dA$.

Thus, $$\iint_R f(x, y)\, dA = \underset{\substack{n\to\infty \\ \delta A \to 0}}{\mathrm{Lt}} \sum_{r=1}^{n} f(x_r, y_r)\delta A_r \quad \ldots (1)$$

The utility of double integral would be limited if it were required to take limit of sums to evaluate them. However, there is another method of evaluating double integral by successive single integrations.

For purpose of evaluation, (1) is expressed as the repeated integral $\int_{x_1}^{x_2}\int_{y_1}^{y_2} f(x, y)\, dxdy$.

Example : $\int_0^5 \int_0^{x^2} x\left(x^2 + y^2\right) dxdy$

Solution : $I = \int_0^5 \int_0^{x^2} \left(x^3 + xy^3\right) dxdy$

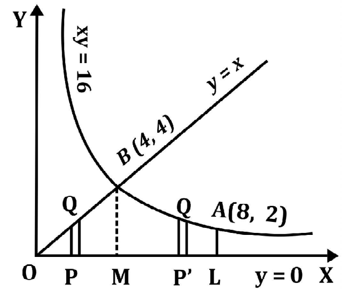
$$= \int_0^5 \left[x^3 y + x.\frac{y^3}{3}\right]_0^{x^2} dx = \int_0^5 \left[x^3.x^2. + x\frac{x^6}{3}\right] dx$$

$$= \int_0^5 \left(x^5 + \frac{x^7}{3}\right) dx = \left|\frac{x^6}{6} + \frac{x^8}{24}\right|_0^5$$

$$= 5^6\left[\frac{1}{6} + \frac{5^2}{24}\right] = 18880.2$$

Example : $\iint_R x^2 dxdy$ where R is region in the first quadrant bounded by the lines x = y, y = 0, x = 8 and the curve xy =16

Solution :



The line AL (x=8) intersects the hyperbola xy = 16 at A(8, 2) while the line $y = x$ intersects this hyperbola at B(4, 4) . Fig. shows the region R of integration which is the area OLAB. To evaluate to given integral, we divide this area into two parts OMB and MLAB.

$$\iint_R x^2 dxdy = \int_0^4\int_0^x x^2 dxdy + \int_4^8\int_0^{16/x} x^2 dxdy = \int_0^4\int_0^x y\Big|_0^x x^2 dx + \int_4^8\int_0^{16/x} y\Big|_0^{16/x} x^2 dx$$

$$= \int_0^4 x^3 dx + \int_4^8 16x dx = \frac{x^4}{4}\Big|_0^4 + 16\frac{x^2}{2}\Big|_4^8 = 448$$

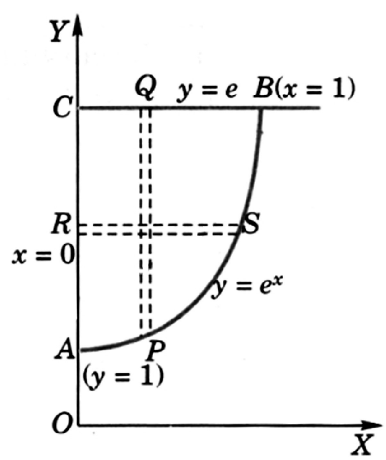
3.8.10.1 CHANGE OF ORDER OF INTEGRATION

In a double integral with variable limits, the change of order of integration changes the limit of integration. While doing so, sometimes it is required to split up the region of integration and the given integral is expressed as the sum of a number of double integrals with changed limits. To fix up the new limits, it is always advisable to draw a rough sketch of the region of integration.

The change of order of integration quite often facilitates the evaluation of a double integral. The following examples will make these ideas clear.

Example : $\int_0^1\int_{e^x}^e \frac{dydx}{\log y}$ by changing the order of integration

Solution :



Here the integration is first w. r. t. y from P on $y = e^x$ to Q on the line y = e. Then the integration is w. r. t. x from x = 1, giving the shaded region ABC On changing the order of integration, we first integrate w. r. t x from R on x = 0 to S on x = logy and then w. r. t y from y = 1 to y = e.

$$\int_0^1\int_{e^x}^{e}\frac{dydx}{\log y}=\int_1^e\int_0^{\log y}\frac{dxdy}{\log y}=\int_1^e\frac{dy}{\log y}x\Big|_0^{\log y}=\int_1^e dy=y\Big|_1^e=e-1$$

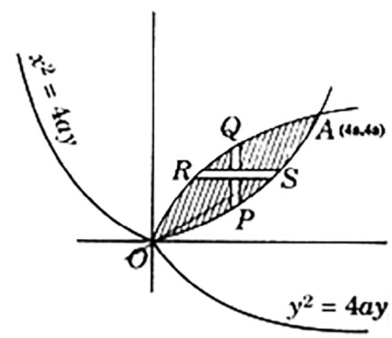
Example : Change the order of integration in $I=\int_0^{4a}\int_{x^2/4a}^{2\sqrt{ax}}dydx$ and hence evaluate.

Solution : Here integration is first w. r. t. y and P on the parabola $x^2 = 4ay$ to Q on the parabola $x^2 = 4ay$ and then w. r. t. x form $x = 0$ to $x = 4a$ giving the shaded region of Integration (fig) On changing the order of integration. We first integrate w. r. t x from R to S then w. r. t. y fromy = 0 toy = 4a

$$I=\int_0^{4a}\int_{\frac{y^2}{4a}}^{2\sqrt{ay}}dxdy=\int_0^{4a}|x|_{\frac{y^2}{4a}}^{2\sqrt{ay}}dy$$

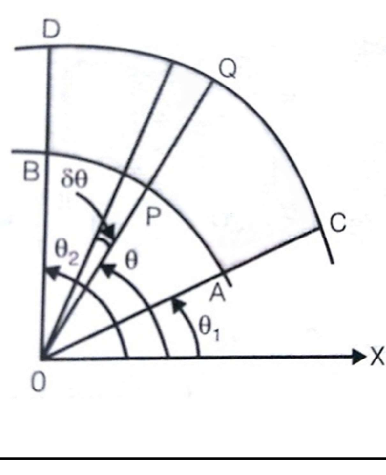
$$=\int_0^{4a}\left(2\sqrt{ay}-y^2/4a\right)dy$$

$$=\left|2\sqrt{a}\,\frac{y^{3/2}}{3/2}-\frac{y^3}{12a}\right|_0^{4a}=\frac{32a^2}{3}-\frac{16a^2}{3}=\frac{16a^2}{3}$$



3.8.10.2 Double Integrals in Polar co-ordinates

To evalute $\int_{\theta_1}^{\theta_2}\int_{r_1}^{r_2} f(r,\theta)dr\,d\theta$, We first integrate w.r.t r between limits $r = r_1$ and $r = r_2$ keeping θ fixed and the resulting expression is integrated w.r.t. θ from θ_1 to θ_2 In this integral r_1, r_2 are function of θ and θ_1, θ_2 are constants.



Here AB and CD are the curves $r_1 = f_1(\theta)$ and $r_2 = f_2(\theta)$ bounded by the lines $\theta = \theta_1$ and $\theta = \theta_2$. PQ is a Wedge of angular thickness $\delta\theta$.

Then $\int_{r_1}^{r_2} f(r,\theta)dr$ indicates that the integration is along PQ from P to Q while the integration w.r.t. θ corresponds to the turning of PQ from AC to BD.

Thus the whole region of integration is the area ACDB .The order of integration may be changed with appropriate changes in the limits.

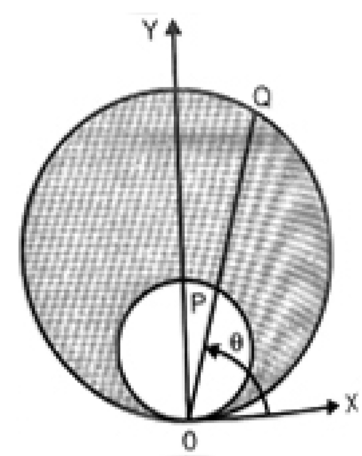
Example : Calculate $\iint r^3 dr d\theta$ over the area included between the circles $r = 2\sin\theta$ and $r = 4\sin\theta$

Solution : 22.5π

Given circles $r = 2\sin\theta$ ....(i)

and $r = 4\sin\theta$ ....(ii)

are shown in Figure below . The shaded area between these circles is the region of integration. If we integrate first w.r.t , then its limits are from $p(r = 2\sin\theta)$ to $Q(r = 4\sin\theta)$ and to cover the whole region θ varies from 0 to π. Thus the required integral is



$$I = \int_0^{\pi} d\theta \int_{2\sin\theta}^{4\sin\theta} r^3 dr = \int_0^{\pi} d\theta \left|\frac{r^4}{4}\right|_{2\sin\theta}^{4\sin\theta}$$

$$= 60\int_0^{\pi} \sin^4\theta\, d\theta = 60 \times 2\int_0^{\pi/2} \sin^4\theta\, d\theta$$

Using reduction formula ,

$$\int_0^{\pi/2} \sin^4\theta\, d\theta = \frac{3\times1}{4\times2}\times\frac{\pi}{2}$$

So , the required integral , $I = 120 \times\frac{3\times1}{4\times2}\times\frac{\pi}{2} = 22.5\pi$

3.8.11 TRIPLE INTEGRALS

Consider a function f(x,y,z)defined at every point of the 3 –dimensional finite region V. Divide V into n elementary volumes $\delta V_1, \delta V_2 \ldots \delta V_n$. Let (x_r, y_r, z_r) be any point within the r^{th} sub- division δV_r. Consider the sum. $\sum_{r=1}^{\infty} f(x_r, y_r, z_r)\delta V_r$

The limit of this sum, if it exists, as $n \to \infty$ and $\delta V_r \to 0$ is called the triple integral of $f(x, y, z)$ over the region V and is denoted by, $\iiint (x, y, z)\,dxdydz$.

If x_1, x_2 are constants: y_1, y_2 are either constants or functions of x and z_1, z_2 are either constants or functions of x and y, then this integral is evaluated as follows:

First is integrated w. r. t. z between the limits z_1 and z_2 keeping x and y fixed. The resulting expression is integrated w. r. t. y between the limits and keeping x constant. The result just obtained is finally integrated w. r. t. x

Thus
$$I = \boxed{\int_{x_1}^{x_2} \boxed{\int_{y_{1(y)}}^{y_{2(x)}} \boxed{\int_{z_{2(x,y)}}^{z_{1(x,y)}} f(x, y, z)\,dz}\,dy}\,dx}$$

Where the integration is carried out from the innermost rectangle to the outermost rectangle. The order of integration may be different for different type of limits.

Example : Evaluate $\int_{-1}^{1}\int_{0}^{z}\int_{x-z}^{x+z} (x + y + z)\,dxdydz$

Solution : Integrating first w. r. t. y keeping x and z constant, we have

$$I = \int_{-1}^{1}\int_{0}^{z} \left| xy + \frac{y^2}{2} + yz \right|_{x-z}^{x+z} dxdz$$

$$= \int_{-1}^{1}\int_{0}^{z} \left[(x+z)(2z) + \frac{1}{2}4xz \right] dxdz$$

$$= 2\int_{-1}^{1} \left(\frac{z^2}{2} + z^3 + \frac{z^3}{2} \right) dz = 4\left| \frac{z^4}{4} \right|_{-1}^{1} = 0$$

3.9 VECTOR CALCULUS

A vector quantity is a quantity that is fully described by both magnitude and direction. On the other hand, a scalar quantity is a quantity that is fully described by its magnitude.

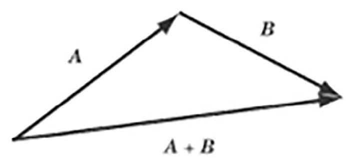
For example, suppose your teacher tells you "A bag of gold is located outside the classroom. To find it, displace yourself 20 meters." This statement may provide yourself enough information to pique your interest; yet, there is not enough information included in the statement to find the bag of gold. The displacement required to find the bag of gold has not been fully described. On the other hand, suppose your teacher tells you "A bag of gold is located outside the classroom. To find it, displace yourself from the center of the classroom door 20 meters in a direction 30 degrees to the west of north." This statement now provides a complete description of the displacement vector - it lists both magnitude (20 meters) and direction (30 degrees to the west of north) relative to a reference or starting position (the center of the classroom door). Vector quantities are not fully described unless both magnitude and direction are listed.

3.9.1 VECTOR LAW OF ADDITION

The easiest way to learn how vector addition works is to look at it graphically. There are two equivalent ways to add vectors graphically: the head-to-tail method and the parallelogram method. Both will get you to the same result, but one or the other is more convenient depending on the circumstances.

1) Head-to-Tail Method

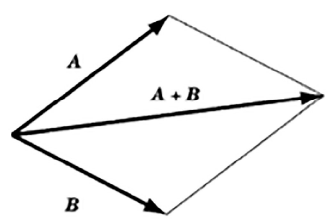
We can add any two vectors, A and B, by placing the tail of B so that it meets the tip of A. The sum, A + B, is the vector from the tail of A to the tip of B.



Note that you will get the same vector if you place the tip of B against the tail of A. In other words, A + B and B + A are equivalent.

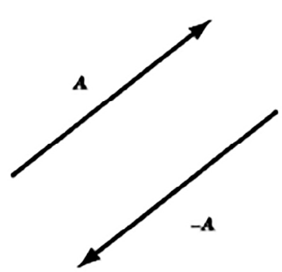
2) Parallelogram Method

To add A and B using the parallelogram method, place the tail of B so that it meets the tail of A. Take these two vectors to be the first two adjacent sides of a parallelogram, and draw in the remaining two sides. The vector sum, A + B, extends from the tails of A and B across the diagonal to the opposite corner of the parallelogram. If the vectors are perpendicular and unequal in magnitude, the parallelogram will be a rectangle. If the vectors are perpendicular and equal in magnitude, the parallelogram will be a square.

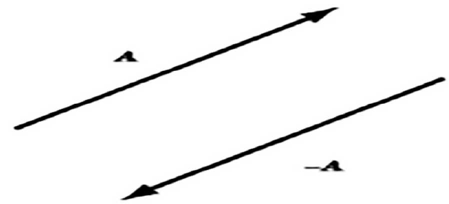


3.9.2 VECTOR SUBTRACTION

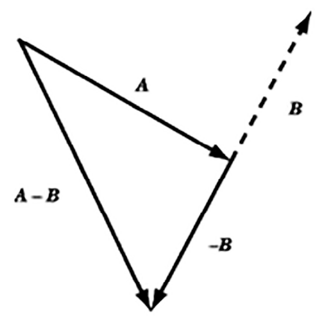
You probably know that subtraction is the same thing as adding a negative: 8 – 5 is the same thing as 8 + (–5). The easiest way to think about vector subtraction is in terms of adding a negative vector. What's a negative vector? It's the same vector as its positive counterpart, only pointing in the opposite direction.



A – B, then, is the same thing as A + (–B). For instance, let's take the two vectors A and B:



To subtract B from A, take a vector of the same magnitude as B, but pointing in the opposite direction, and add that vector to A, using either the tip-to-tail method or the parallelogram method.



3.9.3 DOT PRODUCT

The Dot product of two vectors u and v is given by

$u.v = |u||v|\cos\theta$. Where θ is the angle between the vectors u and v.

The **dot product** is also called the **scalar product**.

Let, $u = x_1\hat{i} + x_2\hat{j} + x_3\hat{k}$ & $v = y_1\hat{i} + y_2\hat{j} + y_3\hat{k}$ are 2 vectors.

The **dot product** of u and v is

$$u.v = x_1y_1 + x_2y_2 + x_3y_3$$

3.9.4 CROSS PRODUCT

The cross product of two vectors u and v is given by

$u \times v = |u|\,|v|\,\sin\theta\ \hat{n}$, where θ is the angle between u and v in counter clock wise direction.

the vector $\hat{n}$ is the unit normal vector perpendicular to both u and v and decided by right hand thumb rule. The cross product of two vectors u and v is a vector that is orthogonal to both u and v and, more precisely, the cross product of and is given by the following formula :

$$u = x_1\hat{i} + x_2\hat{j} + x_3\hat{k} \ \ \& \ \ v = y_1\hat{i} + y_2\hat{j} + y_3\hat{k}$$

$$u \times v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} = \hat{i}\begin{vmatrix} x_2 & x_3 \\ y_2 & y_3 \end{vmatrix} - \hat{j}\begin{vmatrix} x_1 & x_3 \\ y_1 & y_3 \end{vmatrix} + \hat{k}\begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix}$$

3.9.5 UNIT VECTORS

The unit vector of the vector A is $\hat{A} = \frac{\vec{A}}{|\vec{A}|}$

Any vector whose length is 1 is a unit vector. $\hat{i}$, $\hat{j}$ and $\hat{k}$ are the examples of unit vectors

$\left|\hat{i}\right| = \left|\hat{j}\right| = \left|\hat{k}\right| = 1$

3.9.6 DIRECTION OF A VECTOR

Direction of a vector $\hat{A} = \frac{\vec{A}}{|\vec{A}|}$

3.9.7 VECTOR OPERATOR DEL

∇ (Read as Del) is defined by the equation $\nabla = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$

3.9.8 GRADIENT

The vector function ∇f is defined as the gradient of the scalar valued function f and is written as grad f.

Thus $\text{grad} f = \nabla f = \hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z}$

3.9.9 DIRECTIONAL DERIVATIVE

The rate of change of f at any point p in any fixed direction given by a vector b is defined as in calculus.

Example : The magnitude of directional derivative of

$f = \log x + \log y + \log z$ at $\left(\frac{1}{2}, \frac{1}{3}, \frac{1}{4}\right)$ is

(a) $\sqrt{29}$ (b) 3 (c) 29 (d) $\sqrt{3}$

Solution : (a)

$f = \log x + \log y + \log z$

$\nabla f = \frac{1}{x}\hat{i} + \frac{1}{y}\hat{j} + \frac{1}{z}\hat{k}$

$$|\nabla f| = \sqrt{\left(\frac{1}{x}\right)^2 + \left(\frac{1}{y}\right)^2 + \left(\frac{1}{z}\right)^2}$$

$$|\nabla f| \text{ at} \left(\frac{1}{2}, \frac{1}{3}, \frac{1}{4}\right) = \sqrt{29}$$

3.9.10 DIVERGENCE

The divergence of a continuously differentiable vector point function F is denoted by div F and is defined by the equation

$$\operatorname{div}\vec{F} = \nabla.\vec{F} = \hat{i}\frac{\partial \vec{F}}{\partial x} + \hat{j}\frac{\partial \vec{F}}{\partial y} + \hat{k}\frac{\partial \vec{F}}{\partial z}$$

If $\vec{F} = f\hat{i} + \Phi\hat{j} + \Psi\hat{k}$

Then

$$\operatorname{div}\vec{F} = \nabla.\vec{F} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right).\left(f\hat{i} + \Phi\hat{j} + \Psi\hat{k}\right) = \frac{\partial f}{\partial x} + \frac{\partial \Phi}{\partial y} + \frac{\partial \Psi}{\partial z}$$

Example : The divergence of the vector field $\vec{A} = x\hat{a}_x + y\hat{a}_y + z\hat{a}_z$ is

(a) 0 (b) 1/3 (c) 1 (d) 3

Solution : (d)

Given $\vec{A} = x\hat{a}_x + y\hat{a}_y + z\hat{a}_z$

$$\nabla.A = \frac{\partial}{\partial x}(A_x) + \frac{\partial}{\partial y}(A_y) + \frac{\partial}{\partial z}(A_z)$$

$A_x = x, A_y = y, A_z = z,$

$$= \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z)$$

$= 1 + 1 + 1 = 3$

3.9.11 CURL

The curl of a continuously differentiable vector valued function $\vec{F}$ is given by defined by

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f & \Phi & \Psi \end{vmatrix} = \hat{i}\left(\frac{\partial \Psi}{\partial y} - \frac{\partial \Phi}{\partial z}\right) + \hat{j}\left(\frac{\partial f}{\partial z} - \frac{\partial \Psi}{\partial x}\right) + \hat{k}\left(\frac{\partial \Phi}{\partial x} - \frac{\partial f}{\partial y}\right)$$

Note : (1) If the flux entering any element of space is the same as that leaving it, i.e., div V = 0, then such a point function is called a **Solenoidal** vector function.

(2) A motion in which the curl of the velocity vector is zero is said to be **Irrotational.** It is also called conservative field. Otherwise it is rotational.

Example : Curl of vector $\vec{F} = x^2z^2\hat{i} - 2xy^2z\hat{j} + 2y^2z^3\hat{k}$ is

(a) $(4yz^3 + 2xy^2)\hat{i} + 2x^2z\hat{j} - 2y^2z\hat{k}$ (b) $(4yz^3 + 2xy^2)\hat{i} + 2x^2z\hat{j} - 2y^2z\hat{k}$

(c) $2xz^2\hat{i} - 4xyz\hat{j} + 6y^2z^2\hat{k}$ (d) $2xz^2\hat{i} + 4xyz\hat{j} + 6y^2z^2\hat{k}$

Solution : (a)

Given $\vec{F} = x^2z^2\hat{i} - 2xy^2z\hat{j} + 2y^2z^3\hat{k}$

$$\text{curl}\vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2z^2 & -2xy^2z & 2y^2z^3 \end{vmatrix}$$

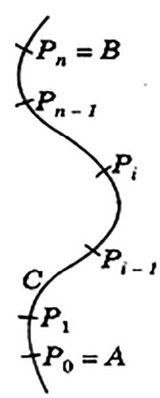
$$= \hat{i}(4yz^3 + 2xy^2) - \hat{j}(0 - 2x^2z) + \hat{k}(-2y^2z - 0) = \hat{i}(4yz^3 + 2xy^2) - \hat{j}(2x^2z) - \hat{k}(2y^2z)$$

3.9.12 LINE INTEGRAL

Consider a continuous vector function F(R) which is defined at each point of curve C in space. Divide C into number of parts at the point's fig. Let their position vectors be $R_0, R_1, \ldots R_{i-1}, R_i \ldots, R_n$. Let U_i be the position vector of any point on the are P_{i-1}, P_i.

Now consider the sum $S = \sum_{i=0}^{n} F(U_i).\delta R_i$, where $dR_i = R_i - R_{i-1}$ the limit of this sum as $n \to \infty$ in such a way that $|\delta R_i| \to 0$, provided it exists is called the tangential line integral of F(R) along C and is symbolically written as

$$\int_C F(R).dR \text{ or } \int_C F.\frac{dR}{dt}dt$$



When the path of integration is a closed curve, this fact is denoted by using $\oint$ in place of ∫.

If $F(R) = if(x,y,z) + j\phi(x,y,z) + k\psi(x,y,z)$
and $dR = idx + jdy + kdz$

Then $\int_C F.(R).dR = \int_C (f\,dx + \phi dy + \psi dz)$

Two other types of line integrals are $\int_C F \times dR$ and $\int_C f\,dR$ which are both vectors.

Example : If $F = 3xy\,\hat{i} - y^2\hat{j}$, evaluate $\int F.dR$ where C is the curve in the xy-plane $y = 2x^2$ from (0,0) to (1,2)

Solution : Since the particle moves in the xy–plane(z=0), we take $R = xi + yj$.

Then. $\int_C F.dR$, where C is the parabola $y = 2x^2$

$\int_C \left(3xyi - y^2 j\right)\left(dxi + dyj\right) = \int_C \left(3xydx - y^2dy\right)$ ...(i) Substituting $y = 2x^2$ where x goes from 0 to 1,

(i) becomes

$$= \int_{x=0}^{1} \left[3x\left(2x^2\right)dx - \left(2x^2\right)^2 d\left(2x^2\right)\right] \quad = \int_0^1 \left(6x^3 - 16x^5\right)dx = -7/6$$

Otherwise, let x=t in $y = 2x^2$. Then the parametric equation of Care x=t, $y = 2t^2$. The points (0, 0) and (1, 2) correspond to t = 0 and t = 1respectively. Then (i) becomes

$$= \int_{t=0}^{1} \left[3t\left(2t^2\right)dt - \left(2t^2\right)^2 d\left(2t^2\right)\right] = \int_0^1 \left(6t^3 - 16t^5\right)dt = -7/6$$

3.9.13 Surface Integral

In mathematics, a surface integral is a generalization of multiple integrals to integration over surfaces. It can be thought of as the double integral analog of the line integral. Given a surface, one may integrate over its scalar fields (that is, functions which return scalars as values), and vector fields (that is, functions which return vectors as values).

3.9.13.1 Stoke's theorem

If S be an open surface bounded by a closed curve C and $\vec{F} = f_1\hat{i} + f_2\hat{j} + f_3\hat{k}$ be any continuously differentiable vector point function, then $\int_C F.dR = \int_C \text{curl } F.Nds$

Where $N = \cos\alpha\hat{i} + \cos\beta\hat{j} + \cos\gamma\hat{k}$ is a unit external normal at any point of S.

3.9.13.2 Green's theorem in the plane

If $\phi(x,y), \psi(x,y), \phi_y$ and ψ_x be continuous in a region E of the xy-plane bounded by a closed curve C, then

$$\int_C \left(\phi dx + \psi dy\right) = \iint_E \left(\frac{\partial\psi}{\partial x} - \frac{\partial\phi}{\partial y}\right)dxdy$$

Example : The value of the line integral $\int_c \overline{F}.\overline{r}'\,ds$ where C is a circle of radius $\frac{4}{\sqrt{\pi}}$ units is __. Here, $\overline{F}(x,y) = y\hat{i} + 2x\hat{j}$ and $\overline{r}'$ is the unit tangent vector on the curve C at an arc length s from a reference point on the curve. $\hat{i}$ and $\hat{j}$ are the basis vectors in the x-y Cartesian reference. In evaluating the line integral, the curve has to be traversed in the counter-clockwise direction

Solution : **(16)** By Green's theorem, $\int_c \overline{F}.\overline{r}'\,ds = \int y\,dx + 2x\,dy = \iint (2-1)\,dx\,dy = \pi\left(\frac{4}{\sqrt{\pi}}\right)^2 = 16$

3.9.14 Vector Volume Integral

Let $\phi(x,y,z)$ be a differentiable scale point function and $\overline{F}(x,y,z)$ be a differentiable vector point function defined over the region bounding a volume ' v ' then the volume integral are

$$\iiint_v \phi(x,y,z)\ \mathrm{dv}$$

$$\iiint_v \overline{F}(x,y,z)\ dv = \hat{i}\iiint_v F_1 dv + \hat{j}\iiint_v F_2 dv + \hat{k}\iiint_v F_3 dv$$

3.9.14.1 Gauss-Divergence Theorem

It is use to convert open surface integral to volume integral

Let ' S ' be closed surface enclosing a volume ' V ' and $\vec{F}(x,y,z) = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$ be a differentiable vector function defined over the surface ' S ' then ,

$$\iint_S \overrightarrow{F}.\ \widehat{N}\ ds = \iiint_V div\overrightarrow{F}\ dxdydz$$

In Cartesian form

$$\iint_S F_1\ dydz + F_2\ dxdz + F_3\ dxdy = \iiint_V \left(\frac{dF_1}{dx} + \frac{dF_2}{dy} + \frac{dF_3}{dz}\right) dxdydz$$

Example : The surface integral $\iint_s F.n\,dS$ over the surface S of the $x^2 + y^2 + z^2 = 9$,

Where $\vec{F} = (x+y)\hat{i} + (x+z)\hat{j} + (y+z)\hat{k}$ and n is the unit outward surface normal yields ___.

Solution : **(226.19)** By gauss divergence theorem

$$\iint \vec{F}.\hat{n}ds = \iiint \nabla.\vec{F}dv$$

$$\nabla.\vec{F} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\left[(x+y)\hat{i} + (x+z)\hat{j} + (y+z)\hat{k}\right] = 1+0+1 = 2$$

$$\iint F.nds = 2\iiint dv = 2\times\frac{4}{3}\pi r^3 = 72\pi = 226.19$$

GATE QUESTIONS

Q.1 $\lim\limits_{x\to 0}\frac{\sin^2 x}{x}$ is equal to

(a) 0 (b) ∞

(c) 1 (d) -1 **[GATE-2003 (ME)]**

Q.2 The value of the function $f(x) = \lim\limits_{x\to 0}\frac{x^3+x^2}{2x^3-7x^2}$ is

(a) 0 (b) $-\frac{1}{7}$

(c) $\frac{1}{7}$ (d) ∞ **[GATE-2004 (CE)]**

Q.3 $x = a(\theta + \sin\theta)$ and $y = a(1-\cos\theta)$ then $\frac{dy}{dx}$ will be equal to

(a) $\sin\frac{\theta}{2}$ (b) $\cos\frac{\theta}{2}$

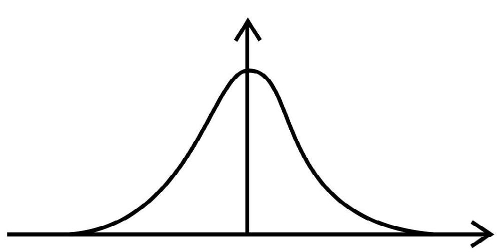
(c) $\tan\frac{\theta}{2}$ (d) $\cot\frac{\theta}{2}$ **[GATE- 2004 (ME)]**

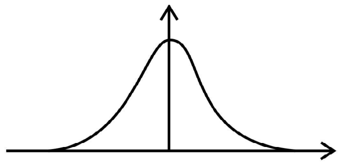
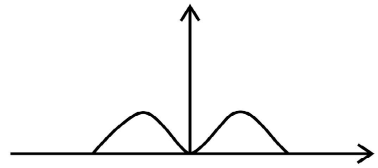
Q.4 The function $f(x)=2x^3-3x^2-36x+2$ has its maxima at

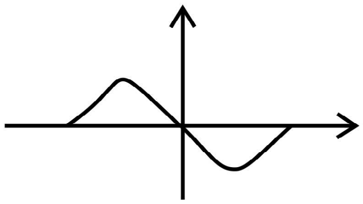
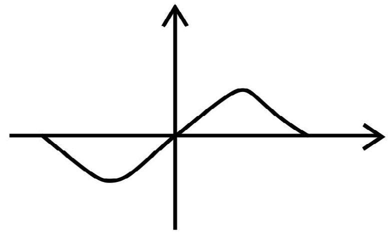
(a) $x = -2$ only (b) x = 0 only

(c) x = 3 only (d) both $x = -2$ and x = 3 **[GATE- 2004 (CE)]**

Q.5 The derivative of the symmetric function drawn in given figure will look like



(a)  (b) 

(c)  (d) 

[GATE- 2004 (ME)]

Q.6 A rail engine accelerates from its stationary position for 8 seconds and travels a distance of 280m. According to the mean value theorem, the speedometer at a certain time during acceleration must read exact.

(a) 0 (b) 8kmph

(c) 75kmph (d) 126kmph **[GATE- 2005(CE)]**

Q.7 As x is increased from $-\infty$ to ∞ the function $f(x)=\frac{e^x}{1+e^x}$

(a) monotonically increases

(b) monotonically decreases

(c) increases to a minimum value and then decreases

(d) decreases to a minimum value and then increases **[GATE- 2006 (E(C)]**

Q.8 A tangent is drawn from the point (–2, 0) to the parabola $y^2 = 8x$ which meets the parabola in first quadrant. The coordinates of point of contact is

(a) (2,4) (b) (4,2)

(c) (1,2) (d) (2,8)

Q.9 The integral $\int_0^{\pi} \sin^3\theta \, d\theta$ is given by

(a) $\frac{1}{2}$ (b) $\frac{2}{3}$

(c) $\frac{4}{3}$ (d) $\frac{8}{3}$ **[GATE- 2006 (ME)]**

Q.10 If $S=\int_1^{\infty} x^{-3}\,dx$, then S has the value

(a) $\frac{-1}{3}$ (b) $\frac{1}{4}$

(c) $\frac{1}{2}$ (d) 1 **[GATE- 2006 (ME)]**

Q.11 $\int_0^{\pi/4} \frac{(1-\tan x)}{(1+\tan x)}dx$ evaluates to

(a) 0 (b) 1

(c) $\ln 2$ (d) $\frac{1}{2}\ln 2$ **[GATE- 2006 (ME)]**

Q.12 Evaluate $\int_0^{\infty} \frac{\sin t}{t}dt$

(a) π (b) $\frac{\pi}{2}$

(c) $\frac{\pi}{4}$ (d) $\frac{\pi}{8}$ **[GATE- 2006 (ME)]**

Q.13 Which of the following integrals is unbounded?

(a) $\int_0^{\pi/4} \tan x\,dx$ (b) $\int_0^{\infty} \frac{1}{x^2+1}dx$

(c) $\int_0^{\infty} xe^{-x}dx$ (d) $\int_0^{1} \frac{1}{1-x}dx$ **[GATE- 2006 (ME)]**

Q.14 The value of the integral $\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$ is

(a) $-\pi$ (b) $-\frac{\pi}{2}$

(c) $\frac{\pi}{2}$ (d) π **[GATE- 2006 (ME)]**

Q.15 The value of the quantity P, where $P=\int_0^1 xe^x dx$, is equal to

(a) 0 (b) 1

(c) e (d) $\frac{1}{e}$ **[GATE- 2006 (ME)]**

Q.16 A parabolic cable is held between two supports at the same level. The horizontal span between the supports is L. The sag at the mid-span is h. The equation of the parabola is $y = 4h\frac{x^2}{L^2}$ where x is the horizontal coordinate and y is the vertical coordinate with the origin at the centre of the cable. The expression for the total length of the cable is

(a) $\int_0^L \sqrt{1+64\frac{h^2x^2}{L^4}}\,dx$ (b) $2\int_0^{L/2} \sqrt{1+64\frac{h^3x^2}{L^4}}\,dx$

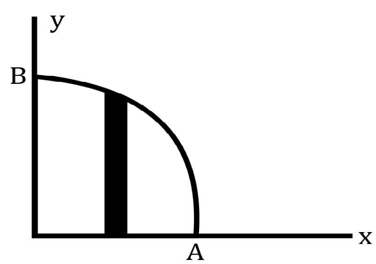
(c) $\int_0^{L/2} \sqrt{1+64\frac{h^2x^2}{L^4}}\,dx$ (d) $2\int_0^{L/2} \sqrt{1+64\frac{h^2x^2}{L^4}}\,dx$ **[GATE- 2006 (ME)]**

Q.17 The length of the curve $y = \frac{2}{3}x^{3/2}$ between x = 0 and x = 1 is

(a) 0.27 (b) 0.67

(c) 1 (d) 1.22 **[GATE- 2006 (ME)]**

Q.18 A path AB in the form of one quarter of a circle of unit radius is shown in the figure. Integration of $(x + y)^2$ on path AB traversed in a counter clockwise sense is



(a) $\frac{\pi}{2}-1$ (b) $\frac{\pi}{2}+1$

(c) $\frac{\pi}{2}$ (d) 1 **[GATE- 2006 (ME)]**

Q.19 The area enclosed between the curves $y^2=4x$ and $x^2=4y$ is

(a) $\frac{16}{3}$ (b) 8

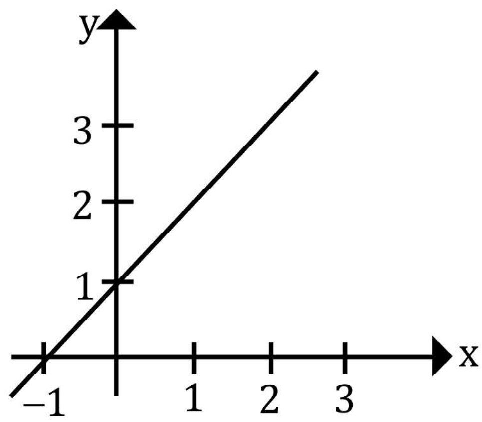
(c) $\frac{32}{3}$ (d) 16 **[GATE- 2006 (ME)]**

Q.20 The parabolic arc $y = \sqrt{x}$, $1 \le x \le 2$ is revolved around the x-axis. The volume of the solid of revolution is

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{2}$

(c) $\frac{3\pi}{4}$ (d) $\frac{3\pi}{2}$ **[GATE- 2006 (ME)]**

Q.21 The following Plot shows a function y which varies linearly with x. The value of the integral $I = \int_1^2 y\,dx$ is



(a) 1.0 (b) 2.5
(c) 4.0 (d) 5.0 **[GATE- 2006 (ME)]**

Q.22 The value of the integral of the function $g(x, y) = 4x^3+10y^4$ along the straight line segment from the point (0, 0) to the point (1, 2) in the x-y plane is

(a) 33 (b) 35
(c) 40 (d) 56 **[GATE- 2006 (ME)]**

Q.23 The minimum value of function $y = x^2$ in the interval [1, 5] is

(a) 0 (b) 1
(c) 25 (d) Undefined **[GATE- 2007 (ME)]**

Q.24 $\lim_{x\to 0} \frac{e^x - \left(1 + x + \frac{x^2}{2}\right)}{x^3} =$

(a) 0 (b) $\frac{1}{6}$
(c) $\frac{1}{3}$ (d) 1 **[GATE-2007 (ME)]**

Q.25 The value of $\lim_{x\to 8} \frac{x^{1/3} - 2}{(x-8)}$ is

(a) $\frac{1}{16}$ (b) $\frac{1}{12}$
(c) $\frac{1}{8}$ (d) $\frac{1}{4}$ **[GATE- 2008 (ME)]**

Q.26 $\lim_{\theta \to 0} \frac{\sin(\theta/2)}{\theta}$ is

(a) 0.5 (b) 1
(c) 2 (d) not defined **[GATE- 2008 (ME)]**

Q.27 $\lim_{x \to \infty} \frac{x - \sin x}{x + \cos x}$ equals

(a) 1 (b) -1
(c) ∞ (d) $-\infty$ **[GATE- 2008 (CS)]**

Q.28 Consider function $f(x)=\left(x^2-4\right)^2$ where x is a real number. Then the function has

(a) only one minimum (b) only two minima
(c) three minima (d) three maxima **[GATE- 2008 (EE)]**

Q.29 Which one of the following function is strictly bounded?

(a) $1/x^2$ (b) e^x
(c) x^2 (d) e^{-x^2} **[GATE- 2008 (EE)]**

Q.30 Consider the function $f(x) = x^2 - x - 2$. The maximum value of f(x) in the closed interval [-4, 4] is

(a) 18 (b) 10
(c) -2.25 (d) indeterminate **[GATE- 2008 (EE)]**

Q.31 Which of the following functions would have only odd powers of x in its Taylor series expansion about the point x = 0 ?

(a) $\sin(x^3)$ (b) $\sin(x^2)$
(c) $\cos(x^3)$ (d) $\cos(x^2)$ **[GATE- 2008 (E(C)]**

Q.32 In the Taylor series expansion of exp(x)+sin(x) about the point $x = \pi$, the coefficient of $(x-\pi)^2$ is

(a) exp (π) (b) 0.5exp (π)
(c) exp (π) + 1 (d) exp (π) – 1 **[GATE- 2008 (E(C)]**

Q.33 The is $\lim_{x \to 0} \frac{\sin\left[\frac{2}{3}x\right]}{x}$

(a) $\frac{2}{3}$ (b) 1
(c) $\frac{3}{2}$ (d) ∞ **[GATE-2010 (CE)]**

Q.34 What is the value of $\lim_{n\to\infty}\left(1-\frac{1}{n}\right)^{2n}$

(a) 0 (b) e^{-2}

(c) $e^{-1/2}$ (d) 1 **[GATE- 2010 (CS)]**

Q.35 The function $y = |2-3x|$

(a) is continuous $\forall x \in R$ and differentiable $\forall x \in R$

(b) is continuous $\forall x \in R$ and differentiable $\forall x \in R$ except at x = 3/2

(c) is continuous $\forall x \in R$ and differentiable $\forall x \in R$ except at x = 2/3

(d) is continuous $\forall x \in R$ except at x = 3 and differentiable $\forall x \in R$ **[GATE- 2010 (ME)]**

Q.36 In the Taylor series expansion of e^x about x = 2 .The coefficient of $(x-2)^4$ is

(a) $\frac{1}{4!}$ (b) $\frac{2^4}{4!}$

(c) $\frac{e^2}{4!}$ (d) $\frac{e^4}{4!}$ **[GATE- 2010 (ME)]**

Q.37 For the function e^{-x}, the linear approximation around x = 2 is

(a) $(3-x)e^{-2}$ (b) $1-x$

(c) $\left[3+2\sqrt{2}-\left(1+\sqrt{2}\right)x\right]e^{-2}$ (d) e^{-2} **[GATE- 2010 (ME)]**

Q.38 Given a function $f(x,y) = 4x^2+6y^2-8x-4y+8$ The optimal value of f (x, y)

(a) is a minimum equal to 10/3 (b) is a maximum equal to 10/3

(c) is a minimum equal to 8/3 (d) is a maximum equal to 8/3 **[GATE-2010 (CE)]**

Q.39 If , $e^y = x^{\frac{1}{x}}$ then y has a

(a) maximum at x = e (b) minimum at x = e

(c) maximum at $x = e^{-1}$ (d) minimum at $x = e^{-1}$ **[GATE-2010 (E(C)]**

Q.40 The point of maxima of |x| in (-1,1) is/are________.

(a) 1 (b) -1, 1

(c) 0 (d) None of these

Q.41 At x=0, the function f (x) = x^3 +1 has

(a) A maximum value (b) A minimum value

(c) A singularity (d) A point of inflection **[ME-GATE-2012]**

Q.42 For the spherical surface, $x^2 + y^2 + z^2 = 1$, the unit outward normal vector at the point $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$ is given by

(a) $\frac{1}{\sqrt{2}}\hat{i}+\frac{1}{\sqrt{2}}\hat{j}$ (b) $\frac{1}{\sqrt{2}}\hat{i}-\frac{1}{\sqrt{2}}\hat{j}$

(c) $\hat{k}$ (d) $\frac{1}{\sqrt{3}}\hat{i}+\frac{1}{\sqrt{3}}\hat{j}+\frac{1}{\sqrt{3}}\hat{k}$ **[ME-GATE-2012]**

Q.43 The area enclosed between the straight line $y=x$ and the parabola $y = x^2$ in the x-y plane is

(a) $\frac{1}{6}$ (b) $\frac{1}{4}$

(c) $\frac{1}{3}$ (d) $\frac{1}{2}$ **[ME-GATE-2012]**

Q.44 Consider the function $f(x) = |x|$ in the interval $-1 \le x \le 1$. At the point $x = 0$, $f(x)$ is

(a) Continuous and differentiable

(b) Non-continuous and differentiable

(c) Continuous and non differentiable

(d) Neither continuous nor differentiable **[ME-GATE-2012]**

Q.45 $\lim_{x\to 0}\left(\frac{1-\cos x}{x^2}\right)$ is

(a) 1/4 (b) 1/2

(c) 1 (d) 2 **[ME-GATE-2012]**

Q.46 Assuming and t is a real number, $\int_0^{\frac{\pi}{3}} e^{it}dt$ is:

(a) $\frac{\sqrt{3}}{2}+i\frac{1}{2}$ (b) $\frac{\sqrt{3}}{2}-i\frac{1}{2}$

(c) $\frac{1}{2}+i\left(\frac{\sqrt{3}}{2}\right)$ (d) $\frac{1}{2}+i\left(1-\frac{\sqrt{3}}{2}\right)$ **[GATE- 2006 (ME)]**

Q.47 For the parallelogram OPQR shown in the sketch, $\overline{OP} = a\hat{i}+b\hat{j}$ and $\overline{OQ} = c\hat{i}+d\hat{j}$. The area of the parallelogram is

(a) ad - bc (b) ac + bd

(c) ad + bc (d) ab − cd **[CE-GATE-2012]**

Q.48 $\lim_{x\to 0^-}\frac{\sin[x]}{[x]}$, [.] is greatest integer funtion , equals______.

Q.49 Consider the function f(x) = sin(x) in the interval $x \in [\pi/4, 7\pi/4]$. The number and location(s) of the local minima of this function are

(a) One, at $\frac{\pi}{2}$ (b) One, at $\frac{3\pi}{2}$

(c) Two, at $\frac{\pi}{2}$ and $\frac{3\pi}{2}$ (d) Two, at $\frac{\pi}{4}$ and $\frac{3\pi}{2}$ **[CS-GATE-2012]**

Q.50 Choose the CORRECT set of functions, which are linearly dependent.

(a) sin x, $\sin^2 x$ and $\cos^2 x$ (b) cosx, sinx and tan x

(c) cos2x, $\sin^2 x$ and $\cos^2 x$ (d) cos2x, sinx and cosx **[ME-GATE-2013]**

Q.51 The following surface integral is to be evaluated over a sphere for the given steady velocity vector field, F = xi + yj + zk defined with respect to a Cartesian coordinate system having i, j, and k as unit base vectors. $\iint_s \frac{1}{4}(F.n)dA$ Where S is the sphere, and n is the outward unit normal vector to the sphere. The value of the surface integral is

(a) π (b) $3\frac{\pi}{4}$

(c) 2π (d) 4π **[ME-GATE-2013]**

Q.52 If $u = \tan^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$ then $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$ equals

(a) $2\cos 2u$ (b) $\frac{1}{4}\sin 2u$

(c) $\frac{1}{4}\tan 4u$ (d) $2\tan 2u$

Q.53 $\int_0^{\pi/2} \log\cot x\,dx =$

(a) 0 (b) $\pi \ln 2$

(c) $-\pi \ln 2$ (d) ∞

Q.54 If $|4x-7|$ = 5 then the value of is $2|x|-|-x|$

(a) 2, $\frac{1}{3}$ (b) $\frac{1}{2}$,3

(c) $\frac{2}{3}$, $\frac{1}{3}$ (d) $\frac{2}{9}$,3 **[CE-GATE-2013]**

Q.55 The value of $\int_0^{\pi/6} \cos^4 3\theta \sin^3 6\theta \, d\theta$ is

(a) 0 (b) $\frac{1}{15}$

(c) 1 (d) $\frac{8}{3}$ **[CE-GATE-2013]**

Q.56 Which one of the following functions is continuous at x = 3?

(a) $f(x) = \begin{cases} 2, & \text{if } x = 3 \\ x-1, & \text{if } x > 3 \\ \frac{x+3}{3}, & \text{if } x < 3 \end{cases}$ (b) $f(x) = \begin{cases} 4, & \text{if } x = 3 \\ 8-x, & \text{if } x \neq 3 \end{cases}$

(c) $f(x) = \begin{cases} x+3, & \text{if } x \leq 3 \\ x-4, & \text{if } x > 3 \end{cases}$ (d) $f(x) = \frac{1}{x^3 - 27}, \text{ if } x \neq 3$ **[CS-GATE-2013]**

Q.57 Consider a vector field $\vec{A}(\vec{r})$. The closed loop line integral $\oint \vec{A} \bullet d\vec{l}$ can be expressed as

(a) $\oiint (\nabla \times \vec{A}) \bullet d\vec{s}$ over the closed surface bounded by the loop

(b) $\oiiint (\nabla \bullet \vec{A}) dv$ over the closed volume bounded by the loop

(c) $\iiint (\nabla \bullet \vec{A}) dv$ over the open volume bounded by the loop

(d) $\iint (\nabla \times \vec{A}) \bullet d\vec{s}$ over the closed surface bounded by the loop **[EC-GATE-2013]**

Q.58 The divergence of the vector field $\vec{A} = x\hat{a}_x + y\hat{a}_y + z\hat{a}_z$ is

(a) 0 (b) $\frac{1}{3}$

(c) 1 (d) 3 **[EC-GATE-2013]**

Q.59 If $f(n) = \int_0^{\pi/4} \tan^n x \, dx$ then $f(3) + f(1)$ equals

(a) 1 (b) $\frac{1}{2}$

(c) 2 (d) $\frac{3}{2}$

Q.60 Value of the integral $I = \int_0^{\pi/4} \cos^2 x\, dx$ is

(a) $\frac{\pi}{8}+\frac{1}{4}$ (b) $\frac{1}{8}-\frac{1}{4}$

(c) $-\frac{\pi}{8}-\frac{1}{4}$ (d) $\frac{\pi}{8}-\frac{1}{4}$

Q.61 A function $y = 5x^2 + 10x$ is defined over an open interval x = (1,2). Atleast at one point in this interval, dy/dx is exactly

(a) 20 (b) 25

(c) 30 (d)35 **[EE-GATE-2013]**

Q.62 For a vector E, which one of the following statements is NOT TRUE?

(a) If $\nabla.E = 0$, E is called solenoidal

(b) If $\nabla \times E = 0$, E is called conservative

(c) If $\nabla \times E = 0$, E is called irrotational

(d) If $\nabla.E = 0$, E is called irrotational **[IN-GATE-2013]**

Q.64 The integral $\oint_c (ydx - xdy)$ is evaluated along the circle $x^2 + y^2 = \frac{1}{4}$ traversed in counter clockwise direction. The integral is equal to

(a) 0 (b) $\frac{-\pi}{4}$

(c) $-\frac{\pi}{2}$ (d) $\frac{\pi}{4}$ **[ME-GATE-2014(1)]**

Q.65 $\lim_{x\to 0}\left(\frac{e^{2x}-1}{\sin(4x)}\right)$ is equal to

(a)0 (b)0.5

(c)1 (d)2 **[ME-GATE-2014(2)]**

Q.66 Curl of vector $\vec{F} = x^2z^2\hat{i} - 2xy^2z\hat{j} + 2y^2z^3\hat{k}$ is

(a) $(4yz^3 + 2xy^2)i + 2x^2z\hat{j} - 2y^2z\hat{k}$ (b) $(4yz^3 + 2xy^2)\hat{i} + 2x^2z\hat{j} - 2y^2z\hat{k}$

(c) $2xz^2\hat{i} - 4xyz\hat{j} + 6y^2z^2\hat{k}$ (d) $2xz^2\hat{i} + 4xyz\hat{j} + 6y^2z^2\hat{k}$ **[ME-GATE-2014(2)]**

Q.67 If a function is continuous at a point,

(a) The limit of the function may not exist at the point

(b) The function must be derivable at the point

(c) The limit of the function at the point tends to infinity

(d) The limit must exist at the point and the value of limit should be same as the value of the function at that point **[ME-GATE-2014(3)]**

Q.68 Divergence of the vector field $x^2z\hat{i}+xy\hat{j}-yz^2\hat{k}$ at $(1,-1,1)$ is

(a) 0 (b) 3

(c) 5 (d) 6 **[ME-GATE-2014(3)]**

Q.69 The value of the integral $\int_0^2 \frac{(x-1)^2\sin(x-1)}{(x-1)^2+\cos(x-1)}dx$ is

(a) 3 (b) 0

(c) -1 (d) -2 **[ME-GATE-2014(4)]**

Q.70 Evaluate : $\lim_{x\to 7}\left[\frac{x^6-7^6}{x-7}+\frac{x^2-7^2}{x-7}\right]$

(a) 327040 (b) 100856

(c) ∞ (d) None of these

Q.71 The value of integral $\int_0^2\int_0^x e^{x+y}dydx$ is

(a) $\frac{1}{2}(e-1)$ (b) $\frac{1}{2}(e^2-1)^2$

(c) $\frac{1}{2}(e^2-e)$ (d) $\frac{1}{2}\left(e-\frac{1}{e}\right)^2$ **[ME-GATE-2014(4)]**

Q.73 A particle moves along a curve whose parametric equations are: $x = t^3 + 2t$, $y = -3e^{-2t}$ and $z = 2\sin(5t)$, where x, y and z show variations of the distance covered by the particle (in cm) with time t (in s). The magnitude of the acceleration of the particle (in cm/s^2) at t = 0 is _______ **[CE-GATE-2014(1)]**

Q.74 Evaluate: $\lim_{x\to 0}\frac{e^{2x}-1}{\log(1+x)}$

(a) 0 (b) 2

(c) 4 (d) $\frac{1}{2}$

Q.75 The expression $\lim_{\alpha \to 0} \frac{X^{\alpha} - 1}{\alpha}$ is equal to

(a) log x (b) 0

(c) x log x (d) ∞ **[CE-GATE-2014(2)]**

Q.76 Let the function

$$f(\theta) = \begin{vmatrix} \sin\theta & \cos\theta & \tan\theta \\ \sin\left(\pi/6\right) & \cos\left(\pi/6\right) & \tan\left(\pi/6\right) \\ \sin\left(\pi/3\right) & \cos\left(\pi/3\right) & \tan\left(\pi/3\right) \end{vmatrix}$$ Where $\theta = \in \left[\frac{\pi}{6}, \frac{\pi}{2}\right]$ and $f'(\theta)$ denote the

derivative of f with respect to è. Which of the following statement is/are TRUE?

(I) There exists $\theta \epsilon \left(\frac{\pi}{6}, \frac{\pi}{3}\right)$ such that $f'(\theta) = 0$.

(II) There exists $\theta \epsilon \left(\frac{\pi}{6}, \frac{\pi}{3}\right)$ such that $f'(\theta) \neq 0$.

(a) I only (b) II only

(c) Both I and II (d)Neither Inor II

Q.77 Evaluate the series : $\lim_{n \to \infty} \frac{1+4+9+16+25+\ldots+n^2}{n^3}$

(a) $\frac{1}{3}$ (b) 1

(c) $\frac{1}{2}$ (d) $\left(\frac{1}{2}\right)^2$

Q.78 A function f(x) is continuous the interval [0, 2]. It is known that f(0) = f(2) = -1 and f(1) = 1. Which one of the following statements must be true?

(a) There exists a y in the interval (0,1) such that f(y) = f(y + 1)

(b) For every y in the interval (0, 1), f(y) = f(2 – y)

(c) The maximum value of the function in the interval (0.2) is 1

(d) There exists a y in the interval (0, 1) such that f(y) = f(2 – y) **[CS-GATE-2014(1)]**

Q.79 A non-zero polynomial f(x) of degree 3 has roots at x = 1,x = 2 and x = 3. Which one of the following must be TRUE?

(a) f(0) f(4) < 0 (b) f(0) f(4) > 0

(c) f(0) + f(4) > 0 (d) f(0) +f(4)< 0 **[CS-GATE-2014(2)]**

Q.80 If $\int_0^{2\pi} |x \sin x| dx = K\pi$, then the value of k is equal to ______. **[CS-GATE-2014(3)]**

Q.81 The value of the integral given below is $\int_0^{\pi} x^2 \cos x\, dx$

(a) -2π (b) π
(c) $-\pi$ (d) 2π **[CS-GATE-2014(3)]**

Q.82 The Taylor series expansion of 3sin x + 2 cos x is

(a) $2+3x-x^2-\frac{x^3}{2}+........$ (b) $2-3x+x^2-\frac{x^3}{2}+.....$

(c) $2+3x+x^2+\frac{x^3}{2}+...$ (d) $2-3x-x^2+\frac{x^3}{2}+......$ **[EC-GATE-2014(1)]**

Q.83 The volume under the surface z(x, y) = x + y and above the triangle in the x-y plane defined by $\{0 \le y \le x \text{ and } 0 \le x \le 12\}$ is_______. **[EC-GATE-2014(1)]**

Q.84 For $0 \le t < \infty$, the maximum value of the function f (t) = $e^{-t} - 2e^{-2t}$ occurs at

(a) t = $\log_e 4$ (b) t = $\log_e 2$
(c) t = 0 (d) t = $\log_e 8$ **[EC-GATE-2014(2)]**

Q.85 The value of $\lim_{x\to\infty}\left(1+\frac{1}{X}\right)^X$ is

(a) ln2 (b) 1.0
(c) e (d) ∞ **[EC-GATE-2014(2)]**

Q.86 The maximum value of the function f(x) = ln(1 + x) - x (where x > - 1) occurs at x = _____ **[EC-GATE-2014(3)]**

Q.87 The maximum value of $f(x)=2x^3-9x^2+12x-3$ in the interval $0 \le x \le 3$ is _______. **[EC-GATE-2014(3)]**

Q.88 The series $\sum_{n=0}^{\infty}\frac{1}{n!}$ converges to

(a) 2 ln2 (b) $\sqrt{2}$
(c) 2 (d) e **[EC-GATE-2014(4)]**

Q.89 For a right angled triangle, if the sum of the lengths of the hypotenuse and a side is kept constant, in order to have maximum area of the triangle, the angle between the hypotenuse and the side is

(a) 12° (b) 36°
(c) 60° (d) 45° **[EC-GATE-2014(4)]**

Q.90 The magnitude of the gradient for the function f (x,y,z) = $x^2 + 3y^2 + z^3$ at the point (1,1,1) is ________. **[EC-GATE-2014(4)]**

Q.91 Given. $\vec{F} = z\hat{a}_x + x\hat{a}_y + y\hat{a}_z$. If S represents the portion of the sphere $x^2 + y^2 + z^2 = 1$ for $z \geq 0$, then $\int_s \nabla \times \vec{F}.\vec{ds}$ is ———— **[EC-GATE-2014(4)]**

Q.92 If $r = x\hat{a}_x + y\hat{a}_y + z\hat{a}_z$ and $|\vec{r}| = r$, then $\text{div}\left(r^2 \nabla (\text{Inr})\right) =$ _____. **[EC-GATE-2014(2)]**

Q.93 Given the vector, $A = (\cos x)(\sin y)\hat{a}_y + (\sin x)(\cos y)\hat{a}_y$, where $\hat{a}_x, \hat{a}_y$ denote unit vectors along x, y directions, respectively. The magnitude of curl of A is ___. **[EC-GATE-2014(3)]**

Q.94 If z = xyln(xy), then

(a) $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 0$ (b) $y\frac{\partial z}{\partial x} = x\frac{\partial z}{\partial y}$

(c) $x\frac{\partial z}{\partial x} = y\frac{\partial z}{\partial y}$ (d) $y\frac{\partial z}{\partial x} + x\frac{\partial z}{\partial y} = 0$ **[EC-GATE-2014(3)]**

Q.95 Let $f(x) = xe^{-x}$ The maximum value of the function in the interval $(0, \infty)$ is

(a) e^{-1} (b) e

(c) $1 - e^{-1}$ (d) $1 + e^{-1}$ **[EE-GATE-2014(1)]**

Q.96 The line integral of function F = yzi, in the counterclockwise direction, along the circle $x^2 + y^2 = 1$ at z =1 is

(a) -2π (b) $-\pi$

(c) π (d) 2π **[EE-GATE-2014(1)]**

Q.97 Minimum of the real valued function $f(x) = (x-1)^{\frac{2}{3}}$ occurs at x equal to

(a) $-\infty$ (b) 0

(c) 1 (d) ∞ **[EE-GATE-2014(2)]**

Q.98 To evaluate the double integral $\int_0^8 \left(\int_{y/2}^{(y/2)+1} \left(\frac{2x-y}{2} \right) dx \right) dy$, we make the substitution $u = \left(\frac{2x-y}{2} \right)$ and $v = \frac{y}{2}$ The integral will reduce to

(a) $\int_0^4 \left(\int_0^2 2udu \right) dv$ (b) $\int_0^4 \left(\int_0^1 2udu \right) dv$

(c) $\int_0^4 \left(\int_0^1 udu \right) dv$ (d) $\int_0^4 \left(\int_0^2 udu \right) dv$ **[EE-GATE-2014(2)]**

Q.99 The minimum value of the function $f(x) = x^3 - 3x^2 - 24x + 100$ in the interval $[-3. 3]$ is

(a) 20 (b) 28

(c) 16 (d) 32 **[EE-GATE-2014(2)]**

Q.100 The value of $\int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x + \cos x} dx$ will lie in the range

(a) $\left(0, \frac{\pi}{2}\right)$ (b) $\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$

(c) $\left(-\frac{\pi}{2}, 0\right)$ (d) $\left(-\frac{\pi}{4}, -\frac{\pi}{2}\right)$

Q.101 The improper integral $I = \int_0^1 \frac{dx}{\sqrt{1-x^2}}$ converges to

(a) ∞ (b) 0

(c) $\frac{\pi}{2}$ (d) π

Q.102 The value of the limit $\lim_{n\to\infty} \frac{1}{n}\left(\sin\frac{\pi}{2n} + \sin\frac{2\pi}{2n} + \ldots + \sin\frac{n\pi}{2n} \right)$

(a) $\frac{2}{\pi}$ (b) 0

(c) 1 (d) None of these

Q.103 $\int_0^{\pi} \sin^5 x \cos^9 x \, dx$

(a) 0 (b) 1

(c) ∞ (d) None of these

Q.104 A vector is defined as $f = y\hat{i} + x\hat{j} + z\hat{k}$ where $\hat{i}\,\hat{j}$and $\hat{k}$ are unit vectors in Cartesian (x, y, z) coordinate system. The surface integral $\oiint f.ds$ over the closed surface S of a cube with vertices having the following coordinates: (0,0,0), (1,0,0), (1,1,0), (0,1,0), (0,0,1), (1,0,1), (1,1,1), (0,1,1) is________. **[IN-GATE-2014]**

Q.105 The value of $\lim_{x \to 0} \frac{1-\cos\left(x^2\right)}{2x^4}$ is

(a) 0 (b) $\frac{1}{2}$

(c) $\frac{1}{4}$ (d) undefined **[ME-GATE-2015(1)]**

Q.106 Consider a spatial curve in three-dimensional space given in parametric form by $x\left(t\right) = \cos t,\ y(t) = \sin,\ z(t) = \frac{2}{\pi}t,\ 0 \le t \le \frac{\pi}{2}$ The length of the curve is _______

[ME-GATE-2015(1)]

Q.107 Consider an ant crawling along the curve $\left(x-2\right)^2 + y^2 = 4,$ where x and y are in meters. The ant starts at the point (4, 0) and moves counter-clockwise with a speed of 1.57 meters per second. The time taken by the ant to reach the point (2, 2) is (in seconds) _______. **[ME-GATE-2015(1)]**

Q.108 Curl of vector V(x,y,z) $= 2x^2i + 3z^2j + y^3k$ at $x = y = z = 1$ is

(a) $-3i$ (b) 3i

(c)$3i - 4j$ (d)$3i - 6k$ **[ME-GATE-2015(2)]**

Q.109 The point of local minima of $y = |x|$ in the Interval $[-1,1]$ is_______.

(a) -1 (b) 1

(c) 0 (d) None of these

Q.110 The surface integral $\int\int_s \frac{1}{\pi}\left(9xi - 3yj\right).ds$ and S over the sphere giveb by $x^2 + y^2 + z^2 = 9$ is_______. **[ME-GATE-2015(2)]**

Q.111 Let ϕ be an arbitrary smooth real valued scalar function and V be an arbitrary smooth vector valued function in a three- dimensional space. Which one of the following is an identity?

(a) $\text{Curl}(\phi\overline{V}) = \nabla(\phi\,\text{Div}\overline{V})$ (b) $\text{Div}\overline{V} = 0$

(c) $\text{Div Curl}\overline{V} = 0$ (d) $\text{Div}(\phi\overline{V}) = \phi\text{Div}\overline{V}$ **[ME-GATE -2015(3)]**

Q.112 The value of $\lim_{x\to 0}\left(\frac{-\sin x}{2\sin x + x\cos x}\right)$ is__ **[ME-GATE -2015(3)]**

Q.113 The value of $\int_c \left[\left(3x-8y^2\right)dx+(4y-6xy)dy\right]$

(where C is boundary of the region bounded by x =0, y =0 and x+y=1) is ____________.

[ME-GATE -2015(3)]

Q.114 $\int_0^{\pi} \sin^3 x \cos^5 x\, dx$

(a) 0 (b) 1

(c) ∞ (d) None of these

Q.115 The directional derivative of the field u(x,y,z) = x^2 – 3yz in the direction of the vector $\left(\hat{i}+\hat{j}-2\hat{k}\right)$ at point (2,-1,4) is _____. **[CE-GATE-2015(1)]**

Q.116 is equal to $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{2x}$

(a) e^{-2} (b) e

(c) 1 (d) e^2 **[CE-GATE-2015(2)]**

Q.117 $\lim_{x\to\infty} x^{1/x}$ is

(a) ∞ (b) 0

(c) 1 (d) Not defined **[CS-GATE-2015(1)]**

Q.118 $\int_{1/\pi}^{2/\pi} \frac{\cos(1/x)}{x^2}dx =$ __________. **[CS-GATE-2015(1)]**

Q.119 Let $f(x) = x^{-(1/3)}$ and A denote the area of the region bounded by f(x) and the X-axis, when x varies from – 1 to 1. Which of the following statements is/are TRUE?

(I) f is continuous in [– 1,1] (II) f is not bounded in [– 1,1]

(III) A is nonzero and finite

(a) II only (b) III only

(c) II and III only (d) I, II and III **[CS-GATE-2015(2)]**

Q.120 Evaluate: $\lim_{x\to 0}\dfrac{5e}{e^2+e^{\frac{1}{x}}}$

(a) Doesn't exit (b) 0

(c) $\dfrac{5}{e}$ (d) None of these

Q.121 In for non-zero x, $af(x)+bf\left(\dfrac{1}{x}\right)=\dfrac{1}{x}-25$ where $a \neq b$ then $\int_1^2 f(x)dx$ is

(a) $\dfrac{1}{a^2-b^2}\left[a(\ln 2-25)+\dfrac{47b}{2}\right]$ (b) $\dfrac{1}{a^2-b^2}\left[a(2\ln 2-25)-\dfrac{47b}{2}\right]$

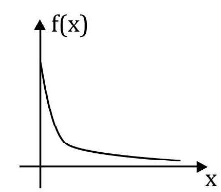
(c) $\dfrac{1}{a^2-b^2}\left[a(2\ln 2-25)+\dfrac{47b}{2}\right]$ (d) $\dfrac{1}{a^2-b^2}\left[a(\ln 2-25)+\dfrac{47b}{2}\right]$ **[CS-GATE-2015(3)]**

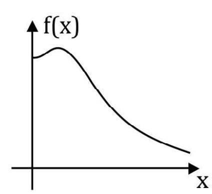
Q.122 A function $f(x)=1-x^2+x^3$ is defined in the closed interval $[-1, 1]$. The value of x, in the open interval $(-1, 1)$ for which the mean value theorem is satisfied, is

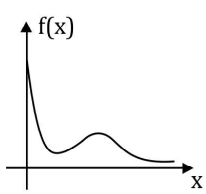
(a) $-1/2$ (b) $-1/3$

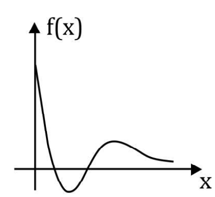
(c) $1/3$ (d) $1/2$ **[EC-GATE-2015(1)]**

Q.123 Which one of the following graphs describes the function f (x) = $e^{-x}\left(x^2+x+1\right)$?

(a) 

(b) 

(c) 

(d) 

[EC-GATE-2015(1)]

Q.124 The maximum area (in square units) of a rectangle whose vertices lie on the ellipse $x^2+4y^2=1$ is ____. **[EC-GATE-2015(1)]**

Q.125 The value of the integral $\int_{-\infty}^{\infty} 12\cos(2\pi t)\dfrac{\sin(4\pi t)}{4\pi t}dt$ is______. **[EC-GATE-2015(2)]**

Q.126 The contour on the x-y plane, where the partial derivative of $x^2 + y^2$ with respect to y is equal to the partial derivative of 6y+4x with respect to x, is

(a) y = 2 (b) x = 2

(c) x + y = 4 (d) x −y = 0 **[EC-GATE-2015(3)]**

Q.127 $f(x)=\begin{cases} \dfrac{\sin x}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$ The function f(x) has

(a) Discontinuity of first kind at x = 0

(b) Discontinuity of second kind at x = 0

(c) Removable discontinuity

(d) Continuous at x = 0

Q.128 The volume enclosed by the surface f(x, y) = e^x over the triangle bounded by the line x = y; x = 0; y = 1 in the xy plane is **[EE-GATE-2015(2)]**

Q.129 If $\vec{a},\vec{b},\vec{c},\vec{d}$ are four co planar vector then angle between $\vec{a}\times\vec{b}$ and $\vec{c}\times\vec{d}$ is

(a) 0 (b) 90

(c) 180 (d) Either 0 or 180

Q.130 Consider the function $f(x) = 2x^3 - 3x^2$ in the domain $[-1, 2]$. The global minimum of f(x) is ________. **[ME-GATE -2016(1)]**

Q.131 The values of x for which the function $f(x) = \dfrac{x^2 - 3x - 4}{x^2 + 3x - 4}$ is NOT continuous are

(a) 4 and -1 (b) 4 and 1

(c) -4 and 1 (d) -4 and -1

Q.132 The magnitude of directional derivative of $f = \log x + \log y + \log z$ at $\left(\dfrac{1}{2},\dfrac{1}{3},\dfrac{1}{4}\right)$ is

(a) $\sqrt{29}$ (b) 3

(c) 29 (d) $\sqrt{3}$

Q.133 Let $\vec{r} = yz\,\vec{i} + zx\,\vec{j} + xy\,\vec{k}$, then the value of divergence and curl of $\vec{r}$ are respectively

(a) 0, 3 (b) 3, 0

(c) 0, 0 (d) 3, 3

Q.134 $\lim_{x\to 0}\dfrac{\log_e(1+4x)}{e^{3x}-1}$ is equal to

(a) 0 (b) $\dfrac{1}{12}$

(c) $\dfrac{4}{3}$ (d) 1 **[ME-GATE -2016(3)]**

Q.135 If $x = r\cos\theta, y = r\sin\theta$ then $\dfrac{\partial^2\theta}{\partial x\partial y} = yz\vec{i} + zx\vec{j} + xy\vec{k}$ is

(a) $\dfrac{1}{r^2}\cos\theta$ (b) $\dfrac{1}{r^2}\sin\theta$

(c) $-\dfrac{1}{r^2}\sin 2\theta$ (d) $-\dfrac{1}{r^2}\cos 2\theta$

Q.136 The value of the line integral $\oint_c \overline{F}\overline{r}.ds$, where C is a circle of radius $\dfrac{4}{\sqrt{\pi}}$ units is ____________.Here, $\overline{F}(x,y) = y\hat{i} + 2x\hat{j}$ and $\overline{r}\,'$ is the UNIT tangent vector on the curve C at an arc length s From a reference point on the curve. $\hat{i}$ and $\hat{j}$ are the basis vectors in the x-y Cartesian reference. In evaluating the line integral, the curve has to be traversed in the counter-clockwise direction **[ME-GATE -2016(3)]**

Q.137 $\lim_{x\to\infty}\sqrt{x^2+x-1}-x$ is

(a) 0 (b) ∞

(c) $\dfrac{1}{2}$ (d) $-\infty$ **[ME-GATE -2016(3)]**

Q.138 The optimum value of the function $f(x) = x^2 - 4x + 2$ is

(a) 2 (maximum) (b) 2(minimum)

(c) -2(maximum) (d)-2(minimum) **[CE-GATE-2016(1)]**

Q.139 The area between the parabola $x^2 = 8y$ and the straight line y = 8 is ________.

[CE-GATE-2016(2)]

Q.140 The quadratic approximation of f (x) = $x^3 - 3x^2 - 5$ at the point x = 0 is

(a) $3x^2 - 6x - 5$ (b) $-3x^2 - 5$

(c) $-3x^2 + 6x - 5$ (d) $3x^2 - 5$ **[CE-GATE-2016(2)]**

Q.141 The maximum value of $\sqrt{x^2+y^2}$ when $x^2+xy+y^2=1$ is

(a) $\frac{\sqrt{6}}{3}$ (b) $\sqrt{2}$

(c) $\frac{\sqrt{3}}{2}$ (d) 1

Q.142 The area of the region bounded by the parabola $y=x^2+1$ and the straight line x + y = 3 is

(a) $\frac{59}{6}$ (b) $\frac{9}{2}$

(c) $\frac{10}{3}$ (d) $\frac{7}{6}$ **[CE-GATE-2016(1)]**

Q.143 $\lim_{x\to 4}\frac{\sin(x-4)}{x-4} =$ ________________. **[CS-GATE-2016(1)]**

Q.144 Let f (x) be a polynomial and g(x) = f '(x) be its derivative. If the degree of (f (x) + f (−x)) is 10, then the degree of (g(x) − g(−x)) is ________. **[CS-GATE-2016(1)]**

Q.145 Given the following statements about a function $f: R \to R$, select the right option.

P : If f(x) is continuous at $x = x_o$, then it is also differentiable at $x = x_o$.

Q : If f(x) is continuous at $x = x_0$, then it may not be differentiable at $x = x_o$.

R : If f(x) is differentiable at $x = x_o$, then it is also continuous at $x = x_o$.

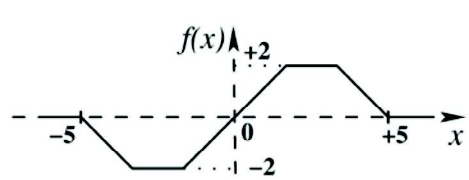
(a) P is true, Q is false, R is false (b) P is false, Q is true, R is true

(c) P is false, Q is true, R is false (d) P is true, Q is fal se, R is true **[EC-GATE-2016(1)]**

Q.146 The integral $\frac{1}{2\pi}\iint_D (x+y+10)dx\,dy$, where D denotes the disc : $x^2+y^2 \le 4$, evaluates to ________. **[EC-GATE-2016(1)]**

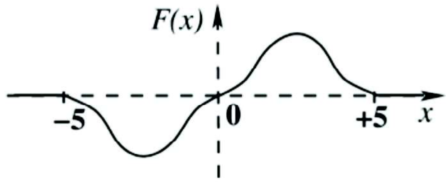
Q.147 The region specified by $\left\{(\rho,\varphi,z): 3 \le \rho \le 5, \frac{\pi}{8} \le \varphi \le \frac{\pi}{4}, 3 \le z \le 4.5\right\}$ in cylindrical coordinates has volume of ________. **[EC-GATE-2016(1)]**

Q.148 Consider the plot of f(x) versus x as shown below.

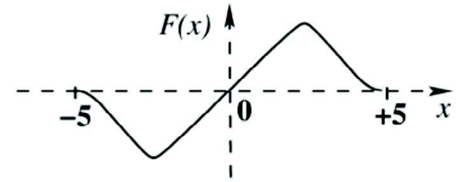


Suppose $F(x)=\int_{-5}^{x} f(y)dy$. Which one of the following is a graph of F(x)?

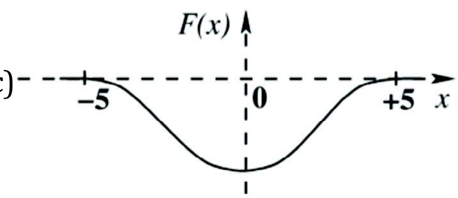
(a)



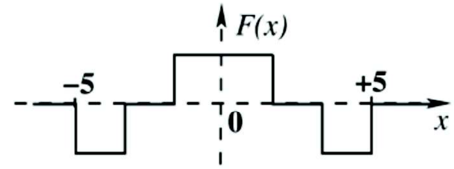
(b)



(c)



(d)



Q.149 Suppose C is the closed curve defined as the circle $x^2 + y^2 = 1$ with C oriented anti-clockwise. The value of $\oint \left(xy^2dx + x^2ydy\right)$ over the curve C equals _________. **[EC-GATE-2016(2)]**

Q.150 As x varies from -1 to +3, which one of the following describes the behavior of the function $f(x) = x^3 - 3x^2 + 1$?

(a) f(x) increase monotonically

(b) f(x) increases, then decreases and increases again

(c) f(x) decreases, then increases and decreases again

(d) f(x) increases and then decreases **[EC-GATE-2016(2)]**

Q.151 How many distinct values of x satisfy the equation sin(x) = x/2, where x is in radians?

(a) 1 (b) 2

(c) 3 (d) 4 or more **[EC-GATE-2016(2)]**

Q.152 Consider the time-varying vector $I = 15\cos(\omega t)\hat{x} + 5\sin(\omega t)\hat{y}$ in Cartesian coordinates, where $\omega > 0$ is a constant. When the vector magnitude $|I|$ is at its minimum value, the angle θ that I makes with the x axis (in degrees, such that $0 \le \theta \le 180$) is _________.

[EC-GATE-2016(2)]

Q.153 The integral $\int_0^1 \frac{dx}{\sqrt{(1-x)}}$ is equal to_________. **[EC-GATE-2016(3)]**

Q.154 Let $f(X,Y)=2x^2+4xy+3y^2$

Then the expression $x^2\frac{\partial^2 f}{\partial x^2}+2xy\frac{\partial^2 f}{\partial x\partial y}+y^2\frac{\partial^2 f}{\partial y^2}$ corresponds to

(a) 2f (b) 4x + 4y

(c) 4x + 6y (d) $\frac{1}{2}f$

Q.155 A triangle in the xy-plane is bounded by the straight lines 2x = 3y, y = 0 and x = 3. The volume above the triangle and under the plane x+y + z = 6 is ____. **[EC-GATE-2016(3)]**

Q.156 Consider the following assertion-reasoning statements :

A: $f(x)=|x|$ has no extremum at x = 0

R: f'(0) does exit i.e. the derivative doesn't exist at x = 0

Which can you conclude from the above statements?

(a) A is true and R is correct explanation of A

(b) A and R both are true but R is not the correct explanation of A

(c) A is true , B is false

(d) A is false, B is true

Q.157 In which of the following functions mean Value theorem is not applicable?

(1) $y=\frac{1}{x}, x\in[-1,1]$ (2) $y=|x|, x\in[-1,1]$

(3) $y = x\sin\frac{1}{x}, x\in\left[+\frac{\pi}{4},\frac{\pi}{2}\right]$ (4) $y=\sin x, x\in\left[0,\frac{\pi}{2}\right]$

(a) (1),(2),(3) (b) (1),(2)

(c) (1),(3) (d) (1),(2),(3),(4)

Q.158 Let $f:[-1,1]\to R$, where f(x) = $2x^3 - x^4 - 10$. The minimum value of f(x) is________.

[IN-GATE-2016]

Q.159 The value of $\lim_{x\to 0}\frac{x^2-\sin(x)}{x}$ is

(a) 0 (b) 3

(c) 1 (d) -1 **[GATE -2017]**

Q.160 A parametric curve defined by $x=\cos\left(\frac{\pi u}{2}\right), y=\sin\left(\frac{\pi u}{2}\right)$ in the rage $0 \le u \le 1$ is rotated about the X- axis by 360 degrees. Area of the surface generated is

(a) $\frac{\pi}{2}$ (b) π

(c) 2π (d) 4π **[GATE -2017]**

Q.161 For the vector $\vec{V}=2yz\hat{i}+3xz\hat{j}+4xy\hat{k}$ the value of $\nabla.(\nabla\times\vec{V})$ is _____. **[GATE -2017]**

Q.162 The improper integral, $I=\int_0^1 \frac{dx}{\sqrt{1-x^2}}$ converges to,

(a) ∞ (b) 0

(c) $\frac{\pi}{2}$ (d) π

Q.163 The surface integral $\iint_s F.ndS$ over the surface S of the sphere $x+y+z=9$ where

$F=(x+y)\hat{i}+(x+z)\hat{j}+(y+z)\hat{k}$ and n is the unit outward surface normal yields ____

[GATE-2017]

Q.164 At the point x = 0, the function f(x) =x has

(A) Local maximum

(B) Local minimum

(C) Both local maximum and minimum

(D) Neither local maximum nor local minimum **[CE-GATE-2018(1)]**

Q.165 The value of the integral $\int_0^\pi x\cos^2 x\, dx$ is **[CE-GATE-2018(1)]**

Q.166 The value (up to two decimal places) of a line integral $\int \vec{F}(\vec{r}).\, d\vec{r}$, for $\vec{F}(\vec{r})=x^2\hat{i}+y^2\hat{j}$ along C which is a straight line joining (0,0) to (1,1) is _____ **[CE-GATE-2018(2)]**

Q.167 Let f be a real valued function of a real variable defined as f (x) =x^2 for x ≥ 0 , and f (x) = $-$ x^2 for x < 0 . Which of the following statement is true?

(A) f (x) is discontinuous at x = 0

(B) f (x) is continuous but not differentiable at x =0

(C) f (x) is differentiable but its first derivative is not continuous at x = 0

(D) f (x) is differentiable but its first derivative is not differentiable at x =0 .

[EE-GATE-2018]

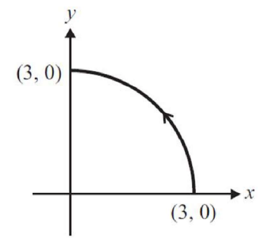
Q.168 The value of the directional derivative of the function $\phi(x,y,z)=xy^2+yz^2+zx^2$ at the point (2, – 1, 1) in the direction of the vector p = i +2 j+2k is

(A)1 (B) 0.95

(C) 0.93 (D) 0.9 **[EE-GATE-2018]**

Q.169 Let $f(x)=3x^3-7x^2+5x+6$.The maximum value of f (x) over the interval [0, 2] is ______ (upto one decimal place) **[EE-GATE-2018]**

Q.170 As shown in the figure, C is the arc from the point (3, 0) to the point (0, 3) on the circle $x^2+y^2=9$, The value of the integral $\int_c \left(y^2+2xy\right)dx+\left(x^2+2xy\right)dy$ is ________(up to 2 decimal places).



[EE-GATE-2018]

Q.171 Taylor series expansion of $\mathrm{f}(x)=\int_0^x e^{-t^2/2}\,e\,\mathrm{dt}$ around $x = 0$ has the form $f(x) = a_0 + a_1 x + a_2 x^2 + \ldots$ The coefficient a_2(correct to two decimal places) is equal to _______. **[EC-GATE-2018]**

Q.172 Given $\vec{\mathrm{F}}=\left(x^2-2y\right)\vec{i}-4yz\,\vec{j}+4xz^2\,\vec{k}$, the value of the line integral $\int_c \vec{\mathrm{F}}.\vec{\mathrm{dl}}$ along the straight line c from (0,0,0) to (1,1,1) is **[IN-GATE-2018]**

Q.173 Consider two functions $f(x) = (x-2)^2$ and $g(x) = 2x-1$, where x is real. The smallest value of x for which f (x) = g(x) is ______. **[IN-GATE-2018]**

Q.174 According to the Mean Value Theorem, for a continuous function f (x) in the interval [a, b] there exists a value ξ in this interval such that $\int_a^b f(\mathrm{x})\,\mathrm{dx} =$

(A) $f(\xi)(b-a)$ (B) $f(b)(\xi-a)$

(C) $f(a)(b-\xi)$ (D) 0 **[ME-GATE-2018(1)]**

Q.175 The value of the integral $\oiint_s \vec{r}\,.\,\vec{n}\,ds$ over the closed surface S bounding a volume V, where $\vec{r}=x\vec{i}+y\vec{j}+z\vec{k}$ is the position vector and $\vec{n}$ is the normal to the surface S, is

(A) V (B) 2V

(C) 3V (D) 4V **[ME-GATE-2018(1)]**

Q.176 The divergence of the vector field $\vec{u}=e^x(\cos y\ \hat{i}+\sin y\ \hat{j})$ is

(A) 0 (B) $e^x\cos y+e^x\sin y$

(C) $2e^x\cos y$ (D) $2e^x\sin y$ **[ME-GATE-2018(2)]**

Q.177 For a position vector $\vec{r} = x\,\vec{i} + y\,\vec{j} + z\,\vec{k}$, the norm of the vector can be defined as $\left|\vec{r}\right| = \sqrt{x^2 + y^2 + z^2}$. Given a function $\phi = \ln\left|\vec{r}\right|$, its gradient $\nabla\phi$ is

(A) $\vec{r}$

(B) $\dfrac{\vec{r}}{\left|\vec{r}\right|}$

(C) $\dfrac{\vec{r}}{\vec{r}.\vec{r}}$

(D) $\dfrac{\vec{r}}{\left|\vec{r}\right|^3}$

[ME-GATE-2018(2)]

Answer Key :

1	2	3	4	5	6	7	8	9	10	11	12	13	14
(a)	(b)	(c)	(a)	(c)	(d)	(a)	(a)	(c)	(c)	(d)	(b)	(d)	(d)
15	16	17	18	19	20	21	22	23	24	25	26	27	28
(b)	(d)	(d)	(b)	(a)	(d)	(b)	(a)	(b)	(b)	(b)	(a)	(a)	(b)
29	30	31	32	33	34	35	36	37	38	39	40	41	42
(d)	(a)	(a)	(b)	(a)	(b)	(c)	(c)	(a)	(a)	(a)	(d)	(d)	(a)
43	44	45	46	47	48	49	50	51	52	53	54	55	56
(a)	(c)	(b)	(a)	(a)	0.84	(d)	(c)	(a)	(b)	(a)	(b)	(b)	(a)
57	58	59	60	61	62	63	64	65	66	67	68	69	70
(d)	(d)	(b)	(a)	(b)	(d)	(a)	(c)	(b)	(a)	(d)	(c)	(b)	(b)
71	72	73	74	75	76	77	78	79	80	81	82	83	84
(b)	(c)	12	(a)	(a)	(c)	-2	(a)	(a)	4	(a)	(a)	864	(a)
85	86	87	88	89	90	91	92	93	94	95	96	97	98
(c)	0	6	(d)	(c)	7	3.14	3	0	(c)	(a)	(b)	(c)	(b)
99	100	101	102	103	104	105	106	107	108	109	110	111	112
(b)	(a)	(c)	(a)	(a)	1	(c)	1.86	2	(a)	(d)	216	(c)	0.333
113	114	115	116	117	118	119	120	121	122	123	124	125	126
1.66	(a)	-5.72	(d)	(c)	-1	(c)	(a)	(a)	(b)	(a)	1	3	(a)
127	128	129	130	131	132	133	134	135	136	137	138	139	140
(c)	0.72	(d)	(-5)	(c)	(a)	(c)	(c)	(d)	16	(c)	(d)	85.3	(b)
141	142	143	144	145	146	147	148	149	150	151	152	153	154
(b)	(b)	1	9	(b)	20	4.71	(c)	0	(b)	(c)	90	2	(a)
155	156	157	158	159	160	161	162	163	164	165	166	167	168
10	(d)	(b)	-13	(d)	(c)	0	(c)	226.19	(c)	(b)	(0.67)	(d)	(a)
169	170	171	172	173	174	175	176	177					
(12)	(0)	(0)	(d)	(1)	(a)	(c)	(c)	(c)					

ESE QUESTIONS

Q.1 What is the maximum value of z, if z = 10x + 6y subject to the constraints $3x+y \le 12,\ 2x+5y \le 34, x \ge 0, y \ge 0$?

(a) 56 (b) 52

(c) 50 (d) 40

ANSWER KEY :

1. (a)

GATE SOLUTIONS

Q.1 **(a)** $\lim_{x\to 0}\frac{\sin^2 x}{x}=\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^2 .x=1\times 0=0$

Q.2 **(b)** $\lim_{x\to 0}\left[\frac{x^3+x^2}{2x^3-7x^2}\right],\frac{0}{0}$ form

Using L'Hospital rule

$$\lim_{x\to 0}\left[\frac{3x^2+2x}{6x^2-14x}\right]=\lim_{x\to 0}\left[\frac{6x+2}{12x-14}\right]=-\frac{1}{7}$$

Q.3 **(c)** $x=a(\theta+\sin\theta), y=a(1-\cos\theta), \frac{dx}{d\theta}=a(1+\cos\theta), \frac{dy}{d\theta}=a\sin\theta$

$$\frac{dy}{dx}=\frac{dy/d\theta}{dx/d\theta}=\frac{a\sin\theta}{a(1+\cos\theta)}=\frac{2a\sin\left(\frac{\theta}{2}\right).\cos\left(\frac{\theta}{2}\right)}{a\times 2\cos^2\left(\frac{\theta}{2}\right)}=\tan\left(\frac{\theta}{2}\right)$$

Q.4 **(a)** $f'(x)=6x^2-6x-36=0$

$\Rightarrow x^2-x-6=0 \quad \Rightarrow x=3,-2$

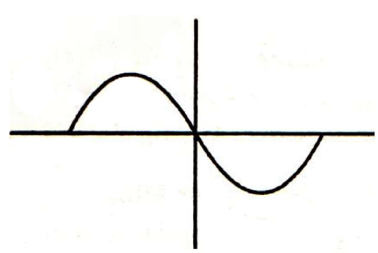
Now $f''(x)=12x-6$

$f''(3)=30>0$ (minima)

$f''(-2)=-30<0$ (maxima)

Hence maxima is at $x=-2$ only

Q.5 **(c)** Given function has negative slope in +ve half & positive slope in –ve half. So its differentiation curve is satisfied by (c)



Q.6 **(d)** Since the position of rail engine A(t) is continuous and differentiable function, according to Lagrange's mean value theorem

$$A'(t) = v(t) = \frac{S(8)-S(0)}{8-0} = \frac{(280-0)}{(8-0)} m/sec = \frac{280}{8}\frac{m}{sec} = \frac{280}{8}\times\frac{360}{1000} kmph = 126 kmph$$

v(t) is the velocity of the rail engine.

Q.7 **(a)** $f'(x) = \frac{e^x(1+e^x)-e^{2x}}{(1+e^x)^2} = \frac{e^x}{(1+e^x)^2}$ Since e^x is +ve for all values of x, $f'(x)$ is +ve for all values of x and hence f(x) is monotonically increasing function.

Q.8 **(a)** Let the point of contact is $(2t^2, 4t)$ to $y^2 = 8x$

$$2y\frac{dy}{dx} = 8,\ \frac{dy}{dx} = \frac{4}{y} = \frac{4}{4t} = \frac{1}{t}$$

Equation of line through $(0,-2)$ is $(y-0) = \frac{1}{t}(x+2)$ for points of intersection

$$4t = \frac{1}{t}(2t^2+2)$$
$$4t^2 = 2t^2+2$$
$$t = \pm 1$$

Since the point is in first quadrant

The co-ordinates of point $= (2t^2, 4t)_{(t=1)} = (2,4)$

Q.9 **(c)** $I = \int_0^{\pi}\sin^3\theta\, d\theta = \int_0^{\pi}(1-\cos^2\theta)\sin\theta\, d\theta = \int_{-1}^{+1}(1-t^2)dt = \left| t - \frac{t^3}{3}\right|_{-1}^{1} = \frac{4}{3}$

Q.10 **(c)** $S = \int_1^{\infty} x^{-3}dx = \left[\frac{x^{-2}}{-2}\right]_1^{\infty} = -\left[\frac{1}{2x^2}\right]_1^{\infty} = -\left[\frac{1}{\infty} - \frac{1}{2}\right] = \frac{1}{2}$

Q.11 **(d)** Since, $\int_0^a f(x)dx = \int_0^a f(a-x)dx$

$\therefore$ x = 5 $\Rightarrow$ y = 25

$$I = \int_0^{\pi/4}\frac{(1-\tan x)}{(1+\tan x)}dx = \int_0^{\pi/4}\tan\left(\frac{\pi}{4}-x\right)dx = -\ln\sec\left(\frac{\pi}{4}-x\right)\Big|_0^{\frac{\pi}{4}} = \frac{1}{2}\ln 2$$

Q.12 **(b)** First Method – Use Laplace Transform

Second Method : Let $I(\alpha)=\int_0^\infty e^{-\alpha x}\frac{\sin x}{x}dx,\ \frac{\partial I}{\partial \alpha}=\int_0^\infty (-x)e^{-\alpha x}\left(\frac{\sin x}{x}\right)dx=\int_0^\infty e^{-\alpha x}\sin x dx$

$$\frac{\partial I}{\partial \alpha}=-\left[\frac{e^{-\alpha x}}{1+\alpha^2}(-\alpha\sin x-1.\cos x)\right]_0^\infty=\frac{1}{1+\alpha^2}\ \therefore\ \frac{\partial I}{\partial \alpha}=\frac{1}{1+\alpha^2},\quad \partial I=\frac{\partial \alpha}{1+\alpha^2}$$

Integrating, $I=-\tan^{-1}\alpha+C$, $I(\infty)=-\frac{\pi}{2}+C$, from (i) $I(\infty)=0$

$$\therefore\ -\frac{\pi}{2}+C=0\ \Rightarrow C=\frac{\pi}{2}\quad \therefore I(\alpha)=-\tan^{-1}\alpha+\frac{\pi}{2}$$

$$\Rightarrow I(0)=\frac{\pi}{2}-\tan^{-1}0\qquad \therefore\ I(0)=\frac{\pi}{2}$$

But from (i),

$$I(0)=\int_0^\infty \frac{\sin x}{x}dx\qquad \therefore\ \int_0^\infty \frac{\sin x}{x}dx=\frac{\pi}{2}$$

Q.13 **(d)** $\int_0^{\frac{\pi}{4}}\tan x\, dx=\log\sqrt{2}$, $\int_0^\infty \frac{dx}{x^2+1}=\frac{\pi}{2}$

$$\int_0^\infty xe^{-x}dx=1\ \int_0^\infty \frac{1}{1-x}dx=-\ln|1-x\|_0^\infty=-\infty+0=-\infty$$

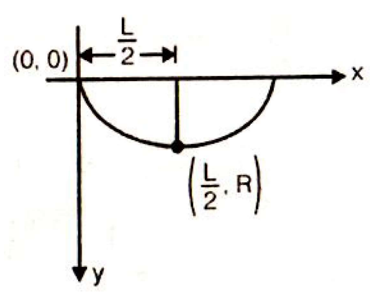
Since, only (d) is unbounded

Q.14 **(d)** $\int_{-\infty}^\infty \frac{dx}{1+x^2}=\left[\tan^{-1}x\right]_{-\infty}^\infty=\tan^{-1}(\infty)-\tan^{-1}(-\infty)=\frac{\pi}{2}-\left[\frac{-\pi}{2}\right]=\pi$

Q.15 **(b)** $\int_0^1 x\, e^x dx=\left[x\, e^x-e^x\right]_0^1=(e-e)-(0-e^0)=1$

Q.16 **(d)** Length of curve $y=f(x)$ between $x=a$ and $x=b$ is given by

$$\int_a^b\sqrt{1+\left(\frac{dy}{dx}\right)^2}dx$$



Here, $y = 4h\dfrac{x^2}{L^2}$ ...(i)

$$\frac{dy}{dx} = 8h\frac{x}{L^2}$$

Since, y = 0 at x = 0 and $y = h$ at $x = \dfrac{L}{2}$

(As can be seen from equation (i), by substituting x=0and $x = L/2$)

$$\frac{1}{2}(\text{Length of cable}) = \int_0^{L/2}\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \int_0^{L/2}\sqrt{1+\left(\frac{8hx}{L^2}\right)^2}\,dx$$

$$\text{Length of cable} = 2\int_0^{L/2}\sqrt{1+64\frac{h^2x^2}{L^4}}\,dx$$

Q.17 **(d)**

$y = \dfrac{2}{3}x^{3/2}$, $\dfrac{dy}{dx} = x^{1/2}$

Length of the curve is given by

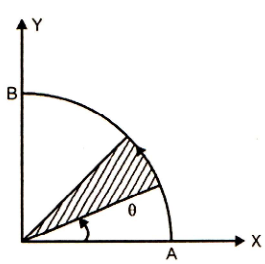
$$\int_0^1\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \int_0^1\sqrt{1+x}\,dx = \left[\frac{2}{3}(1+x)^{3/2}\right]_{x=0}^{x=1} = 1.22$$

Q.18 **(b)** Path AB : $x^2 + y^2 = 1$, $x = \cos\theta$, $y = \sin\theta$

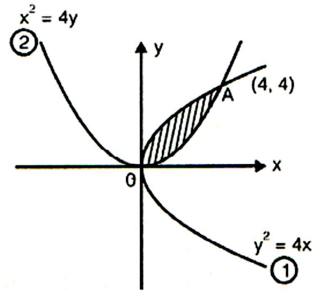
Along path AB, θ varies from 0 to $\dfrac{\pi}{2}$

$$\int_{\text{Path AB}} (x+y)^2(rd\theta) = \int_0^{\pi/2}(\cos\theta+\sin\theta)^2\,d\theta = \int_0^{\pi/2}\left(\sin^2\theta+\cos^2\theta+2\sin\theta\cos\theta\right)d\theta$$

$$= \int_0^{\pi/2}(1+\sin 2\theta)\,d\theta = \theta + \left.\frac{(-\cos 2\theta)}{2}\right|_0^{\pi/2} = \frac{\pi}{2}+1$$



Q.19 **(a)**



Curve 1: $y^2 = 4x$, Curve 2: $x^2 = 4y$

Intersection points of curve 1 and 2 are $A(4,4)$ and $0(0,0)$

The area enclosed between curves 1 and 2 are given by

$$\text{Area} = \int_{x_1}^{x_2} y_1 dx - \int_{x_1}^{x_2} y_2 dx = \int_0^4 2x^{\frac{1}{2}} dx - \int_0^4 \frac{x^2}{4} dx = \frac{16}{3}$$

Method 2 : Required Area $= \int_0^4 \int_{\frac{x^2}{4}}^{2\sqrt{x}} dx\, dy = \int_0^4 \left(2\sqrt{x} - \frac{x^2}{4}\right) dx = \frac{16}{3}$

Q.20 **(d)** The volume of a solid generated by revolution about x axis, of the area bounded by curve y = f(x) the x-axis and the ordinates x = a, y = b is

$$\text{Volume} = \pi \int_a^b y^2 dx = \pi \int_1^2 x\, dx = \pi \left[\frac{x^2}{2}\right]_1^2 = \frac{3\pi}{2}$$

Q.21 **(b)** Equation of line with slope 1 and y – Intercept of 1 is Y = x+1

$$I = \int_1^2 y dx = \int_1^2 (x+1) dx = \left.\frac{(x+1)^2}{2}\right|_1^2 = \frac{1}{2}(9-4) = 2.5$$

Q.22 **(a)** Equation of straight line from point (0, 0) to (1, 2) is

$$y - 0 = \frac{(2-0)}{(1-0)}(x-0), \ y = 2x$$

$$g(x,y) = 4x^3 + 10y^4 = 4x^3 + 10(2x)^4 = 4x^3 + 160x^4$$

$$\int_0^1 \left(4x^3 + 160x^4\right) dx = \left.\left(\frac{4x^4}{4} + \frac{160x^5}{5}\right)\right|_0^1 = 1 + 32 = 33$$

Q.23 **(b)** $y = x^2 \Rightarrow \frac{dy}{dx} = 2x = 0 \text{ at } x = 0$

$$\Rightarrow \left[\frac{d^2y}{dx^2}\right]_{x=0} = 2 \text{ which is } +\text{ve}$$

So we have a local minima at x = 0

At x = 0, y = 0

But since x = 0 does not belong to [1, 5]

It is not a candidate for minima or maxima in that range

At the end point $x = 1 \Rightarrow y = 1$

At second end point $x = 5 \Rightarrow y = 25$

So, absolute minimum value of function in [1, 5] is 1.

Q.24 **(b)** $\lim_{x\to 0}\frac{e^x - \left(1 + x + \frac{x^2}{2}\right)}{x^3}$ Using L'Hospitial rule.

$$\lim_{x\to 0}\frac{e^x - \left(1 + x + \frac{x^2}{2}\right)}{x^3} = \lim_{x\to 0}\frac{e^x - 1}{6x} = \lim_{x\to 0}\frac{e^x}{6x} = \frac{1}{6}$$

Q.25 **(b)** Using L'Hospitial rule $\lim_{x\to 8}\frac{x^{1/3} - 2}{(x-8)} = \lim_{x\to 8}\frac{1}{3}\frac{\left(x^{-2/3} - 0\right)}{(1-0)} = \frac{1}{3}(8)^{-2/3} = \frac{1}{12}$

Q.26 **(a)** $\lim_{\theta\to 0}\frac{\frac{1}{2}\sin\left(\frac{\theta}{2}\right)}{\left(\theta\times\frac{1}{2}\right)} = \frac{1}{2}\lim_{\theta\to 0}\frac{\sin\left(\frac{\theta}{2}\right)}{\left(\frac{\theta}{2}\right)} = \frac{1}{2}$

Q.27 **(a)** $\lim_{x\to\infty}\frac{x - \sin x}{x + \cos x} = \lim_{x\to\infty}\frac{1 - \frac{\sin x}{x}}{1 + \frac{\cos x}{x}} = \frac{1-0}{1+0} = 1$

.28 **(b)** $f(x) = \left(x^2 - 4\right)^2$

$$f'(x) = 2\left(x^2 - 4\right)(2x) = 4x\left(x^2 - 4\right) = 0$$

x = 0 and x = 2 and x = -2 are the stationary points.

$$f''(x) = 4\left[x(2x) + \left(x^2 - 4\right)\times 1\right] = 12x^2 - 16$$

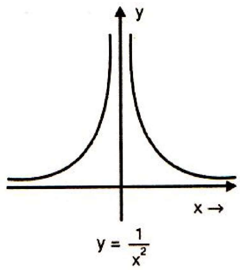
$f''(0) = -16 < 0$, (so maxima at $x = 0$)

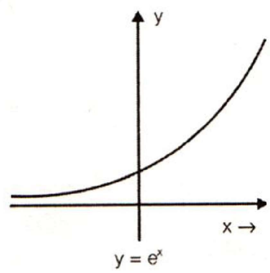
$f''(2) = 32 > 0$, (so minima at $x = 2$)

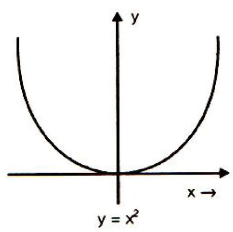
$f''(-2) = 32 > 0$, (so minima at $X = -2$)

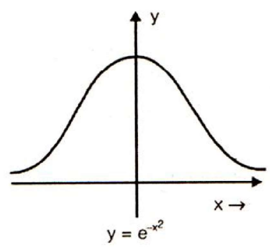
$\therefore$ there are one maxima and two minima for this function.

Q.29 (d) From the graphs below, we can see that only e^{-x^2} is strictly bounded.

(a)


(b)


(c)


(d)


Q.30 **(a)** $f'(x) = 2x - 1 = 0 \Rightarrow x = \frac{1}{2}, f''(x) = 2, f''\left(\frac{1}{2}\right) = 2 > 0$

So, at $x = \frac{1}{2}$ we have a local minima so this is not a candidate for maxima in range $[-4,4]$.

Now f (-4) =10, f(+4) =18,

So maximum value in range $[-4, 4]$ is 18

Q.31 **(a)** $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots,$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$$

From this

$$\sin x^2 = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \ldots$$

$$\cos x^2 = 1 - \frac{x^4}{2!} + \frac{x^8}{4!} - \frac{x^{12}}{6!} + \ldots$$

So, $\sin x^2$ and $\cos x^2$ have only even powers of x

Similarly, $\sin x^3 = x^3 - \frac{x^9}{3!} + \frac{x^{15}}{5!} + \ldots, \ \cos x^3 = 1 - \frac{x^6}{2!} + \frac{x^{12}}{4!} \ldots..$

So only $\sin(x^3)$ has all odd powers of x.

$\therefore$ Correct choice is (a)

Q.32 **(b)** $f'(x) = e^x + \cos x$

$f''(x) = e^x - \sin x$

$f''(\pi) = e^\pi - \sin\pi = e^\pi - 0 = e^\pi$

The coefficient of $(x-\pi)^2 = \dfrac{f''(\pi)}{2!} = \dfrac{e^\pi}{2!} = 0.5\exp(\pi)$

Q.33 **(a)** Using L-Hospital

$$\lim_{x\to 0}\frac{\sin\left[\frac{2}{3}x\right]}{x} = \lim_{x\to 0}\frac{\frac{2}{3}\cos\left[\frac{2}{3}x\right]}{1} = \frac{2}{3}$$

Q.34 **(b)** $\lim\limits_{n\to\infty}\left(1-\dfrac{1}{n}\right)^{2n} = \lim\limits_{n\to\infty}\left[\left(1-\dfrac{1}{n}\right)^{n}\right]^2 = \left[\lim\limits_{n\to\infty}\left(1-\dfrac{1}{n}\right)^{n}\right]^2 = \left(e^{-1}\right)^2 = e^{-2}$

Q.35 **(c)** $y = |x-a|$ is continuous at x=a but not differentiable at x = a

Q.36 **(c)** Using Taylor series

$f(x) = \sum\limits_{n=0}^{\infty} a_n (x-a)^n$, Where $a_n = \dfrac{f^n(a)}{n!}$

Coefficient of $(x-2)^4 = a_4 = \dfrac{f^4(2)}{4!} = \dfrac{e^2}{4!}$

Q.37 **(a)** The Taylor's series expansion of $f(x)$ around $x = 2$ is

$$f(x) = f(2) + (x-2)f'(2) + \frac{(x-2)^2}{2!}f''(2)\ldots$$

For linear approximation we take only the first two terms and get

$f(x) = f(2) + (x-2)f'(2)$

Here, $f(x) = e^{-x}$ and $f'(x) = -e^{-x}$, $\quad\therefore f(x) = e^{-2} + (x-2)\left(-e^{-2}\right) = (3-x)e^{-2}$

Q.38 **(a)** $f(x,y) = 4x^2 + 6y^2 - 8x - 4y + 8$, $\dfrac{\partial f}{\partial x} = 8x - 8$, $\dfrac{\partial f}{\partial y} = 12y - 4y$

Equating, $\dfrac{\partial f}{\partial x} = 0$ and $\dfrac{\partial f}{\partial y} = 0 \Rightarrow x = 1$ and $y = \dfrac{1}{3}$, $\left(1, \dfrac{1}{3}\right)$ is the only stationary point

$r = \left[\dfrac{\partial^2 f}{\partial x^2}\right]_{\left(1,\frac{1}{3}\right)} = 8$, $s = \left[\dfrac{\partial^2 f}{\partial x dy}\right]_{\left(1,\frac{1}{3}\right)} = 0$, $t = \left[\dfrac{\partial^2 f}{\partial y^2}\right]_{\left(1,\frac{1}{3}\right)} = 12$

Since $rt - s^2 = 96 > 0$

We have either a maxima or minima at $\left(1, \frac{1}{3}\right)$

Also since, $r = \left[\frac{\partial^2 f}{\partial x^2}\right]_{\left(1,\frac{1}{3}\right)} = 8 > 0$ the point $\left(1, \frac{1}{3}\right)$ is a point of minima

The minimum value is $f\left(1, \frac{1}{3}\right) = 4 \times 1^2 + 6 \times \frac{1}{3^2} - 8 \times 1 - 4 \times \frac{1}{3} + 8 = \frac{10}{3}$ So the optimal value of f(x, y) is minimum equal to $\frac{10}{3}$.

Q.39 **(a)** $e^y = x^{1/x}$

Taking log on both sides, $y = \frac{1}{x}\log x$

$$\frac{dy}{dx} = \left(\frac{1}{x}\right)\left(\frac{1}{x}\right) + (\log x)\left(-\frac{1}{x^2}\right) = \frac{1}{x^2}(1 - \log\log x)$$

Equating $\frac{dy}{dx} = 0$, $\frac{1}{x^2}(1 - \log x) = 0$

$\Rightarrow \log x = 1 \quad \Rightarrow$ x = e is a stationary point.

$$\frac{d^2y}{dx^2} = \left(\frac{1}{x^2}\right)\left(-\frac{1}{x}\right) + (1 - \log x)\left(-\frac{2}{x^3}\right)$$

$$= -\frac{1}{x^3}\left[1 + 2(1 - \log x)\right] = -\frac{1}{x^3}(3 - 2\log x)\left[\frac{d^2y}{dx^2}\right]_{x=e} = -\frac{1}{e^3}(3 - 2\log e) = -\frac{1}{e^3}$$

So, at x = e, we have a maximum.

Q.40 **(d)**

Q.41 **(d)** As the concavity of the function changes at $x = 0$. The function $f(x) = x^3 + 1$ has a point of inflection at x = 0.

Q.42 **(a)** Given spherical surface is $x^2 + y^2 + z^2 = 1$ and point on the sphere is $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$

Normal vector outward to the sphere at $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$ is

$\frac{1}{\sqrt{2}}i + \frac{1}{\sqrt{2}}j + 0.k = \frac{1}{\sqrt{2}}i + \frac{1}{\sqrt{2}}j$, Which is an unit vector.

Q.43 **(a)** $\text{Area} = \int_{2}^{1}\left(x - x^2\right)dx = \left(\frac{X^2}{2} - \frac{X^3}{3}\right)_0^1 = \frac{1}{6}$

Q.45 **(b)** Use L-hospital rule

Q.46 **(a)** $I = \int_0^{\frac{\pi}{3}} e^{it}dt = \left[\frac{e^{it}}{i}\right]_0^{\pi/3} = \left[\frac{\cos t + i\sin t}{i}\right]_0^{\pi/3} = \frac{1}{i}\left[\frac{1}{2} + i\frac{\sqrt{3}}{2} - 1\right] = \left[\frac{\sqrt{3}}{2} + i\frac{1}{2}\right]$

Q.47 **(a)** $\text{Area} = \left\|\overrightarrow{OP} \times \overrightarrow{OQ}\right\| = \left\|\begin{matrix} i & j & k \\ a & b & 0 \\ c & d & 0 \end{matrix}\right\| = ad - bc$

Q.49 **(d)**

Q.50 **(c)** $\cos 2x = \cos^2 x - \sin^2 x$

$\cos 2x$ is the linear combination of the functions $\cos^2 x$ and $\sin^2 x$

The functions $\cos 2x,\ \sin^2 x$ and $\cos^2 x$ are linearly dependent

Q.51 **(a)** Using divergence theorem

$$\frac{1}{4}\iiint_V \text{div}\overline{F}\ dv = \frac{1}{4}\iiint_V 3dv = \frac{3}{4}\iiint_V dv = \frac{3}{4} \times \text{volume of the sphere}$$

Q.52 **(b)** $\tan u = \sqrt{x}\,\dfrac{1 + \frac{y}{x}}{1 + \sqrt{\frac{y}{x}}}$

$F(u) = \tan u$ is a Homogeneous function of degree $\frac{1}{2}$

By Euler's theorem

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\frac{F(u)}{F'(u)} = \frac{1}{2}\frac{\tan u}{\sec^2 u} = \frac{\sin 2u}{4}$$

Q.53 **(a)** $I = \int_0^{\frac{\pi}{2}} \log\cot x\ dx \qquad ----(1)$

$$I = \int_0^{\pi/2} \log\cot\left(\frac{\pi}{2} - x\right)dx \quad I = \int_0^{\pi/2} \log\tan x\ dx \qquad ----(2)$$

Adding (1) and (2)

$$2I = \int_0^{\pi/2}\left(\log\cot x + \log\tan x\right)dx$$

Q.54 **(b)** $|4x-7|=5 \Rightarrow 4x-7=\pm 5 \Rightarrow x=3, \frac{1}{2}$

$\therefore 2|x|-|-x|= (2\times 3)-3=3$ or

$2|x|-|-x|= \left(2\times\frac{1}{2}\right)-\frac{1}{2}=\frac{1}{2}$

Q.56 **(a)** $\lim\limits_{x\to 3+} f(x)=\lim\limits_{x\to 3+}(x-1)=2=f(3)$

$\lim\limits_{x\to 3-} f(x)=\lim\limits_{x\to 3-}\left(\frac{x+3}{3}\right)=2=f(3)$ $\therefore$ f(x) is continuous at x = 3

Q.57 **(d)** Stoke's Theorem: "The Line Integral of a vector $\overline{A}$ around a closed path L is equal to the integral of curl of $\overline{A}$ over the open surface S enclosed by the closed path L".

$\therefore \oint \overline{A}.\overline{dl}=\iint \left(\nabla\times\overline{A}\right).\overline{ds}$

Q.58 **(d)** Given $\vec{A}=x\hat{a}_x+y\hat{a}_y+z\hat{a}_z,\ \nabla.A=\frac{\partial}{\partial x}(x)+\frac{\partial}{\partial y}(y)+\frac{\partial}{\partial z}(z)=3$

Q.59 **(b)** $f(n)=\int_0^{\pi/4}\tan^n x\,dx$

$f(3)+f(1)=\int_0^{\pi/4}\left(\tan^3 x+\tan x\right)dx=\int_0^{\pi/4}\tan x\sec^2 x\,dx=\left[\frac{\sec^2 x}{2}\right]_0^{\pi/4}=\left[\frac{2}{2}-\frac{1}{2}\right]=\frac{1}{2}$

Q.60 **(a)** $I=\int_0^{\frac{\pi}{4}}\cos^2 x\,dx=\int_0^{\frac{\pi}{4}}\frac{1+\cos 2x}{2}dx=\frac{1}{2}[x]_0^{\frac{\pi}{4}}+\frac{1}{2}\left[\frac{\sin 2x}{2}\right]_0^{\frac{\pi}{4}}=\frac{\pi}{8}+\frac{1}{4}\left[\sin\frac{\pi}{2}-\sin 0\right]=\frac{\pi}{8}+\frac{1}{4}$

Q .61 **(b)**

Q.62 **(d)**

Q.64 **(c)** Given integral

$\oint_c (ydx-xdy)$ where C is $x^2+y^2=\frac{1}{4}$

Applying Green's theorem

$\oint_c Mdx-Ndy=\iint_R\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)dxdy$

Where r is region included in c

Q.65 (b) Using L-Hospitals Rule

Q.66 **(a)** Given $\vec{F} = x^2z^2i - 2xy^2zj + 2y^2z^3k$

$$\text{curl}\,\vec{F} = \nabla\times\vec{F} = \begin{vmatrix} i & j & k \\ \dfrac{\partial}{2x} & \dfrac{\partial}{2y} & \dfrac{\partial}{2z} \\ x^2z^2 & -2xy^2z & 2y^2z^3 \end{vmatrix} = \hat{i}(4yz^3 + 2xy^2) - \hat{j}(0 - 2x^2z) + \hat{k}(-2y^2z - 0)$$

$$= (4yz^3 + 2xy^2)\hat{i} + 2x^2z\hat{j} - 2y^2z\hat{k}$$

Q.67 **(d)** We know that f(x) is continuous at x = a if $\lim\limits_{x\to a} f(x)$ exists and equal to f(a)

Q.68 **(c)** Given $F = x^2ai + xyj - yz^2k$

$$\nabla.\vec{F} = \frac{\partial}{\partial x}(x^2z) + \frac{\partial}{\partial y}(xy) + \frac{\partial}{\partial z}(-yz^2) = -2xy + x - 2yz$$

$$\nabla.\vec{F}\ \text{at}(1,-1,1) = 2 + 1 + 2 = 5$$

Q.69 **(b)** Let $I = \int_0^2 \frac{(x-1)^2\sin(x-1)}{(x-1)^2 + \cos(x-1)}dx$

$$I = \int_0^2 \frac{(2-x-1)^2\sin\ (2-x-1)}{(2-x-1)^2 + \cos\ (2-x-1)}\ dx = \int_0^2 \frac{(1-x)^2\sin\ (1-x)}{(1-x)^2 + \cos\ (1-x)}\ dx = -I$$

$$\Rightarrow I + I = 0 \Rightarrow 2I = 0 \Rightarrow I = 0$$

Q.70 **(b)** $\left(\lim\limits_{x\to a}\dfrac{x^n - a^n}{x-a} = na^{n-1}\right)\lim\limits_{x\to 7}\dfrac{x^6 - 7^6}{x-7} + \lim\limits_{x\to 7}\dfrac{x^2 - 7^2}{x-7} = 6(7)^5 + 2(7) = 100856$

Q.71 **(b)** $\int_0^2\int_0^x e^{x+y}dydx = \int_0^2 e^x\left(\int_0^x e^y dy\right)dx = \int_0^2 e^x\left(e^y\right)_0^x dx = \int_0^2 e^x\left(e^x - 1\right)dx$

$$= \int_0^2\left(e^{2x} - e^x\right)dx = \left(\frac{e^{2x}}{2} - e^x\right)_0^2 = \frac{e^4}{2} - e^2 - \frac{1}{2} + 1 = \frac{e^4}{2} - e^2 + \frac{1}{2}$$

$$= \frac{1}{2}\left(e^4 - 2e^2 + 1\right) = \frac{1}{2}\left(e^2 - 1\right)^2$$

Q.73 **(12 to 12)**

$$x = t^3 + 2t,\ y = -3e^{-2t},\ z = 2\sin\ (5t)$$

$$a_x = \frac{\partial^2 x}{\partial t^2} = 6t,\ a_y = \frac{\partial^2 y}{\partial t^2} = -12e^{-2t}, a_z = \frac{\partial^2 z}{\partial t^2} = -50\sin 5t$$

$$\vec{a} = a_x i + a_y j + a_z k = 6t\,i - 12e^{-2t}\,j - 50\sin 5t\ k$$

at $t = 0$, $\vec{a} = -12j$, $|a| = 12\ cm/s^2$

Q.74 **(a) Using L-Hospital** $\lim\limits_{x\to 0}\frac{e^{2x}-1}{\log(1+x)} = \lim\limits_{x\to 0}\left(\frac{2e^{2x}}{\frac{1}{1+x}}\right) = 2$

Q.75 **(a)** By L' Hospital rule,

$$\lim_{\alpha\to 0}\frac{x^{\alpha}-1}{\alpha} = \lim_{\alpha\to 0}\left(\frac{x^{\alpha}\log x}{1}\right) = \log x$$

Q.85 **(c)** $\lim\limits_{x\to\infty}\left(1+\frac{1}{X}\right)^{X} = e$

Q.86 **(0)** $f'(x) = 0 \Rightarrow \frac{1}{1+x} - 1 = 0 \Rightarrow \frac{-x}{1+x} = 0 \Rightarrow x = 0$

$$f''(x) = \frac{-1}{(1+x)^2}, f''(0) = -1 < 0$$

X=0 is Point of maxima.

Q.87 **(6)** $f'(x) = 6x^2 - 18x + 12 = 0,\ x = 1, 2 \in [0,3]$

Now $f(0) = -3;\ f(3) = 6;\ f(1) = 2;\ f(2) = 1$

Hence, $f(x)$ is maximum at $x = 3$ and the maximum value is 6

Q.88 **(d)** $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \ldots\ldots = \sum\limits_{n=0}^{\infty}\frac{x^n}{n!} \Rightarrow \sum\limits_{n=0}^{\infty}\frac{1}{n!} = 1 + \frac{1}{1!} + \frac{1}{2!} + \ldots$

Q.89 **(c)**

Q.90 **(7)**

Q.91 **(3.14)**

Q.92 **(3)** $\nabla(1nr) = \frac{\vec{r}}{r^2} \Rightarrow \text{div}(r^2\nabla(\text{Inr})) = \text{div}(\vec{r}) = 3$

$$\left[\nabla(\text{In } r) = \sum \hat{a}_x \frac{\partial}{\partial x}(\text{Inr}) = \sum \hat{a}_x\left(\frac{1}{r}\right)\left(\frac{x}{r}\right) = \frac{1}{r^2}\sum \hat{a}_x x = \frac{\vec{r}}{r^2}\right]$$

Q.93 (0) $\text{Curl}\,\vec{A} = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \cos x \sin y & \sin x \cos y & 0 \end{vmatrix} = \vec{0}$

Q.94 **(c)** $\frac{\partial z}{\partial x} = y\left[x \times \frac{1}{xy} \times y + \ln xy\right] = y(1+\ln xy)$ and $\frac{\partial z}{\partial x} = x(1+\ln xy) \Rightarrow x\frac{\partial z}{\partial x} = y\frac{\partial z}{\partial y}$

Q.95 **(a)** $f'(x) = 0 \Rightarrow e^{-x}(1-x) = 0 \Rightarrow x = 1,\ f''(1) < 0$

Maximum value is $f(1) = e^{-1}$

Q.96 **(b)**

Q.97 **(c)**

Q.98 **(b)** $u = \left(\frac{2x-y}{2}\right)$ ... (1) $v = \frac{y}{2}$ ... (2)

$x = \frac{y}{2} \Rightarrow u = 0;\ x = \frac{y}{2} + 1 \Rightarrow u = 1, y = 0 \Rightarrow v = 0;\ y = 8 \Rightarrow v = 4$

From (1) and (2), $x = u + v$ ... (3) $y = 2v$ ....(4)

Jacobian is given by $J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} = 2$

$$\therefore \int_0^8 \left(\int_{y/2}^{(y/2)+1} \left(\frac{2x-y}{2}\right) dx \right) dy = \int_{v=0}^{4} \left(\int_{u=0}^{1} (u)\,|J|\,du \right) dv = \int_0^4 \left(\int_0^1 2u\,du \right) dv$$

Q.99 **(b)** $f'(x) = 0 \Rightarrow x^2 - 2x - 8 = 0 \Rightarrow$ Only $x = -2 \in [-3,3]$

$f(-3) = 118$; $f(3) = 28$; $f(-2) = 128$, f(x) is minimum at x =3 and the minimum value is f(3) = 28

Q.100 **(a)** $I = \int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x + \cos x} dx = \int_0^{\frac{\pi}{2}} \frac{\sin\left(\frac{\pi}{2} - x\right)}{\sin\left(\frac{\pi}{2} - x\right) + \cos\left(\frac{\pi}{2} - x\right)} dx$

$$I=\int_0^{\frac{\pi}{2}}\frac{\cos x}{\cos x+\sin x}dx,\qquad \text{Adding, } 2I=\int_0^{\frac{\pi}{2}}dx=\frac{\pi}{2},\ I=\frac{\pi}{4}$$

Q.101 **(c)** $\int_0^1\frac{1}{\sqrt{1-x^2}}dx=\sin^{-1}x\Big|_0^1=\frac{\pi}{2}$

Q.102 **(a)** $\lim\limits_{n\to\infty}\frac{1}{n}\left(\sin\frac{\pi}{2n}+\sin\frac{2\pi}{2n}+\ldots+\sin\frac{n\pi}{2n}\right)$

$$=\lim_{n\to\infty}\frac{1}{n}\sum_0^n\sin\left(\frac{\pi}{2}.\frac{r}{n}\right)\quad=\int_0^1\sin\left(\frac{\pi}{2}x\right)dx=\frac{2}{\pi}$$

Q.103 **(a)** Using $\int_0^a f(x)dx=\int_0^a f(a-x)dx$

$$I=\int_0^{\pi}\sin^5x\ \cos^9x\ dx=\int_0^{\pi}\sin^5(\pi-x)\cos^9(\pi-x)dx$$

$$=-\int_0^{\pi}\sin^5x\ \cos^9x\ dx=-I\Rightarrow 2I=0\Rightarrow I=0$$

104 **(1)** From Gauss divergence theorem, we have

$$\int_s\bar{f}.\overline{ds}=\int_v \text{div}\bar{f}\ dv=\int_v\left(\nabla.\bar{f}\right)dv$$

$$=\int_v\left(\frac{\partial f_1}{\partial x}+\frac{\partial f_2}{\partial y}+\frac{\partial f_3}{\partial z}\right)dxdydz=\int_{x=0}^{1}\int_{y=0}^{1}\int_{z=0}^{1}(0+0+1)dxdydz=1$$

Q.105 **(C)** Use L-Hospital rule

Q.106 **(1.86)** The length of the curve

$$L=\int_0^{\pi/2}\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2}dt=\int_0^{\pi/2}\sqrt{(-\sin t)^2+(\cos t)^2+\left(\frac{2}{\pi}\right)^2}dt$$

$$L=\int_0^{\pi/2}\sqrt{\sin t^2+\cos^2 t+\frac{4}{\pi^2}}dt=\int_0^{\pi/2}\sqrt{1+\frac{4}{\pi^2}}dt=\frac{1}{2}\sqrt{\pi^2+4}=1.86$$

Q.107 **(2)** $\frac{1}{4}\times\text{circumference}=\frac{1}{4}(4\pi)=\pi,\ \text{time}=\frac{\pi}{1.5}=2\text{ sec}$

Q.108 **(a)** $\nabla \times V = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x^2 & 3z^2 & y^3 \end{vmatrix} = i\left[3y^2 - 6z\right] \Rightarrow \nabla \times V_{(1,1,1)} = -3i$

Q.109 **(d)**

Q.110 **(216) By gauss divergence theorem,**

$$\int_s \bar{F}.\bar{n}\,ds = \int_v \text{div}\bar{F}\,dV$$

$$\bar{F} = 9x\bar{i} - 3\bar{y} \Rightarrow \nabla.\bar{F} = 9 - 3 = 6$$

$$\iint_s \frac{1}{\pi}\left(9x\bar{i} - 3y\bar{j}\right).\bar{n}\,ds = \frac{1}{\pi}\int_v 6dV = \frac{1}{\pi}6V = 216$$

Q.111 **(c)**

Q.112 **(0.333)** $\lim_{x\to 0}\left(\frac{-\sin x}{2\sin x + x\cos x}\right), \left(\frac{0}{0}\text{form}\right)$

$$= \lim_{x\to 0}\left(\frac{-\cos x}{2\cos x + \cos x - x\sin x}\right) = \frac{-1}{3}$$

Q.113 **(1.66)** X = 0 to x = 1 − y & y = 0 to y = 1

By Green's theorem,

$$\int_c \left(3x - 8y^2\right)dx + (4y - 6xy)dy = \int_{y=0}^{1}\int_{x=0}^{1-y}\left[-6y - (-16y)\right]dxdy = \int_{y=0}^{1}\left[\int_{x=0}^{1-y} 10ydx\right]dy = 1.66$$

Q.114 **(a)** Since, we know $\int_0^a f(x)dx = \int_0^a f(a-x)dx \quad \therefore \int_0^\pi \sin^5 x \cos^9 x dx = 0$

Q.115 **(-5.72)** Let $u(x,y,z) = x^2 - 3yz$

$\vec{a} = i + j - 2k$ and $P(2,-1,4)$

$\nabla u = i\frac{\partial u}{\partial x} + j\frac{\partial u}{\partial y} + k\frac{\partial u}{\partial z} = i2x + j(-3z) + k(-3y)$

$\nabla u\,|_{(2,-1,4)} = 4i - 12j + 3k, \ |\vec{a}| = \sqrt{1+1+4} = \sqrt{6}$

directional derivative $= \nabla u.\frac{\vec{a}}{\|\vec{a}\|} = (4i - 12j + 3k).\frac{i + j - 2k}{\sqrt{6}} = \frac{4 - 12 - 6}{\sqrt{6}} = \frac{-14}{\sqrt{6}} = -5.72$

Q.116 **(d)** $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{2x} = \left(\lim_{x\to\infty}\left(x+\frac{1}{x}\right)^{x}\right)^{2} = e^{2}$

Q.117 **(c)** $\lim_{x\to 0} x^{1/x} \Rightarrow \ln y = \lim_{x\to\infty}\frac{\ln x}{x}$ (Taking logarithms)

$$= \lim_{x\to\infty}\frac{1/x}{1} = 0$$

y = 1

Q.118 **(−1)** $\int_{1/\pi}^{2/\pi}\left|\frac{\cos\left(\frac{1}{x}\right)}{x^{2}}\right| dx = \int_{\pi/2}^{x}\cos t\,dt,\ \text{put}\,1/x = t,\ \frac{-1}{x^{2}}dx = dt$

$$= \left(\sin t\right)_{\pi/2}^{\pi} = \sin\pi - \sin(\pi/2) = -1$$

Q.119 **(c)** Since $\phi(0)\to\infty$

$\therefore \phi$ is not bounced in $[-1,\ 1]$ and hence f is not continuous in $[-1,\ 1]$

$$A = \int_{-1}^{1} f\left(x\right)dx = -\int_{-1}^{0} x^{-1/3}dx + \int_{0}^{1} x^{-1/3}dx = -\frac{3}{2}\left(x^{2/3}\right)_{-1}^{0} + \frac{3}{2}\left(x^{2/3}\right)_{0}^{1} = \frac{3}{2}+\frac{3}{2} = 3$$

Which is non zero & finite

$\therefore$ Statement II & III are true

Q.120 **(a)** Let $f(x) = \frac{5e}{e^{2}+e^{\frac{1}{x}}}$, $\lim_{x\to 0^{+}}\frac{5e}{e^{2}+e^{\frac{1}{0^{+}}}} = 0,\ \lim_{x\to 0^{-}}\frac{5e}{e^{2}+e^{\frac{1}{0^{-}}}} = \frac{5}{e}$

Since $\lim_{x\to 0+0} f\left(x\right) \neq \operatorname{Lim}_{x\to 0-0} f\left(x\right)$, $\therefore \lim_{x\to 0} f\left(x\right)$ does not exist

Q.121 **(a)** $af\left(x\right)+bf\left(1/x\right) = \frac{1}{x}-25$ ....(1) $\Rightarrow af\left(1/x\right)+bf\left(x\right) = x-25$ ...(2)

Solving, we get $f\left(x\right) = \frac{1}{a^{2}-b^{2}}\left[a\left(\frac{1}{x}-25\right)-b\left(x-25\right)\right]$

$$\therefore \int_{1}^{2} f\left(x\right)dx = \frac{1}{a^{2}-b^{2}}\left[a\left\{\ln x - 25x\right\}_{1}^{2} - b\left\{\frac{x^{2}}{2}-25x\right\}_{1}^{2}\right] = \frac{1}{a^{2}-b^{2}}\left[a\left\{\ln 2 - 25\right\} - \frac{47}{2}b\right]$$

Q.122 **(b)** By Lagrange's mean value theorem

$$f'(x)=\frac{f(1)-f(-1)}{1-(-1)}=\frac{2}{2}=1,\quad -2x+3x^2=1,\ 3x^2-2x-1=0 \Rightarrow x=1,\ -1/3$$

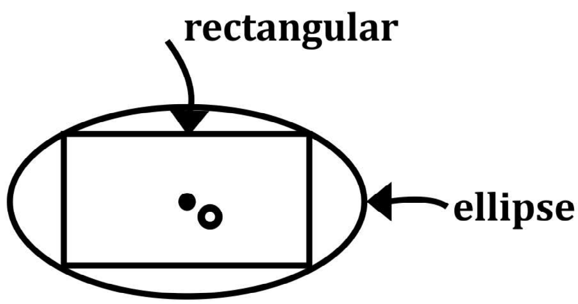
Q.123 **(a)** $f(1)=e^{(-x)}(x^2+x+1)$

f(0) = 1

f(0.5) = 1.067

For positive values of x, function never goes negative.

Q.124 **(1)**



Let 2x, 2y be the length, breadth respectively of the rectangle inscribed in the ellipse

$x^2+4y^2=1$, then

Area of the rectangle = (2x) (2y) = 4xy

Consider, $f=(\text{Area})^2=16\ x^2y^2$

$$=4x^2(1-x^2)\left(\because y^2=\frac{1-x^2}{4}\right)$$

$$f'(x)=0\Rightarrow x(1-2x^2)=0\Rightarrow x=\frac{1}{\sqrt{2}}\qquad \therefore y^2=\frac{1}{8}\Rightarrow y=\frac{1}{\sqrt{8}}$$

$$f''(x)=8-48x^2<0 \text{ when } x=\frac{1}{\sqrt{2}}\Rightarrow f \text{ is maximum at } x=\frac{1}{\sqrt{2}}$$

$$\therefore \text{Area is maxumum and the maximum area is } 4\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{8}}\right)=1$$

Q.125 **(3)** $\displaystyle\int_{-\infty}^{\infty}12\cos 2\pi t\frac{\sin 4\pi t}{4\pi t}dt=\frac{12}{4\pi}\int_{0}^{\infty}\frac{2\cos 2\pi t\sin 4\pi t}{t}dt$

$$=\frac{3}{\pi}\left[\int_0^{\infty}\frac{\sin 6\pi t dt}{t}+\int_0^{\infty}\frac{\sin 2\pi t dt}{t}\right]=\frac{3}{\pi}\left[\int_0^{\infty}e^{\theta t}\frac{6\sin 6\pi t}{t}dt+\int_0^{\infty}e^{\theta t}\frac{\sin 2\pi t}{t}dt\right]$$

$$=\frac{3}{\pi}\left[L\left\{\frac{\sin 6\pi t}{t}\right\}+L\left\{\frac{\sin 2\pi t}{t}\right\}\right]\text{with } s=0$$

$$=\frac{3}{\pi}\left[\int_s^{\infty}\frac{6\pi}{s^2+36\pi^2}ds+\int_s^{\infty}\frac{2\pi}{s^2+4\pi^2}ds\right]\text{ with s=0}$$

$$=\frac{3}{\pi}\left[6\pi\frac{1}{6\pi}\tan^{-1}\left(\frac{s}{6\pi}\right)+2\pi.\frac{1}{2\pi}\tan^{-1}\left(\frac{s}{6\pi}\right)\Big|_s^{\infty}\right]\qquad \text{with}=0$$

$$=\frac{3}{\pi}\left[\tan^{-1}\infty-\tan^{-1}\left(\frac{s}{6\pi}\right)+\tan^{-1}(\infty)-\tan^{-1}\left(\frac{s}{2\pi}\right)\right]\quad=\frac{3}{\pi}\left[\frac{\pi}{2}-\tan^{-1}0+\frac{\pi}{2}-\tan^{-1}0\right]$$

$$=\frac{3}{\pi}\left[\frac{\pi}{2}-0+\frac{\pi}{2}-0\right]=\frac{3}{\pi}\times\pi=3$$

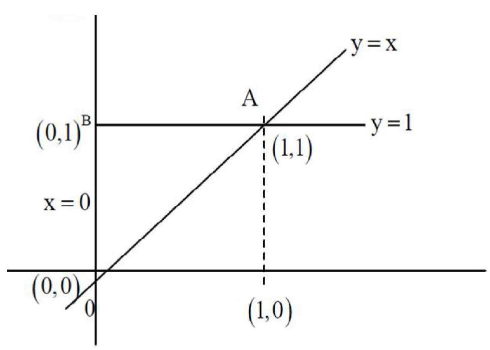
Q.126 **(a)**

Q.127 **(c)** $\lim_{x\to 0^+}\frac{\sin x}{x}=1, \lim_{x\to 0^-}\frac{\sin x}{x}=1, f(0)=0$

$$\lim_{x\to 0^+}f(x)=\lim_{x\to 0^-}f(x)\neq f(0)$$

$\therefore$ Removable discontinuity

Q.128 **(0.72)** Triangle is given by x = y, x = 0, y = 1 is xy plane.



Required volume = $\iint_{OAB} f(x,y)\,dxdy = \int_{x=0}^{1}\int_{y=x}^{1} e^x dxdy = \int_{x=0}^{1} e^x (y)_x^1 dx = e-2=0.72$

Q.129 **(d)** Both a × b and c ×d are perpendicular to plane and parallel to each other, hence angle between them will be either 0 or 180 depending upon the alignment of each vector in the plane.

Q.130 **(-5)**

Q.131 (c) The function $f(x)=\dfrac{x^2-3x-4}{x^2+3x-4}$ is not continuous at the points where

$x^2+3x-4=0 \Rightarrow x=-4 \ \& \ 1$

Q.132 (a) $f=\log x+\log y+\log z \Rightarrow \nabla f=\dfrac{1}{x}i+\dfrac{1}{y}j+\dfrac{1}{z}k$

$$|\nabla f|=\sqrt{\left(\frac{1}{x}\right)^2+\left(\frac{1}{y}\right)^2+\left(\frac{1}{z}\right)^2},\ |\nabla f|\text{at}\left(\frac{1}{2},\frac{1}{3},\frac{1}{4}\right)=\sqrt{29}$$

Q.133 **(c)** $\nabla.\vec{f}=\dfrac{\partial}{\partial x}(yz)+\dfrac{\partial}{\partial y}(zx)+\dfrac{\partial}{\partial z}(xy)=0$

$$\nabla\times\vec{f}=\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & zx & xy \end{vmatrix}=0$$

Q.134 **(c)**

Q.135 **(d)** $x=r\cos\theta,\ y=r\sin\theta,\ \tan\theta=\dfrac{y}{x}$

Taking partial derivative with respect to y

$$\sec^2\theta\frac{\partial\theta}{\partial y}=\frac{1}{x}\Rightarrow\frac{\partial\theta}{\partial y}=\frac{\cos^2\theta}{x}=\frac{x}{x^2+y^2}\Rightarrow\frac{\partial^2\theta}{\partial x\partial y}=\frac{\left(x^2+y^2\right)-x2x}{\left(x^2+y^2\right)^2}=\frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\frac{\partial^2\theta}{\partial x\partial y}=\frac{r^2\left(\sin^2\theta-\cos^2\theta\right)}{r^4}=\frac{-1}{r^2}\cos 2\theta$$

Q.136 **(16)** By Green's theorem $\int_c \overline{F}.\overline{r}'\,ds=\int ydx+2xdy=\iint(2-1)dxdy=\pi\left(\dfrac{4}{\sqrt{\pi}}\right)^2=16$

Q.137 **(c)** $\lim\limits_{x\to\infty}\left(\sqrt{x^2+x-1}-x\right)\left(\dfrac{\sqrt{x^2+x-1}+x}{\sqrt{x^2+x-1}+x}\right)=\lim\limits_{x\to\infty}\dfrac{x^2+x-1-x^2}{\sqrt{x^2+x-1}+x}=\lim\limits_{x\to\infty}\dfrac{x\left(1-\frac{1}{x}\right)}{x\sqrt{1+\frac{1}{x}-\frac{1}{x^2}}+1}=\dfrac{1}{2}$

Q.138 **(d)**

Q.139 **(85.33)**

Q.140 **(b)** The quadratic approximation of f(x) at the point x=0 is $f(x) = f(0) + \frac{x}{1!}f'(0) + \frac{x^2}{2!}f''(0)$

$$=(-5)+x(0)+\frac{x^2}{2}(-6)=-3x^2-5$$

Q.141 **(b)**

Q.142 **(b)** The point of intersection of the curves $y = x^2 + 1$ and $x + y = 3$ are $x = -2, 1$

$$\iint_R dydx = \int_{x=-2}^{1}\left[\int_{y=x^2+1}^{3-x} dy\right]dx = \int_{-2}^{1}\left\{(3-x)-(x^2+1)\right\}dx = \left(\frac{-x^3}{3}-\frac{x^2}{2}+2x\right)_{-2}^{1} = \frac{9}{2}$$

Q.143 **(1)**

Q.144 **(9)**

Q.145 **(b)**

Q.146 **(20)**

$$\frac{1}{2\pi}\iint_D (x+y+10)dxdy = \frac{1}{2\pi}\int_{x=-2}^{2}\int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}}(x+y+10)dxdy = \frac{1}{\pi}\int_{-2}^{2}(x\sqrt{4-x^2}+10\sqrt{4-x^2})dx$$

$$=\frac{1}{\pi}\left[0+10\times 2\int_0^2\sqrt{4-x^2}dx\right] = \frac{20}{\pi}\left[\frac{x}{2}\sqrt{4-x^2}+\frac{4}{2}\sin^{-1}\left(\frac{x}{2}\right)\right]_0^2 = \frac{20}{\pi}\left[0+2\left(\frac{\pi}{2}\right)\right] = 20$$

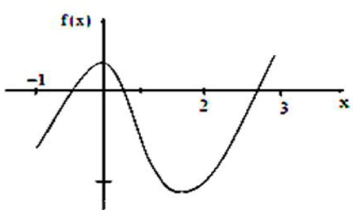
Q.147 **(4.71)** $\text{Volume} = \int_3^5\int_{\pi/8}^{\pi/4}\int_3^{4.5}\rho\, d\rho\, d\phi\, dz = 4.71$

Q.148 **(C)**

Q.149 **(0)** By Green's theorem

$$\oint xy^2dx + x^2ydy = \iint_R (2xy-2xy)dxdy = 0$$

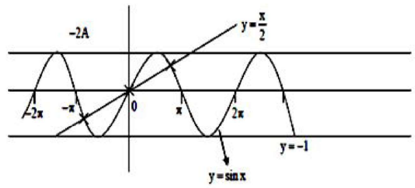
Q.150 **(b)** We can plot for various valve of x



f(x) is increasing, decreasing & again increasing

Q.151 (c) let y = sin x, y = $\frac{x}{2}$ be two curves

The solutions of sin x = $\frac{x}{2}$ are intersected points of two curves y = sin x and y = $\frac{x}{2}$



Therefore, they are intersecting at three points.

Q.152 **(90)**

Q.153 **(2)** $\int_0^1 (1-x)^{-1/2}dx = -\left.\frac{(1-x)^{\frac{-1}{2}+1}}{\frac{-1}{2}+1}\right|_0^1 = -2(1-x)^{1/2}\Big|_0^1 = 2$

Q.154 **(a)**

Q.155 **(10)** Volume $= \iint_R z\,dxdy = \int_{x=0}^{3}\int_{y=0}^{\frac{2}{3}x}(6-x-y)\,dydx = \int_{x=0}^{3}\left[(6-x)y - \frac{y^2}{z}\right]_{y=0}^{\frac{2}{3}x}.dx$

$$= \int_0^3 \left[(6-x)\left(\frac{2x}{3}\right) - \frac{1}{2}\left(\frac{4}{9}x^2\right)\right]dx = \int_0^3\left(4x - \frac{2}{3}x^2 - \frac{2}{9}x^2\right)dx = \int_0^3\left(4x - \frac{8}{9}x^2\right)dx = 10$$

Q.156 **(d)** $\exists$ relative minimum at x = 0, But derivative doesn't exit at corner point

Q.157 **(b)**

Q.158 **(-13)** $f(x) = 2x^3 - x^4 - 10,\ f'(x) = 6x^2 - 4x^3,\ f''(x) = 12x - 12x^2$

$f'(x) = 0 \Rightarrow 6x^2 - 4x^3 = 0,\ x^2(6-4x) = 0 \Rightarrow x = 0,\ x = \frac{3}{2}$ are stationary points.

$f''(0) = 0$ neither maxima nor minima at $x = 0,\ x = \frac{3}{2} \notin [-1,1]$

minimum of f(x) = minimum$\{f(-1), f(1)\}$ = Minimum{-13,-9}= -13

Q.159 **(d)** $\lim_{x\to 0}\frac{x^2 - \sin(x)}{x}$ $\left(\frac{0}{0}\text{from}\right)$

By L'Hospitals rule

$\lim_{x\to 0}\frac{2x - \cos x}{1} = -1$

Q.160 **(c)** $x=\cos\left(\frac{\pi u}{2}\right)$, $y=\sin\left(\frac{\pi u}{2}\right)$, $x^2+y^2=1$

It represent a circle in x-y Plane. for : - $0\le u\le 1\Rightarrow 0\le x\le 1$

So, we will get a quarter circle in x-y plane and by revolving it by 360^{o}, we will get a hemisphere of unit radius

Area of hemisphere = $2\pi r^2=2\pi(1)^2=2\pi$

Q.161 **(0)** Divergence of curl of any vector is always zero. $\therefore \nabla.(\nabla\times\vec{V})=0$

Q.162 **(c)** $\int_0^1 \frac{1}{\sqrt{1-x^2}}dx=\sin^{-1}x\Big|_0^1=\frac{\pi}{2}$

Q.163 **(226.19)** By gauss divergence theorem

$$\therefore\ \iint F.nds=2\iiint dv=2\times\text{volume of the sphere}=2\times\frac{4}{3}\pi r^3=226.19$$

Q.164 **(d)**

Q.165 **(b)** We know that

$$\int_a^b x\ f(x)\,dx=\frac{b-a}{2}\int_a^b x\ f(x)\,dx,\quad \text{If } f(a+b-x)=f(x)$$

Let $f(x)=\cos^2 x\Rightarrow f(\pi-x)=\cos^2(\pi-x)=\cos^2 x=f(x)$

$$\int_0^{\pi} x\cos^2 x\ dx=\frac{\pi}{2}\int_0^{\pi}\cos^2 x\ dx=\frac{\pi}{2}\int_0^{\pi}\left[\frac{1+\cos 2x}{2}\right]dx=\frac{\pi}{4}\left[x+\frac{\sin 2x}{2}\right]_0^{\pi}=\frac{\pi^2}{4}$$

Q.166 **(0.67)** $\int\vec{F}(\vec{r}).d\vec{r}=\int x^2dx+y^2dy$, path is straight line joining (0,0) & (1,1) is $y=x$,

$dy=dx$, so limit remains the same i.e. 0 to 1. Substituting in the integral

$$\therefore\int x^2dx+x^2dx=\int_{x=0}^{1}2x^2dx=\frac{2x^3}{3}\Big|_0^1=\frac{2}{3}=0.67$$

Q.167 **(d)**

Q.168 **(a)** $\phi(x,y,z)=xy^2+yz^2+zx^2$

$$\nabla\phi=\frac{\partial\phi}{\partial x}\hat{i}+\frac{\partial\phi}{\partial y}\hat{j}+\frac{\partial\phi}{\partial z}\hat{k}=(y^2+2xz)\hat{i}+\left(2xy+z^2\right)\hat{j}+\left(2yz+x^2\right)\hat{k}$$

$$\nabla\phi\big|_{(2,-1,1)}=5\hat{i}-3\hat{j}+2\hat{k}$$

$$\nabla\phi.\hat{p}=\nabla\phi.\frac{\vec{p}}{|\vec{p}|}=\left(5\hat{i}-3\hat{j}+2\hat{k}\right)\cdot\frac{\left(\hat{i}+2\hat{j}+2\hat{k}\right)}{3}=\frac{5-6+4}{3}=1$$

Q.169 **(12)** $f(x) = 3x^3 - 7x^2 + 5x + 6$, To find the maximum value of the function

$$f'(x) = 9x^2 - 14x + 5 = 0, \quad x = 1, \frac{5}{9}$$

$$f(0) = 5,\ f\left(\frac{5}{9}\right) = 7.131,\ f(1) = 0,\ f(2) = 12$$

Hence, the function f (x) has the maximum value of **12.**

Q.170 **(0)**

Q.171 **(0)** We have to find Taylor series expansion around x=0 , is given as

$$f(0) = 0,\ f'(x) = e \cdot e^{-\frac{x^2}{2}},\ f''(x) = e \cdot e^{-\frac{x^2}{2}}\left(\frac{-2x}{2}\right) = -e \cdot xe^{-\frac{x^2}{2}},\ a_2 = \frac{f''(0)}{2!} = 0$$

Q.172 **(d)** Given , $\vec{F} = (x^2 - 2y)\vec{i} - 4yz\vec{j} + 4xz^2\vec{k}$

The equation of line joining the points (0,0,0) to (1,1,1) is $\frac{x-0}{1-0} = \frac{y-0}{1-0} = \frac{z-0}{1-0} = t$

$$\int_c \vec{F}.\vec{dl} = \int\left[(x^2 - 2y)dx - 4yzdy + 4xz^2dz\right] = \int_0^1 (t^2 - 2t)dt - 4t^2dt + 4t^3dt$$

$$\int_c \vec{F}.\vec{dl} = \left[\frac{t^3}{3} - t^2\right]_0^1 - 4\left[\frac{t^3}{3}\right]_0^1 + 4\left[\frac{t^4}{3}\right]_0^1 = \frac{1}{3} - 1 - \frac{4}{3} + 1 = -1$$

Q.173 **(1)** Given : $f(x) = (x-2)^2$, $g(x) = 2x - 1$

When, $f(x) = g(x) \Rightarrow (x-2)^2 = 2x - 1 \Rightarrow x^2 - 6x + 5 = 0 \Rightarrow x = 1, 5$

Thus, the smallest value of x is 1.

Q.174 **(a)** The Integral form of mean value theorem in $\xi \in (a,b)$ is

$$\int_a^b f(x)dx = (b-a)f(\xi)$$

Q.175 **(c)** By Gauss Divergence Theorem

$$\oiint_s \vec{r}\,.\,\hat{n}\,ds = \iiint_v \nabla\,.\,\vec{r}\,dv = \iiint_v 3\,dv = 3V$$

Q.176 **(c)** $\vec{u} = e^x \cos y\,\hat{i} + e^x \sin y\,\hat{j}$

$$\nabla.\,\vec{u} = \frac{\partial}{\partial x}(u_1) + \frac{\partial}{\partial y}(u_2) = \frac{\partial}{\partial x}(e^x \cos y) + \frac{\partial}{\partial y}(e^x \sin y) = e^x \cos y + e^x \cos y = 2e^x \cos y$$

Q.177 **(c)** $\nabla\phi = \nabla(\ln\ r) = \frac{\vec{r}}{r^2}$

ESE SOLUTIONS

Q.1 **(a)**

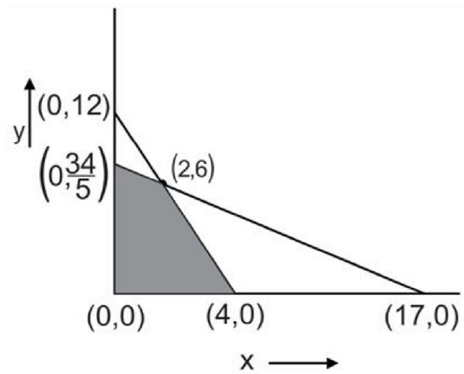
$$Z = 10x + 6y$$

$$Z_{\left(0,\frac{34}{5}\right)} = 10(0) + 6\left(\frac{34}{5}\right)$$

$$Z_{(4,0)} = 10(4) + 6(0) = 40$$

$$Z_{(2,6)} = 10(2) + 6(6) = 56$$

$$Z_{\max} = 56$$



4 DIFFERENTIAL EQUATION

4.1 INTRODUCTION

A differential equation is a mathematical equation that relates some function of one or more variables with its derivatives. Differential equations arise whenever a deterministic relation involving some continuously varying quantities (modeled by functions) and their rates of change in space and/or time (expressed as derivatives) is known or postulated. Differential equations are mathematically studied from several different perspectives, mostly concerned with their solutions —the set of functions that satisfy the equation. Only the simplest differential equations admit solutions given by explicit formulas; however, some properties of solutions of a given differential equation may be determined without finding their exact form.

4.2 ORDER AND DEGREE OF DIFFERENTIAL EQUATION

Order of a differential equation is the order of the highest order derivative (also known as differential coefficient) present in the equation.

Example : $\dfrac{d^3y}{dx^3}+3x\dfrac{dy}{dx}=e^y$

In this equation the order of the highest derivative is 3 hence this is a third order differential equation

Degree of differential equation is represented by the power of the highest order derivative in the given differential equation after removing the radicals from any differential coefficient, if present. The differential equation must be a polynomial equation in derivatives for the degree to be defined.

Example : $\dfrac{d^4y}{dx^4}+\left(\dfrac{d^2y}{dx^2}\right)^2-3\dfrac{dy}{dx}+y=9$

Solution : Here, the exponent of the highest order derivative is one and the given differential equation is a polynomial equation in derivatives. Hence, the degree of this equation is 1.

Example : The order and degree of the differential equation $\dfrac{\left[1+\left(\dfrac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\dfrac{d^2y}{dx^2}}=c$ are

(a)1 and 6 (b)2 and 6
(c)2 and 2 (d)1 and 2

Solution : (c)

$$\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}} = c\frac{d^2y}{dx^2}$$

$$\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3} = c^2\left(\frac{d^2y}{dx^2}\right)^2$$

Order = 2, Degree = 2

4.3 DIFFERENTIAL EQUATIONS OF FIRST ORDER

4.3.1 DEFINITIONS

A Differential equation is an equation which involves derivatives or differential coefficients or differentials. Thus the following are all examples of differential equations .

(a) $x^2dx + y^2dy = 0$

(b) $\frac{d^2x}{dt^2} + a^2x = 0$

(c) $y = x\frac{dy}{dx} + \frac{x^2}{dy/dx}$

(d) $\left[1+\left(\frac{dy}{dx}\right)^2\right]^{5/3} = a\left(\frac{d^2y}{dx^2}\right)$

(e) $\frac{dx}{dt} - wy = a\cos(pt), b\frac{dy}{dt} + wx = a\sin(pt)$

(f) $x^2\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 3z$

(g) $\frac{\partial^2 y}{\partial t^2} = a^2\frac{\partial^2 y}{\partial x^2}$

An ordinary differential equations is that in which all the differential coefficients are with respect to a single independent variable. Thus the equations (a) to (d) are all ordinary differential equations. (e) is a **system** of ordinary differential equations.

A partial differential equations is that in which there or more independent variable and partial differential coefficients with respect to any of them .The equations (f) and (g) are partial differential equations.

The order of a differential equation is the order of the highest derivative appearing in it. The degree of a differential equation is the degree of the highest derivative occurring in its, after the equation has been expressed in a form free from radicals and fraction as far as the derivatives are concerned.

Thus from the example above.

(a) is of the first order and first degree;

(b) is of the second order and first degree;

(c) written as $y\frac{dy}{dx} = x\left(\frac{dy}{dx}\right)^2 + x^2$ is of the order one but of second degree;

(d) After removing radicals is written as $\left[1+\left(\frac{dy}{dx}\right)^2\right]^5 = a^3\left(\frac{d^2y}{dx^2}\right)^3$

It is of the second order and third degree.

4.4 SOLUTION OF DIFFERENTIAL EQUATION

A solution (or integral) of a differential equation is a relation between the variables which satisfies the given differential equation.

For example $x = A\cos(nt+\alpha)$ ...(1)

is a solution of $\frac{d^2x}{dt^2} + n^2x = 0$ ...(2)

1. GENERAL SOLUTION

The general (or complete) solution of a differential equation is that in which the number of arbitrary constants is equal to the order of the differential equation. Thus (1) is a general solution (2) as the number of arbitrary constants (A, α) is the same as the order of (2)

2. PARTICULAR SOLUTION

A particular solution is that which can be obtained from the general solution by giving particular values to the arbitrary constants.

For example, $x = A\cos(nt + \pi/4)$

Is the particular solution of the equation (2) as it can be derived from the general solution (1) by putting $\alpha = \frac{\pi}{4}$.

3. SINGULAR SOLUTION

A differential equation may sometimes have an additional solution which cannot be obtained from the general solution by assigning a particular value to the arbitrary constant. Such a solution is called a singular solution & is not of much engineering interest.

4. LINEARLY INDEPENDENT SOLUTION

Two solution $y_1(x)$ and $y_2(x)$ of the differential equation

$$\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)y = 0 \qquad \text{.... (3)}$$

are said to be linearly independent if $c_1 y_1 + c_2 y_2 = 0$

Such that $c_1 = 0$ and $c_2 = 0$

If c_1 and c_2 are not both zero, then the two solutions y_1 and y_2 are said to be linearly dependent.

If $y_1(x)$ and $y_2(x)$ any two solutions of (3), then their linear combination $c_1 y_1 + c_2 y_2$ where c_1 and c_2 are constants is also a solution of (3).

4.5 EQUATIONS OF THE FIRST ORDER AND FIRST DEGREE

It is not possible to solve such equations in general. We shall, however, discuss some special methods of solution which are applied to the following types of equations:

(i) Equations where variables are separable, **(ii)** Homogeneous equations,

(iii) Linear equations, **(iv)** Exact equations.

In other cases, the particular solution may be determined numerically.

1. VARIABLES SEPARBLE

If in an equation it is possible to collect all functions of x and dx on one side and all the functions of y and dy on the other side, then the variables are said to be separable. Thus the general form of such an equation is $f(y)dy = \phi(x)dx$

Integrating both sides, we get $\int f(y)dy = \int \phi(x)dx + c$ as its solution.

Example : Solve $(1 + x^2)dy - xy\,dx = 0$

Solution : By adjusting the given differential equation.

$$\frac{dy}{y} = \frac{xdx}{1+x^2}$$

Integrating both sides, we get

$$\ln\ y = \frac{1}{2}\ \ln(1 + x^2) + \ln C = \ln C\left(1+x^2\right)^{\frac{1}{2}}$$

Taking exponentials

$$y = C\left(1+x^2\right)^{\frac{1}{2}}$$

Example : Solution of the differential equation $3y\frac{dy}{dx} + 2x = 0$ represents a family of

(a) ellipses (b) Circle (c) parabolas (d) Hyperbolas

Solution : (a) $3y\frac{dy}{dx}+2x=0$

$$\int 3y\,dy + \int 2x\,dx = 0$$

$$\frac{3}{2}y^2 + x^2 = C$$

Which is the equation of a family of ellipses.

2. Differential equation with HOMOGENEOUS function

A funtion $f(x,y)$ is said to be homogeneous of degree n, if it can be written as $f(x,y)=x^n\phi\left(\frac{y}{x}\right)\text{or } y^n\phi\left(\frac{x}{y}\right)$.

$x^2 + 2xy - 2y^2$ is homogeneous of degree 2

$2x^3y + 3x^2y^2 + 5y^4$ is homogeneous of degree 4

$2x + 5y$ is homogeneous of degree 1

A differential equation $\frac{dy}{dx}=f(x,y)=x^n\phi\left(\frac{y}{x}\right)\text{or } y^n\phi\left(\frac{x}{y}\right)$ is called as differential equation with homogeneous function. To solve this we uses the following steps

(i) Put y = vx then $\frac{dy}{dx}=v+x\frac{dv}{dx}$,

(ii) Separate the variables v and x

(iii) Use variable seperable method to solve this.

Example : Solve $(x^2-y^2)dx + 2xy\,dy = 0$

Solution : Substituting $y = vx$ and $dy = v\,dx + x\,dv$

we get $(1-v^2)dx + 2v(v\,dx + x\,dv) = 0$

Separating the variables gives

$$\frac{2v\,dv}{v^2+1} = -\frac{dx}{x}$$

Integrating we get

$\ln(v^2 + 1) = -\ln x + \ln C$

Taking exponentials we obtain

$x(v^2 + 1) = C$

Finally, since $v = y/x$, this becomes

$x^2 + y^2 = Cx$

Example : The solution of differential equation $x^2 \frac{dy}{dx} = \left(x^2 + xy + y^2\right)$ is

(a) $\tan^{-1}\left(\frac{x}{y}\right) = \log y + c$ (b) $\tan^{-1}\left(\frac{y}{x}\right) = \log x + c$

(c) $\tan^{-1}\left(\frac{x}{y}\right) = \log x + c$ (d) $\tan^{-1}\left(\frac{y}{x}\right) = \log y + c$

Solution : (b) Put $y = vx$, $\frac{dy}{dx} = v + x\frac{dv}{dx}$,

$$\frac{dy}{dx} = \frac{x^2 + xy + y^2}{x^2}, \quad v + x\frac{dv}{dx} = \frac{x^2 + vx^2 + v^2x^2}{x^2}, \quad v + x\frac{dv}{dx} = 1 + v + v^2$$

$$\frac{dv}{1+v^2} = \frac{dx}{x}, \quad \tan^{-1}(V) = \ln(x) + c, \quad \tan^{-1}\left(\frac{y}{x}\right) = \ln(x) + c$$

3. LEIBNITZ'S LINEAR EQUATION

A differential equation is said to be linear if the dependent variable and its differential coefficients occur only in the first degree and not multiplied together.

Thus the standard form of a linear equation of the first order, commonly known's as Leibnitz's linear equation is $\frac{dy}{dx} + Py = Q$ ... (1), Where P and Q are the functions of x only.

The solution of equation (1) is given by

$$y(I.F.) = \int Q(I.F.)dx + C, \quad \text{Where I.F.(Integrating Factor)} = e^{\int P dx}$$

Example : The solution of the equation $\frac{dy}{dx} + y\tan x = \sec x$ is

(a) $y = \sin x + c\cos x$ (b) $y = \sin x + c\sec x$

(c) $y = \tan x + \cot x + c$ (d) None of these

Solution : (a) $I.F. = e^{\int \tan x \, dx} = e^{\log(\sec x)} = \sec x$

Solution is given by

$$y\left(\sec x\right) = \int \sec^2 x dx + c = \tan x + c$$

$$y = \sin x + c \cos x$$

Example : The solution of the differential equation $\frac{dy}{dx} + 2xy = e^{-x^2}$ with $y\left(0\right) = 1$ is

(a) $(1+x)e^{x}$ (b) $(1+x)e^{-x^2}$

(c) $(1-x)e^{x}$ (d) $(1-x)e^{x^2}$

Solution : (b) $\text{I.F.} = e^{\int 2x dx} = e^{x^2}$

Solution is $y(\text{I.F.}) = \int Q(\text{I.F.})\,dx + c$, $ye^{x^2} = \int e^{x^{-2}} e^{x^2} dx + c$, $ye^{x^2} = x + c$

at x = 0, y = 1 (given) $\therefore\ 1e^0 = 0 + c \ \Rightarrow\ c = 1$

So, the solution is $ye^{x^2} = x + 1 \quad \Rightarrow y = e^{x^{-2}}(x+1)$

4. BERNOULLI'S EQUATION

The equation $\frac{dy}{dx} + Py = Qy^n$ ...(1) where P, Q are function of x, is reducible to the Leibnitz's linear equation and is usually called as Bernoulli's equation.

To solve (1), divide both sides by y^n, so that $y^{-n}\frac{dy}{dx} + Py^{1-n} = Q$ ...(2)

Put $y^{1-n} = z$ so that $(1-n)y^{-n}\frac{dy}{dx} = \frac{dz}{dx}$

From equation (2)

$$\frac{1}{1-n}\frac{dz}{dx} + Pz = Q$$

$$\frac{dz}{dx} + P(1-n)z = Q(1-n)$$

Which is Leibnitz's linear in z and can be solved easily.

The solution of Bernoulli's Equation is

$$y(\text{I.F.}) = \int (1-n)\,Q\,(\text{I.F.})dx + C$$

$$\text{I.F.} = e^{\int (1-n)P dx}$$

Example : Solve : $\tan y\frac{dy}{dx} + \tan x = \cos y \cos^2 x$

Solution : $\tan y\frac{dy}{dx} + \tan x = \cos y \cos^2 x$

$$\sec y \tan y \frac{dy}{dx} + \sec y \tan x = \cos^2 x$$

Writing $z = \sec y$, so that $\frac{dz}{dx} = \sec y \tan y \frac{dy}{dx}$

The equation becomes $\frac{dz}{dx} + z\tan x = \cos^2 x$

I.F. $= e^{\int \tan x dx} = e^{\log \sec x} = \sec x$

The solution of the equation is

$z\sec x = \int \cos^2 x \sec x dx + C$

$\sec y \sec x = \int \cos x dx + C = \sin x + C$

$\sec y = (\sin x + C)\cos x$

5. EXACT DIFFERENTIAL EQUATIONS

A differential equation of the form $M(x,y)dx + N(x,y)dy = 0$ is said to be exact if its left hand member is the exact differential of some function u(x,y)ie., $du = Mdx + Ndy = 0$. Its solution , therefore, is u(x, y) = c.

The necessary and sufficient condition for the differential equation Mdx + Ndy =0 to be exact is $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$. The solution of exact differential equation $M(x,y)dx + N(x,y)dy = 0$ is given by $\int\limits_{y\text{ constant}} Mdx + \int(\text{terms of N not contiainig x})dy = c$

Example : Test the following equation for exactness and find the solution if it is exact.

$(3x^2y - y)dx + (x^3 - x + 2y)dy = 0$

Solution : $M = 3x^2y - y, \qquad N = x^3 - x + 2y$

$\frac{\partial M}{\partial y} = 3x^2 - 1, \qquad \frac{\partial N}{\partial X} = 3x^2 - 1$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial X}$

Since $\partial M/\partial y = \partial N/\partial x$, the equation is exact i.e. there is a function u(x, y) of which the left- hand side is exactly the total differential.

To find this function we integrate $\partial u/\partial x = M = 3x^2y - y$ with respect to x, holding y constant. We obtain

$$u(x, y) = \int M\,\partial x = \int(3x^2y - y)\partial x = x^3y - yx + \varphi(y) \qquad ...(1)$$

where $\varphi(y)$ consists of terms that are free from x ($\int M\,\partial x$ denotes integration with respect to x, holding y constant).

In the same way, we integrate $\partial u/\partial y = N = x^3 - x + 2y$ with respect to y, holding x constant. We obtain

$$u(x, y) = \int N\, \partial y = \int\left(x^3 - x + 2y\right)\partial y = x^3 y - xy + y^2 + \psi(x) \qquad ...(2)$$

where ø(x) consists of terms that are free from y (i.e. terms containing x only or constants). Comparing (1) and (2) we see that the general solution is

$x^3y - xy + y^2 = C$

Example If $pdx + x\sin ydy = 0$ is exact then p can be

(a) $\sin y + \cos y$ (b) $-\sin y$ (c) $x^2 - \cos y$ (d) $\cos y$

Solution : (d)

$pdx + x\sin ydy = 0$, $M = p$, $N = x$ siny

For differential equation to be exact

$\frac{\partial p}{\partial y} = -\sin y$, By integration $p = \cos y$

4.6 LINEAR DIFFERENTIAL EQUATIONS

4.6.1 DEFINITION

1. Linear differential equations are those in which the dependent variable and its derivatives occur only in the first degree and are not multiplied together. Thus the general linear differential equation of the nth order is of the form

$$\frac{d^n y}{dx^n} + p_1\frac{d^{n-1}y}{dx^{n-1}} + p_2\frac{d^{n-2}y}{dx^{n-2}} + \ldots + p_n y = X$$

Where $p_1, p_2, \ldots p_n$ and X are functions of x only

2. Linear differential equations with constant co-efficients are of the form

$$\frac{d^n y}{dx^n} + k_1\frac{d^{n-1}y}{dx^{n-1}} + k_2\frac{d^{n-2}y}{dx^{n-2}} + \ldots + k_n y = X$$

Where $k_1 k_2, \ldots k_n$ are constants. Such equations are most important in the study of electro-mechanical vibrations and other engineering problems.

4.6.2 THEOREM

If y_1, y_2 are only two solutions of the equation

$$\frac{d^n y}{dx^n} + k_1\frac{d^{n-1}y}{dx^{n-1}} + k_2\frac{d^{n-2}y}{dx^{n-2}} + \ldots + k_n y = 0 \quad ...(1)$$

then $c_1y_1 + c_2y_2\,(= u)$ is also its solution.

Since $y = y_1$ and $y = y_2$ are solution of (1)

$$\therefore \quad \frac{d^n y_1}{dx^n} + k_1\frac{d^{n-1}y_1}{dx^{n-1}} + k_2\frac{d^{n-2}y_1}{dx^{n-2}} + \ldots + k_n y_1 = 0 \qquad ...(2) \qquad \text{and}$$

$$\frac{d^n y_2}{dx^n}+k_1\frac{d^{n-1}y_2}{dx^{n-1}}+k_2\frac{d^{n-2}y_2}{dx^{n-2}}+\ldots+k_n y_2=0 \quad \ldots(3)$$

If c_1, c_2 be two arbitrary constants, then

$$\frac{d^n\left(c_1y_1+c_2y_2\right)}{dx^n}+k_1\frac{d^{n-1}\left(c_1y_1+c_2y_2\right)}{dx^{n-1}}+\ldots+k_n\left(c_1y_1+c_2y_2\right)$$

$$=c_1\left(\frac{d^n y_1}{dx^n}+k_1\frac{d^{n-1}y_1}{dx^{n-1}}+\ldots+k_n y_1\right)$$

$$+c_2\left(\frac{d^n y_2}{dx^n}+k_1\frac{d^{n-1}y_2}{dx^{n-1}}+\ldots+k_n y_2\right)$$

$$c_1\left(0\right)+c_2\left(0\right)=0 \qquad \text{[By (2) and (3)]}$$

i.e., $$\frac{d^n u}{dx^n}+k_1\frac{d^{n-1}u}{dx^{n-1}}+k_2\frac{d^{n-2}u}{dx^{n-2}}+\ldots+k_n y=0 \qquad \ldots.(4)$$

This proves the theorem.

4.6.3 GENERAL SOLUTION

Since the general solution of differential equation of the nth order contain n arbitrary constants, it follows, from above that if $y_1, y_2, y_3 \ldots y_n$ are 'n' independent solutions of (1), then

$c_1y_1+c_2y_2+\ldots+c_ny_n\,(=u)$ is its complete solution.

If y = v be any particular solution of

$$\frac{d^n y}{dx^n}+k_1\frac{d^{n-1}y}{dx^{n-1}}+k_2\frac{d^{n-2}y}{dx^{n-2}}+\ldots+k_n y=X \quad \ldots(5)$$ then

$$\frac{d^n v}{dx^n}+k_1\frac{d^{n-1}v}{dx^{n-1}}+k_2\frac{d^{n-2}v}{dx^{n-2}}+\ldots+k_n v=X \quad \ldots(6)$$ adding (4) and (6),

we have

$$\frac{d^n\left(u+v\right)}{dx^n}+k_1\frac{d^{n-1}\left(u+v\right)}{dx^{n-1}}+k_2\frac{d^{n-2}\left(u+v\right)}{dx^{n-2}}+\ldots+k_n\left(u+v\right)=X$$

This shows that y = u + v is the complete solution of (5)

The part u is called the **complementary function (C.F.)** and the part v is called the **particular integral (P.I.)** of (5).

$\therefore$ The complete solution (C.S.) of (5) is **y = C.F +P.I**

Thus in order to solve the question (5), we have to first find the C.F., i.e., the complete solution of (1), and then the P.I., i.e., a particular solution of (5).

4.6.4 OPERATOR D

Denoting $\frac{d}{dx}, \frac{d^2}{dx^2}, \frac{d^3}{dx^3}$ etc. by D, D^2, D^3etc., so that

$$\frac{dy}{dx} = Dy, \frac{d^2y}{dx^2} = D^2y, \frac{d^3x}{dx^3} = D^3y \text{ etc.,}$$

the equation (5) above can be written in the symbolic form

$$\left(D^n + k_1D^{n-1} + \ldots + k_n\right)y = X, \text{i.e.,} f\left(D\right)y = X,$$

Where $f\left(D\right) = D^n + k_1D^{n-1} + \ldots + k_{n,}$ i.e., a polynomial in D.

Thus the symbol D stands for the operation of differentiation and can be treated much the same as an algebraic quantity i.e., f(D) can be factorized by ordinary rules of algebra and the factors may be taken in any order. For instance

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 3y = \left(D^2 + 2D - 3\right)y = \left(D+3\right)\left(D-1\right)y$$

4.6.5 RULES FOR FINDING THE COMPLEMENTARY FUNCTION

To solve the equation

$$\frac{d^ny}{dx^n} + k_1\frac{d^{n-1}y}{dx^{n-1}} + k_2\frac{d^{n-2}y}{dx^{n-2}} + \ldots + k_ny = 0 \text{ ....(1)}$$

Where k's are constants

The equation (1) in symbolic form is

$$(D^n + k_1D^{n-1} + k_2D^{n-2} + \ldots + k_n)y = 0 \text{ ......(2)}$$

Its symbolic co-efficient equated to zero i.e.

$$D^n + k_1D^{n-1} + k_2D^{n-2} + \ldots + k_n = 0$$

is called the auxiliary equation (A.E.)
Let $m_1, m_{2,} \ldots m_n$ be roots of auxiliary equation.

Case I. If all the roots be real and different, then (2) is equivalent to

$$\left(D - m_1\right)\left(D - m_2\right)\ldots\left(D - m_n\right)y = 0 \quad \text{...(3)}$$

Now, (3) will be satisfied by the solution of

$$\left(D - m_n\right)y = 0, \text{i.e., by} \frac{dy}{dx} - m_ny = 0$$

This is a **Leibnitz's linear** and I.F. = e^{-m_nx}

$\therefore$ Its solution is $ye^{-m_n x} = c_n$, i.e., $y = c_n e^{m_n x}$

Similarly, since the factors in (3) can be taken in any order, it will be satisfied by the solution of

$$(D - m_1)y = 0, (D - m_2)y = 0 \text{ etc. i.e., by}$$
$$y = c_1 e^{m_1 x}, y = c_2 e^{m_2 x} \text{etc.}$$

Thus the complete solution of (1) is $y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \ldots + c_n e^{m_n x}$ ....(4)

Case II. If two roots are equal (i.e., $m_1 = m_2$), then (4) becomes

$$y = (c_1 + c_2)e^{m_1 x} + c_3 e^{m_3 x} + \ldots + c_n e^{m_n x}$$
$$y = Ce^{m_1 x} + c_3 e^{m_3 x} + \ldots + c_n e^{m_n x}$$

$\because$ $c_1 + c_2 =$ one arbitary constant C

It has only n-1 arbitrary constants and is, therefore, not the complete solution of (1). In this case, we processed as follows:

The part of the complete solution corresponding to the repeated root is the complete solution of $(D - m_1)(D - m_1)y = 0$

Putting $(D - m_1)y = z$, it becomes $(D - m_1)z = 0$

This is a Leibnitz's linear in z and I.F. $= e^{-m_1 x}$

$\therefore$ its solution is $ze^{-m_1 x} = c_1$ or $z = c_1 e^{m_1 x}$

Thus $(D - m_1)y = z = c_1 e^{m_1 x}$ ...(5)

Its I.F. being $e^{-m_1 x}$, the solution of (5) is

$$ye^{-m_1 x} = \int c_1 e^{m_1 x} dx + c_2 = c_1 x + c_2 \text{ or } y = (c_1 x + c_2)e^{m_1 x}$$

Thus the complete solution of (1) is

$$y = (c_1 x + c_2)e^{m_1 x} + c_3 e^{m_3 x} + \ldots + c_n e^{m_n x}$$

If, however, the A.E. has three equal roots (i.e., $m_1 = m_2 = m_3$)

then the complete solution is

$$y = (c_1 x^2 + c_2 x + c_3)e^{m_1 x} + c_4 e^{m_4 x} + \ldots + c_n e^{m_n x}$$

Case III. If one pair of roots be complex, i.e., $m_1 = \alpha + i\beta, m_2 = \alpha - i\beta$, then the complete solution is

$$y = c_1 e^{(\alpha + i\beta)x} + c_2 e^{(\alpha - i\beta)x} + c_3 e^{m_3 x} + \ldots + c_n e^{m_n x}$$
$$= e^{\alpha x}\left(c_1 e^{i\beta x} + c_2 e^{-i\beta x}\right) + c_3 e^{m_3 x} + \ldots + c_n e^{m_n x}$$

$$y = e^{\alpha x}\left[c_1\left(\cos\beta x + i\sin\beta x\right) + c_2\left(\cos\beta x - i\sin\beta x\right)\right] + c_3 e^{m_3 x} + ... + c_n e^{m_n x}$$

$$y = e^{\alpha x}\left(C_1\cos\beta x + C_2\sin\beta x\right) + c_3 e^{m_3 x} + \ldots + c_n e^{m_n x} \text{ Where } C_1 = c_1 + c_2 \text{ and } C_2 = i\left(c_1 - c_2\right).$$

Case IV. If complex roots be equal i.e., $m_1 = m_2 = \alpha + i\beta, m_3 = m_4 = \alpha - i\beta$, then by case II, the complete solution is $y = e^{\alpha x}\left[\left(c_1 x + c_2\right)\cos\beta x + \left(c_3 x + c_4\right)\sin\beta x\right] + \ldots + c_n e^{m_n x}$

Example : The solution of the differential equation $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = 0$ with y(0)= y'(0)=1 is

(a) $(2-t)e^{t}$ (b) $(1+2t)e^{-t}$ (c) $(2+t)e^{-t}$ (d) $(1-2t)$

Solution : (b) Differential equation is $(D^2 + 2D + 1)\,y = 0$

Auxiliary equation is

$(D^2 + 2D + 1) = 0 \Rightarrow (D+1)^2 = 0 \Rightarrow D = -1, -1$

$\therefore$ solution is $y(t) = (c_1 + c_2 t)e^{-t}$

$y'(t) = c_2 e^{-t} + (c_1 + c_2 t)e^{-t}$

$y(0) = 1;\ y'(0) = 1$ gives $c_1 = 1$ and $c_2 + c_1(-1) = 1 \Rightarrow c_2 = 2$

$\therefore y(t) = (1 + 2t)e^{-t}$

Example : The complimentary function of the differential equation $\frac{d^4y}{dx^4} + 3\frac{d^2y}{dx^2} = 0$ is

(a) $\left[c_1 + c_2 x + c_3\sin\sqrt{3}x + c_4\cos\sqrt{3}x\right]$ (b) $\left[c_2 x + c_3\sin\sqrt{3}x + c_4\cos\sqrt{3}x\right]$

(c) $\left[c_1 + c_3\sin\sqrt{3}x + c_4\cos\sqrt{3}x\right]$ (d) $\left[c_1 + c_2 x + c_3\sin\sqrt{3}x + c_4\cos\sqrt{3}x\right]$

Solution : (a)

D.E is $\left(D^4 + 3D^2\right)y = 0,$

$A.E : m^4 + 3m^2 = 0 \Rightarrow m^2\left(m^2 + 3\right) = 0\ ;\ m = 0,\ 0,\ \pm\sqrt{3}\,i$

$\therefore$ $C.F. = \left(C_1 + C_2 x\right) + C_3\sin(\sqrt{3}x) + C_4\cos(\sqrt{3}x)$

4.6.6 INVERSE OPERATOR

$\frac{1}{f(D)}X$ is that function of x, not containing arbitrary constants which when operated upon by f(D) gives X.

i.e., $f(D)\left\{\frac{1}{f(D)}X\right\} = X$

Thus $\frac{1}{f(D)}X$ satisfies the equation f(D) y = X and is , therefore, its particular integral.

Obviously, f(D) and $\frac{1}{f(D)}$ are inverse operators.

Case : I $\frac{1}{D}X = \int X dx$

Let $\frac{1}{D}X = y$ ...(i)

Operating by D, $D\frac{1}{D}X = Dy$ i.e. $X = \frac{dy}{dx}$

Integrating both sides w.r.t. x.y = $\int X dx$ no constant being added as (i) does not contain any constant.

$\frac{1}{D}X = \int X\, dx.$

Case : II $\frac{1}{D-a}X = e^{ax}\int Xe^{-ax}dx$

Let, $\frac{1}{D-a}X = y$ ...(ii)

Operating by D-a, (D-a).$\frac{1}{D-a}X = (D-a)y.$

Or $X = \frac{dy}{dx} - ay, \text{i.e.,} \frac{dy}{dx} - ay = X$ which is a Leibnitz's linear equation.

$\therefore$ I.F. being e^{-ax}, its solution is

$y\ e^{-ax}, = \int Xe^{-ax}dx,$ no constant being added as (ii) doesn't contain any constant. Thus

$\frac{1}{D-a}X = y = e^{ax}\int Xe^{-ax}dx.$

4.6.7 RULES FOR FINDING THE PARTICULAR INTEGRAL

Consider the equation

$$\frac{d^n y}{dx^n} + k_1\frac{d^{n-1}y}{dx^{n-1}} + k_2\frac{d^{n-2}y}{dx^{n-2}} + \ldots + k_n y = X$$

Which is symbolic form of $\left(D^n + k_1 D^{n-1} + k_2 D^{n-2} + \ldots + k_n\right) y = X$

$$\text{P.I.} = \frac{X}{D^n + k_1 D^{n-1} + k_2 D^{n-2} + \ldots + k_n}$$

Case I. When X = e^{ax}

Since $D^n e^{ax} = a^n e^{ax}$

$$\therefore \left(D^n + k_1 D^{n-1} + k_2 D^{n-2} + \ldots + k_n\right) e^{ax} = \left(a^n + k_1 a^{n-1} + k_2 a^{n-2} + \ldots + k_n\right) e^{ax},$$

i.e. $f(D) e^{ax} = f(a) e^{ax}$

$$\frac{1}{f(D)} e^{ax} = \frac{1}{f(a)} e^{ax} \text{provided } f(a) \neq 0 \qquad \ldots(1)$$

If f(a) = 0, the above rule fails and we proceed further.

Since a is root of $\text{A.E.} f(D) = D^n + k_1 D^{n-1} + k_2 D^{n-2} + \ldots + k_n = 0$

$\therefore D - a$ is a factor of f(D). Suppose $f(D) = (D - a)\,\Phi(D)$,

where $\Phi(a) \neq 0$. Then

$$\frac{1}{f(D)} e^{ax} = \frac{1}{D-a} \cdot \frac{1}{\phi(D)} e^{ax} = \frac{1}{D-a} \cdot \frac{1}{\phi(a)} e^{ax} \text{ [By (1)]}$$

$$= \frac{1}{\phi(a)} \cdot \frac{1}{D-a} \cdot e^{ax} = \frac{1}{\phi(a)} \cdot e^{ax} \int e^{ax} \cdot e^{-ax} dx$$

$$\left[\text{as } \frac{1}{D-a} X = e^{ax} \int X e^{-ax} dx\right]$$

$$= \frac{1}{\phi(a)} e^{ax} \int dx = x \frac{1}{\phi(a)} e^{ax} \text{ i.e.}$$

$$\frac{1}{f(D)} e^{ax} = x \frac{1}{f'(a)} e^{ax} \qquad \ldots(2)$$

If $f'(a) = 0$, then applying (2) again, we get

$$\frac{1}{f(D)} e^{ax} = x^2 \frac{1}{f''(a)} e^{ax}, \text{ provided } f''(a) \neq 0 \quad \text{and so on.}$$

4.6.8 WORKING PROCEDURE TO SOLVE THE EQUATION

$$\frac{d^n y}{dx^n}+k_1\frac{d^{n-1}y}{dx^{n-1}}+\ldots+k_{n-1}\frac{dy}{dx}+k_n y=X$$

of which the symbolic form is

$$\left(D^n+k_1D^{n-1}+\ldots+k_{n-1}D+k_n\right)y=X$$

Step I. To find the complementary function

(i) Write the A.E.

i.e. $\left(D^n+k_1D^{n-1}+\ldots+k_{n-1}D+k_n\right)=0$ and solve it for D.

(ii) Write the C.F. as follows:

Roots of Auxiliary Equation(A.E.)	Complementary Function(C.F.)
1. $m_1, m_2, m_3\ldots$ (real and different roots)	$c_1e^{m_1x}+c_2e^{m_2x}+c_3e^{m_3x}+\ldots$
2. $m_1, m_1, m_3\ldots$. (two real and equal roots)	$(c_1+c_2x)e^{(m_1x)}+c_3e^{(m_3x)}+\ldots$
3. $m_1, m_1, m_1, m_4, \ldots$.(three real and equal roots)	$(c_1+c_2x+c_3x^2)e^{m_1x}+c_4e^{4x}+\ldots$
4. $\alpha+i\beta, \alpha-i\beta, m_3, \ldots$ (complex roots)	$e^{\alpha x}(c_1\cos\beta x+c_2\sin\beta x)+c_3e^{m_3x}+\ldots$
5. $\alpha\pm i\beta, \alpha\pm i\beta, m_5, \ldots$ (repeated complex roots)	$e^{\alpha x}\left[(c_1+c_2x)\cos\beta x+(c_3+c_4x)\sin\beta x\right]+C_5e^{m_5x}+\ldots$

Step. II To find the particular integral

From symbolic form

$$\text{P.I.}=\frac{1}{D^n+k_1D^{n-1}+\ldots+k_n}X=\frac{X}{f(D)}\text{ or }\frac{X}{\phi(D^2)}$$

(i) When $X=e^{ax}$

$$\text{P.I.}=\frac{1}{f(D)}e^{ax}=\frac{1}{f(a)}e^{ax},\ f(a)\neq 0$$

If $f(a)=0$, then

$$\text{P.I.}=\frac{1}{f(D)}e^{ax}=x\frac{1}{f'(a)}e^{ax},\ f'(a)\neq 0$$

If $f'(a)=0$, then

$$\text{P.I.}=\frac{1}{f(D)}e^{ax}=x^2\frac{1}{f''(a)}e^{ax},\ f''(a)\neq 0\text{ and so on.}$$

Example : For $\frac{d^2y}{dx^2}+4\frac{dy}{dx}+3y=3e^{2x}$, the particular integral is

(a) $\frac{1}{15}e^{2x}$ (b) $\frac{1}{5}e^{2x}$ (c) $3e^{2x}$ (d) $C_1e^{-x}+C_2e^{-3x}$

Solution : (b)

$$\frac{D^2y}{dx^2}+4\frac{dy}{dx}+3y=3e^{2x}$$

$$\Rightarrow (D^2+4D+3)y=3e^{2x}$$

Particular Integral $\text{P.I.}=\frac{1}{D^2+4D+3}\left(3e^{2x}\right)=\frac{1}{5}e^{2x}$

(ii) When X = $\sin(\mathbf{ax}+\mathbf{b})$ or $\cos(\mathbf{ax}+\mathbf{b})$.

$$\text{P.I.}=\frac{1}{\phi(D^2)}\sin(ax+b)\left[\text{or}\cos(ax+b)\right]$$

$$=\frac{1}{\phi(-a^2)}\sin(ax+b)\left[\text{or}\cos(ax+b)\right],\ \left[\phi(-a^2)\neq 0\right]$$

If $\left[\phi(-a^2)=0\right]$, then

$$\text{P.I.}=\frac{1}{\phi(D^2)}\sin(ax+b)\left[\text{or}\cos(ax+b)\right]$$

$$=x\frac{1}{\phi'(-a^2)}\sin(ax+b)\left[\text{or}\cos(ax+b)\right],\ \left[\phi'(-a^2)\neq 0\right]$$

If $\left[\phi'(-a^2)=0\right]$, then

$$\text{P.I.}=\frac{1}{\phi(D^2)}\sin(ax+b)\left[\text{or}\cos(ax+b)\right]$$

$$=x^2\frac{1}{\phi''(-a^2)}\sin(ax+b)\left[\text{or}\cos(ax+b)\right],\ \left[\phi''(-a^2)\neq 0\right]\text{and so on.}$$

(iii) When $\mathbf{X}=\mathbf{x}^m$, m being a positive integer.

$$\text{P.I.}=\frac{1}{f(D)}x^m=\left[f(D)\right]^{-1}x^m$$

To evaluate it, expand $\left[f(D)\right]^{-1}$ in ascending powers of D by Binominal theorem as far as D^m operate on x^m term by term.

(iv) When $X=e^{ax}V$, where V is a function of x.

$$\text{P.I.}=\frac{1}{f(D)}e^{ax}V=e^{ax}\frac{1}{f(D+a)}V$$

and then evaluate $\frac{1}{f(D+a)}V$ as in (i), (ii) and (iii).

(v) When X is any function of x.

$$\text{P.I.} = \frac{1}{f(D)}X$$

Resolve $\frac{1}{f(D)}$ into partial fractions and operate each partial fraction on X remembering

that $\frac{1}{D-a}X = e^{ax}\int Xe^{ax}dx.$

Step III. To find the complete solution

Then the C.S. is y = C.F. + P.I.

Example : The general solution of the differential equation $\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 9y = 0$ is

(a) $y = c_1e^{-2t}\cos(\sqrt{5}t) + e^{-2t}\sin(\sqrt{5}t)$ (b) $y = c_1e^{2t}\cos(\sqrt{5}t) + c_2e^{2t}\sin(\sqrt{5}t)$

(c) $y = c_1e^{2t}\cos(\sqrt{5}t) - c_2e^{2t}\sin(\sqrt{5}t)$ (d) $y = c_1e^{-2t}\cos(\sqrt{5}t) - e^{-2t}\sin(\sqrt{5}t)$

Solution : (b) Differential equation $\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 9y = 0$ have Characteristic equation,

$$m^2 - 4m + 9 = 0, \therefore m = \frac{4 \pm \sqrt{16-36}}{2} = 2 \pm \sqrt{5}i$$

Complementary solution is $y = c_1e^{2t}\cos(\sqrt{5}t) + c_2e^{2t}\sin(\sqrt{5}t)$

Here, $X = 0$, so Particular Integral (P.I.) = 0

Therefore, complete solution is $y = c_1e^{2t}\cos(\sqrt{5}t) + c_2e^{2t}\sin(\sqrt{5}t)$

Example: The P.I. of the differential equation $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = 2x + \cos x$

(a) $\frac{3\cos x + \sin x}{10} - \left(x + \frac{1}{2}\right)$ (b) $\frac{\sin x - 3\cos x}{10} + \left(x + \frac{1}{2}\right)$

(c) $\frac{-(3\cos x - \sin x)}{10} - \left(x + \frac{1}{2}\right)$ (d) none of these

Solution : (c) P.I. for $2x = \frac{2x}{D^2 + D - 2} = \frac{x}{-2\left(1 - \frac{D}{2} - \frac{D^2}{2}\right)} = -\frac{1}{2}\left(1 + \frac{D}{2} + \frac{D^2}{2} + \ldots\right)(2x) = -\left(x + \frac{1}{2}\right)$

$$\text{P.I. for } \cos x = \frac{1}{D^2+D-2}\cos x = \frac{1}{-1+D-2}\cos x = \frac{D+3}{D^2-9}\cos x = \frac{D+3}{-10}\cos x$$

$$= -\frac{1}{10}\left[-\sin x + 3\cos x\right] = \frac{-\left(3\cos x - \sin x\right)}{10},$$

$$\text{P.I.} = -\left(x+\frac{1}{2}\right) - \frac{3\cos x - \sin x}{10}$$

4.7 Second Order Linear PDE Classfication

Classification of Second Order Linear PDEs

The general form of second order partial Differential Equations

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = G$$

If $B^2 - 4AC < 0$, then the equation is called elliptic.

If $B^2 - 4AC = 0$, then the equation is called parabolic.

If $B^2 - 4AC > 0$, then the equation is called hyperbolic

Example : The type of partial differential equation $\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2} + 3\frac{\partial^2 P}{\partial x \partial y} + 2\frac{\partial P}{\partial x} - \frac{\partial P}{dy} = 0$ is

(a) elliptic (b) parabolic (c) hyperbolic (d) none of these

Solution : (c) Comparing the given equation with the general form of second order partial differential equation, we have

$$A = 1,\ B = 3,\ C = 1 \Rightarrow B^2 - 4AC = 5 > 0$$

$\therefore$ P.D.E is Hyperbolic

Example : The type of partial differential equation $\frac{\partial f}{\partial t} = \frac{\partial^2 f}{\partial x^2}$ is

(a) Parabolic (b) Elliptic (c) Hyperbolic (d) Nonlinear

Solution : (a) $\frac{\partial f}{\partial t} = \frac{\partial^2 f}{\partial x^2}$ Here $B^2 - 4AC = 0$, The equation is parabolic

4.8 Method of Separation of Variables

The basic idea of this method is to transform a partial differential equation into as many differential equations as the number of independent by representing the solution as a product of function of each independent variable.

4.8.1 Two dimensional Laplace Equations

Consider the steady - state heat conduction (or potential) problem for the rectangle

$R\left\{0 < x < a,\ 0 < y < b\right\}$: Physically, this problem arises if three edges of a thin isotropic rectangular plate *(R)* are insulated and maintained at zero temperature, while the fourth edge is subjected to a variable temperature f(x) until the steady-state conditions are attained rough R. Then the steady-state value of *u(x,* y) represents distribution of temperature in the interior of the plate and u(x, y) satisfies two dimensional Laplace equation.

$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$, Boundary conditions : $\mathrm{u}(0, y) = 0 = u(a, y)\,; u(x, 0) = 0; u(x, b) = f(x)$

Method of Separation of Variables

Let$(x, y) = X(x)\, Y(y)$ be solution of the equation. Then

$\frac{\partial^2 u}{\partial x^2} = X''Y$ and $\frac{\partial^2 u}{\partial y^2} = X(x)\, Y''(y)$

Putting these values in the equation we get

$\mathrm{X}''(x)Y(y) + X(x)\, Y''(y) = 0$

$\frac{\mathrm{X}''(x)}{\mathrm{X}(x)} = -\frac{Y''(y)}{Y(y)} = k$

$X''(x) - kX = 0\ 0,\ Y''(y) + kY = 0$

When $k = 0 : X''(x) = 0,\ Y''(y) = 0$ (differential equations)

$X = C_1 + C_2\, x \text{ and } Y = C_3 + C_4 y$

$u(x, y) = (C_1 + C_2 x)(C_3 + C_4 y)$

When $K > 0 (k = p^2)$: $X''(x) - p^2 X = 0,\ Y''(y) + p^2 Y = 0$

$u(x, y) = (C_1 e^{-px} + C_2 e^{px})(C_3 \cos py + C_4 \sin py)$

When $K > 0 (k = -p^2)$: $X''(x) - p^2 X = 0,\ Y''(y) + p^2 Y = 0$

$u(x, y) = (\mathrm{C}_1 \cos px + C_2 \sin px)(C_3 e^{-py} + C_4 e^{py})$

4.8.2 One dimensional Heat equation

$\mathrm{C}^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$

Boundary conditions : $u(0, t) = 0$, and $u(L, t) = 0,\ t > 0$ (Ends Kept at Zero temperature)

Initial Conditions : Initial temperature $\mathrm{u}(x, 0) = f(x)$

General solution : $\mathrm{u}(x, t) = \sum_{n=1}^{\infty} a_n e^{-\left(\frac{c^2 n^2 \pi^2}{t^2}\right)t} \sin\left(\frac{n\pi}{L} x\right)$

Where $\mathrm{u}(x, 0) = f(x)$

$\Rightarrow \quad a_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L} x\right) dx$

4.8.3 One Dimensional Wave Equation

Consider an elastic string of length *(l)*, fixed at its ends on the *x* axis at *x* = 0 and *x* = *L*. The displacement *u(x, t)* at point *x* (0 < *x* < *L)* at any time *t* > 0 satisfies One Dimensional Wave equation.

$$c^2\frac{\partial^2 u}{\partial x^2}-\frac{\partial^2 u}{\partial t^2}=0, \qquad 0<x<\text{L};\ \text{t}=0$$

Boundary conditions (BC) : u(0, t) = 0, and u (L., t) = 0, t > 0.

Initial conditions (IC) : lnitial displacement u(x, 0) =f(x)

Initial velocity u_t (x, 0) =g(x).

Method of Separation of Variables

Let $\text{u}(x,t)=X(x)\,T(t)$ be a solution of the equation.

$$\text{c}^2X''(x)\text{T}(t)-\text{X}(x)\ \text{T}''(\text{t})=0$$

$$\frac{\text{X}''(x)}{X(x)}=\frac{T''(t)}{c^2T(t)}=k$$

$$X''-kX=0 \text{ and } \text{T}''-kc^2T=0$$

When $k=0: u(x,t)=(C_1+C_2x)(C_3+C_4t)$

When $\text{K}>0\left(k=p^2\right): \text{X}''-p^2X=0,\ T''(t)-p^2c^2T=0$

$$u(x,y)=\left(C_1e^{-px}+C_2e^{px}\right)\left(C_3e^{-cpt}+C_4e^{cpt}\right)$$

When $\text{K}<0\left(k=-p^2\right): \text{X}''(x)+p^2X=0,\ T''(t)+p^2c^2T=0$

$$u(x,y)=\left(C_1\cos px+C_2\sin px\right)\left(C_3\cos cpt+C_4\sin cpt\right)$$

Using the Boundary & Initial conditions, we get general solution.

$$\text{u}(x,t)=\sum_{n=1}^{\infty}\left[a_n cos\left(\frac{cn\pi}{L}t\right)+b_n\sin\left(\frac{cn\pi}{L}t\right)\right]\sin\left(\frac{n\pi}{L}x\right)$$

$$a_n=\frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)dx$$

$$b_n=\frac{2}{cn\pi}\int_0^L g(x)\sin\left(\frac{n\pi}{L}x\right)dx$$

4.8.4 D' Alembert's Solution

Consider one Dimensional Wave equation $c^2\frac{\partial^2 u}{\partial x^2}-\frac{\partial^2 u}{\partial t^2}=0, \quad 0<x<\text{L};\ \text{t}=0,$

u (0, t) = 0, and u (L, t) = 0, t > 0.

Initial displacement u(x, 0) = f(x) ; Initial velocity u(x, 0) =0.

The general solution is given by $u(x,\ t) = \dfrac{f(x+ct) + f(x-ct)}{2}$

4.8.5 One dimensional Heat Equation or Diffusion Equation

Consider a perfectly insulated thin bar of length L. The temperature distribution *u(x, t)* at any point 0 < x < L at any time t > 0 satisfies one dimensional Heat equation

$$C^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$$

Ends of the Bar Kept at Temperature Zero

Boundary conditions (BC): u(0, t) = 0, and u (L,t) = 0, t > 0.

Initial conditions (IC) : lnitial Temperature u(x, 0) = f(x)

Method of Separation of Variables :

Let u(x, t) = X(x) T(t) he a solution of the equation.

$$c^2 X''(x) T(t) = X(x) T'(t)$$

$$\frac{X''(x)}{X(x)} = \frac{T'(t)}{c^2 T(t)} = k$$

$$X'' - kX = 0 \; and \; T' - kc^2\, T = 0$$

For Heat conduction the solution must be transient solution. i.e., *u(x, t)* decreases with time.

When $K < 0\left(k = -p^2\right) : X^2 + p^2 X = 0 \; T' + p^2 c^2 T = 0$

$$u(x, y) = \left(C_1 \cos px + C_2 \sin px\right) C_3 e^{-c^2 p^2 t}$$

Using the BC's and IC's we get the general solution

$$u(x,t) = \sum_{n=1}^{\infty} a_n e^{-\left(\frac{c^2 n^2 \pi^2}{t^2}\right)t} \sin\left(\frac{n\pi}{L} x\right)$$

$$a_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi}{L} x\right) dx$$

GATE QUESTIONS

Q.1 The degree of the differential equation $\frac{d^2x}{dt^2}+2x^3=0$ is

(a)0 (b)1

(c)2 (d)3 **[GATE-2005 (EC)]**

Q.2 The order and degree of the differential equation $3\frac{d^2y}{dt^2}+4\left(\frac{dy}{dt}\right)^3+y^2+2=x$ are respectively

(a) 3 and 2 (b) 2 and 3

(c) 3 and 3 (d) 3 and 1 **[GATE-2005 (EC)]**

Q.3 The Blasius equation, $\frac{d^3f}{d\eta^3}+\frac{f}{2}\frac{d^2f}{d\eta^2}=0$ is

(a) Second order nonlinear ordinary differential equation

(b) Third order nonlinear ordinary differential equation

(c) Third order linear ordinary differential equation

(d) Mixed order nonlinear ordinary differential equation **GATE-2005 (EC)]**

Q.4 The following differential equation

$$\frac{d^2y}{dt^2}+4\sqrt{\left(\frac{dy}{dt}\right)^3-y}=0$$

(a) Degree = 2, order = 1 (b) Degree = 1, order =2

(c) Degree = 2, order = 2 (d) Degree = 3, order = 3

Statement for linked Answer Questions 5 and 6:

The complete solution of the ordinary differential equation

$$\frac{d^2y}{dx^2}+p\frac{dy}{dx}+qy=0 \text{ is } y=C_1e^{-x}+C_2e^{-3x}$$

Q.5 Then, p and q are

(a) p = 3, q = 3 (b) p = 3, q = 4

(c) p = 4, q = 3 (d) p = 4, q = 4 **[GATE-2005 (ME)]**

Q.6 Which of the following is solution of the differential equation

$$\frac{d^2y}{dx^2}+p\frac{dy}{dx}+(q+1)=0?$$

(a) e^{-3x} (b) xe^{-x}

(c) xe^{-2x} (d) x^2e^{-2x} **[GATE-2005 (ME)]**

Q.7 A solution of the following differential equation is given by $\frac{d^2y}{dx^2}-5\frac{dy}{dx}+6y=0$

(a) $y=e^{2x}+e^{-3x}$ (b) $y=e^{2x}+e^{3x}$

(c) $y=e^{-2x}+e^{3x}$ (d) $y=e^{-2x}+e^{-3x}$ **[GATE-2005 (EC)]**

Q.8 The solution of $\frac{d^2y}{dx^2}+2\frac{dy}{dx}+17y=0;\ y(0)=1, \frac{dy}{dx}\left(\frac{\pi}{4}\right)=0$ in the range $0<x<\frac{\pi}{4}$ is given by

(a) $e^{-x}\left(\cos 4x+\frac{1}{4}\sin 4x\right)$ (b) $e^{x}\left(\cos 4x-\frac{1}{4}\sin 4x\right)$

(c) $e^{-4x}\left(\cos x-\frac{1}{4}\sin x\right)$ (d) $e^{-4x}\left(\cos 4x-\frac{1}{4}\sin 4x\right)$ **[GATE-2005 (CE)]**

Q.9 For $\frac{d^2y}{dx^2}+4\frac{dy}{dx}+3y=3e^{2x}$, the particular integral is

(a) $\frac{1}{15}e^{2x}$ (b) $\frac{1}{5}e^{2x}$

(c) $3e^{2x}$ (d) $C_1e^{-x}+C_2e^{-3x}$ **[GATE-2006 (ME)]**

Q.10 The solution of the differential equation $\frac{dy}{dx}+2xy=e^{-x^2}$ with $y(0)=1$ is

(a) $(1+x)e^{x^2}$ (b) $(1+x)e^{-x^2}$

(c) $(1-x)e^{x^2}$ (d) $(1-x)e^{-x^2}$ **[GATE-2006 (ME)]**

Q.11 The solution of $x\frac{dy}{dx}+y=x^4$ with the condition $y(1) = \frac{6}{5}$ is

(a) $y=\frac{x^4}{5}+\frac{1}{x}$ (b) $y=\frac{4x^4}{5}+\frac{4}{5x}$

(c) $y=\frac{x^4}{5}+1$ (d) $y=\frac{x^5}{5}+1$ **[GATE-2006 (ME)]**

Q.12 A spherical naphthalene ball exposed to the atmosphere loses volume at a rate proportional to its instantaneous surface are due to evaporation. If the initial diameter of the ball is 2 cm and the diameter reduces to 1 cm after 3 months, the ball completely evaporates in

(a)6 months (b) 9 months
(c)12 months (d)infinite time **[GATE-2006 (CE)]**

Q.13 The solution for the differential equation $\frac{dy}{dx} = x^2 y$ with the condition that y=1 at x=0 is

(a) $y = e^{\frac{1}{2x}}$ (b) $\ln(y) = \frac{x^3}{3} + 4$

(c) $\ln(y) = \frac{x^2}{2}$ (d) $y = e^{\frac{x^3}{3}}$ **[GATE-2007 (CE)]**

Q.14 A body originally at $60°C$ cools down to $40°C$ in 15 minutes when kept in air at a temperature of $25°C$. What will be the temperature of the body at the end of 30 minutes

(a) $35.2°C$ (b) $31.5°C$
(c) $28.7°C$ (d) $15°C$ **[GATE-2007 (CE)]**

Q.15 It is given that $y'' + 2y' + y = 0$, y(0) = 0, y(1)=0. What is y(0.5)?

(a)0 (b)0.37
(c)0.62 (d)1.13 **[GATE-2008 (ME)]**

Q.16 Given that $\ddot{x} + 3x = 0$ and x(0) = 1, $\dot{x}(0) = 0$, what is x(1)?

(a)-0.99 (b)-0.16
(c)0.16 (d)0.99 **[GATE-2008 (ME)]**

Q.17 The general solution of $\frac{d^2y}{dx^2} + y = 0$ is

(a) $y = P\cos x + Q\sin x$ (b) $y = P\cos x$
(c) $y = P \sin x$ (d) $y = P\sin^2 x$ **[GATE-2008 (CE)]**

Q.18 The order of the differential equation $\left(\frac{d^2y}{dt^2}\right) + \left(\frac{dy}{dt}\right)^3 + y^4 = e^{-t}$ is

(a) 1 (b)2
(c) 3 (d) 4 **[GATE-2009 (EC)]**

Q.19 Solution of the differential equation $3y\frac{dy}{dx} + 2x = 0$ represents a family of

(a) ellipses (b) Circles
(c) parabolas (d) Hyperbolas **[GATE-2009(CE)]**

Q.20 The solution of $\frac{dy}{dx} = y^2$ with initial value y(0) = 1 bounded in the interval

(a) $-\infty \le x \le \infty$ (b) $-\infty < x < 0$

(c) $x < 1, x > 1$ (d) $-2 \le x \le 2$ **[GATE-2009(CE)]**

Q.21 The solution of the first order differential equation $x'(t) = -3x(t), x(0) = x_0$ is

(a) $x(t) = x_0 e^{-3t}$ (b) $x(t) = x_0 e^{3t}$

(c) $x(t) = x_0 e^{-\frac{t}{3}}$ (d) $x(t) = x_0 e^{-t}$ **[GATE-2009(CE)]**

Q.22 Match List-I with List-II and select the correct answer using the codes given below the lists:

List-I	**List-II**
(A) $\frac{dy}{dx} = \frac{y}{x}$	1)Circles
(B) $\frac{dy}{dx} = -\frac{y}{x}$	2)Straight lines
(C) $\frac{dy}{dx} = \frac{x}{y}$	3)Hyperbolas
(D) $\frac{dy}{dx} = -\frac{x}{y}$	

Codes :	**A**	**B**	**C**	**D**
(a)	2	3	3	1
(b)	1	3	2	1
(c)	2	1	3	3
(d)	3	2	1	2

[GATE-2009(CE)]

Q.23 Transformation to linear form by substituting $v = y^{1-n}$ of the equation

$\frac{dy}{dt} + p(t)y = q(t)y^n;\ n > 0$ will be

(a) $\frac{dv}{dt} + (1-n)pv = (1-n)q$ (b) $\frac{dv}{dt} + (1-n)pv = (1+n)q$

(c) $\frac{dv}{dt} + (1+n)pv = (1-n)q$ (d) $\frac{dv}{dt} + (1+n)pv = (1+n)q$ **[GATE-2009(CE)]**

Q.24 The solution to the ordinary differential equation $\frac{d^2y}{dx^2}+\frac{dy}{dx}-6y=0$ is

(a) $y=c_1e^{3x}+c_2e^{-2x}$ (b) $y=c_1e^{3x}+c_2e^{2x}$

(c) $y=c_1e^{-3x}+c_2e^{2x}$ (d) $y=c_1e^{-3x}+c_2e^{-2x}$ **[GATE-2010 (CE)]**

Q.25 For the differential equation $\frac{d^2x}{dt^2}+6\frac{dx}{dt}+8x=0$ with initial conditions x(0) = 1 & $\left.\frac{dx}{dt}\right|_{t=0}=0$ the solution is

(a) $x(t)=2e^{-6t}-e^{-2t}$ (b) $x(t)=2e^{-2t}-e^{-4t}$

(c) $x(t)=-e^{-6t}+2e^{-4t}$ (d) $x(t)=e^{-2t}+2e^{-4t}$ **[GATE-2010 (EE)]**

Q.26 Consider the differential equation $x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}-4y=0$ with the boundary conditions of y(0) = 0 and y(1) = 1. The complete solution of the differential equation is

(a) x^2 (b) $\sin\left(\frac{\pi x}{2}\right)$

(c) $e^x\sin\left(\frac{\pi x}{2}\right)$ (d) $e^{-x}\sin\left(\frac{\pi x}{2}\right)$ **[ME-GATE-2012]**

Q.27 The solution of the ordinary differential equation $\frac{dy}{dx}+2y=0$ for the boundary condition, y = 5 at x = 1 is

(a) $y=e^{-2x}$ (b) $y=2e^{-2x}$

(c) $y=10.95e^{-2x}$ (d) $y=36.95e^{-2x}$ **[CE-GATE-2012]**

Q.28 The partial differential equation $\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}=\frac{\partial^2 u}{\partial x^2}$ is a

(a) Linear equation of order 2 (b) Non-linear equation of order 1

(c) Linear equation of order 1 (d) Non-linear equation of order 2

[ME-GATE-2013]

Q.29 The value of the definite integral $\int_1^e \sqrt{x}\ln(x)\,dx$ is

(a) $\frac{4}{9}\sqrt{e^3}+\frac{2}{9}$ (b) $\frac{2}{9}\sqrt{e^3}-\frac{4}{9}$

(c) $\frac{2}{9}\sqrt{e^3}+\frac{4}{9}$ (d) $\frac{4}{9}\sqrt{e^3}-\frac{2}{9}$ **[ME-GATE-2013]**

Q.30 The type of partial differential equation $\frac{\partial f}{\partial t} = \frac{\partial^2 f}{\partial x^2}$ is

(a) Parabolic
(b) Elliptic
(c) Hyperbolic
(d) Nonlinear **[IN-GATE-2013]**

Q.31 If $y = f(x)$ is solution of $\frac{d^2y}{dx^2} = 0$ with the boundary conditions y=5 at x=0, and $\frac{dy}{dx} = 2$ at x=10, f(15) = _____ **[ME-GATE-2014(1)]**

Q.32 The general solution of the differential equation $\frac{dy}{dx} = \cos(x+y)$ with c as a constant, is

(a) $y + \sin(x+y) = x + c$
(b) $\tan\left(\frac{x+y}{2}\right) = y + c$
(c) $\cos\left(\frac{x+y}{2}\right) = x + c$
(d) $\tan\left(\frac{x+y}{2}\right) = x + c$ **[ME-GATE-2014(2)]**

Q.33 Consider two solutions $x(t) = x_1(t)$ and $x(t) = x_2(t)$ of the differential equation $\frac{d^2x(t)}{dt^2} + x(t) = 0, t > 0$ such that $x_1(0) = 1, \frac{dx_1(t)}{dt}\Big|_{t=0} = 0,\ x_2(0) = 0, \frac{dx_2(t)}{dt}\Big|_{t=0} = 1$

The Wronskian $W(t) = \begin{vmatrix} x_1(t) & x_2(t) \\ \frac{dx_1(t)}{dt} & \frac{dx_2(t)}{dt} \end{vmatrix}$ at $t = \pi/2$ is

(a) 1
(b) -1
(c) 0
(d) $\pi/2$ **[ME-GATE-2014(3)]**

Q.34 The solution of the initial value problem $\frac{dy}{dx} = -2xy;\ y(0) = 2$ is

(a) $1 + e^{-x^2}$
(b) $2e^{-x^2}$
(c) $1 + e^{x^2}$
(d) $2e^{x^2}$ **[ME-GATE-2014(4)]**

Q.35 If the characteristic equation of the differential equation $\frac{d^2y}{dx^2} + 2\alpha\frac{dy}{dx} + y = 0$ has two equal roots, then the values of α are

(a) ± 1
(b) 0, 0
(c) $\pm j$
(d) $\pm\frac{1}{2}$ **[EC-GATE-2014(2)]**

Q.36 Which ONE of the following is a linear non-homogeneous differential equation, where x & y are the independent & dependent variables respectively?

(a) $\frac{dy}{dx}+xy=e^{-x}$ (b) $\frac{dy}{dx}+xy=0$

(c) $\frac{dy}{dx}+xy=e^{-y}$ (d) $\frac{dy}{dx}+e^{-y}=0$ **[EC-GATE-2014(3)]**

Q.37 If a and b are constants, the most general solution of the differential equation $\frac{d^2x}{dt^2}+2\frac{dx}{dt}+x=0$ is

(a) ae^{-t} (b) $ae^{-t}+bte^{-t}$

(c) $ae^{t}+bte^{-t}$ (d) ae^{-2t} **[EC-GATE-2014(4)]**

Q.38 With initial values y(0) = y'(0)=1 the solution of the differential equation

$\frac{d^2y}{dx^2}+4\frac{dy}{dx}+4y=0$ at x =1 is________. **[EC-GATE-2014(4)]**

Q.39 The solution for the differential equation $\frac{d^2x}{dt^2}=-9x$ with initial conditions x(0)=1 and $\left.\frac{dx}{dt}\right|_{t=0}=1$ is

(a) t^2+t+1 (b) $\sin 3t+\frac{1}{3}\cos 3t+\frac{2}{3}$

(c) $\frac{1}{3}\sin 3t+\cos 3t$ (d) $\cos 3t+t$ **[EE-GATE-2014(1)]**

Q.40 Consider the differential equation $x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}-y=0$. Which one of the following is a solution of this differential equation for x > 0?

(a) e^x (b) x^2

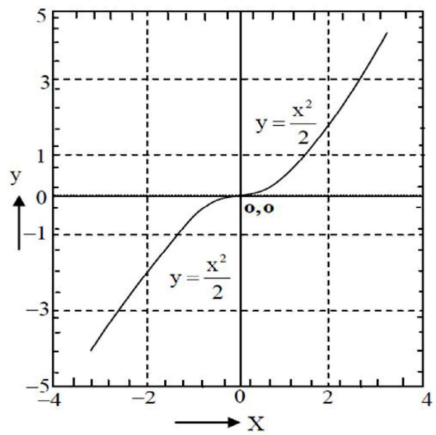
(c) $\frac{1}{x}$ (d) $\ln x$ **[EE-GATE-2014(2)]**

Q.41 The maximum value of the solution y(t) of the differential equation $y(t)+\ddot{y}(t)=0$ with initial conditions $\dot{y}(0)=1$ and y (0) = 2, for t > 0 is

(a) 1 (b) 2

(c) $\sqrt{5}$ (d) $\sqrt{2}$ **[IN-GATE-2014]**

Q.42 The figure shows the plot of y as a function of x



The function shown is the solution of the differential equation (assuming all initial conditions to be zero) is:

(a) $\frac{d^2y}{dx^2} = 1$ (b) $\frac{dy}{dx} = x$

(c) $\frac{dy}{dx} = -x$ (d) $\frac{dy}{dx} = |x|$ **[IN-GATE-2014]**

Q.43 The solution of $\frac{d^2y}{dx^2} = y$ which passes through the origin and the point $\left(\ln 2, \frac{3}{4}\right)$, is

(a) $y = \frac{1}{2}e^x - e^{-x}$ (b) $y = \frac{1}{2}\left(e^x + e^{-x}\right)$

(c) $y = \frac{1}{2}\left(e^x - e^{-x}\right)$ (d) $y = \frac{1}{2}e^x + e^{-x}$ **[ME-GATE-2015]**

Q.44 Consider the following differential equation: $\frac{dy}{dt} = -5y$, initial condition : y = 2 at t = 0.

The value of y at t = 3 is

(a) $-5e^{-10}$ (b) $2e^{-10}$

(c) $2e^{-15}$ (d) $-15e^2$ **[ME-GATE-2015(2)]**

Q.45 Consider the following differential equation: $x\left(ydx + xdy\right)\cos\left(\frac{y}{x}\right) = y\left(xdy - ydx\right)\sin\left(\frac{y}{x}\right)$

Which of the following is the solution of the above equation (c is an arbitrary constant)?

(a) $\frac{x}{y}\cos\left(\frac{x}{y}\right) = c$ (b) $\frac{x}{y}\sin\left(\frac{y}{x}\right) = c$

(c) $xy\cos\left(\frac{y}{x}\right) = c$ (d) $xy\sin\left(\frac{y}{x}\right) = c$ **[CE-GATE-2015(1)]**

Q.46 Consider the following second order linear differential equation $\frac{d^2y}{dx^2} = -12x^2 + 24x - 20$, The boundary conditions are at x= 0, y = 5 and at x = 2, y = 21 The value of at x = 1 is _____.

[CE-GATE-2015(2)]

Q.47 The solution of the differential equation $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = 0$ with y(0)= y'(0)=1 is

(a) $(2-t)e^{t}$ (b) $(1+2t)e^{-t}$

(c) $(2+t)e^{-t}$ (d) $(1-2t)e^{t}$ **[EC-GATE-2015(1)]**

Q.48 The general solution of the differential equation $\frac{dy}{dx} = \frac{1+\cos 2y}{1-\cos 2x}$ is

(a) tan y – cot x = c (b) tan x – cot y = c

(c) tan y + cot x = c (d) tan x + cot y = c **[EC-GATE-2015(2)]**

Q.49 Consider the differential equation $\frac{d^2x(t)}{dt^2} + 3\frac{dx(t)}{dt} + 2x(t) = 0$. Given x(0) = 20 and $x(1) = \frac{10}{e}$ where e = 2.71. The value of x(2) is __________. **[EC-GATE-2015(3)]**

Q.50 A solution of the ordinary differential equation $\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 6y = 0$ is such that $y(0) = 2$ and $y(1) = -\frac{1-3e}{e^3}$ The Value of $\frac{dy}{dt}(0)$ is_________. **[EE-GATE-2015(1)]**

Q.51 If y = f (x) satisfies the boundary value problem

$y'' + 9y = 0,\ y(0) = 0,\ y(\pi/2) = \sqrt{2}$, then $y(\pi/4)$ is ______. **[ME-GATE -2016(1)]**

Q.52 The type of partial differential equation $\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2} + 3\frac{\partial^2 P}{\partial x \partial y} + 2\frac{\partial P}{\partial x} - \frac{\partial P}{dy} = 0$ is

(a) elliptic (b) parabolic

(c) hyperbolic (d) none of these **[CE-GATE-2016(1)]**

Q.53 The solution of the partial differential equation $\frac{\partial u}{\partial t} = \alpha\frac{\partial^2 u}{\partial x^2}$ is of the form

(a) $C\cos(kt)\left[C_1 e^{(\sqrt{k/\alpha})x} + C_2 e^{-(\sqrt{k/\alpha})x}\right]$ (b) $Ce^{kt}\left[C_1 e^{(\sqrt{k/\alpha})x} + C_2 e^{-(\sqrt{k/\alpha})x}\right]$

(c) $Ce^{kt}\left[C_1\cos\left(\sqrt{k/\alpha}\right)x + C_2\sin\left(-\sqrt{k/\alpha}\right)x\right]$ (d) $C\sin(kt)\left[C_1\cos\left(\sqrt{k/\alpha}\right)x + C_2\sin\left(-\sqrt{k/\alpha}\right)x\right]$

[CE-GATE-2016(1)]

Q.54 The respective expressions for complimentary function and particular integral part of the solution of the differential equation $\frac{d^4y}{dx^4}+3\frac{d^2y}{dx^2}=108x^2$ are

(a) $\left[c_1+c_2x+c_3\sin\sqrt{3}x+c_4\cos\sqrt{3}x\right]$ and $\left[3x^4-12x^2+c\right]$

(b) $\left[\ c_2x+c_3\sin\sqrt{3}x+c_4\cos\sqrt{3}x\right]$ and $\left[5x^4-12x^2+c\right]$

(c) $\left[c_1+c_3\sin\sqrt{3}x+c_4\cos\sqrt{3}x\right]$ and $\left[3x^4-12x^2+c\right]$

(d) $\left[c_1+c_2x+c_3\sin\sqrt{3}x+c_4\cos\sqrt{3}x\right]$ and $\left[5x^4-12x^2+c\right]$ **[CE-GATE-2016(1)]**

Q.55 Which one of the following is a property of the solutions to the Laplace equation : $\nabla^2 f=0$?

(a) The solutions have neither maxima nor minima anywhere except at the boundaries.

(b) The solutions are not separable in the coordinates.

(c) The solutions are not continuous.

(d) The solutions are not dependent on the boundary conditions. **[EC-GATE-2016(1)]**

Q.56 The ordinary differential equation $\frac{dx}{dt}=-3x+2$ with x(0)=1 is to be solved using the forward Euler method. The largest time step that can be used to solve the equation without making the numerical solution unstable is ________. **[EC-GATE-2016(2)]**

Q.57 The particular solution of the initial value problem given below is

$$\frac{d^2y}{dx^2}+12\frac{dy}{dx}+36y=0 \text{ with } y(0)=3 \text{ and } \left.\frac{dy}{dx}\right|_{x=0}=-36$$

(a) $(3-18x)e^{-16x}$ (b) $(3+25x)e^{-6x}$

(c) $(3+20x)e^{-6x}$ (d) $(3-12x)e^{-6x}$ **[EC-GATE-2016(3)]**

Q.58 Consider the following partial differential equation u(x, y) with the constant c >1:

$\frac{\partial u}{\partial y}+c\frac{\partial u}{\partial x}=0$ Solution of this equation is

(a) $u(x,y)=f(x+cy)$ (b) $u(x,y)=f(x-cy)$

(c) $u(x,y)=f(cx+y)$ (d) $u(x,y)=f(cx-y)$ **[GATE -2017]**

Q.59 The differential equation $\frac{d^2y}{dx^2}+16y=0$ for y (x) with the two boundary conditions

$\left.\frac{dy}{dx}\right|_{x=0}=1$ and $\left.\frac{dy}{dx}\right|_{x=\frac{\pi}{2}}=-1$ has

(a) No solution (b) Exactly two solutions

(c) Exactly one solution (d) Infinitely many solutions **[GATE -2017]**

Q.60 Consider the differential equation $3y''(x)+27y(x)=0$ with initial Conditions $y(0)=0$ and $y'(0)=2000$. The value of y at x =1 is ____. **[GATE-2017]**

Q.61 The solution (up to three decimal places) at x=1 of the differential equation $\frac{d^2y}{dx^2}+2\frac{dy}{dx}+y=0$ subject to boundary y(0)=1 and $\frac{dy}{dx}(0)$ = 1 is ______.

[CE-GATE-2018(1)]

Q.62 The solution at x=1, t=1 of the partial differential equation $\frac{\partial^2 u}{\partial x^2}=25\frac{\partial^2 u}{\partial t^2}$ subject to initial condition of u(0) = 3x and $\frac{\partial u}{\partial t}(0)=3$ is_______.

(a)1 (b)2

(c)4 (d)6 **[CE-GATE-2018(1)]**

Q.63 The solution of the equation $x\frac{dy}{dx}+y=0$ passing through the point (1,1) is

(a) x (b) x^2

(c) x^{-1} (d) x^{-2} **[CE-GATE-2018(2)]**

Q.64 Let $f(x,y)=\frac{ax^2+by^3}{xy}$, where a and b are constants. If $\frac{\partial f}{\partial x}=\frac{\partial f}{\partial y}$ at x =1 and y = 2, then the relation between a and b is

(a) $a=\frac{b}{4}$ (b) $a=\frac{b}{2}$

(c) a = 2b (d) a = 4b **[EC-GATE-2018]**

Q.65 The position of a particle y(t) is described by the differential equation :

$\frac{d^2y}{dt^2}=-\frac{dy}{dt}-\frac{5y}{4}$.The initial conditions are y(0) = 1 and $\left.\frac{dy}{dt}\right|_{t=0}=0$. The position (accurate to two decimal places) of the particle at $t=\pi$ is__________.

[EC-GATE-2018]

Q.66 Let $r = x^2 + y - z$ and $z^3 - xy + yz + y^3 = 1$. Assume that x and y are independent variables. At (x,y,z)=(2, -1, 1), the value (correct to two decimal places) of $\frac{\partial r}{\partial x}$ is ____________.

[EC-GATE-2018]

Q.67 A curve passes through the point (x = 1, y = 0) and satisfies the differential equation $\frac{dy}{dx} = \frac{x^2 + y^2}{2y} + \frac{y}{x}$. The equation that describes the curve is

(a) $\ln\left(1 + \frac{y^2}{x^2}\right) = x - 1$ (b) $\frac{1}{2}\ln\left(1 + \frac{y^2}{x^2}\right) = x - 1$

(c) $\ln\left(1 + \frac{y}{x}\right) = x - 1$ (d) $\frac{1}{2}\ln\left(1 + \frac{y}{x}\right) = x - 1$ **[EC-GATE-2018]**

Q.68 Consider the following equations $\frac{\partial V(x,y)}{\partial x} = px^2 + y^2 + 2xy$, $\frac{\partial V(x,y)}{\partial y} = x^2 + qy^2 + 2xy$. Where p and q are constants. V(x,y) that satisfies the above equations is

(a) $p\frac{x^3}{3} + q\frac{y^3}{3} + 2xy + 6$ (b) $p\frac{x^3}{3} + q\frac{y^3}{3} + 5$

(c) $p\frac{x^3}{3} + q\frac{y^3}{3} + x^2y + xy^2 + xy$ (d) $p\frac{x^3}{3} + q\frac{y^3}{3} + x^2y + xy^2$ **[IN-GATE-2018]**

Q.69 Consider a function u which depends on position x and time t. The partial differential equation $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ is known as the

(a) Wave equation (b) Heat equation

(c) Laplace's equation (d) Elasticity equation **[ME-GATE-2018(2)]**

Q.70 If y is the solution of the differential equation $y^3\frac{dy}{dx} + x^3 = 0,\ y(0) = 1$, the value of $y(-1)$ is

(a) -2 (b) -1

(c) 0 (d) 1 **[ME-GATE-2018(2)]**

Q.71 Given the ordinary differential equation $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = 0$ With y(0)=0 and $\frac{dy}{dx}(0) = 1$ the value of y(1) is ________ (correct to two decimal places). **[ME-GATE-2018(2)]**

Answer Key :

1	2	3	4	5	6	7	8	9	10	11	12	13	14
b	a	b	b	c	c	b	a	b	b	a	a	d	b
15	**16**	**17**	**18**	**19**	**20**	**21**	**22**	**23**	**24**	**25**	**26**	**27**	**28**
a	b	a	b	a	c	a	a	a	c	b	a	d	d
29	**30**	**31**	**32**	**33**	**34**	**35**	**36**	**37**	**38**	**39**	**40**	**41**	**42**
c	a	36	d	a	b	a	a	b	0.54	c	c	c	d
43	**44**	**45**	**46**	**47**	**48**	**49**	**50**	**51**	**52**	**53**	**54**	**55**	**56**
c	c	c	-2	b	b	0.85	-3	1	c	b	a	a	0.66
57	**58**	**59**	**60**	**61**	**62**	**63**	**64**	**65**	**66**	**67**	**68**	**69**	**70**
a	b	a	94	.368	d	c	d	.21	4.5	a	d	b	c
71													
1.47													

ESE QUESTIONS

Q.1 If $u = \log\left(\frac{x^2 + y^2}{x + y}\right)$, what is the value of $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$?

(a) 0 (b) 1

(c) u (d) e^u **[ESE-2018]**

Q.2 The solution of the differential equation $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = 3e^{2x}$

Where , y(0) =0 and y'(0) = −2 is

(a) $y = e^{-x} - e^{2x} + ex^{2x}$ (b) $y = e^{x} - e^{-2x} - ex^{2x}$

(c) $y = e^{-x} + e^{2x} + xe^{2x}$ (d) $y = e^{x} - e^{-2x} + xe^{2x}$ **[ESE-2018]**

Q.3 If Z = $e^{ax + by}$ F(ax − by); the value of $b\frac{\partial Z}{\partial x} + a\frac{\partial Z}{\partial y}$ is

(a) 2Z (b) 2a

(c) 2b (d) 2abZ **[ESE-2018]**

Q.4 The general integral of the partial differential equation $y^2p - xyq = x(z - 2y)$ is

(a) $\phi(x^2 + y^2, y^2 - yz) = 0$ (b) $\phi(x^2 - y^2, y^2 + yz) = 0$

(c) $\phi(xy, yz) = 0$ (d) $\phi(x + y, \ln x - z) = 0$ **[ESE-2018]**

Q.5 If $\frac{d^2y}{dt^2} + y = 0$ under the conditions y = 1, $\frac{dy}{dt}$ = 0, when t = 0, then y is equal to

(a) sin t (b) cos t

(c) tan t (d) cot t **[ESE-2018]**

ANSWER KEY :

1. (b) **2.** (a) **3.** (d) **4.** (a) **5.** (b)

GATE SOLUTIONS

Q.1 **(b)**

Q.2 **(a)** Order = 2, Degree = 1.

Q.3 **(b)** $\frac{d^3f}{d\eta^3}+\frac{f}{2}\frac{d^2f}{d\eta^2}=0$ is third order$\left(\frac{d^3f}{d\eta^3}\right)$ and it is non-linear due to $\frac{f}{2}\frac{d^2f}{d\eta^2}$

Q.4 **(c)** Order = 2, Degree = 2.

Q.5 **(c)** Given equation is $\frac{d^2y}{dx^2}+p\frac{dy}{dx}+qy=0$

$(D^2+pD+q)y=0 \Rightarrow D^2+pD+q=0$

Its solution is $y=c_1e^{-x}+c_2e^{-3x}$

So the roots of $D^2+pD+q=0$ are $\alpha=-1$ and $\beta=-3$

Sum of roots $=-p=-1-3 \Rightarrow p=4$

Product of roots $=q=(-1)(-3) \Rightarrow q=3$

Q.6 **(c)**Given equation is $\frac{d^2y}{dx^2}+p\frac{dy}{dx}+(q+1)=0 \Rightarrow \left[D^2+pD+(q+1)y=0\right]$

Put p = 4 and q = 3 $\quad \therefore\ (D^2+4D+4)y=0,\quad (D+2)^2=0$

$\therefore\ D=-2,-2 \Rightarrow y=(c_1x+c_2)e^{-2x}$, Out of choices given : $y=xe^{-2x}$

Q.7 **(b)** $\text{A.E.} \Rightarrow D^2-5D+6=0 \Rightarrow (D-2)(D-3)=0 \Rightarrow D=2,\ 3 \Rightarrow y=c_1e^{2x}+c_2e^{3x}$

Q.8 **(a)** $\frac{d^2y}{dx^2}+2\frac{dy}{dx}+17y=0,\ y(0)=1,\ \frac{dy}{dx}\left(\frac{\pi}{4}\right)=0$

$D^2+2D+17=0 \quad \Rightarrow \quad D=-1\pm 4i \quad \Rightarrow \ y=e^{-x}(C_1\cos 4x+C_2\sin 4x)$

$y(0)=1 \Rightarrow C_1=1,\ \frac{dy}{dx}=e^{-x}[(-4C_1-C_2)\sin 4x+(4C_2-C_1)\cos 4x)]$

Q.9 **(b)** $\frac{D^2y}{dx^2}+4\frac{dy}{dx}+3y=3e^{2x} \Rightarrow (D^2+4D+3)y=3e^{2x}$

Particular Integral $\text{P.I.}=\frac{1}{D^2+4D+3}3e^{2x}=\frac{e^{2x}}{5}$

Q.10 **(b)** Given equation $\frac{dy}{dx}+2xy=e^{-x^2}$

This is a Leibnitz's linear equation $\text{I.F.}=e^{\int 2xdx}=e^{x^2}$

Solution is $ye^{x^2}=\int e^{-x^2}e^{x^2}dx+c \Rightarrow ye^{x^2}=x+c$

at x = 0, y = 1 (given) $\therefore\ 1e^{\circ}=0+c0 \Rightarrow\ c=1$

So, the solution is $ye^{x^2}=x+1 \Rightarrow y=e^{-x^2}(x+1)$

Q.11 **(a)** Given differential equation is $x\frac{dy}{dx}+y=x^4,\ \frac{dy}{dx}+\left(\frac{y}{x}\right)=x^3$

$\text{I.F.}=e^{\int\frac{1}{x}dx}=x$, Solution is $y.x=\int x^3.xdx+c \ \Rightarrow\ y=\frac{x^4}{5}+\frac{c}{x}$

Given Condition $y(1)=\frac{6}{5}\Rightarrow c=1$, Solution is $y=\frac{x^4}{5}+\frac{1}{x}$

Q.12 **(a)** $\frac{dV}{dt}=-kA$ ....(i) Where, $V=\frac{4}{3}\pi r^3$ and $A=4\pi r^2$

$\Rightarrow\frac{dV}{dt}=\frac{4}{3}\pi\times 3r^2\frac{dr}{dt}=4\pi r^2\frac{dr}{dt}$

Substituting these in (i) we get, $4\pi r^2\frac{dr}{dt}=-k(4\pi r^2)\Rightarrow\frac{dr}{dt}=-k\Rightarrow dr=-kdt$

Integrating we get $r=-kt+C$ At t = 0, r = 1

$\Rightarrow 1=-k\times 0+C\Rightarrow\Rightarrow C=1$

$\therefore\ r=-kt+1$ .....(ii)

Now at t = 3 months r = 0.5 cm

$\therefore\ 0.5=-k\times 3+1\ \Rightarrow k=\frac{0.5}{3}$

Now substituting this value of R in equation (ii) we get, $r=-\frac{0.5}{3}t+1$

Putting r = 0 (ball completely evaporates)

In above and solving for t gives $0 = -\frac{0.5}{3}t + 1 \Rightarrow t = 6$ months

Q.13 **(d)** $\frac{dy}{dx} = x^2 y$

This is variable separable form $\int \frac{dy}{y} = \int x^2 dx \qquad \Rightarrow \log_e y = \frac{x^3}{3} + C$

$\Rightarrow y = e^{\frac{x^3}{3}+C} = e^C e^{\frac{x^3}{3}} \qquad \Rightarrow y = ke^{\frac{x^3}{3}}$

Now at x =0, y=1 so, $1 = ke^0 \Rightarrow k = 1$

$\therefore\ y = e^{\frac{x^3}{3}}$ is the solution.

Q.14 **(b)** $\frac{d\theta}{dt} = -k(\theta - \theta_0)$ (Newton's law of cooling)

This is in variable separable form separating the variables, we get

$\frac{d\theta}{\theta - \theta_0} = -kdt \Rightarrow \int \frac{d\theta}{\theta - \theta_0} = \int -kdt \quad \Rightarrow \ln(\theta - \theta_0) = -kt + C_1$

$\Rightarrow \theta - \theta_0 = C.e^{-kt} \left(\text{where } C = e^{C1}\right) \Rightarrow \theta - \theta_0 = C.e^{-kt}$

Given, $\theta_0 = 25°C$

Now at $t = 0,\ \theta = 60°C$

60 = 25 + C.e^0 $\qquad \Rightarrow C = 35 \qquad \therefore\ \theta = 25 + 35e^{-kt}$

At t = 15 minutes, $\theta = 40°C$

$\therefore\ 40 = 25 + 35\ e^{(-15k)} \Rightarrow e^{-15k} = \frac{3}{7} \qquad ...(i)$

Now at t = 30 minutes $\theta = 25 + 35e^{-30k} = 25 + 35(e^{-15k})^2$

Now substituting $e^{-15k} = \frac{3}{7}$ from (i), we get, $\theta = 25 + 35\left(\frac{3}{7}\right)^2 = 31.428°C \approx 31.5°C$

Q.15 **(a)** $y'' + 2y' + y = 0$

$\left(D^2 + 2D + 1\right)y = 0 \Rightarrow D^2 + 2D + 1 = 0 \Rightarrow D^2 + 2D + 1 = 0 \Rightarrow \left(D+1\right)^2 = 0 \Rightarrow D = -1, -1$

$\therefore\ y = \left(C_1 + C_2 x\right)e^{-x}$

$y(0) = 0 \Rightarrow 0 = (C_1 + C_2(0))e^{-0} \qquad \Rightarrow C_1 = 0$

$y(1) = 0 \Rightarrow 0 = (C_1 + C_2)e^{-1}$ $\Rightarrow C_1 + C_2 = 0$ $\Rightarrow C_2 = 0$

$\therefore$ $y = (0+0)e^{-x} = 0$ is the solution

$\therefore$ $y(0.5) = 0$

Q.16 **(b)** $\ddot{x} + 3x = 0$

Auxillary equation is $D^2 + 3 = 0$ i.e. $D = \pm\sqrt{3}i$

$\therefore$ $x = A\cos\sqrt{3}t + B\sin\sqrt{3}t$ at $t = 0, x = 1$ $\Rightarrow A = 1$

Now, $\dot{x} = \sqrt{3}(B\cos\sqrt{3}t - A\sin\sqrt{3}t)$ at $t = 0, \dot{x} = 0$ $\Rightarrow B = 0$

So, $x = \cos\sqrt{3}t \Rightarrow x(1) = \cos\sqrt{3} = -0.16$

Q.17 **(a)** $\frac{d^2y}{dx^2} + y = 0$ $\Rightarrow D^2 + 1 = 0$ $\Rightarrow D = \pm i$

$\therefore$ General solution is $y = P\cos(x) + Q\sin(x)$

Q.18 **(b)** Highest derivative of differential equation is 2.

Q.19 **(a)** $3y\frac{dy}{dx} + 2x = 0$ $\Rightarrow \frac{dy}{dx} = \frac{-2x}{3y}$

$\Rightarrow \frac{3}{2}y^2 = -2\frac{x^2}{2} + C$ $\Rightarrow 3y^2 + 2x^2 = C$

$\Rightarrow \frac{x^2}{\left(\frac{1}{2}\right)} + \frac{y^2}{\left(\frac{1}{3}\right)} = C$ $\Rightarrow \frac{x^2}{\left(\frac{1}{2}C\right)} + \frac{y^2}{\left(\frac{1}{3}C\right)} = 1$

Which is the equation of a family of ellipses.

Q.20 **(c)** Given of $dy/dx = y^2$ $\Rightarrow \int\frac{dy}{y^2} = \int dx$ $\Rightarrow -\frac{1}{y} = x + c$ $\therefore$ $y = -\frac{-1}{x+c}$

When $x = 0$ and $y = 1$ $\therefore$ $C = -1 \Rightarrow y = -\frac{1}{x-1}$

y is bounded when $x - 1 \neq 0$, i.e. $x \neq 1$, i.e. $x < 1$ or $x > 1$

Q.21 **(a)** Given, $x'(t) = -3\,x(t)$ i.e. $\frac{dx}{dt} = -3x$

$\frac{dx}{x} = -3dt$ $\Rightarrow$ $\int\frac{dx}{x} = \int -3dt$

$\Rightarrow \ln x = -3t + C$ $\Rightarrow X = e^{-3t+C} = e^c e^{-3t}$

Putting $e^c = C_1 \quad \Rightarrow x = C_1 e^{-3t}$

Now putting initial condition x(0)= x_0

$x_0 = C_1.e^0 = C_1 \quad \therefore \quad C_1 = x_0$

Solution is $x(t) = x_0.e^{-3t}$

Q.22 **(a)**

(A) $\frac{dy}{dx} = \frac{y}{x}$

$\frac{dy}{y} = \frac{dx}{x} \Rightarrow \int \frac{dy}{y} = \int \frac{dx}{x}$

log y = log x + log c = log cx

y = cx ....Equation to straight line

(B) $\frac{dy}{y} = \frac{-dx}{x} \Rightarrow \int \frac{dy}{y} = -\int \frac{dx}{x}$

log y = – log x +log c $\Rightarrow$ log y + log x = log c $\Rightarrow$ logyx = c

$y = c/x$ ....Equation of hyperbola

(C) $\frac{dy}{dx} = \frac{x}{y} \Rightarrow \int y dy = \int x dx \Rightarrow \frac{y^2}{c^2} - \frac{x^2}{c^2} = 1$ ...Equation of hyperbola

(D) $\frac{dy}{dx} = \frac{-x}{y} \Rightarrow \int y dy = -\int x dx \Rightarrow \frac{y^2}{2} + \frac{x^2}{2} = \frac{c^2}{2}$ ...Equation of circle

Q.23 **(a)** This is bernoulli's differential equation

Q.24 **(c)** $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = 0$

$D^2 + D - 6y = 0 \Rightarrow (D+3)(D-2) = 0 \Rightarrow D = -3,\ 2$

$\therefore$ Solution is $y = c_1 e^{-3x} + c_2 e^{2x}$

Q.25 **(b)** Given, $\frac{d^2x}{dt^2} + 6\frac{dx}{dt} + 8x = 0 \Rightarrow x(0) = 1$ and $\left.\frac{dx}{dt}\right|_{t=0} = 0$

$D^2 + 6D + 8 = 0 \Rightarrow (D+4)(D+2) = 0 \Rightarrow D = -2$ and $D = -4$

$\therefore$ Solution is $x = C_1 e^{-2t} + C_2 e^{-4t}$

Since, $x(0)=1$ we have $C_1+C_2=1$ ...(i)

$$\frac{dx}{dt}=-2C_1e^{-2t}-4C_2e^{-4t}$$

Since, $\left.\frac{dx}{dt}\right|_{t=0}=0,$ we have $-2C_1-4C_2=0$ ...(ii)

Solving (i) and (ii) we have, $C_1=2$ and $C_2=-1$ So the solution is $x(t)=2e^{-2t}-e^{-4t}$

Q.26 **(a)** $x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}-4y=0$ Given that, $y(0)=0$ and $y(1)=1$

The auxiliary equation is given by , $D(D-1)+D-4=0$ $\therefore D^2-4=0 \Rightarrow D=\pm 2$

$\therefore$ The required solution is $y=C_1e^{-2z}+C_2e^{2z}$, where $z=\ln x \Rightarrow y=C_1x^{-2}+C_2x^2$

Using boundary conditions $y(0)=0$ and $y(1)=1 \Rightarrow C_1=0$ and $C_2=1$

Hence, solution is x^2

Q.27 **(d)** Given, and y = 5 at x = 1

(D + 2) y = 0 $\Rightarrow$ D = −2

$\therefore$ The solution is $y=C_1e^{-2x}$, Now, $x=1 \Rightarrow = C_1e^2 \Rightarrow 5e^2 = C_1 = 36.95$

$\therefore$ The required solution is y = 3695

Q.28 **(d)**

Q.29 **(c)**

Q.30 **(a)** Here B^2- 4AC = 0 so, The equation is parabolic

Q.31 **(34 to 36)**

Given $\frac{d^2y}{dx^2}=0, y=5$ at $x=0, \frac{dy}{dx}=2$ at $x=10$

$\Rightarrow$ Autxiliary equation is $m^2=0 \Rightarrow m=0,0 \Rightarrow y_c=c_1+c_2x$ and $y_p=0$

General solution $y=y_c+y_p \Rightarrow y=c_1+c_2x \Rightarrow y=5$ at $x=0 \Rightarrow c_1=5$

$\frac{dy}{dx}=2$ at $x=10 \Rightarrow 2=c_2$, $y=5+2x$, $y(15)=5+30=35$

Q.32 **(d)** $\frac{dy}{dx}=\cos(x+y)$

Put $x+y=v \Rightarrow 1+\frac{dy}{dx}=\frac{dv}{dx} \Rightarrow \frac{dy}{dx}=\frac{dv}{dx}-1 \Rightarrow \frac{dv}{dx}-1=\cos v$

$$\Rightarrow \frac{1}{1+\cos v}dv = dx \text{ (variable separable)}$$

$$x+y=v \Rightarrow 1+\frac{dy}{dx}=\frac{dv}{dx} \Rightarrow \frac{dy}{dx}=\frac{dv}{dx}-1$$

$$\Rightarrow \frac{dv}{dx}-1=\cos v \quad \Rightarrow \frac{1}{1+\cos v}dv = dx \quad \text{(variable separable)}$$

$$\int \frac{1}{2\cos 2^{v}_{2}}dv = fdx + c \begin{bmatrix} 1+\cos 2\theta = 2\cos^2\theta \\ 1+\cos\theta = 2\cos^2\dfrac{\theta}{2} \end{bmatrix} = \frac{1}{2}\int \sec^2\frac{v}{2}dv = x+c$$

$$\text{Tan}\frac{v}{2} = x+c \Rightarrow \text{Tan}\left(\frac{x+y}{2}\right) = x+c$$

Q.33 **(a)** Given Differential equation is $\frac{d^2x(t)}{dt^2}+x(t)=0$

Auxilary equation is $m^2+1=0 \Rightarrow m=0\pm i$

Complementary solution is $x_c = c_1\cos t + c_2\sin t$

Particular solution $x_p = 0$

General solution $x = c_1\cos t + c_2\sin t$

Let $x_1(t)=\cos t,\ x_2(t)=\text{sint}$

Clearly $x_1(0)=1,\ \frac{dx_1}{dt}=0$ and $x_2(0)=0, \frac{dx_2}{dt}=1$

$$W = \begin{vmatrix} x_1(t) & x_2(t) \\ \dfrac{dx_1}{dt} & \dfrac{dx_2}{dt} \end{vmatrix}_{t=x/2} = \begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix}_{t=x/2} = \cos^2 t + \sin^2 t = 1$$

Q.34 **(b)** Given

$$\frac{dy}{dx}=-2xy,\ y(0)=2 \Rightarrow \frac{dy}{y}=-2xdx \Rightarrow y=ce^{-x^2} \Rightarrow y=2e^{-x^2}$$

Q.35 **(a)** For equal roots, Discriminant $B^2 - 4AC = 0 \Rightarrow 4\alpha^2-4=0 \Rightarrow \alpha=\pm 1$

Q.36 **(a)** (a) $\frac{dy}{dx}+xy=e^{-x}$ is a first order linear equation (non-homogeneous)

(b) $\frac{dy}{dx} + xy = 0$ is a first order linear equation (homogeneous

(c), (d) are non linear equations

Q.37 **(b)** A.E : – $m^2 + 2m + 1 = 0$, $\therefore$ General solution is $x = (a + bt)e^{-t}$

Q.38 **(0.54)** $A.E : m^2 + 4m + 4 = 0 \Rightarrow m = -2, -2$

solutions is $y = (a + bx)\, e^{-2x}$ .....(1), $y' = (a + bx)(-2e^{-2x}) + e^{-2x}(b)$ ...(2)

using y(0) =1; y' (0)=1, (1) and (2) gives a =1 and b = 3

$\therefore y' = (1 + 3x)e^{-2x}$, $y(1) = 4e^{-2} = 0.541$

Q.39 **(c)** A.E : $m^2+9 = 0 = m = \pm 3i$

$\therefore$ Solution is x = a cos3t + bsin3t ...(1) and $\frac{dx}{dt} = -3a\sin 3t + 3b\cos 3t$ .....(2)

Using x (0) =1 and $\left.\frac{dx}{dt}\right|_{t=0} = 1$ (1) and (2) gives

$1 = a$ and $1 = 3b \Rightarrow b = \frac{1}{3}$ So, $x = \cos 3t + \frac{1}{3}\sin 3t$

Q.40 **(c)** $x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = 0$ using cauchy – Euler equation

$\{D(D-1) + D - 1\}\, y = 0 \Rightarrow (D^2 - 1)y = 0$, where z = log x, x = e^z

$\therefore$ Solution is $y = C_1e^{-z} + C_2e^{z} = \frac{C_1}{x} + C_2x \Rightarrow \frac{1}{x}$ is a solution x

Q.41 $\ddot{y}(t) + y(t) = 0 \Rightarrow D^2 + 1 = 0 \Rightarrow y = c_1\cos x + c_2\sin x$

$y(0) = 2, y'(0) = 1 \Rightarrow y = 2\cos x + \sin x \Rightarrow y_{max} = \sqrt{2^2 + 1^2} = \sqrt{5}$

Q.42 **(d)** By back tracking, from option (D)

$$\frac{dy}{dx} = |x| = \begin{cases} x & \text{for } x \ge 0 \\ -x & \text{for } x \le 0 \end{cases}$$

$$y = \begin{cases} \frac{x^2}{2} & \text{for } x \ge 0 \\ -\frac{x^2}{2} & \text{for } x \le 0 \end{cases}$$

Q.43 **(C)** $\frac{d^2y}{dx^2} = y \Rightarrow (D^2 - 1)y = 0$

$D^2 - 1 = 0 \qquad D = \pm 1$

$y = C_1 e^x + C_2 e^{-x}$

Passes through (0,0) and $\left(\ln 2, \frac{3}{4}\right)$, at, $(0,0) \Rightarrow 0 = C_1 + C_2 \Rightarrow C_1 = -C_2 \quad ....(1)$

So, by going with option we can see in only option (c) we can see .

Q.44 **(c)** $\frac{dy}{dt} = -5y \Rightarrow \frac{dy}{y} = -5dt$

lny = -5t + c ...... (1)

When y = 2 at t = 0 , c = ln2

$\therefore \ln y = -5t + \ln 2 \Rightarrow \ln\left(\frac{y}{2}\right) = -5t \Rightarrow y = 2e^{-5t}, \quad y(3) = 2e^{-15}$

Q.45 **(c)** Given D.E $x(ydx + xdy)\cos\frac{y}{x} = y(xdy - ydx)\sin\frac{y}{x}$

$\therefore \quad (ydx + xdy)\cos\frac{y}{x} + \left(-\sin\frac{y}{x}\right)\frac{y(xdy - ydx)}{x} = 0$

$\therefore \quad (ydx + xdy)\cos\frac{y}{x} + (xy)\left(-\sin\frac{y}{x}\right)\left(\frac{xdy - ydx}{x^2}\right) = 0$

By observing, the above equation is $\frac{d}{dx}\left((xy)\cos\frac{y}{x}\right) = 0$

By integrating $xy\cos\left(\frac{y}{x}\right) = c$

Q.46 **(-2)** Given $\frac{d^2y}{dx^2} = -12x^2 + 24x - 20$

y (0) =5; y(2) =21 then y (1) = ?

Auxiliary equation $m^2 = 0 \Rightarrow m = 0, 0 \Rightarrow y_c = (C_1 + C_2 x)$

$y_p = \frac{1}{D^2}\left(-12x^2 + 24x - 20\right) = -12\frac{x^4}{12} + 24\frac{x^3}{6} - 20\frac{x^2}{2} = -x^4 + 4x^3 - 10x^2$

$y = C_1 + C_2 x + 10x^2 + 4x^3 - x^4$

y(0) = 5, y(2) = 21 $\Rightarrow C_1 = 5, \ C_2 = -20$

y(1) = -2

Q.47 **(b)**

Q.48 **(b)** $\dfrac{dy}{1+\cos 2y} = \dfrac{dx}{1-\cos 2x} \Rightarrow \dfrac{dy}{2\cos^2 y} = \dfrac{dx}{2\sin^2 x}$

$\therefore \int \sec^2 y\, dy = \int \text{cosec}^2 x dx$

$\therefore \tan y = -\cot x + k \quad \therefore \tan y + \cot x = k$

Q.49 **(0.8556)**

Q.50 **(-3)** Roots of auxiliary equation are –3, –2

$y(t) = C_1 e^{-3t} + C_2 e^{-2t} \quad \Rightarrow \quad y(0) = C_1 + C_2 = 2$

$y(1) = -\left(\dfrac{1-3e}{e^3}\right) = -e^{-3} + 3e^{-2} = C_1 e^{-3} + C_2 e^{-2}$

So, $C_1 = -1$, $C_2 = 3$ $\Rightarrow$ So, $y(t) = e^{-3t} + 3e^{-2t}$

$\dfrac{dy(t)}{dt} = 3e^{-3t} - 6e^{-2t}, \dfrac{dy(0)}{dt} = 3 - 6 = -3$

Q.51 **(1)** Given D.E is $y'' + 9y = 0$

A.E is $D^2 + 9 = 0$, $\quad \therefore y = C_1 \cos 3x + C_2 \sin 3x$ .... (i)

Given $y(0) = 0$ & $y(\pi/2) = \sqrt{2}$

$x = 0; y = 0$ and $x = \pi/2; y = \sqrt{2}$

From (i) $0 = C_1 \Rightarrow \sqrt{2} = 0 + C_2(-1) \Rightarrow C_2 = -\sqrt{2}$

$\therefore$ From (1); $y = -\sqrt{2}\sin 3x \quad \therefore y\left(\dfrac{\pi}{4}\right) = -\sqrt{2}\sin\left(\dfrac{3\pi}{4}\right) = -\sqrt{2}\left(\dfrac{1}{\sqrt{2}}\right) = -1$

Q.52 **(c)** A = 1, B = 3, C = 1 $\therefore B^2 - 4AC = 5 > 0$ $\therefore$ P.D.E is Hyperbolic

Q.53 **(b)**

Q.54 **(a)** D.E is $(D^4 + 3D^2)y = 108x^2$, $D = \dfrac{d}{dx}$

A.E : $m^4 + 3m^2 = 0$, $m^2(m^2 + 3) = 0$

$m = 0, 0, \pm i\sqrt{3}, \quad y_c = (C_1 + C_2 x) + C_3 \sin\left(\sqrt{3}x\right) + C_4 \cos\left(\sqrt{3}x\right)$

and $\text{P.I.} = \dfrac{1}{D^4 + 3D^2}\left(108x^2\right) = \dfrac{1}{3D^2\left[1 + \dfrac{D^2}{3}\right]}\left(108x^2\right)$

$$=\frac{36}{D^2}\left[1+\frac{D^2}{3}\right]^{-1}\left(x^2\right)=\frac{36}{D^2}\left[1-\frac{D^2}{3}+..\right]\left(x^2\right)=\frac{36}{D^2}\left[x^2-\frac{1}{3}(2)+0\right]$$

$$=36\iint\left(x^2-\frac{2}{3}\right)dxdx=3x^4-12x^2$$

Q.55 **(a)**

Q.56 **(0.66)**

Q.57 **(a)** $D^2+12D+36=0 \quad \Rightarrow = -6,-6 \quad \Rightarrow y=(C_1+C_2x)\ e^{-6x}$

$$\frac{dy}{dx}=(C_1+C_2x)e^{(-6x)}(-6)+C_2e^{-6x} \quad \Rightarrow \quad y(0)=3 \Rightarrow 3=C_1$$

$$y'(0)=-36 \Rightarrow -6C_1+C_2=-36 \Rightarrow -18+C_2=-36 \Rightarrow C_2=-18 \quad \Rightarrow y=(3-18x)e^{-6x}$$

Q.58 **(b)** Go through given option because it is convenient $\frac{\partial u}{\partial x}=f'(x-cy),\ \frac{\partial u}{\partial y}=-cf'(x-cy)$

So $\frac{\partial u}{\partial y}=-c\frac{\partial u}{\partial x} \quad \Rightarrow \quad \frac{\partial u}{\partial y}+c\frac{\partial u}{\partial x}=0$

Q.59 **(a)** $[D^2+16]y=0..........(1)$

Auxiliary equation

$m^2+16=0 \quad \Rightarrow \quad m=\pm 4i$

Solution of the differential equation $y = A\cos 4x + B\sin 4x$

$$\frac{dy}{dx}=-4A\sin 4x+4B\cos 4x = \left.\frac{dy}{dx}\right|_{X=0}=4B=1\Rightarrow B=\frac{1}{4} \Rightarrow \left.\frac{dy}{dx}\right|_{X=\frac{\pi}{2}}=0+4B=-1\Rightarrow B=\frac{-1}{4}$$

As there are two different values of B, therefore the give differential equation has no solution

Q.60 $y''+9y=0 \quad \Rightarrow (D^2+9)y=0 \quad ....(1)$

The Auxiliary eqn. $m^2+9=0\Rightarrow m=\pm 3i$

Solution of eqn. (1)

$y=A\cos\beta x+B\sin\beta x \quad$ or $\quad y=A\cos 3x+B\sin 3x \quad ....(2)$

By the first condition $\quad y = 0$ at $x = 0 \quad A = 0$ now

$y = B\sin 3x \quad \Rightarrow \quad y' = 3B\cos 3x \quad$ By the 2nd condition

$$y' = 2000 \text{ at } x=0 \Rightarrow B=\frac{2000}{3}\ \therefore\ y=\frac{2000}{3}\sin 3x$$

Now, $y_{\text{at } x=1} = \frac{2000}{3}\sin 3 \Rightarrow y_{\text{at } x=1} = 94.08$

Q.61 (0.368) Given D.E is $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 0 \Rightarrow (D^2+2D+1)y = 0 \quad \Rightarrow (D+1)^2 y = 0 \qquad ....(1)$

The auxillary equation is $(m+1)^2 = 0, \quad m = -1, -1$ [Real and Repeated]

C.F is $y_c = e^{-x}[c_1 + c_2 x]$ & P.I. = 0, Given that y(0) = 1

From $(1) \Rightarrow 1 = C_1 + 0 \therefore C_1 = 1 \Rightarrow \frac{dy}{dx} = -1$ at $x = 0 \Rightarrow -1 = -[1+0] + 1[C_2] \Rightarrow C_2 = 0$

$\therefore y = e^{-x}[1+0] = e^{-x} \Rightarrow y(1) = e^{-1} = 0.368$

Q.62. (d)

Q.63 (c) $x\frac{dy}{dx} + y = 0 \Rightarrow x\frac{dy}{dx} = -y \Rightarrow \frac{1}{y}dy = -\frac{1}{x}dx$

Integrating $\Rightarrow \int \frac{1}{y}dy = -\int \frac{1}{x}dy \Rightarrow \log_e y = -\log_e x + \log_e c \Rightarrow y = \frac{c}{x}$

At (1,1),c=1 $\therefore y = \frac{1}{x} = x^{-1}$

Q.64 (d) $f(x,y) = \frac{ax^2 + by^3}{xy} \qquad \Rightarrow \frac{\partial f}{\partial x} = \frac{\partial f}{\partial y}$ *for* $x = 1, y = 2$

$$\frac{\partial f}{\partial x} = \frac{1}{y}\left[\frac{x(2ax) - (ax^2 + by^3).1}{x^2}\right] \Rightarrow \left.\frac{\partial f}{\partial x}\right|_{x=1,y=2} = \frac{1}{2}\left[\frac{2a - (a+4b)}{1}\right] = \frac{a-4b}{2}$$

$$\frac{\partial f}{\partial y} = \frac{1}{x}\left[\frac{y(2by) - (ax^2 + by^2)\cdot 1}{y^2}\right] \Rightarrow \left.\frac{\partial f}{\partial y}\right|_{x=1,y=2} = \frac{1}{1}\left[\frac{8b - (a+4b)}{4}\right] = \frac{4b-a}{4}$$

From the given condition, equating values of $\frac{\partial f}{\partial x}$ *and* $\frac{\partial f}{\partial y}$ at x = 1,y = 2, we get

$$\frac{a-4b}{2} = \frac{4b-a}{4} \Rightarrow a = 4b$$

Q.65 (0.21) auxillary equation is $D^2 + D + \frac{5}{4} = 0 \Rightarrow 4D^2 + 4D + 5 = 0 \Rightarrow D = -\frac{1}{2} \pm j$

Solution is given as $y = e^{-\frac{t}{2}}(C_1\cos t + C_2\sin t)$, y(0)=1, y'(0)=0 $\Rightarrow c_1 = 1, c_2 = 0$

$\Rightarrow y = e^{-\frac{t}{2}}\cos t \Rightarrow y(\pi) = -e^{-\frac{\pi}{2}}$

Q.66 (4.5) $\frac{\partial r}{\partial x} = \frac{\partial}{\partial x}(x^2 - y + z) = 2x - \frac{\partial z}{\partial x}$ [since y is dependent]...(i)

Now, $z^3 - xy + yz + y^3 = 1,$ Differentiation with respect to x, we get

$$3z^2\frac{\partial z}{\partial x} - y + y\frac{\partial z}{\partial x} = 0 \Rightarrow \frac{\partial z}{\partial x}(3z^2 + y) = y \Rightarrow \frac{\partial z}{\partial x} = \frac{y}{3z^2 + y}.......................(ii)$$

From equation (i) and (ii), $\Rightarrow \frac{\partial r}{\partial x} = 2x - \frac{y}{3z^2 + y}$

$$Now\,at\; x = 2, y = -1, z = 1 \quad = 4 - \frac{(-1)}{3(1)^2 + (-1)} = 4 + \frac{1}{2} \qquad \therefore \left.\frac{\partial r}{\partial x}\right|_{x=2,y=-1,z=1} = 4.5$$

Q.67 (a) $\frac{dy}{dx} = \frac{x^2 + y^2}{2y} + \frac{y}{x} \Rightarrow \frac{dy}{dx} - \left(\frac{1}{x} + \frac{1}{2}\right)y = \frac{x^2}{2}y^{-1}$

Using Bernoulli's differential equation

$$\text{I.F.} = e^{-2\int\left(\frac{1}{x}+\frac{1}{2}\right).dx} = e^{-2\ln x - x} = \frac{e^{-x}}{x^2}$$

Hence, the solution becomes $y^2\frac{e^{-x}}{x^2} = \int x^2.\frac{e^{-x}}{x^2}dx + c \Rightarrow \left(\frac{y^2}{x^2} + 1\right)e^{-x} = c$

Given x =1,y =0 $\therefore c = e^{-1}$

Hence the solution is $\left(\frac{y^2}{x^2} + 1\right)e^{-x} = e^{-1} \Rightarrow \left(\frac{y^2}{x^2} + 1\right) = e^{x-1} \Rightarrow \ln\left(\frac{y^2}{x^2} + 1\right) = x - 1$

Q.68 (d) $\frac{\partial \text{V(x, y)}}{\partial \text{x}} = \text{px}^2 + \text{y}^2 + 2\text{xy} \Rightarrow \text{V(x,y)} = \text{p}\frac{\text{x}^3}{3} + x\text{y}^2 + \text{x}^2\text{y} + \text{f(y)}$ ...(1)

$$\frac{\partial \text{V(x, y)}}{\partial \text{y}} = \text{x}^2 + \text{qy}^2 + 2\text{xy} \Rightarrow \text{V(x,y)} = x^2y + q\frac{\text{y}^3}{3} + \text{xy}^2 + \text{f(x)} \text{ ...(2)}$$

from (1) and (2) $\text{V(x, y)} = \text{p}\frac{\text{x}^3}{3} + q\frac{\text{y}^3}{3} + x\text{y}^2 + \text{x}^2\text{y}$

Q.69 **(b)** The given partial differential equation $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ is form of a one-dimensional heat flow equation $\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2}$.

Q.70 **(c)** $y^3\frac{dy}{dx} = -x^3 \Rightarrow y^3dy = -x^3dx \Rightarrow \frac{x^4 + y^4}{4} = C$

$$y(0) = 1 \Rightarrow C = \frac{1}{4} \Rightarrow x^4 + y^4 = 1 \Rightarrow y(-1) = 0$$

Q.71 **(1.47)** Given that $\frac{d^2y}{dx^2}+\frac{dy}{dx}-6y=0,\ y(0)=0\ \&\ y'(0)=1$ ...(1)

Consider Auxilary Equation $\Rightarrow m^2+m-6=0 \Rightarrow m=2,-3$

The general solution of (1) is $y = C_1e^{2x}+C_2e^{-3x} \Rightarrow y' = 2C_1e^{2x}-3C_2e^{-3x}$ ...(2)

using initial conditions $\Rightarrow 0 = C_1+C_2,\ 1=2C_1-3C_2 \Rightarrow C_1=\frac{1}{5}$ and $C_2=\frac{-1}{5}$

$y=\frac{1}{5}e^{2x}-\frac{1}{5}e^{-3x} \Rightarrow y(1)=\frac{e^2-e^{-3}}{5}=1.47$

ESE SOLUTIONS

Q.1 **(b)** $u = \log\left(\frac{x^2+y^2}{x+y}\right) \Rightarrow e^u = \left(\frac{x^2+y^2}{x+y}\right)$

$f(u) = e^u$ is a Homogenous function of degree 1

by using Euler's theorem $= x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\frac{f(u)}{f'(u)} \Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 1$

Q.2 **(a)** $(D^2 - D - 2)y = 3e^{2x} \Rightarrow (D-2)(D+1)\,y = 3e^{2x} \Rightarrow y_c = c_1e^{2x} + c_2e^{-x}$

$y_p = \frac{3e^{2x}}{(D-2)(D+1)} = \frac{3xe^{2x}}{(2+1)} = xe^{2x} \Rightarrow$ Since $y(0)=0 \Rightarrow (c_1 + c_2) \to (1)$

$\frac{dy}{dx} = 2c_1e^{2x} - c_2e^{-x} + 2xe^{2x} + e^{2x} \Rightarrow$ Since $y'(0) = -2 \Rightarrow -2 = 2c_1 - c_2 + 1$

$\therefore 2c_1 - c_2 = -3$ (From 1), $\therefore$ $c_1 = -1 \Rightarrow c_2 = 1$, $\therefore y = -e^{2x} + e^{-x} + xe^{2x}$

Q.3 **(d)** $b\frac{\partial Z}{\partial x} + a\frac{\partial Z}{\partial y} - 2y = 3e^{2x}$

$\frac{\partial Z}{\partial x} = ae^{(ax+by)}F'(ax-by) + aF(ax-by)e^{(ax+by)}$ ...(1)

$\frac{\partial Z}{\partial y} = -be^{(ax+by)}F'(ax-by) + bF(ax-by)e^{(ax+by)}$ ...(2)

From (1) and (2) $\therefore$ $b\frac{\partial Z}{\partial x} + a\frac{\partial Z}{\partial y} = 2abZ$

Q.4 **(a)** $y^2p - xyq = x(z-2y) \Rightarrow \frac{dx}{y^2} = \frac{dy}{-xy} = \frac{dz}{x(z-2y)}$

$xdx + ydy = 0 \Rightarrow (x^2 + y^2) = c_1 \Rightarrow -\frac{dy}{y} = \frac{dz}{z-2y}$

$ydz + zdy - 2ydy = 0 \Rightarrow d(yz) - 2ydy = 0 \Rightarrow (yz - y^2) = c_2$

Hence, Solution is $\phi(x^2 + y^2, yz - y^2) = 0$

Q.5 **(b)** The A.E. is $(D^2+1) = 0 \Rightarrow y = (c_1\cos t + c_2\sin t)$

with given boundary conditions, solution is y = cost

5 PROBABILITY AND STATISTICS

5. INTRODUCTION :

Probability theory : A branch of mathematics concerned with the analysis of random phenomena. The outcome of a random event cannot be determined before it occurs, but it may be any one of several possible outcomes. The actual outcome is considered to be determined by chance.

Statistics theory : A branch of mathematics concerned with collection, classification, analysis, and interpretation of numerical facts, for drawing inferences on the basis of their quantifiable likelihood (probability). Statistics can interpret aggregates of data too large to be intelligible by ordinary observation because such data (unlike individual quantities) tend to behave in regular, predictable manner.

5.1 BASIC DEFINITIONS :

1. RANDOM EXPERIMENT :

Random experiment is the process to observe the event having an uncertain outcome. When we toss a coin the outcome is uncertain and hence, it can be termed as a random experiment. The result of a random experiment is known as the outcome and the set of all the possible outcomes of an experiment is known as sample space. If we repeat an experiment n number of times, then each time the experiment is done is known as a trial.

2. SAMPLE SPACE AND EVENTS :

Sample space is the Set of all possible outcomes or results of a random experiment. Sample space is represented by letter S. Each outcome in the sample space is called an element of that set.

Example : consider the random experiment of tossing a die. Let us write down the sample space here.

The sample space is all the possible outcomes here. What are the possible outcomes when we toss a die once? As we know a die has 6 faces numbered 1,2,3,4,5,6. When we toss it once, only one of the face will turn up. Hence the sample space is $S = \{1,2,3,4,5,6\}$

Consider one more simple example of tossing two coins. Let us write down the sample space here. $S=\{(H,H),(H,T),(T,H),(T,T)\}$ Here H:Head T : Tail

Can we think of any example which is non-random? That is whose outcome can be predetermined? Consider the example of executing a computer program. If you write the program correctly then we will know what the outcome of the program will be.

3. EVENT :

A subset of a sample space is called as an event. Events are always represented by capital letters A, B, C etc.

Example : consider the random experiment of throwing a die. The sample space here is S={1,2,3,4,5,6}

Consider an event

A: event of selecting an odd number.

Then the different outcomes of the event A is

A = {1,3,5}.

The set A as we see is asubset of the sample space S.

4. COMPLEMENT OF EVENTS :

The non-occurrence of any event is called its complement. The complement of any event A is represented by A^c

Example : consider the random experiment of attending an interview. Then the sample space is S={pass, fail}

That is the two possible outcomes of the interview.

Suppose event A is that the person passes the interview.

Then A= {pass}

But if the person fails in the interview, then that is the non-occurrence of event A, which is nothing but its complement.

A^c= {fail}.

5.2 PROBABILITY :

Sample space and events play important roles in probability. Once we have sample space and event, we can easily find the probability of that event. We have following formula to find the probability of an event.

$$\text{Probability of an event} = \frac{\text{Number of elements in an event of an experiment}}{\text{Number of elements in a sample space of an experiment}}$$

$$P(E) = \frac{n(E)}{n(S)}$$

Where, n(S) represents number of elements in a sample space of an experiment

n(E) represents a number of elements in the event set and P(E) represents the probability of an event.

Let us try to understand this with the help of an example. If a die is tossed, the sample space is {1, 2, 3, 4, 5, 6}. In this set, we have a number of elements equal to 6. Now, if the event is the set of odd numbers in a dice, then we have {1, 3, 5} as an event. In this set, we have 3 elements. So, the probability of getting odd numbers in a single throw of dice is given by

Probability $= \frac{3}{6} = \frac{1}{2}$

Note : **Difference between Sample space and Events :**

As discussed in the beginning sample space is set of all possible outcomes of an experiment and event is the subset of sample space. Let us try to understand this with few examples.

What happens when we toss a coin thrice ? If a coin is tossed three times we get following combinations, HHH, HHT, HTH, THH, TTH, THT, HTT and TTT

All these are the outcomes of the experiment of tossing a coin three times. Hence, we can say the sample space is the set given by,

S = {HHH, HHT, HTH, THH, TTH, THT, HTT, TTT}

Now, suppose the event be the set of outcomes in which there are only two heads. The outcomes in which we have only two heads are HHT, HTH and THH hence the event is given by,

E = {HHT, HTH, THH}

We can clearly see that each element of set E is in set S, so E is a subset of S. There can be more than one event. In this case, we can have an event as getting only one tail or event of getting only one head. If we have more than one event we can represent these events by E_1, E_2, E_3 etc.

We can have more than one event for a Sample space but there will be one and only one Sample space for an Event. If we have Events E_1, E_2, E_3, …… E_n as all the possible subset of sample space then we have,

$$S = E_1 \cup E_2 \cup E_3 \cup \ldots\ldots \cup E_n$$

We can understand this with the help of a simple example. Consider an experiment of rolling a dice. We have sample space,

S = {1, 2, 3, 4, 5, 6}

Now if we have Event E_1 as getting odd number as outcome and E_2 as getting even number as outcome for this experiment then we can represent E_1 and E_2 as the following set,

$E_1 = \{1, 3, 5\}$

$E_2 = \{2, 4, 6\}$

So we have

$\{1, 3, 5\} \cup \{2, 4, 6\} = \{1, 2, 3, 4, 5, 6\}$

Or $S = E_1 \cup E_2$

Hence, we can say union of Events E_1 and E_2 is S.

Example : Find the probability of getting only one head when a coin is tossed twice?

Solution: when a coin is tossed twice the sample space is {HH, HT, TH, TT}. Now, the event is getting one head. So, we have set {HT, TH} as Event set. The number of elements in the sample space is 4 and number of elements in the event set is 2.

Hence the probability is given by, $P(E) = \frac{2}{4} = \frac{1}{2}$

Example : A coin is tossed thrice, what is the probability of getting at least one head?

Solution : When a coin is tossed thrice we have following Sample space,

S = {HHH, HHT, HTH, THH, HTT, THT, TTH, TTT}

Now, the event is getting at least one head, this means we can have one or two or three heads. Hence, the event is,

E = {HHT, HTH, THH, THT, TTH, HTT, HHH}

We can see here n(S)= 8 and n(E)=7. Therefore, the probability is given by,

$$P = \frac{7}{8}$$

Example : What is the probability of getting 2 or 3 when a die is thrown?

Solution : When a die is thrown, we have following sample space

S = {1, 2, 3, 4, 5, 6}

Here event is given by,

E = {2, 3}

So we have n(S) = 6 and n(E) = 2, therefore the probability is given by

$$P(E) = \frac{2}{6} = \frac{1}{3}$$

Example : A dice is rolled twice, what is the probability of getting the sum of 8?

Solution : When a dice is rolled twice, we have following outcomes

S = { (1,1), (1,2), (1,3), (1,4), (1,5), (1,6) (2,1), (2,2), (2,3), (2,4), (2,5), (2,6),

(3,1), (3,2), (3,3), (3,4), (3,5), (3,6), (4,1), (4,2), (4,3), (4,4), (4,5), (46),

(5,1), (5,2), (5,3), (5,4), (5,5), (5,6), (6,1), (6,2), (6,3), (6,4), (6,5), (6,6)}

We can see we have n(S)= 36

Now for sum of 8 we have event,

E = {(2,6), (3, 5), (4,4), (5,3), (6,2)} and n(E) = 8

Now, the probability is given by

$$P(E)=\frac{8}{36}=\frac{2}{9}$$

5.3 TYPES OF EVENTS:

1. INDEPENDENT EVENTS :

consider any two events A and B. they are said to be independent if the occurrence of one event does not dependent on the occurrence.

For example consider the random experiment of tossing a coin. Then the sample space

here is

S= {head, tail}.

Let event A: getting a head

A= {head}

Let event B: getting a tail

B= {tail}.

Here both the events A and B are independent. This is because when we toss a coin for the first time, suppose we get a head. Again when we toss the coin for the second time, it is the outcome does not depend on the previous outcome. Hence the two events A and B are independent. Consider another experiment, of birth of a child. The sample space here is

S={boy, girl}

Suppose event A: a son is born

A={boy}

and event B: a daughter is born

B={girl}.

Here also we see that both the events A and B are independent. This is because the birth of a son in the first delivery does not necessarily ensure that the second child born will be a daughter. Hence both the events A and B are independent.

NOTE : ***If A and B are independent events, the probability of both the events happening together,***

P(A and B) = P(A) P(B)

2. EQUALLY LIKELY EVENTS : When the likelihood of happening of two events are same they are known as equally likely events. If we toss a coin, there are equal chances of getting a head or a tail. Hence, getting a head or a tail by tossing a coin are equally likely events. If a dice is rolled, then getting an odd number and getting an even number are equally likely events, whereas getting an even number and getting a 1 are not equally likely events.

3. EXHAUSTIVE EVENTS : Two events A and B are known as exhaustive events if the union of A and B gives the sample space. For example, if we have deck of 52 cards then the event of getting a black card and the event of getting a red card together will give the total number of cards. So, both these events are exhaustive events.

Example : If a die is rolled, then the event of getting a prime number and an odd number are exhaustive events or not ?

Solution: Event of getting a prime number = {1, 2, 3, 5}

Event of getting an odd number = {1, 3, 5}

Union of both events = {1, 2, 3, 5}

Sample space = {1, 2, 3, 4, 5, 6}

Both events together do not equal to the sample space. Hence, these events are not exhaustive**.**

4. MUTUALLY EXCLUSIVE EVENTS (DISJOINT EVENTS) :

Any toe events A and B are said to be mutually exclusive, if the occurrence of events A and B is a null set.

Example: consider the random experiment of throwing a die. The sample space here is S= {1, 2, 3, 4, 5, 6}

Consider event A: getting an even number A={2, 4, 6}

event B: getting an odd number B= {1, 3, 5}

Now let us look at the occurrence of events A and B together, that is we need to look at the common elements in events A and B. as we see here in both the events A and B there is no common element at all. Hence both the events A and B are mutually exclusive events.

Note : **1.** If Two events A and B are known as mutually exclusive, the probability of A or B happening will be, $P(A \cup B) = P(A) + P(B)$

2. If A,B are any two events are **not mutually exclusive**, then

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

5. IMPOSSIBLE AND SURE EVENTS :

IMPOSSIBLE AND SURE EVENTS:If the probability of an event is zero then it is known as an impossible event. The empty set is known as the impossible event. If the probability of an event is 1 then that event will definitely occur. Such event is known as sure event.

To understand these let us consider the experiment of rolling a die. The associated sample space is S = {1, 2, 3, 4, 5, 6}

Let E be the event "the number appears on the die is a multiple of 7". Can you write the subset associated with the event E? Clearly no outcome satisfies the condition given in the event, i.e., no element of the sample space ensures the occurrence of the event E. Thus, we say that the empty set only correspond to the event E. In other words we can say that it is impossible to have a multiple of 7 on the upper face of the die. Thus, the event E is an impossible event.

Now let us take up another event F "the number turns up is odd or even". Clearly

F = {1, 2, 3, 4, 5, 6,} = S, i.e., all outcomes of the experiment ensure the occurrence of the event F. Thus, the event F = S is a sure event.

6. SIMPLE EVENT :

If an event E has only one sample point of a sample space, it is called a simple or elementary event. In a sample space containing n distinct elements, there are exactly n simple events. For example in the experiment of tossing two coins, a sample space is

S={HH, HT, TH, TT}

There are four simple events corresponding to this sample space. These are

E_1 = {HH}, E_2 = {HT}, E_3 = {TH} and E_4 = {TT}.

7. COMPOUND EVENT :

If an event has more than one sample point, it is called a Compound event. For example, in the experiment of "tossing a coin thrice" the events

E: 'Exactly one head appeared'

F: 'Atleast one head appeared'

G: 'Atmost one head appeared' etc.

Are all compound events. The subsets of S associated with these events are

E={HTT, THT, TTH}

F={HTT, THT, TTH, HHT, HTH, THH, HHH}

G= {TTT, THT, HTT, TTH}

Each of the above subsets contain more than one sample point, hence they are all compound events.

Note : **The Event 'A or B'**:

Recall that union of two sets A and B denoted by $A \cup B$ contains all those elements which are either in A or in B or in both.

When the sets A and B are two events associated with a sample space, then '$A \cup B$' is the event 'either A or B or both'. This event '$A \cup B$' is also called 'A or B'.

Therefore Event 'A or B' = $A \cup B = \{\omega : \omega \in A \text{ and } \omega \in B\}$

The Event 'A and B' we know that intersection of two sets $A \cap B$ is the set of those elements which are common to both A and B. i.e., which belong to both 'A and B'.

If A and B are two events, then the set $A \cap B$ denotes the event 'A and B'.

Thus, $A \cap B = \{\omega : \omega \in A \text{ and } \omega \in B\}$

For example, in the experiment of 'throwing a die twice' Let A be the event 'score on the first throw is six' and B is the event 'sum of two scores is at least 11' then

$A = \{(6, 1), (6, 2\}, (6, 3), (6, 4), (6, 5), (6, 6)\}$, and $B = \{(5, 6), (6, 5), (6, 6)\}$

So $A \cap B = \{(6, 5), (6, 6)\}$

Note that the set $A \cap B$ = {(6, 5), (6, 6)} may represent the event 'the score on the first throw is six and the sum of the scores is at least 11'.

5.4 COMPLEMENTARY PROBABILITY :

$$P(A) = 1 - P(A')$$

Where, P (A')is called the complementary probability of A and P (A') represents the probability that the event A will not happen

Note : 1) $P(A)+P(A')=1$

i.e. A and A' are exclusion as well as collectively exhaustive .

2) $P(A \cap B') = P(A' \cup B') = 1 - P(A \cup B)$

$P(\text{neither A nor B}) = 1 - P\ (\text{either A or B})$

$\{\text{By Demorgan's law } A' \cap B' = (A' \cup B')\}$

TYPE I – QUESTION BASED ON COINS

Sample space (S.S) if we throw 1 coin, 2 coins or 3 coins simultaneously will look like:

1. coin - S.S = {H, T}

2. coin – S.S = {(H, H) (H, T) (T, H) (T, T)}

3. coin – S.S = { (H,H,H) (H,H,T) (H,T,H) (T,H,H) (T,T,T) (T,T,H) (T,H,T) (H,T,T) }

In fact the number of elements in the sample space of coins follows a pattern.

Where 1 coin tossed — S.S elements —> $2^1 = 2$

Where 2 coin tossed — S.S elements —> $2^2 = 4$

Where 3 coin tossed — S.S elements —> $2^3 = 8$

Where 4 coin tossed — S.S elements —> $2^4 = 16$

Where n coin tossed — S.S elements ->— 2^n

Why this is always in powers of '2' only?

This is because on a coin we can only have 2 outputs head or tail.

Remember this concept as in case of dice. We will have 6 output, where it will be in power of 6.

Let's see some examples based on coins.

Example : When 2 coins are tossed simultaneously find the probability of getting exactly 1 head?

Solution : Sample space when 2 coins are tossed will be

S.S = {(H, H) (H, T) (T, H), (T, T)}

So, we want probability of exactly 1 head hence favorable cases are

{H, T} and {T, H} i.e., 2 cases

$\therefore$ Probability of 1 head

$$P(1 \text{ head}) = \frac{2}{4} = \frac{1}{2} = 0.5$$

Example: If 3 coins are tossed simultaneously find the probability of getting exactly 2 tails.

Solution : Sample space of 3 coins will be

S.S = {(H,H,H) (H,H,T) (H,T,H) (T,H,H) (T,T,T) (T,T,H) (T,H,T) (H,T,T) }

Total = 8 elements.

We wish to find probability of exactly 2 tails, so favorable cases are

(T, T, H) (T, H, T) (H, T, T) = 3 elements

Hence required probability = $\frac{3}{8}$

Example : When 2 coins are tossed, find the probability of getting at least 1 tail?

Solution : Sample space of 2 coins = {(H, H) (H, T) (T, H) (T, T)}

Total cases = 4

In case we want at least one tail favorable cases are = {(H, T) (T, H) (T, T)}

Hence required probability = $\frac{3}{4}$ = 0.75

Approach – 2

Mark here that as the sum of total possible probabilities of an event is unity,

Hence we can have either 0 tail of 1 tail or 2 tails when we toss 2 coins.

$\Rightarrow \quad P(\text{at least 1 tail}) = 1 - P(0 \text{ tail})$

Example : If 5 coins are tossed simultaneously find the probability of getting at least one head.

Solution : So, we need not to draw the sample space of 5 coins here.

Why??? Because, we can be smart in identifying that as there are 5 coins, total number of elements in the sample space will be 2^5 = 32.

Also we have learnt that

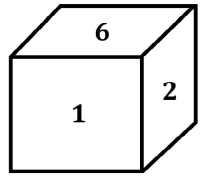
P(at least one head) = 1 – P(Zero head)

P(Zero head) means all the outcomes are tails hence only one such possible case.

$$\therefore \quad P(\text{at least 1 head}) = 1 - \frac{1}{32} = \frac{31}{32}$$

TYPE II – DICE BASED QUESTIONS

We know what is a dice ?
Yes, that on we use in game of Ludo.
How many faces do it have? Yes, 6



When we throw one such dice, we will have 6 elements in the sample space.
Sample space of dice = {1, 2, 3, 4, 5, 6}
Similarly if we throw two dice the sample space will have 6^2 = 36 elements.

$$\left\{\begin{array}{l} (1,1)\ (1,2)\ \ldots\ldots\ldots\ldots\ (1,6) \\ (2,1)\ (2,2)\ \ldots\ldots\ldots\ldots\ (2,6) \\ ------------------ \\ ------------------ \\ ------------------ \\ (6,1)\ (6,2)\ \ldots\ldots\ldots\ldots\ (6,6) \end{array}\right\}$$

If 3 dice are thrown simultaneously it will 6^3 = 216 elements in the sample space.
So, if we throw 'n' dice the total number of sample space elements will be 6^n.
Why this is in power of '6'?
Yes, you got it right, that's because when we throw one dice, the total outcomes are 6.
Let's see some questions on this type.

Example : When a single dice is thrown at random, find the probability of getting
(i) An even number
(ii) A prime number
(iii) A number divisible by 3

Solution : When a single dice is thrown at random the sample space looks like
S.S of 1 dice = {1, 2, 3, 4, 5, 6}

(i) Favorable cases for even number are (2, 4, 6) = 3 cases

$$\therefore \quad P(\text{an even numbers}) \ \frac{3}{6} = \frac{1}{2}$$

(ii) Favorable cases for prime number are {2, 3, 5} = 3 cases

Note : '1' is neither prime nor composite.

$$\therefore \ P(\text{a prime number}) \ = \ \frac{3}{6} = \frac{1}{2}$$

(iii) Favorable cases for number divisible by 3 are {3, 6}

$$\therefore \quad P(\text{number divisible by 3}) \ = \ \frac{2}{6} = \frac{1}{3}$$

Example : Two dice are thrown simultaneously, find the probability of getting

(i) Same digit on both face

(ii) Even number on both face.

Solution : When 2 dice are thrown simultaneously the total number of elements in the sample space will be 36.

(i) Favorable cases for same digit on both the face are

(1,1) (2,2) (3,3) (4,4) (5,5) (6,6) = 6 cases

Hence required probability $\frac{6}{36}=\frac{1}{6}$

(ii) Favorable cases for having event number on both dice.

(2,2) (2,4) (2,6) (4,2) (4,4) (4,6) (6,2) (6,4) (6,6) = 9 cases

Hence P (both even number) = $\frac{9}{36}=\frac{1}{4}$

Example : Two dice are thrown simultaneously, find the probability that sum of digits appearing is

(i) Greater than 10

(ii) Less than 5

Solution : When 2 dice are thrown the total elements in ample space are 36

(i) Favorable cases for sum to be greater than 10 are (5, 6) (6, 5) and (6, 6)= 3 cases

$\therefore$ Probability (sum > 10) = $\frac{3}{36}=\frac{1}{12}$

(ii) Favorable cases for sum to be less than 5 are (1, 1) (1, 2) (1, 3) (2, 1) (2, 2) (3, 1) = 6 cases

$\therefore$ Probability (sum <5) = $\frac{6}{36}=\frac{1}{6}$

TYPE III – COLORED BALLS

These are typical sort of questions which are asked in aptitude entrance exams.

Let's understand this type in details.

Let's say we have a bag or a bucket, or an urn, or a basket. Which contains 3 Red and 5 Green balls and we have to randomly pick one ball. What will be the probability for the ball to be Red? Now, we can easily see, we have 3 favorable Case (as 3 Red balls) And we have 3 + 5 = 8 total balls.

$\therefore \qquad \text{P(Picking a Red ball)} = \frac{3}{8}$

Similarly, if it was asked to pick a green ball from this bag it would give probability as

$\therefore \qquad \text{P(Picking a Green ball)} = \frac{5}{8}$

So, this was an easy task to find probability to pick a single ball, but normally questions will be asked on picking up 2 or more balls.

Let's say in the same example we wish to pick 2 balls from the bag containing 3 red and 5 green balls and we need to find the probability of getting both balls red.

Here, be careful as there can be three ways of picking these balls.

As there are more than one ball, balls can be picked up either simultaneously or with replacement one by one or without replacement one by one.

Let's analyze the above problem in all three cases.

(i) With Replacement Case

We need to pick both red balls so probability to pick first ball red is $\frac{3}{8}$

Now, we have replaced that ball so probability of picking second red ball is again $\frac{3}{8}$

$\therefore \qquad \text{P(Both Red)} = \frac{3}{8} \times \frac{3}{8} = \frac{9}{64}$

Note : ***Multiplication sign in between as we are saying first ball red 'and' second ball also red.***

(ii) Without Replacement case

When we pick the first ball the probability of its being red is $\frac{3}{8}$.

Now as we are not replacing the ball, the probability for second ball to be red will be $\frac{2}{7}$.

Hence $\text{P(Both Red)} = \frac{3}{8} \times \frac{2}{7} = \frac{6}{56} = \frac{3}{28}$

(iii) Simultaneous picking

When two balls are picked up simultaneously (together) then favorable case for both being red will be ${}^{3}C_{2} = 3$ cases

And total case to pick two balls from 8 are ${}^{8}C_{2} = 28$

$\therefore \qquad \text{Probability(Both Red)} = \frac{3}{28}$

Note : ***Without replacement and simultaneously will always get the same answers.***

Example : A bag contains 3 black and 4 white balls. Find the probability that if 2 balls are picked up at random, they will be:

(i) Same Colour

(ii) Different colour

Find answers in all 3 cases (with replacement, without replacement and simultaneous pickings.)

Solution: 3B, 4w

(i) Same Colour :

This means either both black or both white

a) With Replacement

$$P(B,B)+P(w,w)$$
$$=\left(\frac{3}{7}\times\frac{3}{7}\right)+\left(\frac{4}{7}\times\frac{4}{7}\right)=\frac{9+16}{49}=\frac{25}{49}$$

b) Without Replacement

$$P(B,B)+P(w,w)$$
$$=\left(\frac{3}{7}\times\frac{2}{6}\right)+\left(\frac{4}{7}\times\frac{3}{6}\right)=\frac{6+12}{42}=\frac{18}{42}$$
$$=\frac{3}{7}$$

c) Simultaneously

$$P(B,B)+P(w,w)$$
$$=\frac{3C_2+4C_2}{7C_2}=\frac{3+6}{21}=\frac{9}{21}=\frac{3}{7}$$

(ii) Different colour balls :

Different colour means one ball black other white or vice versa.P(B,W) or P(W,B)

Now you will argue that when both these cases are same why we are considering them twice.

See, try to understand, we are talking about probability or possibilities so, we need to take all possible cases. Let me explain you with an example.

In case I say, I eat first and then drink water or I say, I drink first and then eat. Are they same?

No. they are different.

(Don't say sir in stomach it will be all he same)

What I mean to say is that these are two different cases similarly getting first ball black and second white is different from getting first white and second black

(a) With replacement

P (B, W) or P (W, B)

$$\left(\frac{3}{7}\times\frac{4}{7}\right)+\left(\frac{4}{7}\times\frac{3}{7}\right)=\frac{24}{49}$$

(b) Without Replacement

P (B, W) or P (W, B)

$$\left(\frac{3}{7}\times\frac{4}{6}\right)+\left(\frac{4}{7}\times\frac{3}{6}\right)=\frac{24}{42}=\frac{4}{7}$$

(c) Simultaneous picking.

Here we are picking together so no consideration of order but we want one black & one white ball.

So favorable cases will be

$3C_1$ to pick one black ball and

$4C_1$ to pick one white ball

$\therefore$ Probability = $\dfrac{3C_1\times 4C_1}{7C_2}=\dfrac{12}{21}=\dfrac{4}{7}$

Note: **1)** Normally in paper only one case out of with replacement, without replacement and simultaneously will be asked.

2) If nothing is mentioned in the question then by default case is simultaneous picking

TYPE IV – PLAYING CARDS

Do you know how to play cards?

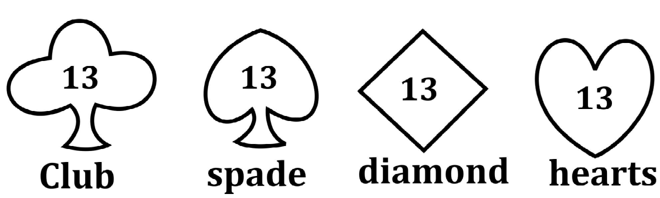
If no,then let's learn it to get marks in exams.

How many cards are there in a pack of cards?

Yes 52!

Do not consider jokers or any other card for playing cards.

Let's first understand what are the different types of cards ?



Club & spade $\rightarrow$ black colour

Diamond & hearts $\rightarrow$ Red colour.

J $\rightarrow$ Jack Q $\rightarrow$ queen

K $\rightarrow$ King A $\rightarrow$ Ace.

 # Total face cards (J, Q, K) which have faces = 3 × 4 = 12

 # Total Number cards (don't say 52 – 12 =40)

Number starts from 2 to 10 = 9 in one suit.

$\therefore$ 9 × 4 =36 Number Cards

 # Ace is neither a face card nor a number card.

So, if we ask number of alphabet cards in a pack of cards?

Don't say 4

As jack, Queen, King are also alphabet cards.

So, total alphabet cards are 16.

Let's see some examples based on this type

Example : A single card is drawn from a well shuffled pack of Cards, find the probability that it is a.

(I) A red Card

(II) A number Card.

(III) A black alphabet card

(IV) A number card of diamond

(V) A king of heart

(VI) A number card divisible by 3.

Solution: For picking up single card from 52 cards

(I) P (Red Card) = 26/52

(II) P (Number card) = 36/52

(III) P (black alphabet card) = 8/52

(IV) P (number card of diamond) = 9/52

(V) P (king of heart) = 1/52

(VI) P (Number card div by 3) = 12/52

Note : When we are picking more than one card, then again we will have those three cases (with replacement, without replacement, and simultaneously).

Example : If 2 cards are drawn from a well shuffled pack of cars, find the probability that

(I) both are red cards.

(II) one is red number card, other is black ace.

(Solve in all three cases for practice)

Solution :

		with Replacement	without Replacement	Simultaneous
(i)	P (both Red)	$\frac{26}{52}\times\frac{26}{52}$	$\frac{26}{52}\times\frac{25}{51}$	$\frac{26C_2}{52C_2}$
(ii)	P (Red Number Card, black Ace) OR P (black Ace, Red Number card)	$\left(\frac{18}{52}\times\frac{2}{52}\right)+$ $\left(\frac{2}{52}\times\frac{18}{52}\right)$	$\left(\frac{18}{52}\times\frac{2}{51}\right)+$ $\left(\frac{2}{52}\times\frac{18}{51}\right)$	$\frac{18C_1\times 2C_1}{52C_2}$

TYPE V - MISCELLANEOUS TYPE

This is the fifth and last type of question available in probability fundamentals.

Let's understand these types with the help of an example.

Example : The Probability of 'A' and 'B' passing an exam are 3/5 and 4/5 respectively. Find the probability of

(I) A and B both passing the exam

(II) Either of them passing the exam

(III) At least one of them passing

(IV) Both fails

Solution: Let P(A) = Probability of 'A' pass = 3/5

$P(\overline{A})$ = Probability of 'A' fail = 1 - 3/5

P(B) = Probability of 'B' pass = 4/5

$P(\overline{B})$ = Probability of 'B' fail = 1 - 4/5

(i) P(Both passes)

= P(A) × P(B)

= 3/5 × 4/5

= 12/25

(ii) P (Either of them passes)

= $P(A) \times P(\overline{B}) + P(B) \times P(\overline{A})$

= 3/5 × 1/4 + 4/5× 2/5

= 11/25

Please note that in such cases we cannot take the case as

P(A) + P(B)

Why??

See try to understand, we wish to get probability of either of them pass. So when we say 'A' pass [P(A)] then 'B' have to fail. And similarly if 'B' passes, 'A' have to fail.

(iii) P(At least 1 passes)

= $P(A)\, P(\overline{B}) + P(B)\, P(\overline{A}) + P(A) \times P(B)$

= 3/5 × 1/5 + 4/3 × 2/5 + 3/5 × 4/5

= 23/25

OR

We could have also said.

P(atleast 1 pass) = 1 – P(Zero)

$\Rightarrow$ P(atleast 1 pass) = 1 – P(both fails)

$\Rightarrow$ P(atleast 1 pass) = 1 – P($\bar{A}$) × P($\bar{B}$)

= 1 – 2/5 × 1/5

= 23/25

(IV)P (Both fail) = P($\bar{A}$) × P($\bar{B}$)

= 2/5 × 1/5 = 2/25

ODDS IN FAVOR

Odds in favor of an event are the ratio of favorable number of cases to the non-favorable number of cases.

$$\text{Odds in favor of an event } = \frac{\text{Favourable number of cases}}{\text{Non- favourable number of cases}}$$

ODDS IN AGAINST

This is the ratio of non-favorable number of cases to the favorable number of cases.

$$\text{Odds in against } = \frac{\text{Non- favourable cases}}{\text{Favourable cases}}$$

Can odds in favor or odds in against be greater than one?

Yes?? No??

The answer is yes.

See try to understand, if odds in favor is greater than one it's ok as probability should not be greater than one.

For instance if odds in favor is 8/5 means we have 8 favorable case and total of 5 + 8 = 13 cases.

Hence probability of that event is 8/13 which is less than one.

Let's say odds in against of an event is 3/1 then its probability will be 1/4.

Hope now you got my point.

Example : Odds in favor of 'A', 'B' and 'C' hitting a target are 2/3, 4/3 and 1/2 respectively.

Find the probability that if all three aims at the same target, then target will be hit?

Solution : Let A = Probability of A hit target = 2/5

$\overline{A}$ = Probability of A not hit target = 3/5

B = Probability of B hit target = 4/7

$\overline{B}$= Probability of B not hit target = 3/7

C = Probability of C hit target = 1/3

$\overline{C}$= Probability of B not hit target = 2/3

So the required probability is

= $AB\overline{C} + A\overline{B}C + \overline{A}BC + \overline{A}\overline{B}C + \overline{A}B\overline{C} + A\overline{B}\overline{C} + ABC$

Or we can also say the required probability is

P(at least one hitting the target) = 1 - $\overline{A}\overline{B}\overline{C}$

= 1 – 3/5 × 3/7 × 2/3

= 1 – 6/35

= 29/35

5.5 CONDITIONAL PROBABILITY:

When probability of an event A is given and we have to find the probability of other event B based on that event A, then the probability obtained is called conditional probability of B given A. It is denoted by P (B/A).

P(B/A) is read as probability of B given A.

Example : Let us consider an experiment of rolling a die. Let A denote the event of getting 1 or 3.The outcomes of tossing a die is {1, 2, 3, 4, 5, 6}. Let B denote the event of getting an odd number. Suppose that the event B has already occurred.

If it is known that we got an odd number, the outcomes favorable are {1, 3, 5}

So the probability of getting 1 or 3 is 2/3. This is under the condition we got an odd number.

Therefore P(A/B) = 2/3 is the conditional probability of event A given event B has already occurred.

CONDITIONAL PROBABILITY FORMULA :

If A and B are two events such that given event B has happened and we have to find the probability of event A, then probability of A given B, denoted by P(A/B) is given by $P(A/B) = P(A \cap B)/P(B)$

Note : ***Multiplication rule of Probability***

Multiplication rule of probability is an important rule based on conditional probability formula.

Multiplication rule for any two events

If A and B are any two events, then the combined probability of both A and B, denoted by P(A)”B) is given by

$P(A \cap B) = P(A/B) \times P(B)$

OR

$P(A \cap B) = P(B/A) \times P(A)$

Multiplication rule for independent events

(i) If A and B are independent(that is events that does not dependent on each other) then

(a) $P(A/B) = P(A)$ or $P(B|A) = P(B)$

(b) $P(A \cap B) = P(A)P(B)$.

(ii) if $P(A \cap B) = P(A)\ P(B)$ then A and B are lndependent.

(iii) In general $A_1, A_2, A_3 \ldots A_n$ are mutually independent events, then

$$P(A_1 \cap A_2 \cap A_3 \cap \ldots \cap A_n) = P(A_1)P(A_2)P(A_3)\ldots P(A_n)$$

(iv) If A and B are independent then

(a) A and B^c (b) A^c and B (c) A^c and B^c

are also independent.

Example : If P(A) = 1/12, P(B) = 1/3 and $P(A \cap B)$ = 1/52 find

(i) P(A/B)

(ii) P(B/A)

Solution : Give that P(A) = 1/12, P(B) = 1/3 and $P(A \cap B)$ = 1/52

We know that for any two events

$P(A/B) = P(A \cap B)/P(B)$ and

$P(B/A) = P(A \cap B)/P(A)$

(i) P(A/B) = (1/52)/(1/3)

= 3/52

(ii) P(B/A) = (1/52)/(1/12)

= 12/52

= 3/13

Example: A box 'A' contains 2 white and 4 black marbles. Another box 'B' contains 5 white and 7 black marbles. A marble is transferred from box 'A' to box 'B' and a ball is drawn at random from box 'B'. What is the probability that it will be black?

Solution: P (drawing a white marble from the box A) = 2/6

P (drawing a black marble from the box A) = 4/6

P(drawing a black marble from the box B if the transferred marble is white) = 7/13

P(drawing a black marble from the box B if the transferred marble is black)= 8/13

P (drawing a black marble from box B)

= P(drawing a black marble from the box B if the transferred marble is white ordrawing a black marble from the box B if the transferred marble is black)

$= 2/6 \times 7/13 \times 4/6 \times 8/13$

$= 14/78 + 32/78$

$= 46/78 \quad = 23/39$

P(drawing a black marble from box B after transferring a marble from box A)= 23/39

Example : What is the conditional probability that a family with two children has two girls, given that they have at least one girl?

Solution : **Step 1**: The all possibilities can be denoted by the set {BB, BG, GB, GG}. The first letter represents the sex of the first child, while the second denotes the sex of the second child. The sample space has four outcomes.

Step 2:

Case 1 : Family has at least one girl will have the possibilities {BG, GB, GG} - 3 favorable outcomes.

Case 2: Two girls will have one favorable outcome {GG}.

Case 3: The intersection $\{(\text{having at least one girl}) \cap (\text{both girls})\}$ will have $\{GG\} \rightarrow 1$ favorable outcome

Step 3:

The set of outcomes for having two girls is the subset of the set of outcomes for having at least one girl.

$$P\left(\frac{\text{two girls}}{\text{at least one girl}}\right) = \frac{P(\text{Two girls} \cap \text{at least one girl})}{P(\text{at least one girl})} = \frac{\frac{1}{4}}{\frac{3}{4}} = \frac{1}{3}$$

$$P\left(\frac{\text{two girls}}{\text{at least one girl}}\right) = \frac{1}{3}$$

Example : Two math workshops were organized by the Math club of the school. The first workshop was attended by 40% of High school students. Both the workshops were attended by 25% of High chool students. What percent of those who attended the first workshop also took part in the second?

Solution : The first workshop was attended by 40% of High school students and both the workshops were attended by 25% of High School students.

Percent of those students who attended the first workshop as well as second = ?

Step 1:

$$P\left(\frac{\text{attended the second}}{\text{attended the first}}\right) = \frac{P(\text{attended both})}{P(\text{attended the first})} = \frac{0.25}{0.4} = 0.625$$

Hence 62.5%of the students who attended the first workshop also took part in the second.

Example : In an exam, two reasoning problems, 1 and 2, are asked. 35% students solved problem 1 and 15% students solved both the problems. How many students who solved the first problem will also solve the second one?

Solution : Probability of student solving problem 1,P(1)=0.35

Probability of student solving both problem, P(1 and 2)=0.15

Probability of solving 2 if 1 is solved,

$$P(2|1) = \frac{P(1 \text{and} 2)}{P(1)} = \frac{0.15}{0.35} = 0.428$$

Example : In a school the third language has to be chosen between Hindi and French. If a student has taken French then what is the probability that he will take Hindi, if the probability of taking Hindi is 0.34?

Solution : Probability of taking French and Hindi, P(A and B)=0 as they are mutually exclusive events.

Probability of taking French, P(B)=0.34

Probability of taking Hindi if French has been opted, $P(A|B) = \frac{P(A \text{ and } B)}{P(B)} = \frac{0}{0.34} = 0$

Compound probability of mutually exclusive events is 0.

Example : The given table shows the data of 10 point holders in a given class of 30. Find the probability that the student getting 10 point is a girl.

	10 pointer	Not 10 pointer
Girl	3	8
Boy	6	13

Solution : From the table we can retrieve the given information,

Total boys = 19

Total girls = 11

Girl getting 10 point = 3

Probability that the person getting 10 point is a girl = $\frac{3}{11}$

It can be explained using the formula $P(A|B) = \frac{P(A \cap B)}{P(B)}$ as given here,

Probability of the student being a girl, $P(B) = \frac{11}{30}$

Probability of the student getting 10 point and being a girl, $P(A \cap B) = \frac{3}{30}$

Probability that the 10 pointer being selected is a girl, $P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{3}{11}$

5.6 TOTAL PROBABILITY :

When we talk about total probability we are talking about a partition space. A partition of any sample space is nothing but simply a collection of events that are disjoint say $C_1, C_2 \ldots, C_k$ and the union of all these events is equal to the sample space.

Such kind of partition will also divide any set say B of the sample space into disjoint pieces. Let us suppose that we have an event B and $A_1, A_2, \ldots, A_k$ are the partitions then, event B is the union of these partitions such that:

$$B = (B \cap A_1) \cup (B \cap A_2) \cup \ldots \cup (B \cap A_k)$$

Since all these partitions are disjoint so we have

$$P(B) = P(B \cap A_1) + P(B \cap A_2) + \ldots + P(B \cap A_k)$$

We know by the multiplication rule that:

$$P(B \cap A_1) = P(A_1)P(B \mid A_1)$$

Where $P(B|A_1)$ is the conditional probability which gives the probability of occurrence of event B when event A_1 has already occurred.

Applying this rule above we get,

$$P(A_1)P(B \mid A_1) + P(A_2)P(B \mid A_2) + \ldots + P(A_k)P(B \mid A_k)$$

This is the law of total probability. This law is very useful in many places. For example: We are drawing two cards from a deck of cards. The first card drawn is a king. To find the probability if the second one is an ace or not we use the law of total probability only which makes it easier to find.

The law of total probability is also referred to as total probability theorem or law of alternatives.

Let us check some examples of problem solving that involves the total probability theorem.

Example : We draw two cards from a deck of shuffled cards without replacement. Find the probability of getting the second card a queen.

Solution : Let 'A' represent the event of getting the first card a queen.

Let 'B' represent the event that the first card is not a queen.

Let 'E' represent the event that the second card is a queen.

Then the probability that the second card will be a queen or not will be represented by the law of total probability as:

$$P(E) = P(A)P(E \mid A) + P(B)P(E \mid B)$$

Where P(E) is the probability that second card is a queen

P(A) is the probability that the first card is a queen

P(E|A) is the probability that the second card is a queen given that first card is a queen.

P(B) is the probability that the first card is not a queen

P(E|B) is the probability that the second card is a queen but the first card drawn is not a queen.

According to question:

$$P(A) = \frac{4}{52},\ P(B) = \frac{48}{52},\ P(E|A) = \frac{3}{51},\ P(E|B) = \frac{4}{51}$$

On substituting we get,

$$P(E) = \frac{4}{52} \times \frac{3}{51} + \frac{48}{52} \times \frac{4}{51}$$

$$= \frac{(3\times4)}{(52\times51)} + \frac{(48\times4)}{(52\times51)}$$

$$= \frac{(4\times(3+48))}{(52\times51)}$$

$$= \frac{(4\times51)}{(52\times51)} = \frac{4}{52} = \frac{1}{13}$$

5.7 BAYES' THEOREM :

Bayes' theorem can be considered as the most important application of conditional probability. Conditional probability is defined considering two events that can occur together. Bayes' theorem extends this to a scenario where an event can occur with any one of a number of mutually exclusive events.

Suppose there are two baskets both containing apples and oranges. Let the first basket contains 10 apples and 12 oranges and in the second basket there are 15 apples and 12 oranges.

Conditional probability problems deal with finding the probability of picking a type of fruit (apple or orange) when it is known which basket it has come from. Bayes' Theorem deals with the inverse probability of the fruit coming from a basket when the type of the fruit is known.

In Bayes' model, the probabilities are calculated expressing the given event as a union several intersections the event makes with the mutually exclusive events.

Suppose an event B can occur in combination with one of the n mutually exclusive events, A_1, A_2, ...A_n, and P(B) > 0, then Bayes' theorem provides a rule for calculating the conditional probability of the occurrence of one of the above n events A_r, given that B has already occurred.

$$P(A_r \mid B) = \frac{P(A_r \cap B)}{P(A_1 \cap B) + P(A_2 \cap B) + \ldots\ldots P(A_n \cap B)}$$

Bayes theorem can be rewritten using the multiplication rule for probabilities,

$P(A_r \cap B) = P(A_r) \,.\, P(B \mid A_r)$

$$P(A_r \mid B) = \frac{P(A_r)P(B|A_r)}{P(A_1)P(B|A_1)+P(A_2)P(B|A_2)+.......P(A_n)P(B|A_n)}$$

As a special case if the event B can occur with two mutually exclusive events A_1 and A_2, then the Baye's formulas are as follows:

$$P(A_1 \mid B) = \frac{P(A_1)P(B|A_1)}{P(A_1)P(B|A_1)+P(A_2)P(B|A_2)}$$

and

$$P(A_2 \mid B) = \frac{P(A_2)P(B|A_2)}{P(A_1)P(B|A_1)+P(A_2)P(B|A_2)}$$

Note : Bayes' theorem relates the conditional and marginal probabilities of stochastic events A and B:

$$P(A|B) = \frac{P(B|A)\,P(A)}{P(B)}$$

Each term in Bayes' theorem has a conventional name:

- P(A) is the prior probability or marginal probability of A. It is "prior" in the sense that it does not take into account any information about B.
- P(A|B) is the conditional probability of A, given B. It is also called the posterior probability because it is derived from or depends upon the specified value of B.
- P(B|A) is the conditional probability of B given A.
- P(B) is the prior or marginal probability of B, and acts as a normalizing constant.

Example: For a hypothetical magazine, the probability that the reader is male given that the reader is at least 35 years old is 0.30. The probability that a reader is male, given that the reader is under 35, is 0.65. If 75% of the reader are under thirty five what is the probability that a randomly chosen reader is

(a) male

(b) Female

(c) Under 35 if it is given the reader is a female

Solution: Let us put the information given in the form of a table.

(a) Let A_1 be the event of the reader being at least 35 years and A_2 the event of the reader being under 35 years old and B the event of the reader being a male.

$P(A_2) = 0.75$ and $P(A_1) = 1 - P(A_2) = 1 - 0.75 = 0.25$

$P(B | A_1) = 0.30$ and $P(B | A_2) = 0.65$

$B = (A_1 \cap B) \cup (A_2 \cap B)$ as the male reader can either under 35 or at least 35 years old.

$$P(B) = P(A_1 \cap B) + P(A_2 \cap B)$$
$$= P(A_1).P(B|A_1) + P(A_2).P(B|A_2)$$

= (0.25)(0.30) + (0.75)(0.65) = 0.5625

The probability the reader is male = 0.5625

(b) P(female) = 1 – P(male) = 1 - 0.5625 = 0.4375

The probability the reader is female = 0.4375.

(c) Let E_1 be the event of the reader is under 35 and E_2 the event of the reader is at least 35 and the F the event of the reader being a female.

We need to find P (E_1 | F) and we have the probabilities

$P(F | E_1) = 1 - 0.65 = 0.35$, $P(F | E_2) = 1 - 0.30 = 0.70$, $P(E_1) = 0.75$ and $P(E_2) = 0.25$

$$P(E_1 | F) = \frac{P(E_1)P(F|E_1)}{P(E_1)P(F|E_1)+P(E_2)P(F|E_2)}$$
$$= \frac{(0.75)(0.35)}{(0.75)(0.35)+(0.25)(0.70)}$$
$$= 0.6$$

Probability the reader is under 35 given that the reader is a female = 0.6.

Example : The probability the event A will occur is 0.4 the conditional probability of another event B occurring given A is 0.6 and the conditional probability of B occurring given A has not occurred is 0.3. Find the probability of

(a) The event **A** occurring given **B**

(b) The event of **A** not occurring given **B**

Solution: Expressing the given probabilities using notation

$P(A) = 0.4$. Hence $P(A^c) = 1 - 0.4 = 0.6$

$P(B | A) = 0.6$ and $P(B | A^c) = 0.3$

(a) We need to compute $P(A | B)$

Using Baye's Theorem

$$P(A | B) = \frac{P(A)P(B|A)}{P(A)P(B|A)+P(A^c)P(B|A^c)}$$

$$= \frac{(0.4)(0.6)}{(0.4)(0.6)+(0.6)(0.3)}$$

Probability of event A occurring given B = 0.571

(b) To find $P(A^c \mid B)$

$P(A^c \mid B) = 1 - P(A \mid B)$

$= 1 - 0.571 = 0.429$

Probability of event A not occurring given B = 0.429

5.8 PERMUTATIONS AND COMBINATIONS : (Fundamental Principles of Counting)

1. Multiplication Theorem

If an operation can be performed in m different ways and following which a second operation can be performed in n different ways, then the two operations in succession can be performed in m × n different ways.

2. Addition Theorem

If an operation can be performed in m different ways and a second independent operation can be performed in n different ways, either of the two operations can be performed in (m+n) ways.

3. Factorial

Let n be a positive integer. Then n factorial can be defined as

$$n! = n(n-1)(n-2)\cdots 1$$

Example : 5!=5 × 4 × 3 × 2 × 1=120

3! = 3×2×1=6

Special Cases

0! = 1

1! = 1

5.8.1 PERMUTATIONS :

Permutations are the different arrangements of a given number of things by taking some or all at a time.

Example : All permutations (or arrangements) that can be formed with the letters a, b, c by taking three at a time are (abc, acb, bac, bca, cab, cba)

All permutations (or arrangements) that can be formed with the letters a, b, c by taking two at a time are (ab, ac, ba, bc, ca, cb)

1. Permutation of n different objects

(a) The number of permutations of n objects taken all at a time, denoted by the symbol ${}^{n}p_{n}$, is given by ${}^{n}p_{n} = n!$

(b) The number of permutations of n objects taken r at a time, where $0 < r \leq n$, denoted by

${}^{n}p_{r}$, is given by ${}^{n}p_{r} = \dfrac{n!}{(n-r)!}$

Special Cases

${}^{n}P_{0} = 1,\quad {}^{n}P_{r} = 0$ for $r > n$

2. When repetition of objects is allowed:

REPETITION : The term repetition is very important in permutations and combinations. Consider the same situation described above where we need to find out the total number of possible samples of two objects which can be taken from three objects P, Q, R.

If repetition is allowed, the same object can be taken more than once to make a sample. i.e., PP, QQ, RR can also be considered as possible samples.

If repetition is not allowed, then PP, QQ, RR cannot be considered as possible samples.

Normally repetition is not allowed unless mentioned specifically.

Note : (a) The number of permutations of n things taken all at a time, when repetition of objects is allowed is n^n.

(b) The number of permutations of n objects, taken r at a time, when repetition of objects is allowed, is n^r.

3. Permutations when the objects are not distinct:

The number of permutations of n objects of which p_1 are of one kind, p_2 are of second kind,...,P_k are of k^{th} kind and the rest if any, are of different kinds is $\dfrac{n!}{p_1!.p_2!.\ ....p_k!}$

5.8.2 COMBINATIONS:

On many occasions we are not interested in arranging but only in selecting r objects from given n objects. A combination is a selection of some or all of a number of different objects where the order of selection is immaterial.

Example : Suppose we want to select two out of three girls P, Q, R. Then, possible combinations are PQ, QR and RP. (Note that PQ and QP represent the same selection.)

Suppose we want to select three out of three girls P, Q, R. Then, only possible combination is PQR.

The number of selections of r objects from the given n objects is denoted by ${}^{n}C_{r}$, and is given by

$${}^{n}C_{r} = \frac{n!}{r!(n-r)!}$$

Special Cases

${}^{n}C_{0} = 1$

${}^{n}C_{n} = 1$

${}^{n}C_{1} = n$

${}^{n}C_{r} = 0$ for $r > n$

Some important results

Let n and r be positive integers such that r = n. Then

(i) ${}^{n}C_{r} = {}^{n}C_{n-r}$

(ii) ${}^{n}C_{r} + {}^{n}C_{r-1} = {}^{n+1}C_{r}$

Difference between Permutations and Combinations and How to identify them

Sometimes, it will be clearly stated in the problem itself whether permutation or combination is to be used. However if it is not mentioned in the problem, we have to find out whether the question is related to permutation or combination.

Consider a situation where we need to find out the total number of possible samples of two objects which can be taken from three objects P, Q, R. To understand if the question is related to permutation or combination, we need to find out if the order is important or not.

If order is important, PQ will be different from QP, PR will be different from RP and QR will be different from RQ

If order is not important, PQ will be same as QP, PR will be same as RP and QR will be same as RQ

Hence,

If the order is important, problem will be related to permutations.

If the order is not important, problem will be related to combinations.

For permutations, the problems can be like "What is the number of permutations the can be made", "What is the number of arrangements that can be made", "What are the different number of ways in which something can be arranged", etc.

For combinations, the problems can be like "What is the number of combinations the can be made", "What is the number of selections the can be made", "What are the different number of ways in which something can be selected", etc.

PQ and QP are two different permutations, but they represent the same combination.

Mostly problems related to word formation, number formation etc. will be related to

Permutations. Similarly most problems related to selection of persons, formation of geometrical figures, distribution of items (there are exceptions for this) etc. will be related to combinations.

Note : 1. Use permutations if a problem calls for the number of arrangements of objects and different orders are to be counted.

2. Use combinations if a problem calls for the number of ways of selecting objects and the order of selection is not to be counted.

Example: Out of 7 consonants and 4 vowels, how many words of 3 consonants and 2 vowels can be formed?

Number of ways of selecting 3 consonants from 7 $= {}^{7}C_{3}$

Number of ways of selecting 2 vowels from 4 $= {}^{4}C_{2}$

Number of ways of selecting 3 consonants from 7 and 2 vowels from $= {}^{7}C_{3} \times {}^{4}C_{2}$

$$= \left(\frac{7\times6\times5}{3\times2\times1}\right)\times\left(\frac{4\times3}{2\times1}\right)=210$$

It meanswe can have 210 groups where each group contains total 5 letters (3 consonants and 2 vowels).

Number of ways of arranging 5 letters among themselves= 5! = 5 × 4 × 3 × 2 × 1 = 120

Hence, required number of ways = 210 × 120 = 25200

Example : In how many different ways can the letters of the word 'MATHEMATICS' be arranged such that the vowels must always come together?

Solution : The word 'MATHEMATICS' has 11 letters. It has the vowels 'A','E','A','I' in it and these 4 vowels must always come together. Hence these 4 vowels can be grouped and considered as a single letter. That is, MTHMTCS(AEAI).

Hence we can assume total letters as 8. But in these 8 letters, 'M' occurs 2 times, 'T' occurs

2 times but rest of the letters are different.

Hence, number of ways to arrange these letters

$$=\frac{8!}{(2!)(2!)}=\frac{(8\times7\times6\times5\times4\times3\times2\times1)}{(2\times1)(2\times1)}=10080$$

In the 4 vowels (AEAI), 'A' occurs 2 times and rest of the vowels are different.

Number of ways to arrange these vowels among themselves

$$=\frac{4!}{2!}=\frac{4\times3\times2\times1}{2\times1}=12$$

Hence, required number of ways

$$=10080\times12=120960$$

Example : A bag contains 2 white balls, 3 black balls and 4 red balls. In how many ways can 3 balls be drawn from the bag, if at least one black ball is to be included in the draw?

Solution: From 2 white balls, 3 black balls and 4 red balls, 3 balls are to be selected such that at least one black ball should be there.

Hence we have 3 choices as given below

We can select 3 black balls...(option 1)

We can select 2 black balls and 1 non-black ball ...(option 2)

We can select 1 black ball and 2 non-black balls ...(option 3)

Number of ways to select 3 black balls $= {}^{3}C_{3}$

Number of ways to select 2 black balls and 1 non-black ball $= {}^3C_2 \times {}^6C_1$

Number of ways to select 1 black ball and 2 non-black balls $= {}^3C_1 \times {}^6C_2$

Total number of ways

$$= {}^3C_3 + {}^3C_2 \times {}^6C_1 + {}^3C_1 \times {}^6C_2$$

$$= {}^3C_3 + {}^3C_1 \times {}^6C_1 + {}^3C_1 \times {}^6C_2 \quad [\because {}^nC_r = {}^nC_{(n-r)}]$$

$$= 1 + 3 \times 6 + 3 \times \left(\frac{6 \times 5}{2 \times 1}\right)$$

$$= 1 + 18 + 45$$

$$= 64$$

Example : How many 3 digit numbers can be formed from the digits 2, 3, 5, 6, 7 and 9 which are divisible by 5 and none of the digits is repeated?

Solution : A number is divisible by 5 if its last digit is 0 or 5

We need to find out how many 3 digit numbers can be formed from the 6 digits (2, 3, 5, 6, 7, 9)which are divisible by 5

Since the 3 digit number should be divisible by 5, we should take the digit 5 from the 6 digits(2,3,5,6,7,9) and fix it at the unit place. There is only 1 way of doing this.

		1

Since the number 5 is placed at unit place, we have now five digits (2, 3, 6, 7, 9) remaining. Any of these 5 digits can be placed at tens place

	5	1

Since the digit 5 is placed at unit place and another one digit is placed at tens place, we have now four digits remaining. Any of these 4 digits can be placed at hundreds place.

4	5	1

Required Number of three digit numbers

$= 4 \times 5 \times 1 = 20$

Example : How many 6 digit telephone numbers can be formed if each number starts with 35 and no digit appears more than once?

Solution : The first two places can only be filled by 3 and 5 respectively and there is only 1 way for doing this.

Given that no digit appears more than once. Hence we have 8 digits remaining (0,1,2,4,6,7,8,9)

So, the next 4 places can be filled with the remaining 8 digits in ${}^{8}P_4$ ways.

Total number of ways = ${}^{8}P_4 = 8\times7\times6\times5 = 1680$

Example : How many 3-letter words with or without meaning, can be formed out of the letters of the word, 'LOGARITHMS', if repetition of letters is not allowed?

Solution : The word 'LOGARITHMS' has 10 different letters.

Hence, the number of 3-letter words (with or without meaning) formed by using these letters

$= {}^{10}P_3 = 10 \times 9 \times 8 = 720$

5.9 STATISTICS :

5.9.1 RANDOM VARIABLE :

The outcome of an experiment need not be a number, for example, the outcome when a coin is tossed can be 'heads' or 'tails'. However, we often want to represent outcomes as numbers. A random variable is a function that associates a unique numerical value with every outcome of an experiment. The value of the random variable will vary from trial to trial as the experiment is repeated.

There are two types of random variable - discrete and continuous.

A random variable has either an associated probability distribution (discrete random variable) or probability density function (continuous random variable).

Example : A coin is tossed ten times. The random variable X is the number of tails that are noted. X can only take the values 0, 1, ..., 10, so X is a discrete random variable.

A light bulb is burned until it burns out. The random variable Y is its lifetime in hours. Y can take any positive real value, so Y is a continuous random variable.

1. DISCRETE RANDOM VARIABLE :

A discrete random variable is one which may take on only a countable number of distinct values such as 0, 1, 2, 3, 4, ... Discrete random variables are usually (but not necessarily) counts. If a random variable can take only a finite number of distinct values, then it must be discrete. Examples of discrete random variables include the number of children in a family, the Friday night attendance at a cinema, the number of patients in a doctor's surgery, the number of defective light bulbs in a box of ten.

2. CONTINUOUS RANDOM VARIABLE :

A continuous random variable is one which takes an infinite number of possible values. Continuous random variables are usually measurements. Examples include height, weight, the amount of sugar in an orange, the time required to run a mile.

5.9.2 CENTRAL TENDENCY (MEASURES OF POSITION):

Central tendency (sometimes called "measures of location," "central location," or just "center") is a way to describe what's typical for a set of data. Central tendency doesn't tell you specifics about the individual pieces of data, but it does give you an overall picture of what is going on in the entire data set. There are three major ways to show central tendency: mean, mode and median.

1. MEAN : Also known as the average. The mean is found by adding up all of the given data and dividing by the number of data entries.

Example : the grade 10 math class recently had a mathematics test and the grades were as follows: 78, 66, 82,89,75,74

$$\text{mean} = \frac{78+66+82+89+75+74}{6} = \frac{464}{6} = 77.3$$

Hence, 77.3 is the mean average of the class.

2. MEDIAN : The median is the middle number. First you arrange the numbers in order from lowest to highest, then you find the middle number by crossing off the numbers until you reach the middle.

Example : use the below data to find the median:

66, 74, 75, 78, 82, 89

Solution : As you can see we have two numbers, there is no middle number. What do we do?

It is simple; we take the two middle numbers and find the average, (or mean).

75 + 78 = 153

153 / 2 = 76.5

Hence, the middle number is 76.5.

3. MODE : this is the number that occurs most often.

Example : find the mode of the following data:

78 , 56 ,68 , 92, 84 , 76, 74, 56, 68, 66, 78, 72, 66, 65, 53, 61, 62, 78, 84, 61, 90, 87, 77, 62, 88, 81.

Solution : 78is the number that occurs most often

Hence, the mode is 78.

5.9.3 MEASURES OF DISPERSION (MEASURES OF SPREAD OR VARIATION):

The degree to which numerical data tend to spread about an average value is called the variation or dispersion of the data.

Example : variance and standard deviation

1.VARIANCE : The variance is the measure of how far a set of number is spread out. It is the average squared difference of the data from the mean score of a distribution.

$$\sigma^2 = \frac{\sum\left(x_i - \mu\right)^2}{n} = \frac{\sum x_i^2}{n} - \mu^2$$

Example : The heights (in cm) of students of a class is given to be 163, 158, 167, 174, 148. Find the variance.

Solution : To find the variance, we need to find the mean of the given data and total members in the data set. Total number of elements, N = 5

$$\mu=\frac{163+158+167+174+148}{5}=162$$

The formula for variance is $V=\frac{\sum(x-\mu)^2}{n}$

Now putting the values in the formula we get,

$$V=\frac{(162-163)^2+(158-163)^2+(167-163)^2+(174-163)^2+(148-163)^2}{5}=77.6$$

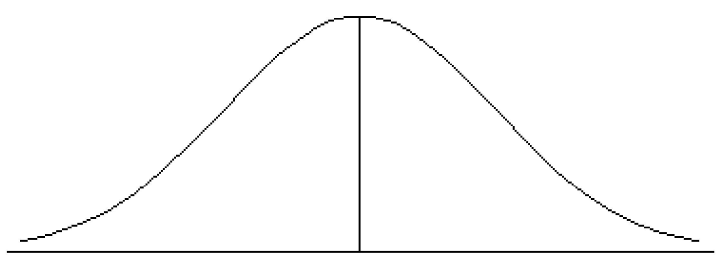
2. STANDARD DEVIATION : The square root of the variance is the standard deviation. While var. gives you a rough idea of spread, the standard deviation is more concrete, giving you exact distances from the mean.

$$\sigma=\sqrt{\frac{\sum(x-\mu)^2}{n}}$$

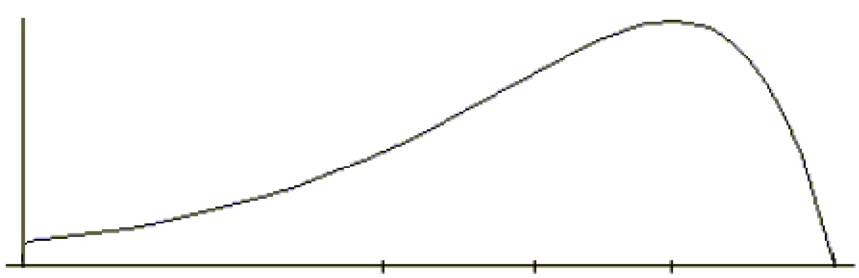
5.9.4 SKEWNESS :

Skewness is a measure of degree of asymmetry of the distribution.

1. SYMMETRIC : Mean, median, and mode are all the same here; the distribution is mound shaped, and no skewness is apparent. The distribution is described as symmetric.

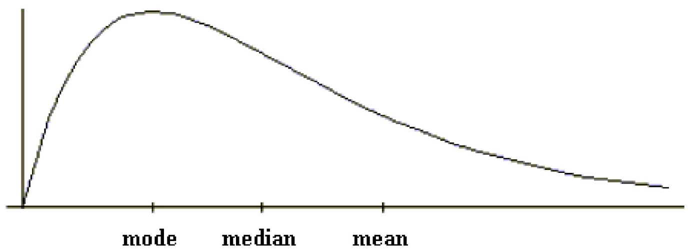


2. SKEWED LEFT : Mean to the left of the median, long tail on the left.



The above distribution is skewed to the left.

3. SKEWED RIGHT : Mean to the right of the median, long tail on the right.



Note : When one has very skewed data, it is better to use the median as measure of central tendency since the median is not much affected by extreme values.

5.9.5 EXPECTED VALUE OF RANDOM VARIABLE X:

The expected value (or population mean) of a random variable indicates its average or central value.

1. The expected value of a random variable X is symbolized by E(X) or μ.

$$E[X] = \begin{cases} \sum xp(x) & \text{X is discrete Random variable} \\ \int_{-\infty}^{\infty} x\,f(x)\,dx & \text{X is continuous Random variable} \end{cases}$$

2. Expected value of X^2 is

$$E[X^2] = \begin{cases} \sum x^2p(x) & \text{X is discrete Random variable} \\ \int_{-\infty}^{\infty} x^2f(x)\,dx & \text{X is continuous Random variable} \end{cases}$$

Example : Find the expected value of X from the given probability distribution table.

X	1	2	3
P(x)	0.8	0.6	0.2

Solution : The expected value of X, $E(X) = \sum(XP(X))$

E(X) = 1 × 0.8 + 2 × 0.6 + 3 × 0.2

= 0.8 + 1.2 + 0.6 = 2.4

Note : **Properties of Expectation**

(i) $E[c] = c$ c is a constant

(ii) $E[aX] = aE(X)$

(iii) $E(aX + b) = a\,E(X) + b$

(iv) If X & Y are random variables $E[X \pm Y] = E(X) \pm E(Y)$

(v) If X & Y random variables $E(XY) = E(X) \cdot E(Y/X)$

(vi) If X & Y independent random Variables $E(XY) = E(X) \cdot E(Y)$

5.9.6 MEAN OF RANDOM VARIABLE X :

$$\text{Mean} = \mu = E[X]$$

Example : If height of four women is given to be 150 cm, 165 cm, 135 cm and 170 cm, then what will be the expected value of the height of a randomly chosen women?

Solution : The expected value is the mean of all the values.

$$E(X) = \frac{150+165+135+170}{4} = \frac{620}{4} = 155$$

The expected value is found to be 155 cm.

Example : The probability density function of a random variable, x is

$$f(x) = \frac{x}{4}(4-x^2) \quad \text{for } 0 \le x \le 2$$
$$= 0 \qquad \text{otherwise}$$

The mean μ of the random variable is ___________.

Solution : $f(x)\ \frac{x}{4} = (4-x^2) \text{ for } 0 \le x \le 2$

$$\text{mean } = \mu_x = E(x)$$

$$\int_0^2 x f(x)\, dx$$

$$\int_0^2 x\left(\frac{x}{4}\right)(4\text{-}x^2)\, dx$$

$$= \int_0^0 (4x^2\text{-}x^4)\, dx$$

$$= \frac{1}{4}\left[\frac{4x^3}{3} - \frac{x^5}{5}\right]_0^2$$

$$= \frac{1}{4}\left[4.\frac{8}{3} - \frac{32}{5}\right]$$

$$= \frac{32}{4}\left[\frac{1}{3} - \frac{1}{5}\right]$$

$$= 8\left[\frac{2}{15}\right] = \frac{16}{15} = 1.0667$$

5.9.7 VARIANCE OF RANDOM VARIABLE X:

The (population) variance of a random variable is a non-negative number which gives an idea of how widely spread the values of the random variable are likely to be; the larger the variance, the more scattered the observations on average.

$\operatorname{Var}(X) = E[(X-\mu)^2]$

$\operatorname{Var}(X) = E\left[X^2\right]-\mu^2$

Note : **Properties of Variance**

(i) $\operatorname{Var}[C]=0$. C is constant

(ii) $\operatorname{Var}(aX) = a^2 V(X)$ where X is random variable and 'a' constant

$Var(-X) = (-1)^2 Var(X) = Var(X)$ Variance is always positive

(iii) $\operatorname{Var}(ax+b) = a^2 \operatorname{Var}(X) + 0$

(iv) If X & Y are independent random variables

$\operatorname{Var}(X+Y) = \operatorname{Var}(X) + \operatorname{Var}(Y)$

$\operatorname{Var}(X-Y) = \operatorname{Var}(X) + \operatorname{Var}(Y)$

(v) $\operatorname{Cov}(X, Y) = E(XY) - E(X)E(Y)$

$\operatorname{Cov}(X, X) = \operatorname{Var}(X)$

$\operatorname{Cov}(a, b) = 0$

(vi) If X and Y are random variables

$\operatorname{Var}(X+Y) = \operatorname{Var}(X) + \operatorname{Var}(Y) + 2\operatorname{Cov}(X, Y)$

$\operatorname{Var}(X-Y) = \operatorname{Var}(X) + \operatorname{Var}(Y) - 2\operatorname{Cov}(X, Y)$

Example : In the following table, x is a discrete random variable and p(x) is the probability density. The standard deviation of x is

X	1	2	3
P(x)	0.3	0.6	0.1

Solution :

X	1	2	3
P(x)	0.3	0.6	0.1

$\text{mean}(\mu) = E(x) = 1\times 0.3 + 2\times 0.6 + 3\times 0.1 = 0.3 + 1.2 + 0.3 = 1.8$

$E(x^2) = \sum x^2 P(x)$

$= 1\times 0.3 + 4\times 0.6 + 9\times 0.1 \quad = 0.3 + 2.4 + 0.9 \quad = 3.6$

$\text{Variance} = E(x^2) - \{E(x)\}^2$

$= 3.6 - (1.8)^2 = 0.36$

$\text{Standard deviation} = \sqrt{\text{variance}} = \sqrt{0.36} = 0.6$

5.10 PROBABILITY DISTRIBUTION:

The probability distribution of a discrete random variable is a list of probabilities associated with each of its possible values. It is also sometimes called the probability function or the **probability mass function**.

The probability mass function *p(x)* of a discrete random variable *X* taking values $x_1, x_2, .. x_n$ is defined such that

$$(\text{i})\ \text{p}\left(x_i\right) \geq 0$$

$$(\text{ii})\ \sum_{i=1}^{n} \text{p}\left(x_i\right) = 1$$

$$(\text{iii})\ \text{p}\left(x_i\right) = \text{p}\left(X = x_i\right)$$

5.11 PROBABILITY DENSITY FUNCTION (PDF):

The probability density function of a continuous random variable is a function which can be integrated to obtain the probability that the random variable takes a value in a given interval.

The pdf *f(x)* of a continuous random variable X is defined such that

$$(i)\ f\left(x\right) \geq 0$$

$$(ii)\ \int_{-\infty}^{\infty} f\left(x\right) dx = 1$$

$$(\text{iii})\ P\left(a < X \leq b\right) = \int_{a}^{\text{b}} f\left(x\right) dx$$

Example : A continuous random variable X has a probability density function

$$\text{f(x)} = \text{e}^{(-\text{x})},\ 0 < \infty \text{ Then } \text{P}\{\text{X} > 1\} \text{ is}$$

Solution : $\text{p}\left(\text{x} > 1\right) = \int_{1}^{\infty} \text{e}^{-\text{x}} \text{dx} = \left[-\text{e}^{-\text{x}}\right]_1^{\infty} = \text{e}^{-1} = 0.368$

5.12 BINOMIAL DISTRIBUTION:

A discrete random variable

$\text{X} \sim \text{B}\left(\text{n, P}\right)$ [X follows binomial distribution with parameters n and p]

if its probability mass function $P\left(x\right)$ is defined as

$$\text{P}\left(x\right) = {}^{n}\text{C}_{\text{x}}\ (\text{p})^{x}\ (\text{q})^{n-x} \qquad \text{where } x = 0, 1, 2, 3, \ldots n$$

Where, n = Number of Bernoulli's trails A is an event

$p = P(A)$ probability of success of the event in each trail

$q = 1 - p = P(A^C)$ probability of failure of the event in each trail

Random variable x represents Number of times event A happens.

The trials must meet the following requirements:

$\Rightarrow$ the total number of trials is fixed in advance;

$\Rightarrow$ there are just two outcomes of each trial; success and failure;

$\Rightarrow$ the outcomes of all the trials are statistically independent;

$\Rightarrow$ all the trials have the same probability of success

Note : If $X \sim B(n, p)$ then

Mean = np

Variance = npq

Standard deviation $= \sqrt{npq}$

Example : The probability that a thermistor randomly picked up from a production unit is defective is 0.1. The probability that out of 10 thermistors (randomly picked up), 3 are defective is

Solution : p= Probability of a thermistor is defective=0.1

q = 1 - p = 1 - 0.1 = 0.9

n= 10

Let X be random variable which is no. of defective pieces.

Required probability = p(x=3)= ${}^{10}C_3 P^3 q^7$

= ${}^{10}C_3 (0.1)^3 (0.9)^7$

=0.057

5.13 POISSON DISTRIBUTION:

Typically, a Poisson random variable is a count of the number of events that occur in a certain time interval or spatial area.

For example, the number of cars passing a fixed point in a 5 minute interval, or the number of calls received by a switchboard during a given period of time.

A discrete random variable $X:P(\lambda)$ $[X$ follows Poisson distribution with parameters $\lambda]$

if its probability mass function *p(x)* is defined as

$$P(x) = \frac{e^{-\lambda}\lambda^x}{x!} \quad \text{Where } x = O, 1, 2, 3....$$

The following requirements must be met:

$\Rightarrow$ The length of the observation period is fixed in advance;

$\Rightarrow$ The events occur at a constant average rate;

$\Rightarrow$ The number of events occurring in disjoint intervals are statistically independent.

Note :

(i) if $X \sim P(\lambda)$ *then*

(a) Mean = λ

(b) Variance = λ

(c) Standard deviation = $\sqrt{\lambda}$

(ii) The Poisson distribution can sometimes be used to approximate the Binomial distribution with parameters n and p. When the number of observations n is large, and the success probability p is small, the Bi(n,p) distribution approaches the Poisson distribution with the parameter given by

m = np.

Example : The number of accidents occurring in a plant in a month follows Poisson distribution with mean as 5.2. The probability of occurrence of less than 2 accidents in the plant during a randomly selected month is

Solution : Given $\lambda = 5.2$

Let x be randomvariablewhichfollowsPoisson'sdistribution

$$P(x<2) = P(x=0) + P(x=1) = \frac{e^{-\lambda}\lambda^0}{0!} + \frac{e^{-\lambda}}{1!}\lambda^1 = e^{-5.2}(6.2) \qquad = 0.0055 \times 6.2 = 0.034$$

5.14 UNIFORM DISTRIBUTION:

A continuous random variable $X \sim U(a, b)$ [X follows uniform distribution over the interval [a, b] if its pdf *f(x)* is defined as

$$f(x) = \frac{1}{b - a} \text{ where } a < x < b$$

If $X \sim U(a, b)$ *then*

Note : (i) Mean $= \dfrac{b+a}{2}$

(ii)varience $= \dfrac{(b-a)^2}{12}$

Example : The standard deviation of a uniformly distributed random variable between 0 and 1 is

Solution : $\sigma = \sqrt{\dfrac{(b-a)^2}{12}} = \sqrt{\dfrac{(1-0)^2}{12}} = \dfrac{1}{\sqrt{12}}$

5.15 EXPONENTIAL DISTRIBUTION :

A continuous random variable $X \sim E(\lambda)$[X follows exponential distribution with parameter $\lambda > 0$] if its probability density functions is given by

$f(x) = \lambda e^{-\lambda x}$ *where* $x \geq 0$

Note:

If $X \sim E(\lambda)$ then

(I) Mean $= 1/\lambda$

(ii) Variance $= 1/\lambda^2$

Example : Arrival times to a busy yogurt shop during their peak arrival time are well modeled by a poison process with a rate of 30 customers per hour. The probability that the time be tween the arrivals of two customers in between 2 and 4 minutes is ___% .

Solution : $(\lambda) = \frac{30}{60} \frac{\text{customers}}{\text{minutes}}$

$$\lambda = \frac{1}{2} \frac{\text{customers}}{\text{minutes}}$$

Since, time between the arrivals of two customers follow exponential distribution. Hence, its probability density function is define as

Where x = time between arrivals

$$f(x) = \frac{1}{2} e^{-\frac{x}{2}}$$

$$P[2 < x < 4] = \int_2^4 f(x)\,dx$$

$$= \int_2^4 \frac{1}{2} e^{-\frac{x}{2}} dx$$

$$= \frac{1}{2}\left[\frac{e^{-\frac{x}{2}}}{-1/2}\right]_2^4$$

$$= -\left[e^{-2} - e^{-1}\right]$$

$$= 0.2325 \simeq 23.25\%$$

5.16 NORMAL DISTRIBUTION :

A continuous random variable X $\sim N(\mu, \sigma^2)$[X follows normal distribution with mean μ and variance σ^2] if its pdf $f(x)$is defined as

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

Note :

(i) The normal curve is bell – shaped and is symmetric about the mean

(i.e., symmetric about the line $x = \mu$)

(ii) Mean = Median = Mode

(iii) $P(X < \mu) = P(X > \mu) = \frac{1}{2}$

(iv) The standard normal variate Z is defined as $Z = \frac{X-\mu}{\sigma}$

(v) The standard normal variate $Z \sim N(\mu = 0, \sigma^2 = 1)$ and its pdf (z) is

is defined as

$$f(z) = \frac{1}{\sqrt{2\pi}} e^{\left(\frac{-z^2}{2}\right)} \quad where\ -\infty < z < \infty$$

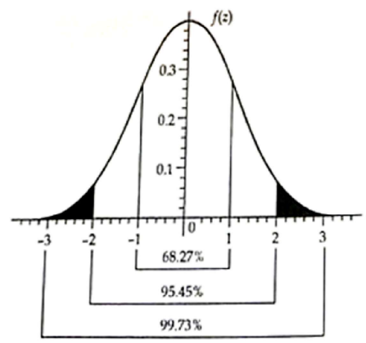
(vi) The standard normal variate curve is symmetrical about the liner z = 0

$(\text{vii})\ P(Z < 0) = P(Z > 0) = \frac{1}{2}$

(a) $P(-1 < Z < 1) = 0.6827$

(b) $P(-2 < Z < 2) = 0.9545$

(c) $P(-3 < Z < 3) = 0.9973$



Example : A nationalized bank has found that the daily balance available in its savings accounts fol lows a normal distribution with a mean of Rs. 500 and a standard deviation of Rs. 50. The percentage of savings account holders, who maintain an average daily balance more than Rs 500 is ______

Solution : Given $\mu = 500$

$\sigma = 50$

$P(X > 500) = ?$

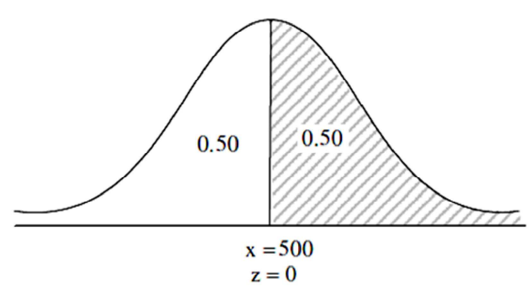
WhereX followsnormaldistribution

Weknowthat standardnormalvariable

$z = \frac{x-\mu}{\sigma}$

$z = \frac{500-500}{50} = 0$

$\therefore (X > 500) = P(z > 0) = 0.50$ (see figure)



GATE QUESTIONS

Q.1 Let P (E) denote the probability of the event E. Given P(A)=1, P(B)=1/2, the values of P(A/B) and P(B/A) respectively are

(a)1/4 , 1/2 (b)1/2 , 1/4

(c)1/2 , 1 (d)1 , 1/2 **[GATE-2003 (CS)]**

Q.2 Aishwarya studies either computer science or mathematics every day. If she studies computer science on a day, then the probability that she studies mathematics the next day is 0.6. If she studies mathematics on a day, then the probability that she studies computer science the next day is 0.4. Given that Aishwarya studies computer science on Monday, what is the probability that she studies computer science on Wednesday?

(a)0.24 (b) 0.36

(c) 0.4 (d) 0.6 **[GATE-2003 (CS)]**

Q.3 Three companies, X, Y and Z supply computers to a university. The percentage of computers supplied by them and the probability of those being defective are tabulated below.

Company	% of computers supplied	Probability of being defective
X	60%	0.01
Y	30%	0.02
Z	10%	0.03

Given that a computer is defective, the probability that it was supplied by Y is:

(a) 0.1 (b) 0.2

(c) 0.3 (d) 0.4 **[GATE-2003 (CS)]**

Q.4 Consider a company that assembles computers. The probability of a faulty assembly of any computer is p. The company therefore subjects each computer to a testing process. This testing process gives the correct result for any computer with a probability of q. What is the probability of a computer being declared faulty?

(a) $pq+(1-p)(1-q)$ (b) $(1-q)p$

(c) $(1-p)q$ (d) pq **[GATE-2003 (CS)]**

Q.5 A box contains 10 screws, 3 of which are defective. The screws are drawn at random with replacement. The probability that none of the two screws is defective will be

(a) 100 % (b) 50 %

(c)49 % (d)none of these **[GATE-2003 (CE)]**

Q.6 From a pack of regular playing cards. Two cards are drawn at random. What is the probability that both cards will be Kings, if first card in Not replaced?

(a) $\frac{1}{26}$ (b) $\frac{1}{52}$

(c) $\frac{1}{169}$ (d) $\frac{1}{221}$ **[GATE-2004 (ME)]**

Q.7 The value of the integral $I = \frac{1}{\sqrt{2\pi}}\int_0^\infty \exp\left(-\frac{x^2}{8}\right)dx$ is

(a) 1 (b) π

(c) 2 (d) 2π **[GATE-2005 (EC)]**

Q.8 A lot has 10% defective items. Ten items are chosen randomly from this lot. The probability that exactly 2 of the chosen items are defective is

(a)0.0036 (b)0.1937

(c)0.2234 (d)0.3874 **[GATE-2005 (ME)]**

Q.9 A fair dice is rolled twice. The probability that an odd number will follow an even number is

(a) $\frac{1}{2}$ (b) $\frac{1}{6}$

(c) $\frac{1}{3}$ (d) $\frac{1}{4}$ **[GATE-2005 (EC)]**

Q.10 A single die is thrown twice. What is the probability that the sum is neither 8 nor 9?

(a) $\frac{1}{9}$ (b) $\frac{5}{36}$

(c) $\frac{1}{4}$ (d) $\frac{3}{4}$ **[GATE-2005 (ME)]**

Q.11 A hydraulic structure has four gates which operate independently. The probability of failure of each gate is 0.2. Given that gate 1 has failed, the probability that both gates 2 and 3 will fail is

(a) 0.240 (b)0.200

(c)0.040 (d)0.008 **[GATE-2005 (ME)]**

Q.12 For each element in a set of size 2n, an unbiased coin is tossed. The 2n coins tossed are independent. An element is chosen if the corresponding coin toss were head. The probability that exactly n elements are chosen is

(a) $\frac{^{2n}C_n}{4^n}$ (b) $\frac{^{2n}C_n}{2^n}$

(c) $\frac{1}{^{2n}C_n}$ (d) $\frac{1}{2}$ **[GATE-2006 (CS)]**

Q.13 There are 25 calculators in a box. Two of them are defective. Suppose 5 calculators are randomly pocked for inspection (I.e., each has the same chance of being selecte(d), what is the probability that only one of the defective calculators will be included in the inspection?

(a) $\frac{1}{2}$ (b) $\frac{1}{3}$

(c) $\frac{1}{4}$ (d) $\frac{1}{5}$ **[GATE-2006 (CE)]**

Q.14 A class of first year B. Tech. students is composed of four batches. A, B, C & D, each consisting of 30 students.It is found that the sessional marks of students in Engineering Drawing in batch C have a mean of 6.6 and standard deviation of 2.3. The mean and standard deviation of the marks for the entire class are 5.5 and 4.2, respectively. It is decided by the course instructor to normalize the marks of the students of all batches to have the same mean and standard deviation as that of the entire class. Due to this, the marks of a student in batch C are changed from 8.5 to

(a) 6.0 (b)7.0

(c) 8.0 (d) 9.0 **[GATE-2006 (CE)]**

Q.15 A box contains 20 defective items and 80 non-defective items. If two items are selected at random without replacement, what will be the probability that both items are defective?

(a) $\frac{1}{5}$ (b) $\frac{1}{25}$

(c) $\frac{20}{99}$ (d) $\frac{19}{495}$ **[GATE-2006 (ME)]**

Q.16 Consider a continuous random variable probability density function

$$f(t) = \begin{cases} 1+t & -1 \le t \le 0 \\ 1-t & 0 \le t \le 1 \end{cases}$$

The standard deviation of the random variable is:

(a) $\frac{1}{\sqrt{3}}$ (b) $\frac{1}{\sqrt{6}}$

(c) $\frac{1}{3}$ (d) $\frac{1}{6}$ **[GATE-2006 (ME)]**

Q.17 A probability density function is of the form $p(x) = Ke^{-\alpha|x|}, x \in (-\infty, \infty)$. The value of K is

(a) 0.5 (b) 1

(c) 0.5α (d) α **[GATE-2006 (E(C)]**

Q.18 Suppose we uniformly and randomly select a permutation from the 20! Permutations of 1, 2, 3… 20. What is the probability that 2 appears at anearlier position than any other even number in the selected permutation?

(a) $\frac{1}{2}$ (b) $\frac{1}{10}$

(c) $\frac{9!}{20!}$ (d) None of these **[GATE-2007(EE)]**

Q.19 An examination consists of two papers, Paper 1 and Paper 2. The probability of failing in Paper 1 is 0.3 and that in Paper 2 is 0.2. Given that a student has failed in Paper 2, the Probability of failing in Paper 1 is 0.6. The probability of a student failing in both the papers is

(a) 0.5 (b) 0.18

(c) 0.12 (d) 0.06 **[GATE-2007(EE)]**

Q.20 If P and Q are two random events, then the following is TRUE :

(a) Independence of P and Q implies that probability $(P \cap Q) = 0$

(b) Probability $(P \cup Q) \geq$ Probability (P) + Probability (Q)

(c) If P and Q are mutually exclusive, then they must be independent

(d) Probability $(P \cap Q) \leq$ Probability (P) **[GATE-2007(EE)]**

Q.21 A loaded dice has following probability distribution of occurrences

Dice Value	1	2	3	4	5	6
Probability	¼	1/8	1/8	1/8	1/8	1/4

If three identical dice as the above are thrown, the probability of occurrence of values 1, 5 and 6 on the three dice is

(a) same as that of occurrence of 3,4,5

(b) same as that of occurrence of 1,2,5

(c) 1/128

(d) 5/8 **[GATE-2007(EE)]**

Q.22 If the standard deviation of the spot speed of vehicles in a highway is 8.8 kmph and the mean speed of the vehicles is 33 kmph, the coefficient of variation in speed is

(a) 0.1517 (b) 0.1867

(c) 0.2666 (d) 0.3646 **[GATE-2007 (CE)]**

Q.23 Let X and Y be two independent random variables .Which one of the relations between expectations (E), variance (Var) and covariance (Cov) given below is FALSE?

(a) $E(XY) = E(X)E(Y)$ (b) $Cov(X, Y) = 0$

(c) $Var(X+Y) = Var(X) + Var(Y)$ (d) $E\left(X^2Y^2\right) = \left(E(X)\right)^2\left(E(Y)\right)^2$ **[GATE-2007 (ME)]**

Q.24 A person on a trip has a choice between private car and public transport. The probability of using a private car is 0.45 .While using the public transport, further choices available are bus and metro out of which the probability of commuting by a bus is 0.55. In such a situation, the probability (rounded upto 2 decimals) of using a car, bus and metro, respectively would be

(a)0.45, 0.30 and 0.25 (b)0.45, 0.25 and 0.30

(c)0.45, 0.55 and 0.00 (d)0.45, 0.35 and 0.2 **[GATE-2008 (CE)]**

Q.25 If probability density function of a random variable X is

$$f(x) = \begin{cases} x^2, & -1 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$$

then, the percentage probability $P\left(-\frac{1}{3} \le x \le \frac{1}{3}\right)$ is

(a)0.247 (b)2.47

(c) 24.7 (d) 247 **[GATE-2008 (CE)]**

Q.26 The standard deviation of a uniformly distributed random variable between 0 and 1 is

(a) $\frac{1}{\sqrt{12}}$ (b) $\frac{1}{\sqrt{3}}$

(c) $\frac{5}{\sqrt{12}}$ (d) $\frac{7}{\sqrt{12}}$ **[GATE-2008 (CE)]**

Q.27 Let X be a random variable following normal distribution with mean +1 and variance 4. Let Y be another normal variable with mean -1 and variance unknown. If $P(X \le -1) = P(Y \ge 2)$ the standard deviation of Y is

(a)3 (b)2

(c) $\sqrt{2}$ (d)1 **[GATE-2008 CS)]**

Q.28 A fair coin is tossed 10 times .What is the probability that ONLY the first two tosses will yield heads.

(a) $\left(\frac{1}{2}\right)^2$ (b) ${}^{10}C_2\left(\frac{1}{2}\right)^2$

(c) $\left(\frac{1}{2}\right)^{10}$ (d) ${}^{10}C_2\left(\frac{1}{2}\right)^{10}$ **[GATE-2009 (E(C)]**

Q.29 A box contains 4 white balls and 3 red balls. In succession, two balls are randomly selected and removed from the box. Given that the first removed ball is white, the probability that the second removed ball is red is

(a) $\frac{1}{3}$ (b) $\frac{3}{7}$

(c) $\frac{1}{2}$ (d) $\frac{4}{7}$ **[GATE-2010 (EE)]**

Q.30 A fair coin is tossed independently four times. The probability of the event "the number of time heads shown up is more than the number of times tails shown up" is

(a) $\frac{1}{16}$ (b) $\frac{1}{8}$

(c) $\frac{1}{4}$ (d) $\frac{5}{16}$ **[GATE-2010 (E(C)]**

Q.31 A deck of 5 cards (each carrying a distinct number from 1 to 5) is shuffled thoroughly. Two cards are then removed one at a time from the deck. What is the probability that the two cards are selected with the number on the first card being one higher than the number on the second card?

(a) 1/5 (b) 4/25

(c) 1/4 (d) 2/5 **[CS-GATE-2011]**

Q.32 Consider a finite sequence of random values X = $[x_1, x_2, \ldots x_n]$. Let μ_x be the mean and σ_x be the standard deviation of X .Let another finite sequence Y of equal length be derived from this as $y_i = a^* x_i + b$, where a and b are positive constants. Let μ_y be the mean and σ_y be the standard deviation of this sequence.Which one of the following statements is INCORRECT?

(a) Index position of mode of X in X is the same as the index position of mode of Y in Y.

(b) Index position of median of X in X is the same as the index position of median of Y in Y.

(c) $\mu_y = a\mu_x + b$

(d) $\sigma_y = a\sigma_x + b$ **[CS-GATE-2011]**

Q.33 A box contains 4 red balls and 6 black balls. Three balls are selected randomly from the box one after another without replacement. The probability that the selected set contains one red ball and two black balls is

(a) $\frac{1}{20}$ (b) $\frac{1}{12}$

(c) $\frac{3}{10}$ (d) $\frac{1}{2}$ **[ME-GATE-2012]**

Q.34 In an experiment, positive and negative values are equally likely to occur. The probability of obtaining at most one negative value in five trials is

(a) $\frac{1}{32}$ (b) $\frac{2}{32}$

(c) $\frac{3}{32}$ (d) $\frac{6}{32}$ **[CE-GATE-2012]**

Q.35 Consider a random variable X that takes values + 1 and -1 with probability 0.5 each. The values of the cumulative distribution function F(x) at x = -1 and +1 are

(a) 0 and 0.5 (b) 0 and 1

(c) 0.5 and 1 (d) 0.25 and 0.75 **[CS-GATE-2012]**

Q.36 Suppose a fair six-sided die is rolled once. If the value on the die is 1, 2, or 3, the die is rolled a second time. What is the probability that the sum total of values that turn up is at least 6 ?

(a) 10/21 (b) 5/12

(c) 2/3 (d) 1/6 **[CS-GATE-2012]**

Q.37 Let X be a nominal variable with mean 1 and variance 4. The probability $p(X<0)$ is

(a) 0.5

(b) Greater than zero and less than 0.5

(c) Greater than 0.5 and less than 1

(d) 1.0 **[ME-GATE-2013]**

Q.38 The probability that a student knows the correct answer to a multiple choice question is $\frac{2}{3}$. If the student does not know the answer, then the student guesses the answer. The probability of the guessed answer being correct is $\frac{1}{4}$. Given that the student has answered the question correctly, the conditional probability that the student known the correct answer is

(a) $\frac{2}{3}$ (b) $\frac{3}{4}$

(c) $\frac{5}{6}$ (d) $\frac{8}{9}$ **[ME-GATE-2013]**

Q.39 The annual precipitation data of a city is normally distributed with mean and standard deviation as 1000 mm &200 mm, respectively.The probability that the annual precipitation will be more than 1200 mm is

(a) < 50% (b) 50%

(c) 75% (d) 100% **[CE-GATE-2013]**

Q.40 A 1 hour rainfall of 10 cm has return period of 50 year. The probability that 1 hour of rainfall 10 cm or more will occur in each of two successive years is

(a) 0.04 (b) 0.2

(c) 0.2 (d) 0.0 **[CE-GATE-2013]**

Q.41 Suppose p is number of cars per minute passing through a certain road junction between 5 PM and 6PM, and p has a Poisson distribution with mean 3. What is the probability of observing fewer than 3 cars during any given minute in this interval?

(a) $8 / (2e^3)$ (b) $9 / (2e^3)$

(c) $17 / (2e^3)$ (d) $26 / (2e^3)$ **[CS-GATE-2013]**

Q.42 A continuous random variable X has a probability density function

$f(x) = e^{-x}, 0 < x < \infty$. Then P{X > 1} is

(a) 0.368 (b) 0.5

(c) 0.632 (d) 1.0 **[EE-GATE-2013]**

Q.43 A continuous random variable X has a probability density function

$f(x) = e^{-x}, 0 < x < \infty$. Then P{X > 1} is

(a) 0.368 (b) 0.5

(c) 0.632 (d) 1.0 **[IN-GATE-2013]**

Q.44 In the following table, x is a discrete random variable and p(x) is the probability density. The standard deviation of x is

X	1	2	3
P(x)	0.3	0.6	0.1

(a) 0.18 (b) 0.36

(c) 0.54 (d) 0.6 **[ME-GATE-2014(1)]**

Q.45 A box contains 25 parts of which 10 are defective. Two parts are being drawn simultaneously in a random manner from the box. The probability of both the parts being

(a) $\frac{7}{20}$ (b) $\frac{42}{125}$

(c) $\frac{25}{29}$ (d) $\frac{5}{9}$ **[ME-GATE-2014(2)]**

Q.46 Consider an unbiased cubic dice with opposite faces coloured identically and each face coloured red, blue or green such that each colour appears only two times on the dice. If the dice is thrown thrice, the probability of obtaining red colour on top face of the dice at least twice is _______ **[ME-GATE-2014(2)]**

Q.47 A group consists of equal number of men and women. Of this group 20% of the men and 50% of the women are unemployed. If a person is selected at random from this group, the probability of the selected person being employed is ______ **[ME-GATE-2014(3)]**

Q.48 A machine produces 0, 1 or 2 defective pieces in a day with associated probability of 1/6, 2/3 and 1/6, respectively. The mean value and the variance of the number of defective pieces produced by the machine in a day, respectively, are

(a)1 and 1/3 (b) 1/3 and 1

(c) 1 and 4/3 (d) 1/3 and 4/3 **[ME-GATE-2014(3)]**

Q.49 A nationalized bank has found that the daily balance available in its savings accounts follows a normal distribution with a mean of Rs. 500 and a standard deviation of Rs. 50. The percentage of savings account holders, who maintain an average daily balance more than Rs 500 is ______ **[ME-GATE-2014(4)]**

Q.50 The number of accidents occurring in a plant in a month follows Poisson distribution with mean as 5.2. The probability of occurrence of less than 2 accidents in the plant during a randomly selected month is

(a) 0.029 (b) 0.034

(c) 0.039 (d) 0.044 **[ME-GATE-2014(4)]**

Q.51 An observer counts 240veh/h at a specific highway location. Assume that the vehicle arrival at the location is Poisson distributed, the probability of having one vehicle arriving over a 30-second time interval is ________ **[CE-GATE-2014(1)]**

Q.52 The probability density function of evaporation on any day during a year in a watershed is given by

$$f(E) = \begin{cases} \frac{1}{5}, & 0 \le E \le 5\text{mm/day} \\ 0, & \text{otherwise} \end{cases}$$

The probability that E lies in between 2 and 4 mm/day in a day in the watershed is (in decimal) **[CE-GATE-2014(1)]**

Q.53 A fair (unbiase(d) coin was tossed four times in successionand resulted in the following outcomes: (i) Head, (ii) Head, (iii) Head, (iv) Head. The probability of obtaining a 'Tail' when the coin is tossed again is

(a) 0 (b) $\frac{1}{2}$

(c) $\frac{4}{5}$ (d) $\frac{1}{5}$ **[CE-GATE-2014(2)]**

Q.54 If {x} is a continuous, real valued random variable defined over the interval $(-\infty,+\infty)$ and its occurrence is defined by the density function given as: f (x) = $\frac{1}{\sqrt{2\pi}*b}e^{-\frac{1}{2}\left(\frac{x-a}{b}\right)^2}$ where 'a' and 'b'are the statistical attributes of the random variable {x}. The value of the integral $\int_{-\infty}^{a}\frac{1}{\sqrt{2\pi}*b}e^{-\frac{1}{2}\left(\frac{x-a}{b}\right)^2}dx$ is

(a) 1 (b) 0.5

(c) π (d) $\frac{\pi}{2}$ **[CE-GATE-2014(2)]**

Q.55 Suppose you break a stick of unit length at a point chosen uniformly at random. Then the expected length of the shorter stick is________. **[CS-GATE-2014(1)]**

Q.56 For fair six-sided dice are rolled. The probability that the sum of the results being 22is $\frac{X}{1296}$. The value of X is____________. **[CS-GATE-2014(1)]**

Q.57 The security system at an IT office is composed of 10 computers of which exactly four are working. To check whether the system is functional, the officials inspect four of the computers picked at random (without replacement). The system is deemed functional if at least three of the four computers inspected are working. Let the probability that the system is deemed functional be denoted by p Then 100p= ____________.

[CS-GATE-2014(2)]

Q.58 The probability that a given positive integer lying between 1 and 100 (both inclusive) is NOT divisible by 2, 3 or 5 is ______ . **[CS-GATE-2014(2)]**

Q.59 Let S be a sample space and two mutually exclusive events A and B be such that $A \cup B = S$. If P(.) denotes the probability of the event, the maximum value of P((A)P((B) is ______

[CS-GATE-2014(3)]

Q.60 A fair coin is tossed n times. The probability that the difference between the number of heads and tails is (n —3) is

(a) 2^{-n} (b) 0

(c) ${}^{n}C_{n-3}2^{-n}$ (d) 2^{-n+3} **[EE-GATE-2014(1)]**

Q.61 Consider a dice with the property that the probability of a face with n dots showing up is proportional to n. The probability of the face with three dots showing up is

[EE-GATE-2014(2)]

Q.62 Let X be a random variable with probability density function

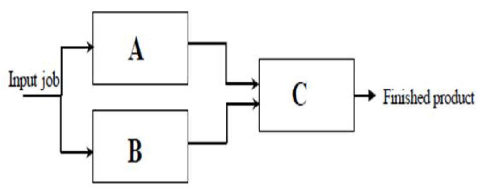
$$f(x) = \begin{cases} 0.2, & |x| \le 1 \\ 0.1, & 1 < |x| \le 4 \\ 0, & \text{otherwise} \end{cases}$$

The probability p(0.5 < x < 5) is **[EE-GATE-2014(2)]**

Q.63 Lifetime of an electric bulb is a random variable with density f(x) $= kx^2$, where x is measured in years. If the minimum and maximum lifetimes of bulb are 1 and 2 years respectively, then the value of k is______________. **[EE-GATE-2014(3)]**

Q.64 Given that x is a random variable in the range $[0, \infty]$ with a probability density Function $\frac{e^{-\frac{x}{2}}}{K}$ the value of the constant K is __________. **[IN-GATE-2014]**

Q.65 The figure shows the schematic of a production process with machines A, B and C. An input job needs to be pre-processed either by A or by B before it is fed to C, from which the final finished product comes out. The probabilities of failure of the machines are given as: PA=0.15, PB=0.05, PC=0.1



Assuming independence of failures of the machines, the probability that a given job is successfully processed (up to the third decimal place) is __________. **[IN-GATE-2014]**

Q.66 In a housing society, half of the families have a single child per family, while the remaining half have two children per family. The probability that a child picked at random, has a sibling is _____ **[EC-GATE-2014(1)]**

Q.67 An unbiased coin is tossed an infinite number of times. The probability that the fourth head appears at the tenth toss is

(a) 0.067 (b) 0.073

(c) 0.082 (d) 0.091 **[EC-GATE-2014(3)]**

Q.68 A fair coin is tossed repeatedly till both head and tail appear at least once. The average number of tosses required is __________. **[EC-GATE-2014(3)]**

Q.69 Parcels from sender S to receiver R pass sequentially through two post-offices. Each post-office has a probability $\frac{1}{5}$ of losing an incoming parcel, independently of all other parcels. 5 Given that a parcel is lost, the probability that it was lost by the second post-office is_______. **[EC-GATE-2014(4)]**

Q.70 The probability of obtaining at least two "SIX" in throwing a fair dice 4 time is

(a) 425/432 (b) 19/144

(c) 13/144 (d) 125/432 **[ME-GATE-2015(1)]**

Q.71 There vendors were asked to supply a very high precision component. The respective probabilities of their meeting the strict design specification are 0.8, 0.7 and 0.5. Each vendor supplies one component. The probability that out of total three components supplied by the vendors, at least one will meet the design specification is ________________.

[ME-GATE-2015(2)]

Q.72 The chance of a student passing an exam is 20%. The chance of a student passing the exam and getting above 90% marks in it is 5% Given that a student passes the examination, the probability that the student gets above 90% marks is

(a) $\frac{1}{18}$ (b) $\frac{1}{4}$

(c) $\frac{2}{9}$ (d) $\frac{5}{18}$ **[ME-GATE-2015(2)]**

Q. 73 Two coins are tossed simultaneously. The probability (upto two decimal points accuracy) of getting at least one tail is ____. **[CE-GATE-2015(1)]**

Q.74 Consider the following probability mass function (pmf) of a random variable X:

$$P(x, q) = \begin{cases} q & \text{if} \quad X=0 \\ 1\text{-}q & \text{if} \quad X=1 \\ 0 & \text{otherwise} \end{cases}$$

If q = 0.4, the variance of X is ______. **[CE-GATE-2015(1)]**

Q.75 The probability density function of a random variable, x is

$$f(x) = \frac{x}{4}\left(4 - x^2\right) \text{ for } 0 \le x \le 2$$

$= 0$ otherwise

The mean μ of the random variable is _____________. **[CE-GATE-2015(2)]**

Q.76 Suppose X_i for i=1,2,3 are independent and identically distributed random variables whose probability mass functions are $Pr[X_i=0]=Pr[X_i=1]=1/2$ for i= 1,2,3. Define another random variable $Y = X_1X_2 \oplus X_3$, where $\oplus$ denotes XOR. Then $Pr\left[y=0 \middle| X_3=0\right] =$ _________ .

[CS-GATE-2015(3)]

Q.77 Two players, A and B, alternately keep rolling a fair dice. The person to get a six first wins the game. Given that player A starts the game, the probability that A wins the game is

(a) 5/11 (b) 1/2

(c) 7/13 (d) 6/11 **[EE-GATE-2015(1)]**

Q.78 The probability that a thermistor randomly picked up from a production unit is defective is 0.1. The probability that out of 10 thermistors(randomly picked up), 3 are defective is

(a) 0.001 (b) 0.057

(c) 0.0107 (d) 0.3 **[IN-GATE-2015]**

Q.79 An urn contains 5 red and 7 green balls. A ball is drawn at random and its colour is noted. The ball is placed back into the urn along with another ball of the same colour. The probability of getting a red ball in the next draw is

(a) $\frac{65}{156}$ (b) $\frac{67}{156}$

(c) $\frac{79}{156}$ (d) $\frac{89}{156}$ **[IN-GATE-2015]**

Q.80 Suppose A and B are two independent events with probabilities $P(A) \neq 0$ and $P(B) \neq 0$. Let $\overline{A}$ and $\overline{B}$ be their complements. Which one of the following statements is FALSE ?

(a) $P(A \cap B) = P(A)\,P(B)$ (b) $P(A \setminus B) = P(A)$

(c) $P(A \cup B) = P(A) + P(B)$ (d) $P(\overline{A} \cap \overline{B}) = P(\overline{A})\,P(\overline{B})$ **[EC-GATE-2015(1)]**

Q.81 Consider a Poisson distribution for the tossing of a biased coin. The mean for this distribution is μ. The standard deviation for this distribution is given by

(a) $\sqrt{\mu}$ (b) μ^2

(c) μ (d) $1/\mu$ **[ME-GATE -2016(1)]**

Q.82 The probability that a screw manufactured by a company is defective is 0.1. The company sells screws in packets containing 5 screws and gives a guarantee of replacement if one or more screws in the packet are found to be defective. The probability that a packet would have to be replaced is ______________. **[ME-GATE -2016(2)]**

Q.83 The area (in percentage) under standard normal distribution curve of random variable Z within limits from -3 to +3 is _______ **[ME-GATE -2016(3)]**

Q.84 Three cards were drawn from a pack of 52 cards. The probability that they are a king, a queen, and a jack is

(a) $\frac{16}{5525}$ (b) $\frac{64}{2197}$

(c) $\frac{3}{13}$ (d) $\frac{8}{16575}$ **[ME-GATE -2016(3)]**

Q.85 Probability density function of a random variable X is given below

$$f(x)=\begin{cases}0.25 & 1\le X\le 5\\ 0 & otherwise\end{cases}$$

P (X $\le$ 4) is

(a) $\frac{3}{4}$ (b) $\frac{1}{2}$

(c) $\frac{1}{4}$ (d) $\frac{1}{8}$ **[CE-GATE-2016(1)]**

Q.86 The spot speeds (expressed in km/hr) observed at a road section are 66, 62, 45, 79, 32, 51, 56, 60, 53, and 49. The median speed (expressed in km/hr) is______.(Note: answer with one decimal accuracy) **[CE-GATE-2016(2)]**

Q.87 X and Y are two random independent events. It is known that P(X) = 0.40 and $P(X \cup Y^C) = 0.7$. Which one of the following is the value of $P(X \cup Y)$?

(a) 0.7 (b) 0.5

(c) 0.4 (d) 0.3 **[CE-GATE-2016(2)]**

Q.88 If f(x) and g(x) are two probability density functions,

$$f(x)=\begin{cases}\frac{x}{a}+1 & ;\ -a\le x<0\\ -\frac{x}{a}+1 & ;\ 0\le x\le a\\ 0 & ;\ otherwise\end{cases} \quad ; \quad g(x)=\begin{cases}-\frac{x}{a} & ;-a\le x<0\\ \frac{x}{a} & ;0\le x\le a\\ 0 & ;otherwise\end{cases}$$

Which one of the following statements is true ?

(a)Mean of f(x) and g(x) are same; Variance of f (x) and g(x) are same

(b) Mean of f(x) and g (x) are same; Variance of f (x) and g (x) are different

(c) Mean of f(x) and g(x) are different; Variance of f (x) and g (x) are same

(d) Mean of f(x) and g(x) are different; Variance of f (x) and g(x) are different

[CE-GATE-2016(2)]

Q.89 A probability density function on the interval [a, 1] is given by $\frac{1}{x^2}$ and outside this interval the value of the function is zero. The value of a is____. **[CS-GATE-2016(1)]**

Q.90 Consider the following experiment.

Step 1. Flip a fair coin twice.

Step 2. If the outcomes are (TAILS, HEADS) then output Y and stop.

Step 3. If the outcomes are either (HEADS, HEADS) or (HEADS, TAILS), then output N and stop.

Step 4. If the outcomes are (TAILS, TAILS), then go to Step 1.

The probability that the output of the experiment is Y is (up to two decimal places)

[CS-GATE-2016(1)]

Q.91 Suppose that a shop has an equal number of LED bulbs of two different types. The probability of an LED bulb lasting more than 100 hours given that it is of Type 1 is 0.7, and given that it is of Type 2 is 0.4. The probability that an LED bulb chosen uniformly at random lasts more than 100 hours is _________. **[CS-GATE-2016(2)]**

Q.92 Let the probability density function of random variable, X, be given as:

$$f_x\ (x) = \frac{3}{2}e^{-3x}u(x) + ae^{4x}u(-x)$$

where u(x) is the unit step function. Then the value of 'a' and Prob$\{X \le 0\}$, respectively, are

(a) $2, \frac{1}{2}$ (b) $4, \frac{1}{2}$

(c) $2, \frac{1}{4}$ (d) $4, \frac{1}{4}$ **[EE-GATE-2016(2)]**

Q.93 Two random variables X and Y are distributed according to

$f_{X,Y}(x,y) = \begin{cases} (x+y), 0 \le x,y \le 1 \\ 0, \quad \text{otherwise} \end{cases}$. The probability $P(X + Y \le 1)$ is _______.

[EC-GATE-2016(2)]

Q.94 The second moment of a Poisson-distributed random variable is 2. The mean of the random variable is _____. **[EC-GATE-2016(1)]**

Q.95 The probability of getting a "head" in a single toss of a biased coin is 0.3. The coin is tossed repeatedly till a "head" is obtained. If the tosses are independent, then the probability of getting "head" for the first time in the fifth toss is __________.

[EC-GATE-2016(3)]

Q.96 A six – face fair dice is rolled number of times. The mean value of the outcomes is _____.

[ME-GATE -2017(1)]

Q.97 Two coins are tossed simultaneously. The probability (upto two decimal points accuracy) of getting at least one head is ____. **[ME-GATE-2017(2)]**

Q.98 A sample of 15 data is as follows :

17,18,17,17,13,18,5,5,6,7,8,9,20,17,3. The mode of the data is

(a) 4 (b) 13

(c) 17 (d) 20 **[ME-GATE-2017(2)]**

Q.99 Four red balls, four green balls and four blue balls are put in a box. Three balls are pulled out of the box at random one after another without replacement. The probability that all the three balls are red is

(a) $\frac{1}{72}$ (b) $\frac{1}{55}$

(c) $\frac{1}{27}$ (d) $\frac{1}{27}$ **[ME-GATE -2018(1)]**

Q.100 A six faced fair dice is rolled 5 times. The probability (in %) of obtaining "ONE" at least four times is

(A) 33.3 (B)3.33

(C)0.33 (D)0.0033 **[ME-GATE -2018(1)]**

Q.101 Let X_1, X_2 be two independent normal random variables with means μ_1, μ_2 and standard deviations σ_1, σ_2, respectively. Consider $Y = X_1 - X_2$; $\mu_1 = \mu_2 = 1$, $\sigma_1 = 1$, $\sigma_2 = 2$. Then,

(A) Y is normally distributed with mean 0 and variance 1

(B) Y is normally distributed with mean 0 and variance 5

(C) Y has mean 0 and variance 5, but is NOT normally distributed

(D) Y has mean 0 and variance 1, but is NOT normally distributed

[ME-GATE -2018(1)]

Q.102 Let X_1 and X_2 be two independent exponentially distributed random variables with means 0.5 and 0.25, respectively. Then Y = min (X_1, X_2) is

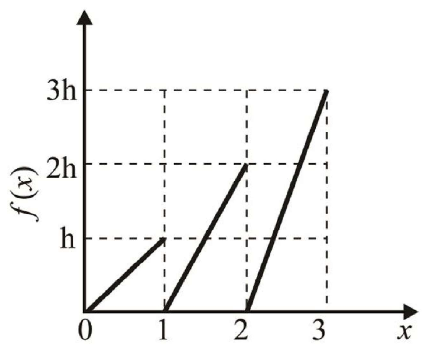
(A) exponentially distributed with mean $\frac{1}{6}$

(B) exponentially distributed with mean 2

(C) normally distributed with mean $\frac{3}{4}$

(D) normally distributed with mean $\frac{1}{6}$ **[ME-GATE -2018(2)]**

Q.103 The graph of a function f(x) is shown in the figure.

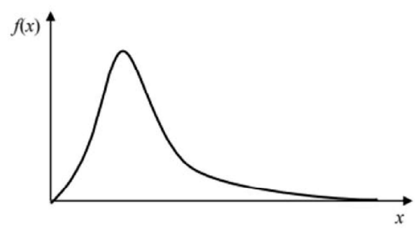


For f(x) to be a valid probability density function, the value of h is

(A) 1/3 (B) 2/3

(C) 1 (D) 3 **[CE-GATE -2018(2)]**

Q.104 A probability distribution with right skew is shown in the figure.



The correct statement for the probability distribution is

(A) Mean is equal to mode

(B) Mean is greater than median but less than mode

(C) Mean is greater than median and mode

(D) Mode is greater than median **[CE-GATE -2018(2)]**

Q.105 Probability (up to one decimal place) of consecutively picking 3 red balls without replacement from a box containing 5 red balls and 1 white ball is ______.

[CE-GATE -2018(2)]

Q.106 Let X_1, X_2, X_3 and X_4 be independent normal random variables with zero mean and unit variance. The probability that X_4 is the smallest among the four is _______.

[EC-GATE-2018]

Q.107 X and Y are two independent random variables with variances 1 and 2, respectively. Let Z = X– Y. The variance of Z is

(A) 0 (B) 1

(C) 2 (D) 3 **[EC-GATE-2018]**

Q.108 Two people, P and Q, decide to independently roll two identical dice, each with 6 faces, numbered 1 to 6. The person with the lower number wins. In case of a tie, they roll the dice repeatedly until there is no tie. Define a trial as a throw of the dice by P and Q. Assume that all 6 numbers on each dice are equi-probable and that all trials are independent. The probability (rounded to 3 decimal places) that one of them wins on the third trial is ____. **[CS-GATE-2018]**

Q.109 Consider Guwahati (G) and Delhi (D) whose temperatures can be classified as high (H),medium (M) and low (L). Let $P(H_G)$ denote the probability that Guwahati has hightemperature. Similarly, $P(M_G)$ and $P(L_G)$ denotes the probability of Guwahati havingmedium and low temperatures respectively. Similarly, we use $P(H_D)$, $P(M_D)$ and $P(L_D)$ forDelhi.The following table gives the conditional probabilities for Delhi's temperature givenGuwahati's temperature.

	H_D	M_D	L_D
H_G	0.40	0.48	0.12
M_G	0.10	0.65	0.25
L_G	0.01	0.50	0.49

Consider the first row in the table above. The first entry denotes that if Guwahati has high temperature (H_G) then the probability of Delhi also having a high temperature (H_D) is 0.40; i.e., $P(H_D|H_G) = 0.40$. Similarly, the next two entries are $P(M_D|H_G) = 0.48$ and$P(L_D|H_G) = 0.12$. Similarly for the other rows.

If it is known that $P(H_G) = 0.2$, $P(M_G) = 0.5$, and $P(L_G) = 0.3$, then the probability (correct to two decimal places) that Guwahati has high temperature given that Delhi has high temperature is ______. **[CS-GATE-2018]**

Q.110 Consider a sequence of tossing of a fair coin where the outcomes of tosses are independent.The probability of getting the head for the third time in the fifth toss is

(A) 5/16 (B) 3/16

(C) 3/5 (D) 9/16 **[IN -GATE-2018]**

Q.111 Two bags A and B have equal number of balls. Bag A has 20% red balls and 80% green balls. Bag B has 30% red balls, 60% green balls and 10% yellow balls. Contents of Bags A and B are mixed thoroughly and a ball is randomly picked from the mixture. What is the chance that the ball picked is red ?

(A) 20% (B) 25%

(C) 30% (D) 40% **[IN-GATE-2018]**

ANSWER KEY:

1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**
d	c	d	c	c	d	a	b	d	d	c	a	b	d
15	**16**	**17**	**18**	**19**	**20**	**21**	**22**	**23**	**24**	**25**	**26**	**27**	**28**
d	b	c	d	c	d	c	c	d	a	b	a	a	c
29	**30**	**31**	**32**	**33**	**34**	**35**	**36**	**37**	**38**	**39**	**40**	**41**	**42**
c	d	a	d	d	d	c	b	b	d	a	d	c	a
43	**44**	**45**	**46**	**47**	**48**	**49**	**50**	**51**	**52**	**53**	**54**	**55**	**56**
a	d	a	0.27	0.66	a	51	b	0.3	0.4	b	b	0.25	10
57	**58**	**59**	**60**	**61**	**62**	**63**	**64**	**65**	**66**	**67**	**68**	**69**	**70**
11.95	0.27	0.25	b	(1/7)	0.4	0.43	2	0.9	0.667	c	3	0.44	b
71	**72**	**73**	**74**	**75**	**76**	**77**	**78**	**79**	**80**	**81**	**82**	**83**	**84**
0.97	b	0.75	0.24	1.07	0.75	d	b	a	c	a	0.41	99.8	a
85	**86**	**87**	**88**	**89**	**90**	**91**	**92**	**93**	**94**	**95**	**96**	**97**	**98**
a	54.5	a	b	0.5	0.33	0.55	a	0.33	1	0.07	3.5	0.75	c
99	100	101	102	103	104	105	106	107	108	109	110	111	
b	c	b	a	a	c	0.5	0.25	d	0.023	0.6	b	b	

ESE QUESTIONS

Q.1 In a sample of 100 students, the mean of themarks (only integers) obtained by them in atest is 14 with its standard deviation of 2.5(marks obtained can be fitted with a normal distribution). The percentage of studentsscoring 16 marks is

(A) 36 (B) 23

(C) 12 (D) 10

(Area under standard normal curve between z = 0 and z = 0.6 is 0.2257; and between z = 0 and z = 1.0 is 0.3413)

Q.2 Consider a random variable to which aPoisson distribution is best fitted. It happensthat $P_{(x=1)} = \frac{2}{3}P_{(x=2)}$ on this distribution plot.The variance of this distribution will be

(A) 3 (B) 2

(C) 1 (D) $\frac{2}{3}$

ANSWER KEY :

1. (c)

2. (a)

GATE SOLUTIONS

Q.1 **(d)** Given that P(A) = 1 $\Rightarrow$ P(B) =1/2

Both events are independent

$$\therefore\ P(A\cap B)=P(A).P(B)=(1).(1/2)=1/2$$

$$P(A/B)=\frac{P(A\cap B)}{P(B)}=\frac{1/2}{1/2}=1$$

$$P(B/A)=\frac{P(A\cap B)}{P(A)}=\frac{1/2}{1}=1/2$$

Q.2 **(c)** C $\Rightarrow$ computer science study

M $\Rightarrow$ maths study.

P(C on Monday & C on Wednesday)

= P(C on Monday, M on Tuesday and C on Wednesday)+p(C on Monday , C on Tuesday and C on Wednesday)

$$=1\times0.6\times0.4+1\times0.4\times0.4 \quad =0.24+0.16 \quad =0.40$$

Q.3 **(d)** S $\Rightarrow$ computer supplied by y, d $\Rightarrow$ defective computer

Probability that the computer was supplied by y, if the product is defective

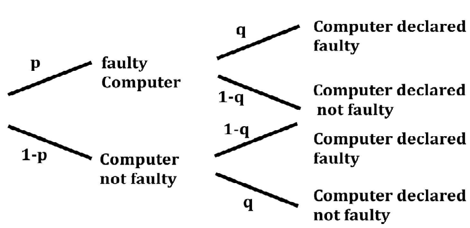
$$P(s/d)=\frac{P(s\cap d)}{P(d)}$$

$$P(s\cap d)=0.3\times0.02=0.006$$

$$P(d)=0.6\times0.01+0.3\times0.02+0.1\times0.03=0.015$$

$$P\left(\frac{s}{d}\right)=\frac{0.006}{0.015}=0.4$$

Q.4 **(c)**



From above tree diagram, the total probability, is

P (computer declared faulty) $= pq + (1-p)(1-q)$

Q.5 **(c)** It can be solved by using binomial distribution, since although population is finite, sampling is done with replacement and so probability does not change from trial to trial.

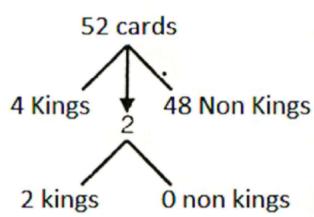
Here, $n = 2$

$x = 0$(no defective)

$$P = P(\text{defective}) = \frac{3}{10}$$

$$P(x=0) = 2C_0\left(\frac{3}{10}\right)^0\left(1-\frac{3}{10}\right)^2 = 0.49 = 49\%$$

Q.6 **(d)** Probability that both cards are kings



$$P(X=2) = \frac{{}^4C_1 \times {}^3C_1}{{}^{52}C_1 \times {}^{51}C_1} = \frac{1}{221}$$

Q.7 **(a)** $I = \frac{1}{\sqrt{2\pi}}\int_0^{\infty} e^{\left(-x^2/8\right)}dx$ Comparing with $\frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{(x-\mu)^2}{2\sigma^2}}dx = 1$

Putting $\mu = 0$, $\sigma = 2$

$$\frac{1}{2\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{x^2}{8}}dx = 1 \Rightarrow \frac{1}{\sqrt{2\pi}}\int_0^{\infty} e^{-\frac{x^2}{8}}dx = 1$$

Q.8 **(b)** As no of items are finite the problem can be solved using binomial distribution Probability of defective item is p = 0.1

Probability of non-defective item is $q = 1 - p = 1 - 0.1 - 0.9$

Probability that exactly 2 of the chosen items are defective

$$= {}^{10}C_2(p)^2(q)^8 = {}^{10}C_2(0.1)^2(0.9)^8 = 0.1937$$

Q.9 **(d)** $P_{odd} = \frac{3}{6} = \frac{1}{2} \Rightarrow P_{even} = \frac{3}{6} = \frac{1}{2}$

Since both events are independent of each other $P_{(odd/even)} = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$

Q.10 **(d)** Total possible outcomes (Sample space) $= (6)^2 = 36$

Total different ways in which sum is either 8 or 9 is

(2,6),(3,5),(3,6),(4,4)(4,5),(5,3),(5,4),(6,2),(6,3) = 9 different ways

$\therefore$ Probability of sum 8 or $9 = 1 - \frac{1}{4} = \frac{3}{4}$

Q.11 **(c)**Since all three gates are independent p(gate 2 and gate 3 fail /gate 1 failed)

= p (gate 2 and gate 3 fail)

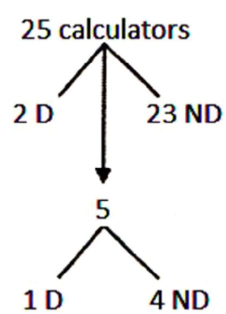
= p(gate 2) $\times$ p(gate3)[gate 2 and 3 fail independently]

Q.12 **(a)** The probability of choosing exactly n elements

= The probability of getting n heads out of 2n coin tosses

$$= {}^{2n}C_n (1/2)^n (1/2)^{2n-n} = {}^{2n}C_n (1/2)^n (1/2)^n = \frac{{}^{2n}C_n}{2^{2n}} = \frac{{}^{2n}C_n}{(2^2)^n} = \frac{{}^{2n}C_n}{4^n}$$

Q.13 **(b)** Since number of calculators is finite, hyper geometric distribution is applicable



P(1 defective in 5 calculators) $= \frac{2C_1 \times 23C_4}{25C_5} = \frac{1}{3}$

Q.14 **(d)** Let the mean and standard division of the students of batch C be μ_C and σ_C respectively, and the mean and standard deviation of entire class of first year students be μ and σ respectively.

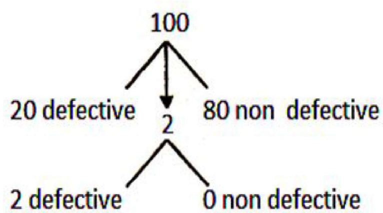
Now given, $\mu_C = 6.6$, $\sigma_C = 2.3$ & $\mu = 5.5$, $\sigma = 4.2$

In order to normalize batch C to entire class. The normalized score (z scores) must be equated.

Since $z = \dfrac{x-\mu}{\sigma} = \dfrac{x-5.5}{4.2}$; $z_C = \dfrac{x_c - \mu_c}{\sigma_c} = \dfrac{8.5-56.6}{2.3}$

Equating these two and solving, we get $\dfrac{8.5-6.6}{2.3} = \dfrac{x-5.5}{4.2} \Rightarrow x = 8.969 \simeq 9.0$

Q.15 **(d)** Problem can be solved by hyper geometric distribution



$$P(X=2) = \frac{20C_2 \times 80C_0}{100C_2} = \frac{19}{495}$$

Q.16 **(b)** Mean $\mu_t = E(t) = \int_{-\infty}^{\infty} t.f(t).dt = \int_{-1}^{1} t.f(t).dt$

$$= \int_{-1}^{0} t.(1+t)\,dt + \int_{0}^{1} t(1-t)\,dt = \left[\frac{t^2}{2} + \frac{t^3}{3}\right]_{-1}^{0} + \left[\frac{t^2}{2} - \frac{t^3}{3}\right]_{0}^{1} = -\left[\frac{1}{2} - \frac{1}{3}\right] + \left[\frac{1}{2} - \frac{1}{3}\right] = 0$$

$$\text{Variance} = E\left(t^2\right) - \left[E(t)\right]^2$$

$$= \int_{-\infty}^{\infty} t^2 f(t).dt - \left[E(t)\right]^2 = \int_{-\infty}^{\infty} t^2 f(t).dt - (0)^2$$

$$= \int_{-\infty}^{\infty} t^2 f(t).dt = \int_{-1}^{0} t^2 (1+t)\,dt + \int_{0}^{1} t^2 (1-t)\,dt \quad = \int_{-1}^{0} \left(t^2 + t^3\right).dt + \int_{0}^{1} t^2 (1-t)\,dt$$

$$= \left[\frac{t^3}{3} + \frac{t^4}{4}\right]_{-1}^{0} + \left[\frac{t^3}{3} - \frac{t^4}{4}\right]_{0}^{1} = \frac{1}{12} + \frac{1}{12} = \frac{1}{6}$$

$$\text{Standard deviation} = \sqrt{\text{variance}} = \frac{1}{\sqrt{6}}$$

Q.17 **(c)** Total probability $= \int_{-\infty}^{\infty} p(x).dx = 1 \Rightarrow \int_{-\infty}^{\infty} K.e^{-\alpha|x|}.dx = 1$

$$\int_{-\infty}^{0} K.e^{\alpha x}.dx + \int_{0}^{\infty} K.e^{-\alpha x} = 1 \Rightarrow \frac{K}{\alpha}\left(e^{\alpha x}\right)_{-\infty}^{0} + \frac{K}{-\alpha}\left(e^{-\alpha x}\right)_{0}^{\infty} = 1 \Rightarrow \frac{K}{\alpha} + \frac{K}{\alpha} = 1$$

$$2K = \alpha \Rightarrow K = 0.5\alpha$$

Q.18 **(b)** Number of permutations with '2' in the first position =19!

Number of permutations with '2' in the second position $= 10 \times 18!$

(fill the first space with any of the 10 odd numbers and the 18 spaces after the 2 with 18 of the remaining numbers in 18!ways)

Number of permutations with '2' in 3rd position $= 10 \times 9 \times 17!$

(fill the first 2 places with 2 of the 10 odd numbers and then the remaining 17 places with remaining 17 numbers)

And so on until '2' is in 11th place. After that it is not possible to satisfy the given condition. Since there are only 10 odd numbers available to fill before the '2'.

So the desired number of permutations which satisfies the given condition is

$19! + 10 \times 18! + 10 + 9 \times 17! + 10 \times 9 \times 8 \times 16! + \ldots + 10! \times 9!$

Now the probability of this happening is given by $\dfrac{19! + 10 \times 18! + 10 \times 9 \times 17! \ldots + 10! \times 9!}{20!}$

$= (1/10)$

Q.19 **(c)** $A \Rightarrow$ event of failing in paper 1 ; $B \Rightarrow$ event of failing in paper 2

Given that P(A) = 0.3 , P(B) =0.2, P(A/B) =0.6

Probability of failing in both $P(A \cap B) = P(B)\ P(A/B) = 0.2 \times 0.6 = 0.12$

Q.20 **(d)** (a) false as $P(P \cap Q) = P(P)\ P(Q)$ if P and Q are independent

(b) false as $P(P \cup Q) = P(P) + P(Q) - P(P \cap Q) \quad \therefore\ P(P \cup Q) \le P(P) + P(Q)$

(c) false since independence and mutually exclusion are unrelated properties

(d) true

$$P \cap Q \subseteq P \Rightarrow n(P \cap Q) \le n(P) \Rightarrow P(P \cap Q) \le P(P)$$

Q.21 **(c)**

DiceValue	1	2	3	4	5	6
Probability	1/4	1/8	1/8	1/8	1/8	1/4

Since the dice are independent

$$\text{probability}(1,5,6) = \frac{1}{4} \times \frac{1}{8} \times \frac{1}{4} = \frac{1}{128}$$

$$\text{probability}(3,4,5)=\frac{1}{8}\times\frac{1}{8}\times\frac{1}{8}=\frac{1}{512}$$

$$\text{probability}(1,2,5)=\frac{1}{4}\times\frac{1}{8}\times\frac{1}{8}=\frac{1}{256}$$

$\therefore$ Option (c)

probability $(1,\ 5 \text{ and } 6)=\dfrac{1}{128}$ is correct.

Q.22 **(c)** coefficient of variation(CV) $=\dfrac{\sigma}{\mu}=\dfrac{8.8}{33}=0.2666$

Q.23 **(d)** d is false.

Since $E\left(X^2Y^2\right)=E(X^2)E(Y^2)$

But since X is not independent of X,

$$E(X^2)\neq\left[E(X)\right]^2$$

$$\therefore E\left(X^2Y^2\right)=E(X^2)E(Y^2)\quad\neq\left[E(X)\right]^2\left[E(Y)\right]^2$$

Q.24 **(a)** Given, Probability(private car)=0.45

P (bus / public trport) = 0.55

Since, a person has a choice between private car and public trport

P (public trport) = 1 — P(private car) = 1 – 0.45 = 0.55

P (bus) = P (bus $\cap$ public trport) = P(bus/public trport) $\times$ P(public trport)

$= 0.55 \times 0.55 = 0.3025 \simeq 0.30$

Now, P(metro) = 1-[P(private car) + P(bus)] $=1-(0.45+0.30)=0.25$

$\therefore$ P (private car) = 0.45, P (bus) = 0.30, P (metro) = 0.25

Q.25 **(b)** Given $f(x)\begin{cases}=x^2 & -1\le x\le 1,\\ =0 & \text{elsewhere}\end{cases}$

$$P\left(-\frac{1}{3}\le x\le\frac{1}{3}\right)=\int_{-\frac{1}{3}}^{-\frac{1}{3}}f(x)dx=\int_{-\frac{1}{3}}^{-\frac{1}{3}}x^2dx=\left[\frac{x^3}{3}\right]_{-\frac{1}{3}}^{\frac{1}{3}}=\frac{2}{81}$$

The probability expressed in percentage, $p=\dfrac{2}{81}\times 100=2.469\%=2.47\%$

* This is not a valid Probability distribution.

Q.26 **(a)** $\sigma = \sqrt{\frac{(b-a)^2}{12}} = \sqrt{\frac{(1-0)^2}{12}} = \frac{1}{\sqrt{12}}$

Q.27 **(a)** Given $\mu_x = 1, \sigma_x^2 = 4 \Rightarrow \sigma_x = 2$

Also $\mu_y = -1$ and σ_y is known

Given, $P(X \le -1) = P(Y \ge 2)$

Converting into standard normal varites

$$P\left(z \le \frac{-1-\mu_x}{\sigma_x}\right) = P\left(z \ge \frac{2-\mu_y}{\sigma_y}\right) \Rightarrow P\left(z \le \frac{-1-1}{2}\right) = P\left(z \ge \frac{2-(-1)}{\sigma_y}\right)$$

$$P(z \le -1) = P\left(z \ge \frac{3}{\sigma_y}\right) \quad \text{...(i)}$$

Now since us know that in standard normal distribution,

$$P(z \le -1) = P(z \ge 1) \quad \text{...(ii)}$$

Comparing (i) and (ii) we can say that $\frac{3}{\sigma_y} = 1 \Rightarrow \sigma_y = 3$

Q.28 **(c)** Probability (only first two tosses are heads) $= P(H, H, T, T, T \ldots T)$

Now, each coin toss is independent.

So required probability $= P(H) \times P(H) \times [P(T)]^8$

$$= \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^8 = \left(\frac{1}{2}\right)^{10}$$

Q.29 **(c)** $P(2^{nd} \text{ red} / 1^{st} \text{ white}) = \frac{P(2^{nd} \text{ red and } 1^{st} \text{ white})}{P(1^{st} \text{ white})} = \frac{\frac{4}{7} \times \frac{3}{6}}{\frac{4}{7}} = \frac{3}{6} = \frac{1}{2}$

Q.30 **(d)** Coin is tossed 4 times

P(no of heads > no of tails) $= P(4H \;\&\; 0T \text{ or } 3H \;\&\; 1T)$

$= P(\text{Exactly 4 no of Heads}) + P(\text{Exactly 3 no of Heads})$

Q.31 **(a)** (2,1), (3,2), (4,3), (5,4) $\Rightarrow$ Required probability $= \frac{4}{5 \times 4} = \frac{4}{20} = \frac{1}{5}$

Q.32 **(d)**

Q.33 **(d)** Given, $\frac{\text{Red}}{4}\ \frac{\text{Black}}{6}$

The selection will be RBB or BBR or BRB

Probability of selecting RBB $=\frac{4}{10}\times\frac{6}{9}\times\frac{5}{8}$

Probability of selecting BBR = $\frac{6}{10}\times\frac{5}{9}\times\frac{4}{8}$

Probability of selecting BRB $=\frac{6}{10}\times\frac{4}{9}\times\frac{5}{8}$

P(Red=1)= sum of above three probabilities = 0.5

Q.34 **(d)** $p = q = \frac{1}{2}$

Probability $= {}^{n}C_{r}P^{r}q^{n-r} = {}^{5}C_{0}\left(\frac{1}{2}\right)^{0}\left(\frac{1}{2}\right)^{55} + C_{1}\left(\frac{1}{2}\right)^{1}\left(\frac{1}{2}\right)^{4} = \frac{6}{32}$

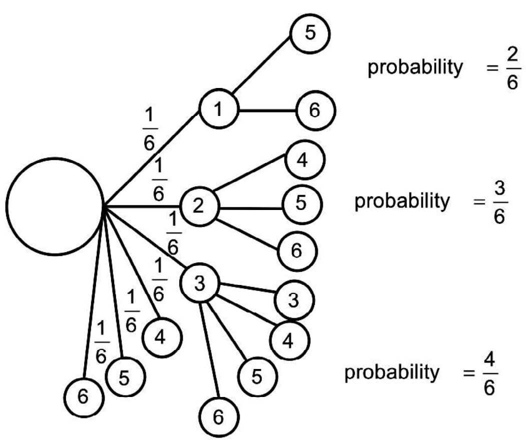
Q.35 **(c)** The cumulative distribution function

$F(x) = P(X \leq x)$

$F(-1) = P(X \leq -1) = P(X = -1) = 0.5$

$F(+1)=P(X\leq +1)= P(X= -1) + P(X= +1) = 0.5 + 0.5 = 1$

Q.36 **(b)**



$\therefore$ required probability $= \frac{1}{6}\times\frac{2}{6}+\frac{1}{6}\times\frac{3}{6}+\frac{1}{6}\times\frac{4}{6}+\frac{1}{6}=\frac{15}{36}=\frac{5}{12}$

Q.37 **(b)** $P(x<0)=P\left(\frac{x-\mu}{\sigma}<\frac{0-\mu}{\sigma}\right)-P(Z<-0.5)$

$= P(Z > 0.5) = 0.5 - P(0 < Z < 0.5) \Rightarrow$ Which is greater than zero and less than 0.5

Q.38 **(d)** A = The student answer the question correctly

E_1= Student knows the correct answer, E_2 = Student guesses the correct answer

$$P(E_1) = \frac{2}{3},\ P(E_2) = \frac{1}{3}$$

$$P(A) = P(E_1) \times P\left(\frac{A}{E_1}\right) + P(E_2) \times P\left(\frac{A}{E_2}\right) = \frac{2}{3} \times 1 + \frac{1}{3} \times \frac{1}{4} = \frac{3}{4}$$

Using Bayes theorem, $P\left(\frac{E_1}{A}\right) = \dfrac{P(E_1) \times P\left(\frac{A}{E_1}\right)}{P(A)} = \dfrac{\frac{2}{3} \times 1}{\frac{3}{4}} = \dfrac{8}{9}$

Q.39 **(a)** $1 - P(7) = 1 - P\left(\frac{1200 - 1000}{200}\right) = 1 - P\left(\frac{1}{T}\right) = 0.159 < 50\%$

Q.40 **(d)** Return period of rainfall, T = 50 years

$\therefore$ Probability of occurrence once in

50 years, $p = \frac{1}{50} = 0.02$

Probability of occurrence in each of 2 successive years = $p^2 = (0.02)^2 = 0.0004$

Q.41 **(c)** $P(r < 3) = P(r = 0) + P(r = 1) + P(r = 2)$

$$= \frac{e^{-\lambda}\lambda^0}{0!} + \frac{e^{-\lambda}\lambda^1}{1!} + \frac{e^{-\lambda}\lambda^2}{2!} \text{ (where } \lambda = 3)$$

$$= e^{-3} + e^{-3} \times 3 + \frac{e^{-3} \times 9}{2} = e^{-3}\left(1 + 3 + \frac{9}{2}\right) = \frac{17}{2e^3}$$

Q.42 **(a)** $P(x > 1) = \int_1^\infty e^{-x} dx = \left[-e^{-x}\right]_1^\infty = e^{-1} = 0.368$

Q.43 **(a)** $P(x > 1) = \int_1^\infty e^{-x} dx = \left[-e^{-x}\right]_1^\infty = e^{-1} = 0.368$

Q.44 **(d)**

X	1	2	3
P(x)	0.3	0.6	0.1

$\text{mean}(\mu) = \exp(x) = 1\times 0.3 + 2\times 0.6 + 3\times 0.1 = 0.3 + 1.2 + 0.3 = 1.8$

$E(x^2) = \sum x^2 P(x) = 1\times 0.3 + 4\times 0.6 + 9\times 0.1 = 0.3 + 2.4 + 0.9 = 3.6$

Variance $= E(x^2) - \{E(x)\}^2 = 3.6 - (1.8)^2 = 0.36$

Standard deviation $= \sqrt{\text{variance}} = \sqrt{0.36} = 0.6$

Q.45 **(a)** Two parts can be selected in the following way $15C_2 \times 10C_0$

Good	Bad
15	10
2	0
1	1
0	2

Total no. of parts = 25

Two parts can be selected in ${}^{25}C_2$ ways

Required probability $= \dfrac{{}^{15}C_2 \times {}^{10}C_0}{{}^{25}C_2} = \dfrac{105}{300} = \dfrac{7}{20}$

Q.46 **(0.25 to 0.27)**

$p = \dfrac{2}{6} = \dfrac{1}{3}\; q = 1 - \dfrac{1}{3} = \dfrac{2}{3}$

Using Binomial distribution $P(x \geq 2) = 3_{C_2}\left(\dfrac{1}{3}\right)^2\left(\dfrac{2}{3}\right)^2 + 3_{C_3}\left(\dfrac{1}{3}\right)^3\left(\dfrac{2}{3}\right)^0 = \dfrac{6}{27} + \dfrac{1}{27} = \dfrac{7}{27}$

Q.47 **(0.64 to 0.66)**

Let $M \to \text{men},\quad W \to \text{women},\; E \to \text{employed},\; U \to \text{unemployed}$

Given $P(M) = 0.5,\; P(W) = 0.5,\; P\left(U/M\right) = 0.20,\; P\left(U/M\right) = 0.50$

By total probability, $P(U) = P(M)P\left(U/M\right) + P(W)P\left(U/M\right) = 0.5\times 0.20 + 0.5\times 0.50 = 0.35$

Required probability $= P(E) = 1 - P(U) = 1 - 0.35 = 0.65$

Q.48 **(a)** Let 'x' be no. of defective pierces.

x	1	2	3
P(x)	1/6	2/3	1/6

Mean $(\mu) = E(x) = \Sigma x P(x)$

$$= \left(0 \times \frac{1}{6}\right) + \left(1 \times \frac{1}{3}\right) + \left(2 \times \frac{1}{6}\right) = 0 + \frac{2}{3} + \frac{1}{3} = 1$$

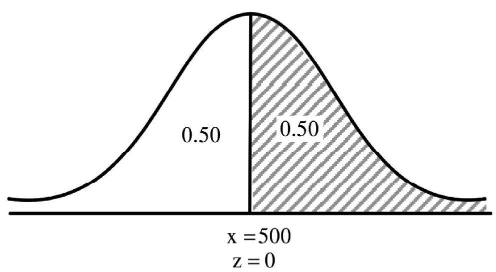
$$E(x^2) = \sum x^2 p(x) = \left(0 \times \frac{1}{6}\right) + \left(1 \times \frac{1}{3}\right) + \left(4 \times \frac{1}{6}\right) = 0 + \frac{2}{3} + \frac{2}{3} = \frac{4}{3}$$

Variance $V(x) = E(x^2) - \{E(x)\}^2 = \frac{4}{3} - 1 = 1/3$

Q.49 **(49 to 51)**

Given $M = 500$, $\sigma = 50$ $P(X > 500) = ?$
Where X follows normal distribution
We know that standard normal variable

$$z = \frac{x - M}{\sigma} = \frac{500 - 500}{50} = 0 \therefore (X > 500) = P(z > 0) = 0.50 \text{ (see figure)}$$



Q.50 **(b)** Given $\lambda = 5.2$

Let x be random variable which follows Poisson's distribution

$$P(x < 2) = P(x = 0) + P(x = 1) = \frac{e^{-\lambda}\lambda^0}{0!} + \frac{e^{-\lambda}}{1!}\lambda^1 = e^{-5.2}(6.2) = 0.0055 \times 6.2 = 0.034$$

Q.51 **(0.27)** Average no. of vehicles per hour,

$$\lambda = 240 \text{ / hour } = \frac{240}{60} \text{ /min } = 4\text{/min} = 2\text{/30 sec}$$

$$P\ (x = 1) = \frac{e^{-2} \cdot 2}{1} = 0.27$$

Q.52 **(0.4 to 0.4)** $P(2 < E < 4) = \int_2^4 f(E)\,dE = \frac{1}{5} E\Big|_2^4 = \frac{1}{5}(4 - 2) = \frac{2}{5} = 0.4$

Q.53 **(b)** $P(T) = \frac{1}{2}$

Q.54 **(b)**

Q.55 **(0.25)** The smaller sticks, therefore, will range in length from almost 0 meters up to a maximum of 0.5 meters, with each length equally possible. Thus, the average length will be about 0.25 meters, or about a quarter of the stick.

Q.56 **(10)** 22 occurred in following ways

6 6 6 4 $\rightarrow$ 4ways

6 6 5 5 $\rightarrow$6ways

Required probability $= \frac{6+4}{2296} = \frac{10}{2296} \Rightarrow x = 10$

Q.57 **(11.85 - 11.95)** P = P [at least three computers are working]

= P (3 or 4 computers working) $= \frac{\left(4_{C_3}\right)\times\left(6_{C_1}\right)}{10_{C_4}} + \frac{4_{C_4}}{10_{C_4}} = \frac{5}{42} \Rightarrow 100p = 11.9$

Q.58 **(0.259 to 0.261)** Let A = divisible by 2, B = divisible by 3 and C = divisible by 5, then

n(A) = 50, n(B) = 33, n(C) = 20, $n(A \cap B) = 16$, $n(B \cap C) = 6$,

$n(A \cap C) = 10$, $n(A \cap B \cap C) = 3$

$P(A \cup B \cup C) = P(A)+P(B)+P(C) - P(A \cap B) - P(B \cap C) - P(A \cap C) + P(A \cap B \cap C) = 74/100$

$\therefore$ Required probability is $P(\overline{A} \cap \overline{B} \cap \overline{C}) = 1 - P(A \cup B \cup C) = 0.26$.

Q.59 **(0.25)** $A \cup B = S \quad \Rightarrow P(A \cup B) = P(S) = 1$

$\Rightarrow P(A) + P(B) = 1$ (A & B are mutually exclusive)

$\Rightarrow P(B) = 1 - P(A)$

Maximum value of P(A)P(B) = ? $\Rightarrow$ Maximum value of P (A) [1- P (A)] =?

Let P(A) = X, Let $f(x) = x(1-x) = x - x^2$

for f(x) maximum $\Rightarrow f'(x) = 0 \Rightarrow 1-2x = 0 \Rightarrow x = \frac{1}{2} \Rightarrow f''(x) = -2; f''\left(\frac{1}{2}\right) < 0$

$\therefore$ f (x) has maximum, At $x = \frac{1}{2}$ and maximum value

$= f\left(\frac{1}{2}\right) = \frac{1}{2}\left(1-\frac{1}{2}\right) = 0.25$

Q.60 **(b)** Let X=difference between the number of heads and tails.

Take n = 2, S = { HH, HT, TH, TT) and X = $-2,0,2$; Here, $n-3 = -1$ is not possible

Take n = 3, S = { HHH, HHT, HTH, HTT, THH, THT, TTH, TTT) and X = $-3,-1,1,3$

Here $n-3 = 0$ is not possible

Similarly, if a coin is tossed n times then the difference between heads and tails is $n-3$ is not possible $\therefore$ required probability is θ

Q.61 **(1/7)** P(n) = k.n where n =1 to 6

We know $\sum_x P(x) = 1 \Rightarrow K[1+2+3+4+5+6] = 1 \Rightarrow K = \frac{1}{21}$

$\therefore$ required probability is P(3) = 3K = $\frac{1}{7}$

Q.62 **(0.4)** $P(0.5 < x < 5) = \int_{0.5}^{5} f(x)\,dx = (0.2)(x)_{0.5}^{1} + (0.1)(x)_{1}^{4}$ 0.1+ 0.=0.4

Q.63 **(0.43)**

Q.64 **(2)**

Q.65 **(0.890 to 0.899)**
Let X, Y, Z be respectively that machines A, B, C success (works) then

P(x) = 0.85, P(y) = 0.95, P(z) = 0.9

$\therefore$ P[Job to be completed successful] =

P[(machines A or machine B success) and (machine C success)]

$$\Rightarrow P((X \cup Y) \cap Z) = P(X \cap Z) + P(Y \cap Z) - P(X \cap Y \cap Z)$$

$$= (0.85)(0.9) + (0.95)(0.9) - (0.85)(0.95)(0.9) = 089325 \simeq 0.893$$

Q.66 **(0.667)** Let E_1 = one children family, E_2 = two children family and
A = picking a child then by Baye's theorem, required probability is

$$P\left(E_2/A\right) = \frac{\frac{1}{2}\cdot x}{\frac{1}{2}\cdot\frac{x}{2} + \frac{1}{2}.x} = \frac{2}{3} = 0.667$$ (Here 'x' is number of families)

Q.67 **(c)** P[fourth head appears at the tenth toss] = P [getting 3 heads in the first 9 tosses and one head at tenth toss]

$$= \left[{}^9C_3 \cdot \left(\frac{1}{2}\right)^9\right] \times \left[\frac{1}{2}\right] = \frac{21}{256} = 0.0825.$$

Q.68 Let x denotes the number of tosses.

We can summarize the event as :

Event	**x**	**Probability = p(x)**
HT or TH	**2**	**1/2**
HHT or TTH	**3**	**1/4**
HHHT or TTTH	**4**	**1/8**
And so on		

$$E(x) = \sum x_1 P(x_i)$$
$$E(x) = 2 \times \frac{1}{2} + 3 \times \frac{1}{4} + 4 \times \frac{1}{8} + ...$$
$$\frac{1}{2}E(x) = 2 \times \frac{1}{4} + 3 \times \frac{1}{8} + ...$$

On subtracting the above equations

$$\frac{1}{2}E(x) = 1 + \left(\frac{1}{4} + \frac{1}{8} + \frac{1}{16} + ...\right), \ \frac{1}{2}E(x) = 1 + \frac{\frac{1}{4}}{1 - \frac{1}{2}} = \frac{3}{2} \Rightarrow E(x) = 3$$

Q.69 **(0.44)** Parcel will be lost if

a. it is lost by the first post office

b. it is passed by first post office but lost by the second post office

P(Parcel lost by second post if it passes first post office)= P (Parcel passed by first post office) x P(Parcel lost by second post office) = $\frac{4}{5} \times \frac{1}{5} = \frac{4}{25}$

P(parcel lost by 2nd post office I parcel lost) = $\frac{4/25}{9/25} = \frac{4}{9} = 0.44$

Q.70 **(b)** $n = 4;\ p = \frac{1}{6}$

$$P(x \ge 2) = 1 - P(x < 2) = 1 - [P(x = 0) + P(x = 1)]$$

$$= 1 - \left[4C_0\left(\frac{1}{6}\right)^0\left(\frac{5}{6}\right)^4 + 4C\left(\frac{1}{6}\right)^1\left(\frac{5}{6}\right)^3\right] = \frac{19}{144}$$

Q.71 **(0.97)** Probability (at least one will meet specification) = 1 –probability (none will meet specification) **=** $1 - (1 - 0.8) \times (1 - 0.7) \times (1 - 0.5) = 0.97$

Q.72 **(b)** Let $A \rightarrow$ student passes the exam, $B \rightarrow$ student gets above 90% marks

Given P(A)=20%; P($A \cap B$) = 5 %

Required probability is P (B/A)= $\frac{P(A \cap B)}{P(A)} = \frac{5\%}{20\%} = \frac{1}{4}$

Q.73 **(0.75)** Total possibilities [HH, HT, TH, TT] **,** Favorable possibility = [TT, HT, TH]

Probability = $\frac{3}{4} = 0.75$

Q.74 **(0.24)**

Q.75 **(1.0667)** $f(x) = \frac{x}{4}(4-x^2)$ for $0 \le x \le 2$

mean = $\mu_x = E(x) = \int_0^2 xf(x)dx = \int_0^2 x\left(\frac{x}{4}\right)(4-x^2)dx = \frac{1}{4}\int_0^0 (4x^2 - x^4)dx$

$$= \frac{1}{4}\left[\frac{4x^3}{3} - \frac{x^5}{5}\right]_0^2 = \frac{1}{4}\left[4.\frac{8}{3} - \frac{32}{5}\right] = \frac{32}{4}\left[\frac{1}{3} - \frac{1}{5}\right] = 8\left[\frac{2}{15}\right] = \frac{16}{15} = 1.0667$$

Q.76 **(0.75)** Given $X_3 = 0$, (X_1, X_2, X_3 are independent)

$\therefore Y = X_1X_2 \oplus X_3 = 0$ gives $X_1X_2 = 0$

$\Rightarrow X_1 = 1, X_2 = 0$ or $X_1 = 0, X_2 = 1$ or $X_1 = 0, X_2 = 0$

$$\therefore P[Y = 0 | X_3 = 0] = \frac{P[(Y=0) \cap (X_3 = 0)]}{P(X_3 = 0)} = \frac{\begin{pmatrix} P[X_1 = 1, X_2 = 0, X_3 = 0] \\ + P[X_1 = 0, X_2 = 1, X_3 = 0] \\ + P[X_1 = 0, X_2 = 0, X_3 = 0] \end{pmatrix}}{P[X_3 = 0]}$$

$$= \frac{\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2} + \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2} + \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}}{\frac{1}{2}} = \frac{3}{4} = 0.75$$

Q.77 **(d)** Probability of getting 6 is $= \frac{6}{36} = \frac{1}{6}$

i.e,. Probability of A wins the game $= -\frac{1}{6}$

Probability of A not wins the game $= 1 - \frac{1}{6} = \frac{5}{6}$

Probability of B wins the game $= \frac{1}{6}$

Probability of B not win the game $= \frac{5}{6}$

If A starts the game, Probability A win the game

=P(A) + P(A) P(B) P(A) + P(A) P(B) P(A) P(B) P(A) +.....

$$=\frac{1}{6}+\frac{5}{6}\frac{5}{6}\frac{1}{6}+\frac{5}{6}\frac{5}{6}\frac{5}{6}\frac{5}{6}\frac{1}{6}+\ldots\ldots\ldots\ldots$$

$$=\frac{1}{6}\left[1+\frac{5}{6}\frac{5}{6}+\frac{5}{6}\frac{5}{6}\frac{5}{6}\frac{5}{6}+\ldots\ldots\right]=\frac{1}{6}\left[1+\left(\frac{5}{6}\right)^2+\left(\frac{5}{6}\right)^4+\ldots\ldots\ldots\right]$$

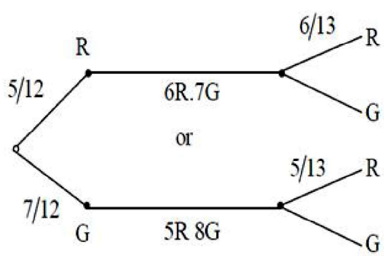
$$=\frac{1}{6}\left[\frac{1}{1-\left(\frac{5}{6}\right)^2}\right]=\frac{1}{6}\times\frac{3}{6}=\frac{6}{11}$$

Q.78 **(b)** p = Probability of a thermistor is defective = 0.1, q=1 – p =1 – 0.1 = 0.9, n = 10

Let X be random variable which is no. of defective pieces.

Required probability P(x = 3) = $^{10}C_3p^3q^7$ = $^{10}C_3\left(0.1\right)^3\left(0.9\right)^7$ = 0.057

Q.79 **(a)** $\frac{5}{12}\times\frac{6}{13}+\frac{7}{12}\times\frac{5}{13}=\frac{65}{12\times13}=\frac{65}{156}$



Q.80 **(c)** We know that A and B are independent

Then $P\left(A\cap B\right)=P\left(A\right)P(B)$

A and B are independent then P(A/B) = P(A) and P(B/A) = P(B)

Also if A and B are independent then $\overline{A}$ and $\overline{B}$ are also independent

i.e., P ($\overline{A}\cap\overline{B}$) = P ($\overline{A}$) P ($\overline{B}$)

$\therefore$ (A), (B), (D) are correct and (C) is false

Since $P(A\cup B)=P(A)+P(B)-P(A\cup B)=P(A)+P(B)-P(A)P(B)$

Q.81 **(a)** For Poisson distribution : Mean =Variance

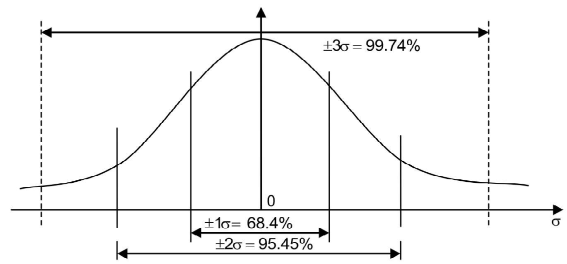
$\therefore$ Standard deviation = $\sqrt{\text{variance}}=\sqrt{\mu}$

Q.82 **(0.40951)** $\Rightarrow P\left[X\geq1\right]=1-P\left[X<1\right]=1-P\left[X=0\right]$

$= 1 - 5C_0(0.1)^0(0.9) = 1 - (0.9)^5 \approx 0.40951$

Binomial distribution : P = probability of defective screw.

Q.83 **(99. 74)**



Q.84 **(a)** Required probability $= \dfrac{{}^4C_1 \times {}^4C \times {}^4C}{{}^{52}C_3} = \dfrac{64}{22100} = \dfrac{16}{5525}$

Q.85 **(a)** $P(x \le 4) = \int_{-\infty}^{4} f(x)\,dx = \dfrac{3}{4}$

Q.86 **(54.5)** Median speed is the speed at the middle value in series of spot speeds that are arranged in ascending order. 50% of speed values will be greater than the median, 50% will be less than the median.

Ascending order of spot speed studies are 32, 39, 45, 51, 53, 56, 60, 62, 66, 79

Median speed$=\dfrac{53+56}{2}$s = 54.5km/ hr

Q.87 **(a)** P $(X \cup Y^C)$ = 0.7 $\Rightarrow$ P(x)+ P(y^c) – P(x).P(y^c) = 0.7

(Since, x,y are independent events)

$\Rightarrow$ P(x)+1 – P(y) –P(x){1 –P(y)} = 0.7 $\Rightarrow$ P(y)—P($x \cap y$) = 0.3 —— (1)

P($x \cup y$)= P(x) + P(y) – P($x \cap y$) = 0.4 + 0.3 = 0.7

Q.88 **(b)** Mean of f(x) is E(x) =

$$\int_{-a}^{0} x\left(\frac{x}{a}+1\right)dx + \int_{0}^{a} x\left(\frac{-x}{a}+1\right)dx = \left(\frac{x^3}{3a}+\frac{x^2}{2}\right)_{-a}^{0} + \left(\frac{-x^3}{3a}+\frac{x^2}{2}\right)_{0}^{a} = 0$$

Variance of f(x) is $E(x^2) - \{E(x)\}^2$

$$\text{where } E(x^2) = \int_{-a}^{0} x^2\left(\frac{x}{a}+1\right)dx + \int_{0}^{a} x^2\left(\frac{-x}{a}+1\right)dx = \left(\frac{x^4}{4a}+\frac{x^3}{3}\right)_{-a}^{0} + \left(\frac{-x^4}{4a}+\frac{x^3}{3}\right)_{0}^{a} = \frac{a^3}{6}$$

$$\Rightarrow \text{Variance} = \frac{a^3}{6}$$

$\Rightarrow$ Next, mean of $g(x)$ is $E(x)=\int_{a}^{0} x\left(\frac{-x}{a}\right)dx+\int_{0}^{a} x\left(\frac{x}{a}\right)dx=0$

Variance $g(x)$ is $E\left(x^2\right)-\{E(x)\}^2$ where $E(x^2)=\int_{-a}^{0} x^2\left(\frac{-x}{a}\right)dx+\int_{0}^{a} x^2\left(\frac{x}{a}\right)dx=\frac{a^3}{2}$

$\Rightarrow$ variance is $\frac{a^3}{2}$

$\Rightarrow$ Mean of f(x) and g(x) are same but variance of f(x) & g(x) are different.

Q.89 **(0.5)** Given $f(x)=\frac{1}{x^2}\ x\in[a,1]=0$ otherwise

We know that $\int_{a}^{1} f(x)dx=1\Rightarrow\int_{a}^{1}\frac{1}{x^2}dx=1\Rightarrow\left(\frac{-1}{x}\right)_a^1=1\Rightarrow\frac{1}{a}-1=1\Rightarrow a=0.5$

Q.90 **(0.33)** From the given steps we can observe that probabilities of y are

$\frac{1}{4},\left(\frac{1}{4}\right)\left(\frac{1}{4}\right),\left(\frac{1}{4}\right)^2\frac{1}{4},\ldots.$

Required probability = $\frac{1}{4}+\left(\frac{1}{4}\times\frac{1}{4}\right)+\left(\left(\frac{1}{4}\right)^2\times\frac{1}{4}\right)+\ldots\ldots$

$=\frac{1}{4}+\left(\frac{1}{4}\right)^2+\left(\frac{1}{4}\right)^3+\ldots.=\frac{1}{4}\left(1+\frac{1}{4}+\left(\frac{1}{4}\right)^2+\ldots..\right)=\frac{1}{4}\left(\frac{1}{1-\frac{1}{4}}\right)=0.33$

Q.91 **(0.55)** E_1 – event of selecting type-I bulb
E_2 – event of selecting type-II bulb
A - Event of selecting a bulb lasts more than 100 hours
Given $P(E_1) = 0.5$, $P(E_2) = 0.5$, $P(A/E_1) = 0.7$, $P(A/E_2) = 0.4$
Required probability, $P(A) = P(E_1)P(A/E_1) + P(E_2)P(A/E_2) = 0.5\times0.7 + 0.5\times0.4 = 0.55$

Q.92 **(a)** we have $\int_{-\infty}^{\infty} f_x(x)dx=1$

$f_x(x)=\frac{3}{2}e^{-3x}u(x)+ae^{4x}u(-x)$

$\Rightarrow\int_{-\infty}^{0} ae^{4x}dx+\int_{0}^{\infty}\frac{3}{2}e^{-3x}dx=1\Rightarrow a\left.\frac{e^{4x}}{4}\right|_{-\infty}^{0}-\left.\frac{e^{-3x}}{2}\right|_{0}^{\infty}=1\Rightarrow\frac{a}{4}+\frac{1}{2}=1\Rightarrow a=2$

Also, $P(X\le 0)=\int_{-\infty}^{0} ae^{4x}dx=\int_{-\infty}^{0} 2e^{4x}dx=\frac{1}{2}$

Q.93 **(0.333)** $P\{x+y\le 1\} = P\{x\le 1-y\} = \int_0^1\int_0^{1-x} f_{xy}(x,y)\,dx\,dy = \int_0^1\left\{\int_0^{1-x}(x+y)\,dy\right\}dx$

Solving we get $P\{x+y\le 1\} = 1/3 = 0.333$

Q.94 **(1)** We know that if λ is parameter of Poisson's distribution

Then, First moment = λ

Second moment = $\lambda^2+\lambda$

Given that $\lambda^2+\lambda = 2 \Rightarrow \lambda^2+\lambda-2=0 \Rightarrow (\lambda+2)(\lambda-1)=0$

$\lambda=-2$ or $1(\lambda\ne -2)$, $\therefore \lambda = 1 \Rightarrow$ First moment = 1

Q.95 **(0.072)** Required probability = TTTTH

= (0.7)(0.7)(0.7)(0.7)(0.3) = 0.07203

Q.96 $\text{mean} = E(x) = \sum xp(x)$

$$=1\times\frac{1}{6}+2\times\frac{1}{6}+3\times\frac{1}{6}+4\times\frac{1}{6}+5\times\frac{1}{6}+6\times\frac{1}{6}=\frac{1+2+3+4+5+6}{6}=\frac{21}{6}=3.5$$

Q.97 Total possibilities [HH, HT, TH, TT]

Favorable possibility = [HH, HT, TH]

Probability $=\frac{3}{4}=0.75$

Q.98 **(c)** Since Mode refers the most frequently appeared value.

$\therefore$ The data which is repeated for maximum number of times is 17.

Q.99 **(b)** *The probability that all the three balls are red is given by* $=\frac{^{4}C_3}{^{12}C_3}=\frac{1}{55}$

Q.100 **(c)** Probability of getting 1 in single throw $p=\frac{1}{6}$

Probability of NOT getting 1 in single throw $q=1-\frac{1}{6}=\frac{5}{6}$

Probability of getting 1 at least 4 tomes in rolled is given by

$$= {}^{5}C_4p^4q^1+{}^{5}C_5p^5q^0 = 5+\left(\frac{1}{6}\right)^4\left(\frac{5}{6}\right)+1\times\left(\frac{1}{6}\right)^5=\frac{25}{6^5}+\frac{1}{6^5}=0.0033$$

Now if you are in haste you will put option(D) as correct,
bit wait it is asking % , hence we have
Ans = $0.0033\times 100 = 0.33$

Q.101 **(b)** $Y = X_1 - X_2$ .... (i)

Taking mean on both sides of (i), we have

$\mu(\mathrm{Y}) = \mu(X_1) - \mu(X_2) \Rightarrow \mu(\mathrm{Y}) = \mu_1 - \mu_2 = 1 - 1 = 0$

Taking variance on both sides of (i), we have
$Var(Y) = Var(X_1 - X_2) = Var(X_1) + Var(X_2) - Cov(X_1 X_2)$

Since X_1, X_2 be two independent normal andom varibles
$Var(Y) = Var(X_1) + Var(X_2) \Rightarrow Var(Y) = \sigma_1^2 + \sigma_2^2 = 1 + 4 = 5$

Q.102 **(a)** Mean of Exponential random variable is given by $E(x) = \frac{1}{\lambda}$

So we have $E(X_1) = \frac{1}{\lambda_1} = 0.5,\ E(X_2) = \frac{1}{\lambda_2} = 0.25$

$Where,\ \lambda_1 = 2\ \&\ \lambda_2 = 4$
So mean of Y is given as $E(Y) = \frac{1}{\lambda} = \frac{1}{\lambda_1 + \lambda_2} = \frac{1}{6}$

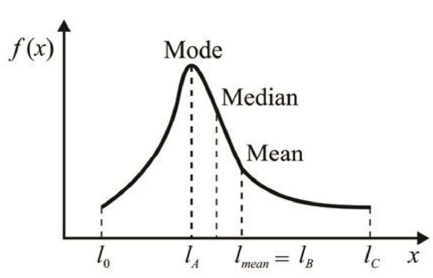
Q.103 **(a)** Probability density function:It is defined as a function of continuous random variables whose integral across an interval gives the probability that the value of variable lies within the same interval.

According to the drawn graph (in question), h lies between 0 and 1, 2hlies between 1 and 2 and 3h between 2 and 3.

$$\underset{1^{st}\,Interval}{\int_0^1 f(x)dx} + \underset{2^{nd}\,Interval}{\int_1^2 f(x)dx} + \underset{3^{rd}\,Interval}{\int_2^3 f(x)dx} = 1$$

$$\frac{h}{2} + \frac{2h}{2} + \frac{3h}{2} = 1 \Rightarrow 0.5h + 1h + 1.5h = 1 \Rightarrow 3h = 1 \Rightarrow h = \frac{1}{3}$$

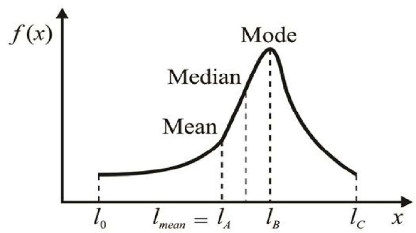
Q.104 In a right skew distribution curve, mean is always right of the peak. Mode comes at the peak and median comes in between mean and mode.



Therefore, for a right skew curve (from the graph) $l_B > l_A$

Mean > Mode $\Rightarrow$ Mean is greater than mode and median.

Note : Similarly, in a left skew distribution curve, mean is always left of the peak. Mode comes at the peak and median comes in between mean and mode.



For a left skew curve (from the graph) $l_A < l_B$

Mean < Mode $\Rightarrow$ Mean is lesser than mode and median.

Q.105 **(0.5)** Given : Number of red balls in the box = 5

Number of white balls in the box = 1

To find : Probability of consecutively picking 3 red balls without replacement.

Probability of first ball to be red= 5/6

Probability of second ball to be red (out of remaining 5 balls) = 4/5

Probability of third ball to be red (out of remaining 4 balls) = 3/4

Probability of consecutively picking 3 red balls without replacement $= \frac{5}{6} \times \frac{4}{5} \times \frac{3}{4} = 0.5$

Q.106 **(0.25)** $P(X_4 \text{ is smallest}) = \frac{3!}{4!} = \frac{1}{4} = 0.25$

Q.107 **(d)** Given : Two independent random variables X and Y with various 1 and 2

$$V(z) = V(X - Y) = (1)^2 V(X) + (-1)^2 V(Y) = 1^2 \times 1 + (-1)^2 \times 2 = 3$$

Q.108 **(0.023)** P(one of them wins in 3rd trial) = P(Ist trial is Tie) x P(IInd trial is Tie) x P(one of them wins 3rd trial)

P(Tie in any trial) = P(P=1 and Q=1) + P(P=2 and Q = 2) +............... + P(P=6 and Q=6)

$$= \frac{1}{36} + \frac{1}{36} + \frac{1}{36} + \frac{1}{36} + \frac{1}{36} + \frac{1}{36} = \frac{3}{36} = \frac{1}{6}$$

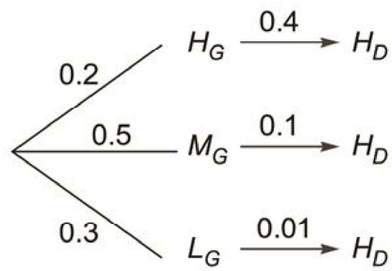
$$\text{P(one of them wins)} = 1 - \frac{1}{6} = \frac{5}{6}$$

$$\text{So required probability} = \frac{1}{6} \times \frac{1}{6} \times \frac{5}{6} = \frac{15}{216} = 0.023 \text{(round to 3 decimal place)}$$

Q.109 **(0.60)** The condition probability table given is

	H_D	M_D	L_D
H_G	0.40	0.48	0.12
M_G	0.10	0.65	0.25
L_G	0.01	0.50	0.49

$P(H_G)$ =02, $P(H_M) = 0.5$, $P(H_L) = 0.3$
Drawing the tree diagram for H_D we get ;



From diagram $P(H_G \cap H_D) = 0.2 \times 0.4$

$$P(H_D) = 0.2 \times 0.4 + 0.5 \times 0.1 + 0.3 \times 0.01 = 0.133$$

Required Probability $P(H_G I H_D) = \dfrac{0.2 \times 0.4}{0.133} = 0.60$(roundingupto 2 decimal)

Q.110 **(b)** Probability of getting the head for the third time in the fifth toss = probability of getting two heads out of 4 tosses and one head in 5th toss.

The required probability (P) = $^n c_r\, p^r q^{n-r} \times \dfrac{1}{2}$

Where , $n = 4, r = 2, p = \dfrac{1}{2}, q = \dfrac{1}{2}$

$$P = {}^4c_2 \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^2 \times \frac{1}{2} = \frac{3}{16}$$

Q.111 **(b)**Given:

For bag A	For bag B
Red balls = 20 %	Red balls = 30 %
Green balls = 80 %	Green balls = 60 %
	Yellow balls = 10 %

Let, both bags contains 10 balls each.

Hence,

In bag A	In bag B
No. of red balls = 2	No. of red balls = 3
No. of green balls = 8	No. of Green balls = 6
	No. of Yellow balls = 1

If balls of bags A and B are mixed, then,

Total no. of balls = 10 + 10 = 20

Total no. of red balls = 2 +3 = 5

Hence, if one ball is picked at random. Then the probability of this ball being of red color is

$$P(\text{Red}) = \frac{\text{No. of red balls}}{\text{Total no. of balls}} \times 100\% = \frac{5}{20} \times 100\% = 25\%$$

ESE SOLUTIONS

Q.1 (c)

Mean=μ=14
S.D=σ=2.5
$\mathrm{P}(X=16)=P(15.5<X=16<16.5)$
(normal approximation to Binomial distribution)

$$P\left(\frac{15.5-\mu}{\sigma}<\frac{X-\mu}{\sigma}<\frac{16.5-\mu}{\sigma}\right)$$
$$=P\left(\frac{1.5}{2.5}<Z<\frac{2.5}{2.5}\right)$$
$$=P(0.6<Z<1)$$
$$=0.3413-0.2257=12\%$$

Q.2 (c)

$$P(X=1)=\frac{2}{3}P(X=2)$$
$$3P(X=1)=2P(X=2)$$
$$3\lambda e^{-\lambda}=\frac{2\lambda^2e^{-\lambda}}{2}$$
$$\therefore \lambda=3$$
$$\therefore Variance=3$$

6 COMPLEX VARIABLE FUNCTION

6.1 INTRODUCTION

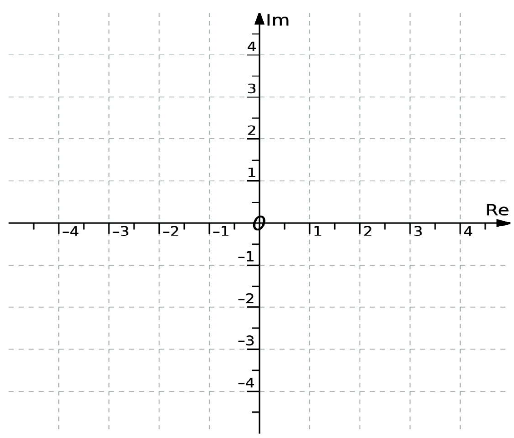
Complex analysis is particularly concerned with the analytic functions of complex variables. Because the separate real and imaginary parts of any analytic function must satisfy Laplace's equation, complex analysis is widely applicable to two-dimensional problems in physics.

6.2 COMPLEX VARIABLE FUNCTION

DEFNITION : If for each value of complex variable z= (x + iy), $i=\sqrt{-1}$ in a given region R,we have one or more values of w =(u + iv), then w is said to be a **complex variable function** of z and we write w = u (x, y) + iv(x , y) =f(z) where u, v are real functions of x and y.

If for each value of z, there corresponds one and only one value of w, then w is said to be single-valued function of z otherwise a multi-valued function.

6.3 GEOMETRICAL REPRESENTATION OF COMPLEX PLANE :



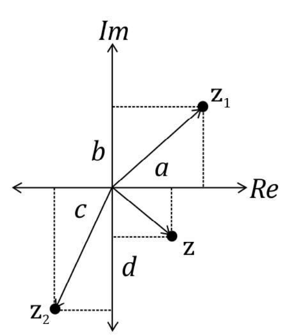
6.3.1 ADDITION AND SUBSTRACTION OF COMPLEX NUMBERS

Addition and substraction of complex numbers is easy in rectangular form

$z_1 = a + ib \qquad z_2 = c + id$

$z = z_1 \pm z_2 = a + ib \pm c + id$

$z = (a+c) \pm i\,(b+d)$



6.3.2 POWER OF i

Some time we need various powers of i.

We know that $i = \sqrt{-1}$

On squaring both sides, we get

$i^2 = -1, i^4 = 1$

Similarly, $i^{4n} = 1$, $i^{4n+1} = i$, $i^{4n+2} = -1$ and $i^{4n+3} = -i$

6.3.3 MULTIPLICATION

$z_1 = a + ib \quad z_2 = c + id$

$z_1 z_2 = (a + ib)(c + id) = ac - bd + i(ad + bc)$

6.3.4 CONJUGATE OF A COMPLEX NUMBER

Two complex numbers which differ only in the sign of imaginary parts are called conjugate of each other. The conjugate of $z = \bar{z}$

A pair of complex number a+ ib and a – ib are said to be conjugate of each other

Theorem : Show that the sum and product of a complex number and its conjugate complex are both real.

Proof : Let x + iy be a complex number and x- iy its conjugate complex.

Sum = $(x+iy) + (x-iy) = 2x$ (Real)

Product = $(x + iy)(x-iy) = x^2+y^2$ (Real) Proved

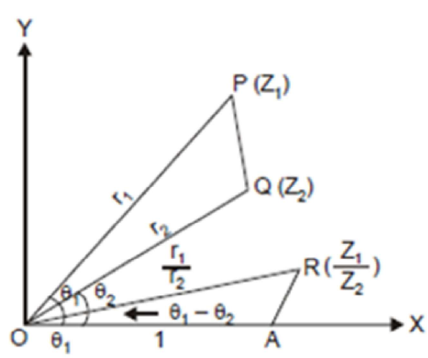
Example : Find out the conjugate of a complex number 7 + 6i.

Solution : Let z = 7 + 6i

To find the conjugate complex number of 7 + 6i we change the sign of imaginary number. Conjugate of z = 7 – 6i

To divide a complex number a + ib by c + id, we write it as $\dfrac{a+ib}{c+id}$

6.3.5 DIVISION OF COMPLEX NUMBERS BY GEOMETRY



Let P and Q represent the complex numbers.

$z_1 = x_1 + iy_1 = r_1 \ (\cos\theta_1 + i\sin\theta_1)$

$z_2 = x_2 + iy_2 = r_2 \ (\cos\theta_2 + i\sin\theta_2)$

Cut off OA = 1, construct ΔOAR on OA similar to ΔOQP.

So that $\dfrac{OR}{OA} = \dfrac{OP}{OQ} \Rightarrow \dfrac{OR}{1} = \dfrac{OP}{OQ}$

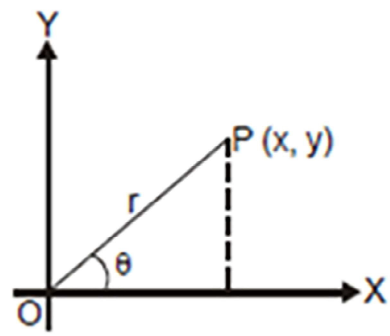
$OR = \dfrac{OP}{OQ} = \dfrac{r_1}{r_2}$

$\angle AOR = \angle QOP = \angle AOP - \angle AOQ = \theta_1 - \theta_2$

$\therefore$ represents the complex number $\dfrac{z_1}{z_2} = \dfrac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\,\sin(\theta_1 - \theta_2)]$

Hence the complex number $\dfrac{z_1}{z_2}$ is represented by the point R.

6.3.6 MODULUS & ARGUMENT



Let x + iy be a complex number.

Putting $x = r\cos\theta$ and $y = r\sin\theta$ so that $r = \sqrt{x^2 + y^2}$

$\text{Cos}\,\theta = \dfrac{x}{\sqrt{x^2 + y^2}}$ and $\sin\theta = \dfrac{y}{\sqrt{x^2 + y^2}}$

Modulus of $z = |Z| = \sqrt{x^2 + y^2}$

Argument of $z = \tan^{-1}\left(\dfrac{y}{x}\right)$

1. $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$

2. $-\pi < \text{Arg}(z) \le \pi$

3. $\text{Arg}(z_1 z_2) = \text{Arg}(z_1) + \text{Arg}(z_2) + 2k\pi, \ k \in I$

6.4 EXPONENTIAL FUNCTION OF COMPLEX VARIABLE

When x is real, we are already familiar with the exponential function

$$e^x = 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} + \ldots$$

Similarly, we define the exponential function of the complex variable z = x + iy, as

$$e^z = 1 + z + \frac{z^2}{2!} + \ldots \frac{z^n}{n!} + \ldots$$

Putting x=0 in (i), we get, z=iy and

$$e^{iy} = 1 + \frac{iy}{1!} + \frac{(iy)^2}{2!} + \ldots \frac{(iy)^n}{n!} + \ldots$$

$$= \left(1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \ldots\right) + i\left(y - \frac{y^3}{3!} + \frac{y^5}{5!} - \ldots\right)$$

= cos y + i sin y

$e^z = e^x e^{iy} = e^x$ (cosy + i siny)

$$x + iy = r(\cos\theta + i\sin\theta) = re^{i\theta}$$

$\therefore$ Exponential form of $z = re^{i\theta}$

Polar form of $z = r(\cos\theta + i\sin\theta)$

6.4.1 CIRCULAR FUNCTIONS OF COMPLEX VARIABLE

Since $e^{iy} = \cos y + i\sin y$ & $e^{-iy} = \cos y - i\sin y$.

$\therefore$ The circular functions of real angles can be written as

$$\sin y = \frac{e^{iy} - e^{-iy}}{2i}, \ \cos y = \frac{e^{iy} + e^{-iy}}{2} \text{ and so on,}$$

It is therefore, natural to define the circular function of the complex variable z by the equations:

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}, \cos z = \frac{e^{iz} + e^{-iz}}{2}, \tan z = -i\,\frac{e^{iz} - e^{-iz}}{e^{iz} + e^{-iz}}$$

6.4.2 HYPERBOLIC FUNTIONS

(i) $\dfrac{e^x - e^{-x}}{2}$ is define as **hyperbolic sine of x** and is written as **sinh x.**

(ii) $\dfrac{e^x + e^{-x}}{2}$ is defined as **hyperbolic cosine of x** and written as **cosh x.**

Thus

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \coth x = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$

$$\cosh x = \frac{e^x + e^{-x}}{2} \qquad \text{sech } x = \frac{2}{e^x + e^{-x}}$$

$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}} \qquad \text{cosech } x = \frac{2}{e^x - e^{-x}}$$

6.4.2.1 RELATION BETWEEN HYPERBOLIC & CIRCULAR FUNCTIONS

sin ix = i sinh x　　　sinh ix = i sin x

cos ix = cosh x　　　cosh ix = cos x

tan ix = i tanh x　　　tanh ix =i tan x

6.4.3 INVERSE HYPERBOLIC FUNCTIONS

If sinh u = z, then u is called the hyperbolic sine inverse of z and is written as $u = \sinh^{-1} z$

Similarly we define $\cosh^{-1} z, \tanh^{-1} z$ etc.

The inverse hyperbolic function like other inverse functions are many-valued, but we shall consider only their principal values.

6.4.4 LOGARITHMIC FUNCTION OF A COMPLEX VARIABLE

If z=(x+iy) and w=(u+iv) be so related that e^w = z, then w is said to be a logarithm of z to the base e and it is written as w= $\log_e z$

$$w = \log_e z$$

$$w = u + iv = \log_e(x + iy) = \log_e(re^{i\theta}) = \log_e r + i\theta$$

$$= \frac{1}{2}\log(x^2 + y^2) + i\tan^{-1}\frac{y}{x}$$

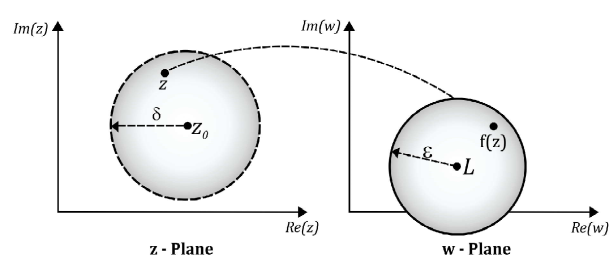
Hence, $u = \frac{1}{2}\log(x^2 + y^2)$, $v = \tan^{-1}\frac{y}{x}$

6.5 LIMIT OF A COMPLEX FUNCTION

A function w = f (z) is said to have **limit** L as z approaches a point z_0, if for every real ε, we can find a positive real δ such that

$$|f(z) - L| < \varepsilon \quad \text{for} \quad |z - z_0| < \delta$$

i.e., for every $z \neq z_0$ in the δ-disc of z – plane, f (z) has a value lying in the $\in -$disc of w-plane. In symbols, we write $\underset{z \to z_0}{\text{Lt}} f(z) = L$



Example : Prove that $\lim_{z \to 1-i} \dfrac{(z^2+4z+3)}{z+1} = 4-i$

Solution : $\lim_{z \to 1-i} \dfrac{(z^2+4z+3)}{z+1} = \lim_{z \to 1-i} \dfrac{(z+1)(z+3)}{z+1}$

$$= \lim_{z \to 1-i} (z+3) = (1-i)+3 = 4-i$$

Example : Find the limit of the following $\lim_{z \to 1+i} \dfrac{z^2-z+1-i}{z^2-2z+2}$

Solution : $\lim_{z \to 1+i} \dfrac{z^2-z+1-i}{z^2-2z+2} = \lim_{z \to 1+i} \dfrac{(z+i)(z-i)-1(z+i)}{(z-1-i)(z-1+i)}$

$$= \lim_{z \to 1+i} \frac{(z+i)-(z-i-1)}{(z-1-i)(z-1+i)} = \lim_{z \to 1+i} \frac{(z+i)}{(z-1+i)}$$

$$= \frac{1+i+i}{1+i-1+i} = \frac{1+2i}{2i} = \frac{(1+2i)(-i)}{2(i)(-i)}$$

$$= \frac{-i+2}{2(1)} = \frac{2-i}{2} = 1-\frac{i}{2}$$

6.5.1 CONTINUITY OF f(z)

A function w = f (z) is said to be **continuous** at z = z_0, if $\underset{z \to z_0}{\text{Lt}} f(z) = f(z_0)$. Further f (z) is said to be continuous in any region R of the z-plane, if it is continuous at every point of that region.

Also if w=f(z)= u (x, y) + iv (x, y) is continuous at $z = z_0$, then u(x, y) and v(x, y) are also continuous at $z = z_0$.

i.e., at $x = x_0$ and $y = y_0$. Conversely if u(x, y) and v(x, y) are continuous at (x_0, y_0), then f (z) will be continuous at $z = z_0$

Example : Examine the continuity of the following

$$f(z) = \begin{cases} \dfrac{z^3 - iz^2 + z - i}{z-i}, & z \neq i \\ 0, & z = i \end{cases}$$

Solution : $\lim_{z \to i} \dfrac{z^3-iz^2+z-i}{z-i} = \lim_{z \to i} \dfrac{z^2(z-i)+1(z-i)}{z-i} = \lim_{z \to i} \dfrac{(z^2+1)(z-i)}{z-i} = \lim_{z \to i}(z^2+1) = -1+1 = 0$

Also from question f(i)=0

Hence, $\lim_{z \to i} f(z) = f(i)$

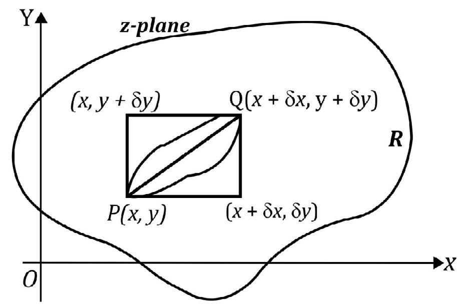
Therefore f(z) is continuous

6.6 DERIVATIVE OF A COMPLEX FUNCTION

Let w = f (z) be a Complex valued function of the variable z = x + iy .

Then the derivative of w = f (z) is defined to be

$$\frac{dw}{dz} = f'(z) = \underset{\delta z \to 0}{Lt} \frac{f(z+\delta z) - f(z)}{\delta z}$$



provided the limit exists and has the same value for all the different ways in which δz approches zero.

Suppose P(z) is fixed and $Q(z+\delta z)$ is a neighbouring point(Fig.) . The point Q may approach P along any straight or curved path in the given region, i.e. may tend to zero in any manner and dw/dz may not exist .It, therefore becomes a fundamental problem to determine the necessary and sufficient conditions for dw/dz to exist. The fact is settled by the following theorem.

Theorem : The necessary and sufficient conditions for the derivative of the function

w = u (x, y) + iv(x, y) = f (z) to exist for all values of z in a region R, are

(i) $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are continuous function of x and y in R :

(ii) $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

The relations (ii) are known as **Cauchy-Riemann equations** or briefly C-R equations.

6.7 ANALYTIC FUNCTIONS

A function f (z) which is complex-valued and possesses a unique derivative with respect to z at all points of a region R, is called an **analytic function** of z in that region. An analytic function is also called a regular function or a holomorphic function.

A function which is analytic everywhere in the complex plane is known as an **entire function.** As derivative of a polynomial exists at every point, a polynomial of any degree is an entire function.

A point at which an analytic function ceases to possess a derivative is called a **singular point** of the function.

Thus if u and v are real valued functions of x and y such that $\delta u/\delta x$, $\delta u/\delta y$, $\delta v/\delta x$, $\delta v/\delta y$ are continuous throughout a region R, then the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

are both necessary and sufficient conditions for the function f (z) = u + iv to be analytic in R. The derivative of f(z) is then, given by

$$f'(z) = f'(z) = \underset{\delta x \to 0}{\text{Lt}} \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \right) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = u_x + iv_x$$

$$\text{or } f'(z) = \underset{\delta y \to 0}{\text{Lt}} \left(\frac{\partial u}{i\delta x} + i\frac{\partial v}{\partial \delta y} \right)$$

$$= \frac{1}{i}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} = v_y - iu_y$$

The real and imaginary parts of an analytic function are called conjugate functions. The relation between two conjugate functions is given by C-R equation.

Example : Is f(z) $=z^3$ analytic ?

Solution : Z = x + iy

$$\Rightarrow \quad z^2 = (x + iy)(x - iy) = x^2 - y^2 + 2ixy$$

$$\Rightarrow \quad z^3 = (x^2 - y^2 + 2ixy)(x + iy)$$

$$= (x^3 - 3xy^2) + i(3x^2y - y^3)$$

Here $u = x^3 - 3xy^2$

$v = 3x^2y - y^3$

$u_x = 3x^2 - 3y^2, \ v_y = 3x^2 - 3y^2$

$u_y = -6xy, \quad v_x = 6xy$

so, $\quad u_x = v_y$ and $u_y = -v_x$

So, C-R equations are satisfied and also the partial derivatives are continuous at all points. Hence, z^3 is analytic for every z.

Example : If w=log z, find dw/dz and determine where w is non-analytic.

Solution : We have $\quad w = u + iv = \log(x + iy)$

$$= \frac{1}{2}\log(x^2 + y^2) + i\tan^{-1} y/x$$

So that $\quad u = \frac{1}{2}\log(x^2 + y^2), \ v = \tan^{-1} y/x$

$$\therefore \frac{\partial u}{\partial x} = \frac{x}{x^2 + y^2} = \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y} = \frac{y}{x^2 + y^2} = -\frac{\partial v}{\partial x}$$

Since the C-R equations are satisfied and the partial derivatives are continuous except at (0,0). Hence w is analytic everywhere except at z = 0 and negative Real axis.

$$\therefore \frac{dw}{dz} = \frac{\partial(\log z)}{\partial z} = \frac{1}{z}(z \neq 0).$$

6.7.1 ZEROS OF AN ANALYTIC FUNTION

A zero of an analytic function f(z) is that value of z for which f(z) = 0.

If f(z) is analytic in the neighbourhood of a point z = a, then by Taylor's theorem

$$f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \ldots + a_n(z-a)^n + \ \ldots \text{where} a_n = \frac{f^n(a)}{n!}$$

If $a_0 = a_1 = a_2 = \ldots = a_{m-1} = 0$ but $a_m \neq 0$, then f(z) is said to have a zero of order m at z = a.

When m = 1, the zero is said to be simple. In the neighbourhood of zero (z=a) of order m,

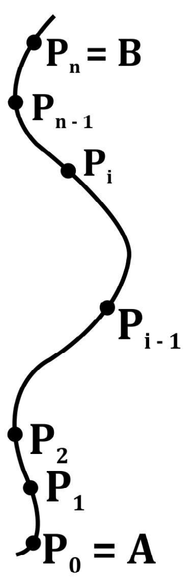
$$f(z) = a_m(z-a)^m + a_{m+1}(z-a)^{m+1} + \ \ldots\ldots = (z-a)^m \Phi(z)$$

Where $\Phi(z) = a_m + a_{m+1}(z-a) + \ldots$

Then Φ(z) is analytic and non-zero in the neighborhood of z = a.

6.8 COMPLEX INTEGRATION

Consider a continuous function f (z) of the complex variable z = x + iy defined at all points of a curve C having end points A and B. Divide C into n parts at the points



$$A = P_0(z_0),\ P_1(z_1),\ \ldots\ldots,\ P_i(z_i), \ldots\ldots, P_n(z_n) = B.$$

The limit of the sum $\sum_{i=1}^{n} f(\zeta_i)\delta z_i$ as $n \to \infty$ in such a way that the length of the chord δz_i approaches zero, is called the **line integral of f (z)** taken along the path C, i.e., $\int_c f(z)\,dz$.

Writing $f(z) = u(x, y) + iv(x, y)$ and $dz = dx + idy$,

$$\int_c f(z)\,dz = \int_c (udx - vdy) + i\int_c (vdx + udy)$$

which shows that the evaluation of the line integral of a complex function can be reduced to the evaluation of two line integrals of real functions.

6.9 SINGULARITIES

A singular point of a function is the point at which the function ceases to be analytic.

(1) Isolated Singularity: If z=a is a singularity of f(z) such that f(z) is analytic at each point in its neighborhood, but not at that point (i.e., there exist a circle with center a, which has no other singularity then a), t1hen z = a is called an **isolated singularity.**

f(z) can be expanded in a Laurent's series around z=a

$$f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \ldots + a_{-1}(z-a)^{-1} + a_{-2}(z-a)^{-2} + \ldots (1)$$

(2) Removable Singularity : If all the negative powers of (z–a) in (1) are zero, then it is called Removable Singularity. Here the singularity can be removed by defining f(z) at z = a in such a way that it become analytic at z=a. Such a singularity is called a **removable singularity.**

Thus if $\lim_{z\to a} f(z)$ exists finitely, then z = a is a removable singularity.

(3) Poles : If all the negative powers of (z - a) in (1) after the nth term are missing, then the singularity at z = a is called a **pole of order n.** A pole of first order is called a **simple pole.**

(4) Essential singularity : If the number of negative powers of (z-a) in (1) is infinite , then z = a is called an essential singularity.

In this case, $\lim_{z\to a} f(z)$ does not exist.

Example : What type of singularity have the following functions : (I) $\frac{1}{1-e^z}$ (II) $\frac{e^{2z}}{(z-1)^4}$

Solution :

(**I**) Poles of $f(z) = 1/(1 - e^z)$ are found by equating to zero $1 - e^z = 0$ or $e^z = 1 = e^{2n\pi i}$

$\therefore \quad z = 2n\pi i\,(n = 0, \pm 1, \pm 2, \ldots\ldots..)$

Clearly $f(z)$ has simple pole at $z = 2n\pi i$.

(II) $\frac{e^{2z}}{(z-1)^4} = \frac{e^{2(t+1)}}{(z-1)^4} = \frac{e^2}{t^4}.e^{2t}$, *where* $t = z - 1$

$$=\frac{e^2}{t^4}\left\{1+\frac{2t}{1!}+\frac{(2t)^2}{2!}+\frac{(2t)^3}{3!}+\frac{(2t)^4}{4!}+\frac{(2t)^5}{5!}+.....\right\}$$

$$=e^2\left\{\frac{1}{t^4}+\frac{2}{t^3}+\frac{2}{t^2}+\frac{4}{3t!}+\frac{2}{3}+\frac{4t}{15}+.....\right\}$$

$$=e^2\left\{\frac{1}{(z-1)^4}+\frac{2}{(z-1)^3}+\frac{2}{(z-1)^2}+\frac{4}{3(z-1)}+\frac{2}{3}+\frac{4}{15}(z-1)+.....\right\}$$

Since there is a terms containing 4th negative power of (z-1),

$\therefore$ z=1 is a pole of 4^{th} order.

Example : The analytic function $f(z)=\dfrac{z-1}{z^2+1}$ has singularities at

(a) 1 and -1 (b) 1 and i (c) 1 and $-i$ (d) i and $-i$

Solution : (d)

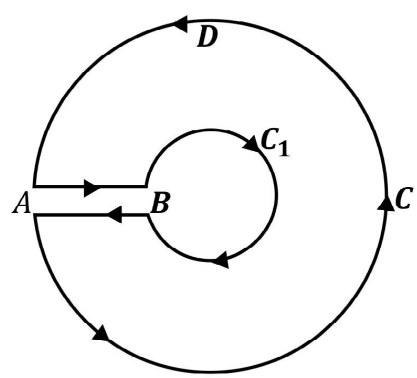
$$f(z)=\frac{z-1}{z^2+1}=\frac{z-1}{z^2+i^2}=\frac{z-1}{(z-i)(z+i)}$$

$\therefore$ The sigularities arc $z = i$ and $-i$

6.10 CAUCHY'S THEOREM

If f(z) is an analytic function and f'(z) is continuous at each point within and on a closed curve C, then $\oint_c f(z)dz=0$

Extension of Cauchy's theorem: If f(z) is analytic in the region D between two simple closed curves C and C_1, then $\oint_c f(z)dz = \oint_{c_1} f(z)dz$



6.10.1 CAUCHY'S INTEGRAL FORMULA

If f(z) is analytic within and on a closed curve and if a is any point within C, then

$$\oint_c \frac{f(z)dz}{z-a}=2\pi i\, f(a) \qquad \oint_c \frac{f(z)}{(z-a)^{n+1}}dz=\frac{2\pi i}{n!}f^n(a)$$

Example : Using Cauchy's integral theorem, the value of the integral (integration being taken counterclockwise direction) $\oint_c \frac{z^3-6}{3z-i}\,dz$ is

(a) $\frac{2\pi}{81}-4\pi i$ (b) $\frac{\pi}{8}-6\pi i$ (c) $\frac{4\pi}{81}-6\pi i$ (d) 1

Solution : (a) Cauchy's integral theorem is

$$f(a)=\frac{1}{2\pi i}\oint_c \frac{f(z)}{z-a}\,dz$$

i.e. $$\oint_c \frac{f(z)}{z-a}\,dz = 2\pi i\ f(a)$$

Now, $$\oint_c \frac{z^3-6}{3z-i}\,dz = \frac{1}{3}\oint_c \frac{z^3-6}{\left(z-\frac{i}{3}\right)}dz$$

Applying Cauchy's ingegral theorem, using $f(z)=z^3-6$.

$$=\frac{1}{3}\left(2\pi i\ f\left(\frac{i}{3}\right)\right)=\frac{1}{3}\left(2\pi i\left[\left(\frac{i}{3}\right)^3-6\right]\right)$$

$$=\frac{1}{3}\left[2\pi i\left(\frac{i^3}{27}-6\right)\right]=\frac{2\pi}{81}i^4-4\pi i=\frac{2\pi}{81}-4\pi i$$

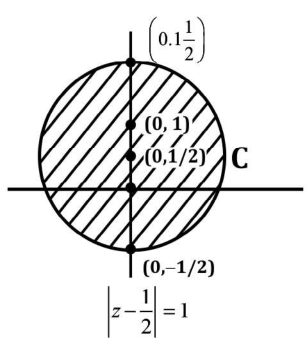
Example : The value of $\oint_c \frac{dz}{(1+z^2)}$ where C is the contour $|z-i/2|=1$ is

(a) $2\pi i$ (b) π (c) $\tan^{-1}z$ (d) $\pi i\tan^{-1}z$

$$\frac{1}{z^2+1}=\frac{1}{(z-i)(z+i)}$$

Poles are at i and $-i$, i.e. $(0, 1)$ and $(0,-1)$

From figure of the given region $|z-i/2|=1,$ we see that pole $(0, 1)$ i.e. $-i$ is outside C.



So, $$I = 2\pi i\ \mathrm{Res}\,f(i) = 2\pi i.\frac{1}{(i-i)(i+i)}=\pi$$

6.11 RESIDUES : The coefficient of $(z-a)^{-1}$ in the expansion of f(z) around an isolated singularityis called the residue of f(z) at that point . Thus is the Laurent's series expansion of f(z) around z= a i.e.

$f(z)=a_0+a_1(z-a)+a_2(z-a)^2+\ldots+a_{-1}(z-a)^{-1}+a_{-2}(z-a)^{-2}+\ \ldots,$

the residue of f (z) at z = a is a_{-1}.

Since, $a_n=\dfrac{1}{2\pi i}\displaystyle\int\frac{f(z)}{(z-a)^{n+1}}dz$

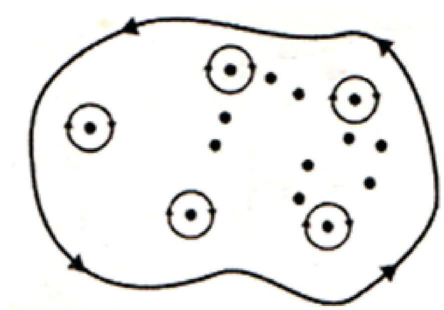
$\therefore\ a_{-1}=f(a)=\dfrac{1}{2\pi i}\displaystyle\int_c f(z)dz \Rightarrow \int_c f(z)dz=2\pi i\, f(a)$

6.11.1 RESIDUE THEOREM

If f(z) is analytic in a closed curve C except at a finite number of singular points within C,

then $\displaystyle\int_c f(z)dz=2\pi i$ (sum of Residue at the singular points within C)

Let us surround each of the singular points $a_1, a_n, \ldots\ldots, a_n$ by a small circle such that it encloses no other singular point. Then these circle $C_1, C_2, \ldots\ldots, C_n$ together with C, from a multiple connected region in which f(z) is analytic



$\therefore$ Applying Cauchy's theorem, we have

$\displaystyle\int_c f(z)dz=\int_{c_1}^{1} f(z)dz+\int_{c_2}^{2} f(z)dz+\ldots\ldots\int_{cn} f(z)dz$

$=2\pi i[\text{Res } f(a_1)+\text{Res } f(a_2)+\ldots\ldots+\text{Res} f(a_n)]$

Which is desired result.

6.11.2 CALCULATION OF RESIDUES

1) If f (z) has a simple pole at z = a, then

$\text{Res } f(z)\ =\lim_{z\to a}[(z-a)f(z)]=C_{-1}=\ f(a)$

Laurent's series in this case is

$f(z)=C_0+C_1(z-a)+C_2(z-a)^2\ldots\ldots+C_{-1}(z-a)^{-1}$

Multiplying by $(z-a)$, we have

$(z-a)\ f(z)=C_0(z-a)+C_1(z-a)^2\ldots\ldots+C_{-1}$

Taking limits $z\to a$ we get

$\lim_{z\to a}[(z-a)f(z)]=C_{-1}=\ f(a).$

2) Another formula for Res f(a):

Let $f(z)=\dfrac{\phi(z)}{\psi(z)}$

Where $\psi(z)=(z-a)F(z), F(a)\neq 0$

Then

$$\lim_{z\to a}\left[(z-a)\frac{\phi(z)}{\psi(z)}\right]=\lim_{z\to a}\frac{(z-a)[\phi(a)+(z-a)\phi'(a)+\ldots\ldots]}{\psi(a)+(z-a)\psi'(a)+\ldots\ldots}$$

since $\psi(a)=0$

$$=\lim_{z\to a}\frac{\phi(a)+(z-a)\phi'(a)+\ldots\ldots]}{\psi'(a)+(z-a)\psi''(a)+\ldots\ldots}$$

Thus , $\text{Res } f(a)=\dfrac{\phi(a)}{\psi'(a)}$

3) If f(z) has a pole of order n at z = a , then

$$\text{Res } f(a)=\frac{1}{(n-1)!}\left\{\frac{d^{n-1}}{dz^{n-1}}\left[(z-a)^n f(z)\right]\right\}_{z=a}$$

Example : Given $f(z)=\dfrac{1}{z+1}-\dfrac{2}{z+3}$

If C is a counterclockwise path in the z-plane such that $|z+1|=1$,

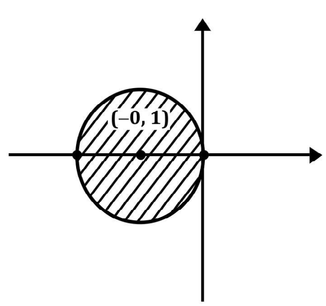
the value of $\dfrac{1}{2\pi j}\oint_C f(z)\,dz$ is

(a)-2 (b) -1 (c) 1 (d) 2

Solution: (C) Given, $f(z)=\dfrac{1}{z+1}-\dfrac{1}{z+3}=\dfrac{(z+3)-2(z+1)}{(z+1)(z+3)}=\dfrac{-z+1}{(z+1)(z+3)}$

Poles are at -1 and -3 i.e. (-1, 0) and (-3, 0).

From figure below of |z + 1| = 1, we see that (-1. 0) is inside the circle and (-3, 0) is outside the circle.



Residue theorem says,

$$\frac{1}{2\pi j}\oint_C f(z)\,dz = \text{Residue of those poles which are Inside C.}$$

This residue is $= \dfrac{-1(-1)+1}{(-1+3)} = \dfrac{2}{2} = 1$

$$\frac{1}{2\pi j}\oint_C f(z)\,dz = 1$$

Example : The integral $\oint f(z)\,dz$ evaluated around the unit circle on the complex plane for $f(z) = \dfrac{\cos z}{z}$ is

(a) $2\pi i$ (b) $4\pi i$ (c) $-2\pi i$ (d) 0

Solution : (a)

$$f(z) = \frac{\cos z}{z}$$

has simple pole at z = 0 and z = 0 is inside unit circle on complex plane

$\therefore$ Residue of $f(z)$ at z = 0

$$\lim_{z\to 0} z\,f(z) = \lim_{z\to 0} \cos z = 1$$

$$\int_C f(z)\,dz = 2\pi i(\text{Residue at } z = 0) = 2\pi i$$

Example : The residue of the function $f(z) = \dfrac{1}{(z+2)^2(z-2)^2}$ at z = 2 is

(a) $-\dfrac{1}{32}$ (b) $-\dfrac{1}{16}$ (c) $-\dfrac{1}{16}$ (d) $\dfrac{1}{32}$

Solution: (a) Since $\underset{z\to 2}{\text{Lt}}\left[(z-2)^2 f(z)\right]$ is finite and non-zero, $f(z)$ has a pole of order two at z = 2.

The residue at z = a is given for a pole of order n as

$$\text{Res } f(a) = \frac{1}{(n-1)!}\left\{\frac{d^{n-1}}{dz^{n-1}}\left[(z-a)^n f(z)\right]\right\}_{z=2}$$

Here n = (2 pole of order 2) and a = 2

$$\therefore \quad \text{Res } f(a) = \frac{1}{1!}\left\{\frac{d}{dz}\left[(z-a)^2\frac{1}{(z+2)^2(z-2)^2}\right]\right\}_{z=2}$$

$$= \left\{\frac{d}{dz}\left[(z-a)^2\frac{1}{(z+2)^2(z-2)^2}\right]\right\}_{z=2} = \left\{\frac{d}{dz}\left[\frac{1}{(z+2)^2}\right]\right\}_{z=2}$$

$$= \left[-2(z+2)^{-3}\right]_{z=2} = \frac{-2}{(2+2)^3} = -\frac{1}{32}$$

6.12 HARMONIC FUNCTION

Any function winch satisfies the Laplace's equation is known as a harmonic function .

Theorem : If *f (z) = u + iv* is an analytic function, then u and v are both harmonic functions.

Proof : Let *f (z) = u + iv*, be an analytic function, then we have

$$\left.\begin{array}{ll}\dfrac{\partial u}{\partial x}=\dfrac{\partial v}{\partial x} & ....(1)\\ \dfrac{\partial u}{\partial y}=-\dfrac{\partial v}{\partial x} & ....(2)\end{array}\right] C-R \text{ equations.}$$

Differentiating (1) with respect to x, we get $\frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 v}{\partial x\partial y}$ ....(3)

Differentiating (2) w.r.t. 'y' we have $\frac{\partial^2 u}{\partial y^2}=\frac{\partial^2 v}{\partial y\partial x}$ ....(4)

Adding (3) and (4) we have $\frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 v}{\partial y^2}=\frac{\partial^2 u}{\partial x\partial y}-\frac{\partial^2 v}{\partial y\partial x}$

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0 \qquad \left(\frac{\partial^2 v}{\partial x\partial y}=\frac{\partial^2 v}{\partial y\partial x}\right)$$

Similarly $$\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=0$$

Therefore both u and v are harmonic functions.

Such functions u, v are called Conjugate harmonic functions if u + iv is also analytic function.

6.13 METHOD TO FIND THE CONJUGATE FUNCTION

An analytic function f(z) = u + iv can be determined if any of u or v is given.

These are the methods to find the analytic function

- Direct method
- Milne-Thomson's method
- Exact differential equation method
- KK's method

DIRECT METHOD : Let f(z) = u + iv is an analytic function.

Case-I If u is given then v can be finding out in following steps

Step – I Evaluate $\frac{\partial u}{\partial x}$

Step – II Evaluate $\frac{\partial v}{\partial y}$ $\left[\text{by } C-R \text{ equation Evaluate } \frac{\partial v}{\partial y}=\frac{\partial u}{\partial x}\right]$

Step – III Integrate $\frac{\partial v}{\partial y}$ with respect to y to find v with taking integrating constant $f(x)$

Step – IV Differentiate v (from step – III) with respect to x. $\left[\text{Evaluate} \frac{\partial v}{\partial x} \text{containing } f'(x)\right]$

Step – V Evaluate $\frac{\partial u}{\partial y}$

Step – VI Evaluate $\frac{\partial v}{\partial x}$ $\left[\text{by C – R equation Evaluate } \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}\right]$

Step – VII Equate the values of $\frac{\partial v}{\partial x}$ from Step – IV and Step – VI and find $f'(x)$.

Step – VIII Integrate $f'(x)$ with respect to x to find $f(x)$.

Step – IX Now, substitute the value of $f(x)$ in Step – III to evaluate v.

Case-II If v is given then u can be finding out in following steps

Step – I Evaluate $\frac{\partial v}{\partial y}$

Step – II Evaluate $\frac{\partial u}{\partial x}$ $\left[\text{by C – R equation Evaluate } \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}\right]$

Step – III Integrate $\frac{\partial u}{\partial x}$ with respect to x to find u with taking integrating constant $f(y)$

Step – IV Differentiate u (from step – III) with respect to y. $\left[\text{Evaluate } \frac{\partial u}{\partial y} \text{ containing} f'(y)\right]$

Step – V Evaluate $\frac{\partial u}{\partial x}$

Step – VI Evaluate $\frac{\partial u}{\partial y}$ $\left[\text{by C – R equation Evaluate } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}\right]$

Step – VII Equate the values of $\frac{\partial u}{\partial y}$ from Step – IV and Step – VI and find $f'(y)$.

Step – VIII Integrate $f'(y)$ with respect to y to find $f(y)$.

Step – IX Now, substitute the value of $f(y)$ in Step – III to evaluate u.

MILNE-THOMSON'S METHOD :

Let a complex variable function is given by $f(z) = u(x, y) + iv(x, y)\ldots\ldots\ldots(1)$

Here $z = x + iy$.

Now, waiting x and y in terms of z and

$$x = \frac{z+\bar{z}}{2} \text{ and } y = \frac{z-\bar{z}}{2}$$

On differentiating $f(z)$ with respect to x –

$$\frac{\partial f(z)}{\partial x} = \frac{\partial u(x, y)}{\partial x} + i\frac{\partial v(x, y)}{\partial x}$$

Now, by considering $z \Longrightarrow x = z$ and $y = 0$

Therefore,

$$\frac{\partial f(z)}{\partial z} = \frac{\partial u(z, 0)}{\partial z} + i\frac{\partial v(z, 0)}{\partial z}$$

On integrating

$F(z) = u(z,0) + iv(z,0)$

This equation is same if we will replace x with z and y with 0 in equation (1).

EXACT DIFFERENTIAL EQUATION METHOD:

i) If u is given

$$v = \int_{y\,=\,\text{constant}} \left(\frac{\partial u}{\partial y}\right) dx + \int (\text{terms of } \left(\frac{\partial u}{\partial x}\right) \text{ not containing x) dy}$$

ii) If v is given then u can be finding out by:

$$u = \int_{y\,=\,\text{constant}} \left(\frac{\partial v}{\partial y}\right) dx + \int \left(\text{terms of } \left(-\frac{\partial v}{\partial x}\right) \text{ not containing x}\right) \text{dy}$$

KK'S METHOD :

Case-I $\left[f(z) = 2u\left(\frac{z}{2}, \frac{z}{2i}\right) - u(0,0) + ic \right]$

The above expression is directly used to find the analytic function if u(x,y) is given, provided that u(0,0) and v(0,0) exists.

Case-II $\left[f(z) = 2iv\left(\frac{z}{2}, \frac{z}{2i}\right) + iv(0,0) + c \right]$

The above expression is directly used to find the analytic function if v(x,y) is given, provided that u(0,0) and v(0,0) exists

Note : This method is not applicable, if either u(x,y) or v(x,y) does not exists at (0,0).

Reference : www.ijsres.com

6.14 TAYLOR SERIES

The **Taylor series** of a function f(z) the complex analog of the real Taylor series is

(1) $$f(z) = \sum_{n=1}^{\infty} a_n (z - z_0)^n \quad where \quad a_n = \frac{1}{n!} f^{(n)}(z_0)$$

or, by (1),

(2) $$a_n = \frac{1}{2\pi i} \oint_C \frac{f(z^*)}{(z^* - z_0)^{n+1}} dz^*.$$

In (2) we integrate counterclockwise around a simple closed path C that contains z_0 in its interior and is such that f(z) is analytic in a domain containing C and every point inside C.

The remainder of Taylor series (1) after term $a_n (z - z_0)^n$ is

(3) $$R_n(z) = \frac{(z-z_0)^{n+1}}{2\pi i} \oint_C \frac{f(z^*)}{(z^*-z_0)^{n+1}(z^*-z)} dz^*$$

(proof below). Writing out the corresponding partial sum of (1), we thus have

(4) $$f(z) = f(z_0) + \frac{(z-z_0)}{1!} f'(z_0) + \frac{(z-z_0)^2}{2!} f^2(z_0) + ...$$

$$+ \frac{(z-z_0)^n}{n!} f^{(n)}(z_0) + R_n(z).$$

This is called **Taylor's Formula** with remainder.

Given the power series

$$f(z) = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + a_3(z-z_0)^3 + ...$$

Then $f(z_0) = a_0$

$f'(z) = a_1 + 2a_2(z-z_0) + 3a_3(z-z_0)^2 + ... ,$ thus $f'(z_0) = a_1$

$f''(z) = 2a_2 + 6a_3(z-z_0) + ... ,$ thus $f''(z_0) = 2!\, a_2$

6.14.1 GEOMETRIC SERIES

Let Let $f(z) = 1(1-z)$. Then we have $f^{(n)}(z) = n!/(1-z)^{n+1}, f^{(n)}(0) = n!$ Hence the Maclaurin expansion of $1/(1-z)$ is the geometric series.

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n = 1 + z + z^2 + ... \qquad (|z| < 1).$$

(z) is singular at z = 1 ; this point lies on the circle of convergence.

6.14.2 EXPONENTIAL FUNCTION

We know that the exponential function e^z is analytic for all z, we obtain the Maclaurin series

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2!} .....,$$

6.14.3 TRIGONOMETRIC & HYPERBOLIC FUNCTIONS

$$\cosh z = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!} = 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + ...$$

$$\sinh z = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!} = z + \frac{z^3}{3!} + \frac{z^5}{5!} + ...,$$

6.15 LAURENT SERIES

Laurent series is generalize form of Taylor series. If in an application, we want to develop a function f(z) in powers of $z - z_0$ when f(z) is singular, we cannot use a Taylor series. Instead we can use a new kind of series, called **Laurent series**, consisting of positive integer powers of $z - z_0$ (and a constant) as well as **negative integer powers** of $z - z_0$ this is the new feature in Laurent series. Laurent series are also used for classifying singularities and in a powerful integration method ("residue integration").

A Laurent series converges in an annulus (in the "hole" of which may have singularities).

6.15.1 Laurent's Theorem

Let f(z) be analytic in a domain containing two concentric circles C_1 and C_2 with center z_0 and the annulus between them (in Fig. Below). Then can be represented by the Laurent series

(1) $$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}$$

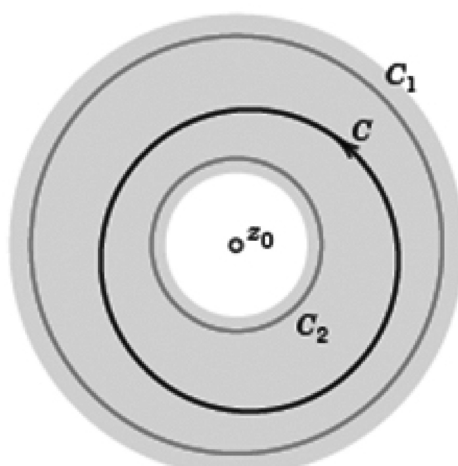
$$= a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \ldots\ldots$$

$$\ldots\ldots + \frac{b_1}{z-z_0} + \frac{b_2}{(z-z_0)^2} + \ldots\ldots$$

Consisting of nonnegative and negative powers. The coefficients of this Laurent series are given by the integrals

(2) $$a_n = \frac{1}{2\pi i}\oint_C \frac{f(z^*)}{(z^*-z_0)^{n+1}} dz^*, \quad bn = \frac{1}{2\pi i}\oint_C (z^*-z_0)^{n-1} f(z^*) dz^*,$$

taken counterclockwise around any simple closed path C that lies in the annulus and encircles the inner circle, as in Fig. [The variable of integration is denoted by since z* is used in (1).]



Laurent's Theorem

This series converges and represents f(z) in the enlarged open annulus obtained from the given annulus by continuously increasing the outer circle C_1 and decreasing C_2 until each of the two circles reaches a point where f(z) is singular.

In the important special case that z_0 is the only singular point of f(z) inside C_2, this circle can be shrunk to the point z_0 , giving convergence in a disk except at the center. In this case the series (or finite sum) of the negative powers of (1) is called the **principal part** of f (z) at z_0 [or of that Laurent series (1)].

6.16 IMPROPER INTEGRAL IN COMPLEX INTEGRATION

When evaluating real integrals of the form $\int_a^b f(x)dx,\ x$ real, it is usually assumed that f(x) is well defined at all points x satisfying $a \le x \le b$. If there exists a point ξ satisfying $a \le \xi \le b$ where f(x) is undefined, then one can introduce small ix4tivc quantities ε and η and write the integral as

$$\int_a^b f(x)dx = \lim_{\in \to \infty} \int_a^{\xi - e} f(x)dx + \lim_{n \to 0} \int_{\xi - \eta}^b f(x)dx \qquad (1)$$

provided that the limits exist. If the limits exist, then the integral is called convergent, otherwise it is called divergent. Another way to define the integral is to introduce just one small positive quantity $\in$ and write

$$\int_a^b f(x)dx = \lim_{\in \to 0} \int_a^{\xi - e} f(x)dx + \int_{\xi + e}^b f(x)dx \qquad (2)$$

provided the hunts exist. The limits in equations (1) and (2) do not always produce the same results. Whenever the integral given by equation (2) exists. then the value of the integral is called the Cauchy principal value. Note that the Cauchy principal value is symmetric with the limits decreasing at the same rate, while in comparison the limits in equation (1) can decrease at different rates.

The same idea integration involving infinite limits. The integration of rational functions f(x) over the interval $[0, \infty)$ are called improper improper integrals and are defined by a limiting process

$$\int_0^\infty (x)\ dx = \lim_{R \to \infty} \int_0^R f(x)dx$$

if this limit exists, Similarly, the integration of real rational functions f(x), defined for all real values of x, over the interval $(-\infty,\ \infty)$ are calculated from the limiting process

$$\int_{-\infty}^\infty f(x)\ dx = \lim_{R_1 \to \infty} \int_{-R}^R f(x)dx + \lim_{R_2 \to \infty} \int_0^{R_2} f(x)\ dx$$

provided that both limits exist. The limiting process given by equation (3) is not the same as the limiting process

$$\int_{-\infty}^\infty f(x)dx = \lim_{R\ \to \infty} \int_{-R}^R\ f(x)dx \qquad (4)$$

because in one case the limiting process is symmetric and non symmetric in the other case. If time limit in equation (3) exists, then the same value can be obtained from the limiting process given by equation (4). However, if the Limit given by equation (4) exists, then the limit given by equation (3) may or may not exist. This can be illustrated with a simple example. Consider the limiting process given by equation (4) in the special case f(x) = x

$$\lim_{R_1\to\infty}\int_{-R}^{R} x\,dx = \lim_{R_1\to\infty}\left.\frac{x^2}{2}\right|_{-R}^{R} = \lim_{R_1\to\infty}\left[\frac{R^2}{2}-\frac{(-R)^2}{2}\right] = 0 \qquad (5)$$

However, the limiting process defined by equation (3) for the function f(x) = x produces

$$\lim_{R_1\to\infty}\int_{-R_1}^{0} x\,dx + \lim_{R_2\to\infty}\int_{0}^{R_2} x\,dx = \lim_{R_1\to\infty}\left.\frac{x^2}{2}\right|_{-R_1}^{0} + \lim_{R_2\to\infty}\left.\frac{x^2}{2}\right|_{0}^{R_2} = \lim_{R_1\to\infty}\frac{R_1^2}{2} + \lim_{R_2\to\infty}\frac{R_2^2}{2} \qquad (6)$$

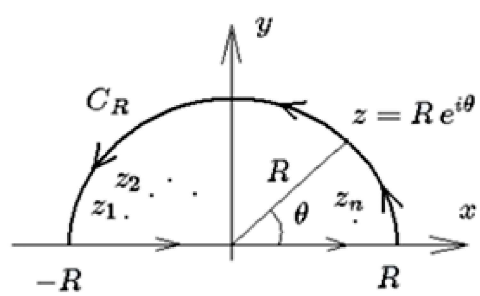
which becomes an indeterminate form in the limit as R1 and R2 increase without bound. The limiting process defined by equation (4) is used to define the Cauchy principal value of an improper integral.

The Cauchy principal value of the improper integral $\int_{-\infty}^{\infty} f(x)\,dx$, of a real rational function f(x) is defined by the limiting process

Cauchy principal value $$\int_{-\infty}^{\infty} f(x) = \lim_{R\to\infty}\int_{-R}^{R} f(x)\,dx \qquad (7)$$

provided that the limit exists. If the limit exists, then the integral is said to be a convergent integral in the Cauchy sense, otherwise it is called divergent.

Consider the contour integral of a complex rational function f(z) taken around the special contour illustrated in the figure. Assume that the rational function f(z) has poles at the points $z_1, z_2, \ldots z_n$ lying in the upper half plane. The contour of integration consists of two parts. The first part is the upper semicircle C_R centered at the origin and having a radius R large enough to enclose all the poles of f(z) in the upper half plane. The second part of the contour is the straight line segment from (-R, 0) to (R, 0). The integration of f(z) around this special contour C is performed in the positive sense. The semicircular part of the path of integration is given the notation C_R to emphasize that this portion of the path depends upon the radius R.



Contour path C consisting of semicircle C_R and straight line path from -R to R on the x-axis. It is assumed that f(z) has poles at points $z_1, z_2, \ldots z_n$ in the upper half-plane which lie inside the contour C.

Example : The value of the integral $\int_{-\infty}^{\infty} \frac{\sin x}{x^2+2x+2} dx$

evaluated using contour integration and the residue theorem is

a) $-\pi \sin(1)/e$ b) $-\pi \cos(1)/e$ c) $\sin(1)/e$ d) $\cos(1)/e$

Solution: (a)

We know that sin x is the imaginary part of e^{ix}

$\therefore$ We consider the function

$$f(z) = \frac{e^{iz}}{z^2+2z+2}$$

Now, the poles of f(z) are given by

$$z^2+2z+2=0$$

$$\Rightarrow \quad z = \frac{-2 \pm \sqrt{4-4(2)}}{2} = \frac{-2 \pm \sqrt{-4}}{2}$$

$$= \frac{-2 \pm i2}{2} = -1 \pm i$$

But $Z = -1+i$ is the only pole (simple) lie in the upper half of the $Z-$plane.

$$\therefore \operatorname{Res}_{z=-1+i} f(z) = \lim_{z \to -1+i} [Z-(-1+i)] \frac{e^{iz}}{[z-(-1+i)][z-(-1-i)]}$$

$$= \lim_{z \to -1+i} \frac{e^{iz}}{z-(-1-i)} = \frac{e^{i(-1+i)}}{-1+i+1+i} = \frac{e^{-i-1}}{2i} = \frac{e^{-i}}{2ie}$$

$$\text{Thus } \int_c \frac{e^{iz}}{z^2+2z+2} dz = 2\pi i \left[\frac{e^{-i}}{2ie}\right] = \frac{\pi e^{-i}}{e}$$

Equating imaginary parts on both sides we get

$$\int_c \frac{\sin z}{z^2+2z+2} dz = \frac{\pi(-\sin(1))}{e} = \frac{-\pi \sin(1)}{e}$$

$$\int_{-\infty}^{\infty} \frac{\sin x}{x^2+2x+2} dx = \frac{-\pi \sin(1)}{e}$$

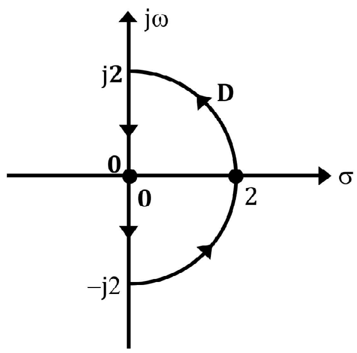
GATE QUESTIONS

Q.1 The value of the contour integral $\oint_{|z-i|=2} \frac{1}{z^2+4} dz$ in a positive sense is

(a) $i\pi/2$ (b) $-i\pi/2$

(c) $i\pi/2$ (d) $\pi/2$ **[GATE-2006 (EC)]**

Q.2 If the semicircular contour D of a radius 2 is as shown in the figure, then the value of integral $\oint_D \frac{1}{(s^2-1)} ds$ is



(a) $j\pi$ (b) $-j\pi$

(c) $-\pi$ (d) π **[GATE-2006 (EC)]**

Q.3 The residue of the function $f(z) = \frac{1}{(z+2)^2(z-2)^2}$ at $z = 2$ is

(a) $-\frac{1}{32}$ (b) $-\frac{1}{16}$

(c) $\frac{1}{16}$ (d) $\frac{1}{32}$ **[GATE-2006 (EC)]**

Q.4 If $f(z) = c_0 + c_1 z^{-1}$, then $\oint_{\text{unit circle}} \frac{1+f(z)}{z} dz$ is given by

(a) $2\pi c_1$ (b) $2\pi(1+c_0)$

(c) $2\pi j c_1$ (d) $2\pi j(1+c_0)$ **[GATE-2006 (EC)]**

Q.5 The residues of a complex function $X(z) = \frac{1-2z}{z(z-1)(z-2)}$ at its poles are

(a) $\frac{1}{2}, -\frac{1}{2}$ and 1 (b) $\frac{1}{2}, \frac{1}{2}$ and -1

(c) $\frac{1}{2}, 1$ and $-\frac{3}{2}$ (d) $\frac{1}{2}, -1$ and $\frac{3}{2}$ **[GATE-2006 (EC)]**

Q.6 The value of $\oint_c \frac{dz}{(1+z^2)}$ where C is the contour $|z - i/2| = 1$ is

(a) $2\pi i$ (b) π

(c) $\tan^{-1} z$ (d) $\pi i \tan^{-1} z$ **[GATE-2007 (EE)]**

Q.7 The Integral $\oint f(z)dz$ evaluated around the unit circle on the complex plane for

$f(z) = \frac{\cos z}{z}$ is

(a) $2\pi i$ b) $4\pi i$

(c) $-2\pi i$ d) 0 **[GATE-2008 (ME)]**

Q.8 An analytic function of a complex variable z = x + iy is expressed as (z) = u(x, y)+ iv(x, y) where $i = \sqrt{-1}$. If u= xy the expression of v should be

(a) $\frac{(x+y)^2}{2} + k$ (b) $\frac{x^2 - y^2}{2} + k$

(c) $\frac{y^2 - x^2}{2} + k$ (d) $\frac{(x-y)^2}{2} + k$ **[GATE-2009 (ME)]**

Q.9 The modulus of the complex number $\left(\frac{3+4i}{1-2i}\right)$ is

(a) 5 (b) $\sqrt{5}$

(c) $\frac{1}{\sqrt{5}}$ (d) $\frac{1}{5}$ **[GATE-2009 (ME)]**

Q.10 The analytic function $f(z) = \frac{z-1}{z^2+1}$ has singularities at

(a) 1 and – 1 (b) 1 and i

(c) 1 and – i (d) i and – i **[GATE-2009 (ME)]**

Q.11 Using Cauchy's Integral theorem, the value of the integral (integration being taken in counter clockwise direction) $\oint_c \frac{z^3-6}{3z-i}dz$ is

(a) $\frac{2\pi}{81}-4\pi i$ (b) $\frac{\pi}{8}-6\pi i$

(c) $\frac{4\pi}{81}-6\pi i$ (d) 1 **[GATE-2009 (ME)]**

Q.12 The value of integral $\int_c \frac{\cos(2\pi z)}{(2z-1)(z-3)}dz$ (where c is a closed curve given by |z|=1) is

(a) $-\pi i$ (b) $\frac{\pi i}{5}$

(c) $\frac{2\pi i}{5}$ (d) πi **[GATE-2009 (ME)]**

Q.13 Square roots of -i, where $i=\sqrt{-1}$, are

(a) $i,-i$

(b) $\cos\left(-\frac{\pi}{4}\right)+i\sin\left(-\frac{\pi}{4}\right),\cos\left(3\frac{\pi}{4}\right)+i\sin\left(3\frac{\pi}{4}\right)$

(c) $\cos\left(\frac{\pi}{4}\right)+i\sin\left(3\frac{\pi}{4}\right),\ \cos\left(3\frac{\pi}{4}\right)+i\sin\left(\frac{\pi}{4}\right)$

(d) $\cos\left(3\frac{\pi}{4}\right)+i\sin\left(-3\frac{\pi}{4}\right),\ \cos\left(-3\frac{\pi}{4}\right)+i\sin\left(\frac{3\pi}{4}\right)$ **[EE-GATE-2013]**

Q.14 $\oint \frac{z^2-4}{z^2+4}dz$ evaluated anticlockwise around the circle $|z-i|=2,$ where $i=\sqrt{-1}$, is

(a) -4π (b) 0

(c) $2+\pi$ (d) 2+2i **[EE-GATE-2013]**

Q.15 The complex function tanh(s) is analytic over a region of the imaginary axis of the complex s-plane if the following is TRUE everywhere in the region for all integers n ?

(a) Re (s) = 0 (b) Im(s) $\neq n\pi$

(c) Im (s) $\neq \frac{n\pi}{3}$ (d) Im (s) $\neq \frac{(2n+1)\pi}{2}$ **[IN-GATE-2013]**

Q.16 The argument of the complex number $\frac{1+i}{1-i}$ where $i=\sqrt{-1}$ is

(a) $-\pi$ (b) $\frac{-\pi}{2}$

(c) $\frac{\pi}{2}$ (d) π **[ME-GATE-2014(1)]**

Q.17 An analytic function of a complex variable z = x + i y is expressed as $f(z)=u(x,y)+iv(x,y)$, where $i=\sqrt{-1}$. If $u(x,y)=2xy$, then $v(x, y)$ must be

(a) $x^2+y^2+\text{constant}$ (b) $x^2-y^2+\text{constant}$

(c) $-x^2+y^2+\text{constant}$ (d) $-x^2-y^2+\text{constant}$ **[ME-GATE-2014(2)]**

Q.18 An analytic function of a complex variable z = x + iy is expressed as $f(z)=w(x, y)+iv(x,y)$, where $i=\sqrt{-1}$.If $u(x,y)= x^2- y^2$, then expression for $v(x,y)$ in terms of x, y and a general constant c would be

(a) $xy+c$ (b) $\frac{x^2+y^2}{2}+c$

(c) $2xy+c$ (d) $\frac{(x-y)^2}{2}+c$ **[ME-GATE-2014(3)]**

Q.19 If z is a complex variable, the value of $\int_5^{3i}\frac{dz}{z}$ is

(a) $-0.511-1.57i$ (b) $-0.511+1.57i$

(c) $0.511-1.57i$ (d) $0.511+1.57i$ **[ME-GATE-2014(4)]**

Q.20 $z=\frac{2-3i}{-5+i}$ can be expressed as

(a) $-0.5-0.5i$ (b) $-0.5+0.5i$

(c) $0.5-0.5i$ (d) $0.5+0.5i$

[CE-GATE-2014(2)]

Q.21 The real part of an analytic function f(z) where z =x + jy is given by $e^{-y}\cos(x)$. The imaginary part of f(z) is

(a) $e^{y}\cos(x)$ (b) $e^{-y}\sin(x)$

(c) $-e^{y}\sin(x)$ (d) $-e^{-y}\sin(x)$ **[EC-GATE-2014(2)]**

Q.22 Let S be the set of points in the complex plane corresponding to the unit circle. (That is, S={z:|z|=1} Consider the function f(z) = zz* where z* denotes the complex conjugate of z. The f (z) maps S to which one of the following in the complex plane

(a) Unit circle

(b) Horizontal axis line segment from origin to (1, 0)

(c) The point (1, 0)

(d) The entire horizontal axis **[EE-GATE-2014(1)]**

Q.23 All the values of the multi-valued complex function 1^i where i = $\sqrt{-1}$, are

(a) purely imaginary

(b) on the unit circle.

(c) real and non-negative

(d) equal in real and imaginary parts **[EE-GATE-2014(2)]**

Q.24 Given two complex numbers $z_1 = 5 + \left(5\sqrt{3}\right)i$ and $z_2 = \frac{2}{\sqrt{3}} + 2i,$ the argument of $\frac{z_1}{z_2}$ in degree is

(a) 0 (b) 30

(c) 60 (d) 90 **[ME-GATE-2015(1)]**

Q.25 Consider the following complex function: $f(z) = \frac{9}{(z-1)(z+2)^2}$ Which of the following one of the residues of the above function?

(a) -1 (b) 9/16

(c) 2 (d) 9 **[CE-GATE-2015(1)]**

Q.26 Given $i = \sqrt{-1}$ the value of the definite integral, $I = \int_0^{\frac{\pi}{2}} \frac{\cos x + i\sin x}{\cos x - i\sin x} dx$ is :

(a) 1 (b) -1

(c) i (d) $-i$ **[CE-GATE-2015(2)]**

Q.27 Let $f(z) = \frac{az+b}{cz+d}$. If $f(z_1) = f(z_2)$ for all $z_1 \neq z_2, a = 2, b = 4$ and c = 5, then d should be equal to ______. **[EC-GATE-2015(2)]**

Q.28 If C denotes the counterclockwise unit circle, the value of the contour integral $\frac{1}{2\pi j}\oint_e \mathrm{Re}(z)dz$ is _________. **[EC-GATE-2015(2)]**

Q.29 Let z = x + iy be a complex variable. Consider that contour integration is performed along the unit circle in anticlockwise direction. Which one of the following statements is NOT TRUE?

(a) The residue of $\frac{z}{z^2-1}$ at z =1 is 1/2

(b) $\oint_c z^2 dz = 0$

(c) $\frac{1}{2\pi i}\oint_c \frac{1}{z} dz = 1$

(d) $\bar{z}$ (complex conjugate of z) is an analytical function **[EC-GATE-2015(1)]**

Q.30 If C is a circle of radius r with centre Z_0, in the complex z-plane and if n is a non-zero integer, then $\oint \frac{dz}{(z-z_0)^{n+1}}$ equals

(a) $2\pi nj$ (b) 0

(c) $\frac{nj}{2\pi}$ (d) $2\pi n$ **[EC-GATE-2015(3)]**

Q.31 Given f (z) = g(z) + h(z), where f, g, h are complex valued functions of a complex variable z. Which one of the following statements is TRUE?

(a) If f(z) is differential at z_0, then g(z) and h(z) are also differentiable at z_0.

(b) If g(z) and h(z) are differentiable at z_0, then f(z) is also differentiable at z_0.

(c) If f(z) is continuous at z_0, then it is differentiable at z_0.

(d) If f(z) is differentiable at z_0, then so are its real and imaginary parts.

[EE-GATE-2015(2)]

Q.32 The value of $\oint \frac{1}{z^2} dz$, where the contour is the unit circle traversed clockwise, is

(a) $-2\pi i$ (b) 0

(c) $2\pi i$ (d) $4\pi i$ **[IN-GATE-2015]**

Q.33 $f(z) = u(x,y) + iv(x,y)$ is an analytic function of complex variable $z = x + iy$ where $i = \sqrt{-1}$ If $u(x,y) = 2xy$, then v(x,y) may be expressed as

(a) $-x^2 + y^2 + \text{constant}$ (b) $x^2 - y^2 + \text{constant}$

(c) $x^2 + y^2 + \text{constant}$ (d) $-(x^2 + y^2) + \text{constant}$ **[ME-GATE -2016(1)]**

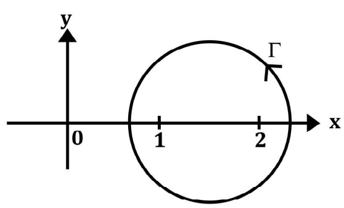
Q.34 The value of the integral $\int_{-\infty}^{\infty} \frac{\sin x}{x^2+2x+2}dx$ evaluated using contour integration and the residue theorem is

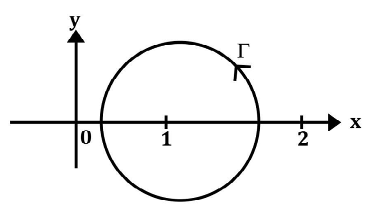
(a) $\frac{-\pi \sin(1)}{e}$ (b) $\frac{-\pi \cos(1)}{e}$

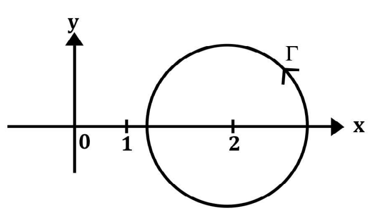
(c) $\frac{\sin(1)}{e}$ (d) $\frac{\cos(1)}{e}$ **[ME-GATE -2016(1)]**

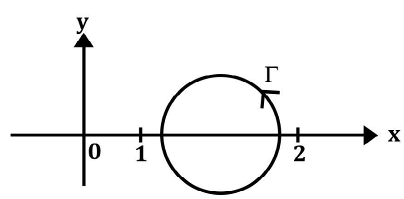
Q.35 A function f of the complex variable z = x + iy, is given as f(x, y) = u (x, y) + iv(x, y), where u(x,y)=2kxy and $v(x,y) = x^2 - y^2$. The value of k, for which the function is analytic, is ____________. **[ME-GATE -2016(2)]**

Q.36 The value of $\oint_{\Gamma} \frac{3z-5}{(z-1)(z-2)}dz$ along a closed path Γ is equal to $(4\pi i)$, where z = x + iy and $i = \sqrt{-1}$. The correct path Γ is

(a) 

(b) 

(c) 

(d) 

[ME-GATE -2016(2)]

Q.37 In the following integral , the contour C encloses the points $2\pi j$ and $-2\pi j$. The value of the integral $-\frac{1}{2\pi}\oint_c \frac{\sin z}{(z-2\pi j)^3}dz$ is _________ **[EC-GATE-2016(1)]**

Q.38 Consider the complex valued function $f(z) = 2z^3 + b|z|^3$ where z is a complex variable. The value of b for which the function f(z) is analytic is _________. **[EC-GATE-2016(2)]**

Q.39 For $f(z) = \frac{\sin(z)}{z^2}$, the residue of the pole at z = 0 is ___________. **[EC-GATE-2016(3)]**

Q.40 The values of the integral $\frac{1}{2\pi j}\oint_c \frac{e^z}{z-2}dz$ along a closed contour c in anti-clockwise direction for

(i) The point z_o=2 inside the contour c, and

(ii) The point z_o=2 outside the contour c, Respectively, are

(a) (i) 2.72, (ii) 0 (b) (i) 7.39, (ii) 0

(c) (i) 0, (ii) 2.72 (d) (i) 0, (ii) 7.39 **[EC-GATE-2016(3)]**

Q.41 In the neighborhood of z = 1, the function f(z) has a power series expansion of the form $f(z) = 1 + (1-z) + (1-z)^2 + \ldots\ldots$ Then f(z) is

(a) $\frac{1}{z}$ (b) $\frac{-1}{z-2}$

(c) $\frac{z-1}{z+1}$ (d) $\frac{1}{2z-1}$ **[IN-GATE-2016]**

Q.42 The value of the integral $\frac{1}{2\pi j}\oint_C \frac{z^2+1}{z^2-1}dz$ where z is a complex number and C is a unit circle with center at 1+ 0j in the complex plane is ______. **[IN-GATE-2016]**

Q.43 If $f(z) = (x^2 + ay^2) + ibxy$ is a complex analytic function of z = x + iy, where $i = \sqrt{-1}$,then

(a) $a = -1$ $b = -1$ (b) $a = -1$, $b = 2$

(c) a = 1, b = 2 (d) a = 2, b = 2 **[GATE-2017]**

Q.44 The value of the integral $\oint \frac{z+1}{z^2-4}dz$ in counter clockwise direction around a circle C of radius 1 with center at the point $z = -2$ is

(a) $\frac{\pi i}{2}$ (b) $2\pi i$

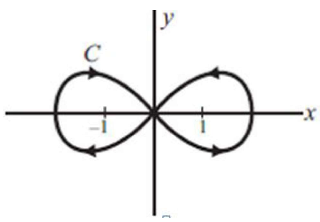
(c) $-\frac{\pi i}{2}$ (d) $-2\pi i$ **[EE-GATE-2018]**

Q.45 If C is a circle $|z| = 4$ and $f(z) = \frac{z^2}{(z^2-3z+2)^2}$ then $\oint_c f(z)dz$ is

(a) 1 (b) 0

(c) –1 (d) –2 **[EE-GATE-2018]**

Q.46 The contour *C* given below is on the complex plane $z = x + jy$, where $j = \sqrt{-1}$.



The value of the integral $\frac{1}{\pi j}\oint_C \left(\frac{1}{z^2-1}\right)dz$ is__________. **[EC-GATE-2018]**

Q.47 Let $f_1(z) = z^2$ and $f_2(z) = \bar{z}$ be two complex variable functions. Here $\bar{z}$ is the complex conjugate of z. Choose the correct answer.

(a) Both f_1 (z) and f_2 (z) are analytic

(b) Only f_1 (z) is analytic

(c) Only f_2 (z) is analytic

(d) Both f_1 (z) and f_2 (z) are not analytic **[IN-GATE-2018]**

Q.48 F(z) is a function of the complex variable z = x+iy given by F(z) = iz + k Re(z)+I Im(z). For what value of k will F(z) satisfy the Cauchy-Riemann equations?

(a) 0 (b) 1

(c) −1 (d) y **[ME-GATE-2018(1)]**

Q.49 Let z be a complex variable. For a counter-clockwise integration around a unit circle C, centred at origin, $\oint_c \frac{1}{5z-4}dz = A\pi i$ the value of A is

(a) 2/5 (b) 1/2

(c) 2 (d) 4/5 **[ME-GATE-2018(2)]**

ANSWER KEY:

1	2	3	4	5	6	7	8	9	10	11	12	13	14
(d)	(a)	(a)	(d)	(c)	(b)	(a)	(c)	(b)	(d)	(a)	(c)	(b)	(a)
15	**16**	**17**	**18**	**19**	**20**	**21**	**22**	**23**	**24**	**25**	**26**	**27**	**28**
(d)	(c)	(c)	(c)	(b)	(b)	(b)	(c)	(c)	(a)	(a)	(b)	10	0.5
29	**30**	**31**	**32**	**33**	**34**	**35**	**36**	**37**	**38**	**39**	**40**	**41**	**42**
(d)	(b)	(b)	(b)	(a)	(a)	-1	(b)	133.87	0	1	(b)	(a)	1
43	**44**	**45**	**46**	**47**	**48**	**49**							
(b)	(a)	(b)	2	(b)	(b)	(a)							

ESE QUESTIONS

Q.1 What is the residue of the function $\frac{1-e^{2z}}{z^4}$ at its pole?

(a) $\frac{4}{3}$ (b) $-\frac{4}{3}$

(c) $-\frac{2}{3}$ (d) $\frac{2}{3}$ **[ESE-2018]**

Q.2 In the Laurent expansion of $f(z)=\frac{1}{(z-1)(z-2)}$ valid in the region $1<|z|<2$ the coefficient of $\frac{1}{z^2}$ is

(a) 0 (b) $\frac{1}{2}$

(c) 1 (d) -1 **[ESE-2018]**

Q.3 Evaluate $\oint_c \frac{1}{(z-1)^3(z-3)}dz$ where c is the rectangular region defined by x = 0, x = 4, y = 1, y = –1

(a) 1 (b) 0

(c) $\frac{\pi}{2}i$ (d) $\pi(3+2i)$ **[ESE(EE)-2018]**

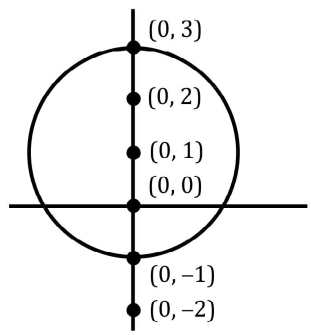
ANSWER KEY

1. (b)
2. (d)
3. (b)

GATE SOLUTIONS

Q.1 **(d)** $\dfrac{1}{z^2+4}=\dfrac{1}{(z+2i)(z-2i)}$

Pole z = 2i lies inside the circle |z-i| = 2, While pole $z=-2i$ is outside the circle $|z-i|=2$ as shown in the figure below:



$\int_c f(z)\,dz = 2\pi i$ [Residue at those poles which are inside C] $=2\pi i \text{ Res } f(2i)=\dfrac{\pi}{2}$

Q.2 **(a)** $I=\oint\dfrac{1}{(s^2-1)}ds=\oint\dfrac{1}{(s+1)(s-1)}ds=2\pi j(\text{sum of Residues})$

Pole $s=-1$ is not inside the contour D, but s = 1 is inside D. residue at pole s = 1 is,

$$\text{residue}=\lim_{s\to1}\frac{(s-1)}{(s-1)(s+1)}=\frac{1}{2},\quad \oint\frac{1}{(s^2-1)}ds=2\pi j\times\frac{1}{2}=\pi j$$

Q.3 **(a)** Since $\lim_{z\to2}\left[(z-2)^2 f(z)\right]$ is finite and non-zero, f(z) has a pole of order two at z= 2

The residue at z = a is given for a pole of order n as

$$\text{Res } f(a)=\frac{1}{(n-1)!}\left\{\frac{d^{n-1}}{dz^{n-1}}\left[(z-a)^n f(z)\right]\right\}$$ Here n=2 (pole of order 2) and a=2

$$\therefore \text{Res } f(2)=\left\{\frac{d}{dz}\left[\frac{1}{(z+2)^2}\right]\right\}_{z=2}=\left[-2(z+2)^{-3}\right]_{z=2}=\frac{-2}{(2+2)^3}=-\frac{1}{32}$$

Q.4 (d) $f(z) = c_0 + c_1 z^{-1}$, $\oint \frac{1+f(z)}{z} dz = \int_{|z|=1} \frac{z + c_0 z + c_1}{z^2}\, dz$

There is a pole of degree 2 at origin, which is inside unit circle

So, [Residue at z=0]=$1 + c_0$

$$\oint \frac{1+f(z)}{z} dz = 2\pi j(1+c_0)$$

Q.5 **(c)** $x(z) = \frac{1-2z}{z(z-1)(z-2)}$

Poles are z = 0, z = 1 and z = 2

$$\text{Residue at } z = 0 = \text{value of } \frac{1-2z}{(z-1)(z-2)} \text{ at } z = 0 = \frac{1-2\times 0}{(0-1)(0-2)} = \frac{1}{2}$$

$$\text{Residue at } z = 1 = \text{ value of } \frac{1-2z}{z(z-2)} \text{ at } z = 1 = \frac{1-2\times 1}{1(1-2)} = 1$$

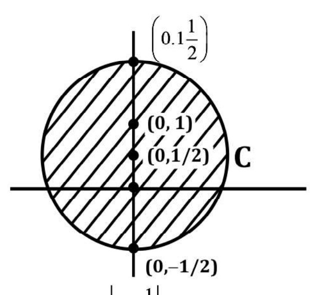
$$\text{Residue at } z = 2 = \text{ value of } \frac{1-2z}{z(z-1)} \text{ at } z = 2 = \frac{1-2\times 2}{2(2-1)} = -\frac{3}{2}$$

hence residues are $\frac{1}{2}, 1 \text{ and } -\frac{3}{2}$

Q.6 **(b)** $\frac{1}{z^2+1} = \frac{1}{(z-i)(z+i)}$

Poles at i and –i, i.e. (0,1) and (0,-1)

From figure of $\left|z - \frac{i}{2}\right| = 1$ below we see that pole(0,1) i.e. i is inside C while pole (0,-1) i.e. –i is outside C



$\left|z - \frac{1}{2}\right| = 1$

$$C : \left|z - \frac{i}{2}\right| = 1,\ I = 2\pi i \text{ Res } f(i) = 2\pi i \frac{1}{(i+i)} = \pi$$

Q.7 **(a)** $f(z)=\dfrac{\cos z}{z}$ has simple pole at z = 0 inside unit circle on complex plane

Residue of f(z) at $z=0=\lim\limits_{z\to 0} z\,f(z)=\lim\limits_{z\to 0}\cos z=1$

$\int_c f(z)\,dz = 2\pi i$ (Residue at $z = 0$) $= 2\pi i$

Q.8 **(c)** Only Option C satisfies Cauchy's -Riemann equation.

Q.9 **(b)** $\left|\dfrac{3+4i}{1-2i}\right|=\dfrac{5}{\sqrt{5}}=\sqrt{5}$

Q.10 **(d)** $f(z)=\dfrac{z-1}{z^2+1}=\dfrac{z-1}{z^2-i^2}=\dfrac{z-1}{(z-i)(z+i)}$. The singularities are at z = i and –i

Q.11 **(a)** $\oint_c \dfrac{z^3-6}{3z-i}dz=\dfrac{1}{3}\oint_c \dfrac{z^3-6}{\left(z-\dfrac{i}{3}\right)}dz$

Applying Cauchy's integral theorem, using $f(z)=z^3-6$

$$=\frac{1}{3}\left(2\pi i f\left(\frac{i}{3}\right)\right)=\frac{1}{3}\left(2\pi i\left[\left(\frac{i}{3}\right)^3-6\right]\right)=\frac{1}{3}\left[2\pi i\left(\frac{i^3}{27}-6\right)\right]=\frac{2\pi}{81}i^4-4\pi i=\frac{2\pi}{81}-4\pi i$$

Q.12 **(c)** Since, $z=\dfrac{1}{2}$ is a point within |z|=1, z = 3 is outside |z|=1.

$$I=\int_c \frac{\cos(2\pi z)}{(2z-1)(z-3)}dz=\frac{1}{2}\int_c \frac{\left[\dfrac{\cos(2\pi z)}{(z-3)}\right]}{\left[z-\dfrac{1}{2}\right]}dz=\frac{1}{2}\frac{\cos(2\pi z)}{(z-3)}\bigg|_{z=\frac{1}{2}}=\frac{2\pi i}{5}$$

Q.13 **(b)**

Q.14 **(a)**

Q.15 **(d)** $\tanh(s)=\dfrac{e^s-e^{-s}}{e^s+e^{-s}}$, $e^s+e^{-s}\neq 0$

$$\Rightarrow e^{2s}\neq -1 \Rightarrow s\neq i\frac{(2n+1)\pi}{2},\quad \therefore\quad I_m(s)\neq\frac{(2n+1)}{2}\pi$$

Q.16 **(c)** Given $z = \frac{1+i}{1-i} \Rightarrow z = \frac{(1+i)(1+i)}{(1-i)(1-i)} = \frac{1+2i+i^2}{1+1} = \frac{1+2i-1}{2} = i$

$$\text{Arg}(z) = \text{Arg}(i) = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\infty = \frac{\pi}{2}$$

Q.17 **(c)** Only Option C satisfies CR-Equation.

Q.18 **(c)** Only Option C satisfies CR-Equation.

Q.19 **(b)** $\int_5^{3i} \frac{dz}{z} = \ln(z)\Big|_5^{3i} = \ln 3i - \ln 5 = (\ln 3 - \ln 5) + i\frac{\pi}{2} = \left(\ln\frac{3}{5}\right) + i\frac{\pi}{2} = -0.511 + 1.57i$

Q.20 **(b)** $z = \frac{2-3i}{-5+i} = \frac{(2-3i)(-5-i)}{(-5+i)(-5-i)} = \frac{(-10-3)-(2i-15i)}{5^2-i^2} = \frac{-13+13i}{26} = -\frac{1}{2} + \frac{i}{2}$

Q.21 **(b)** Only Option B satisfies CR-Equation.

Q.22 **(c)** f (Z)= ZZ* = $|Z|^2$ = 1 = 1 + i0, where Z* is conjugate of Z

hence , f (Z) maps S to the point (1,0) in the complex plane

Q.23 **(c)** $1 = \cos(2k\pi) + i\sin(2k\pi) = e^{i(2k\pi)},\ k \in I$

$1^i = \left(e^{i(2k\pi)}\right)^i = e^{-(2k\pi)}$, All values are real & non negative

Q.24 **(a)** $z_1 = 5 + \left(5\sqrt{3}\right)i,\ \arg z_1 = \tan^{-1}\left(\frac{5\sqrt{3}}{5}\right) = \tan\sqrt{3} = 60^\circ$

$$z_2 = \frac{2}{\sqrt{3}} + 2i,\ \arg z_2 = \tan^{-1}\left(\frac{2}{\frac{2}{\sqrt{3}}}\right) = \tan\left(\sqrt{3}\right) = 60^\circ$$

$$\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2 = 60^\circ - 60^\circ = 0$$

Q.25 **(a)** $f(z) = \frac{9}{(z-1)(z+2)^2}$, poles at $z = 1$ & $z = -2$

$$\left[\text{Res}\,f(z)\right]_{z=1} = \lim_{z\to1}(z-1)\frac{9}{(z-1)(z+2)^2} = \frac{9}{9} = 1$$

$$\left[\text{Res}\,f(z)\right]_{z=-2} = \frac{1}{1!}\lim_{z\to-2}\frac{d^2}{dz^2}\left[(z+2)^2\frac{9}{(z-1)(z+2)^2}\right] = \lim_{z\to-2}\frac{-9}{(z-1)^2} = -1$$

Q.26 **(b)** $I = \int_0^{\frac{\pi}{2}} \frac{\cos x + i\sin x}{\cos x - i\sin x}dx = \int_0^{\pi/2} \frac{e^{ix}}{e^{-ix}}dx = \int_0^{\pi/2} e^{2ix}\,dx = \frac{1}{2}\left[e^{\pi i} - e^{0}\right] = -1$

Q.27 **(10)** $f(z) = \frac{az+b}{cz+d}$, If $f(z_1) = f(z_2)$ for $z_1 \neq z_2$, a =2, b=4, c=5

$$f(z_1) = f(z_2) \Rightarrow \frac{2z_1+4}{5z_1+d} = \frac{2z_2+4}{5z_2+d}$$

$$\Rightarrow 10z_1z_2 + 20z_2 + 2dz_1 + 4d = 10z_1z_2 + 20z_1 + 2dz_2 + 4d$$
$$\Rightarrow 2d(z_2 - z_1) = 20(z_2 - z_1) \Rightarrow d = 10$$

Q.28 **(0.5)**

Q.29 **(d)** $\frac{\partial \bar{z}}{\partial \bar{z}} = 1 \neq 0,\ f(z) = \bar{z}$ is analytic.

Q.30 **(b)** $\oint_c \frac{f(z)}{(z-z_0)^{n+1}}dz = \frac{2\pi i f^n(z_0)}{n!}$

$$\oint_c \frac{dz}{(z-z_0)^{n+1}} = \frac{2\pi i}{n!} \times 0 = 0 \qquad \left(\begin{array}{c} \because f(z) = 1 \\ f^n(z) = 0 \text{ at any } z_0, n \geq 1 \end{array}\right)$$

Q.31 **(b)**

Q.32 **(b)** same as Q.30

Q.33 **(a)** Only Option (a) satisfies CR-Equations.

Q.34 **(a)** We know that sin x is the imaginary part of e^{ix}

consider the function $f(z) = \frac{e^{iz}}{z^2+2z+2}$

Now, the poles of f(z) are given by $z^2 + 2z + 2 = 0,\ z = -1 \pm i$

But $Z = -1+i$ is the only pole (simple) lie in the upper half of the Z-plane.

$$\therefore \operatorname{Res} f(z = -1+i) = \lim_{z \to -1+i}\left[Z - (-1+i)\right]\frac{e^{iz}}{[z-(-1+i)][z-(-1-i)]} = \frac{e^{-i}}{2ie}$$

Thus $\int_c \frac{e^{iz}}{z^2+2z+2}dz = 2\pi i\left[\frac{e^{-i}}{2ie}\right] = \frac{\pi e^{-i}}{e}$

Equating imaginary parts on both sides, we get

$$\int_c \frac{\sin x}{x^2+2x+2}dx = \text{Im}\left(\int_c \frac{\sin z}{z^2+2z+2}dz\right) = \text{Im}\left(\frac{\pi e^{-i}}{e}\right)$$

$$= \text{Im}\left(\frac{\pi}{e}\left(\cos(1) - i\sin(1)\right)\right) = -\frac{\pi\sin(1)}{e}$$

Q.35 (-1) Using Cauchy's Riemann eq^n.

Q.36 **(b)** Since if $z = 1$ lies inside the closed path Γ and $z = 2$ lies outside of the closed path Γ then by cauchy's formula.

$$\oint_\Gamma \frac{3z-5}{(z-1)(z-2)}dz = \oint_\Gamma \frac{\left(\frac{3z-5}{z-2}\right)}{z-1}dz = 2\pi i\left[\frac{3z-5}{z-2}\right]_{\text{at } z=1} = 2\pi i\left[\frac{-2}{-1}\right] = 4\pi i$$

Q.37 **(133.87)** (by Cauchy integral formula)

$$-\frac{1}{2\pi}\int_c \frac{\sin z}{(z-2\pi j)^3}dz = -\frac{1}{2\pi}2\pi j\frac{f''(2\pi j)}{2!} = -\frac{1}{2\pi}2\pi j\left(\frac{-\sin 2\pi j}{2}\right) = -\frac{1}{2}\sin h2\pi = -133.87$$

Q.38 **(0)** $f(z) = 2z^3 + b|z|^3 = 2z^3 + b(z\bar{z})^{\frac{3}{2}},\ \frac{\partial f}{\partial \bar{z}} = \frac{3}{2}b(z)^{\frac{3}{2}}\left(\bar{z}\right)^{\frac{1}{2}} = 0 \Rightarrow b = 0$

Q.39 **(1)** $f(z) = \frac{\sin(z)}{z^2} = \frac{1}{z^2}\left[z - \frac{z^3}{3!} + \frac{z^5}{5!} - \ldots\right] = \frac{1}{z} - \frac{z}{3!} + \frac{z^2}{5!} - \ldots$

Residue = coefficient of $\frac{1}{z} = 1$

Q.40 **(b)** $(\text{i})\ \frac{1}{2\pi j}\oint \frac{e^2}{z-2} = \frac{1}{2\pi j}\left[2\pi j f(2)\right] = e^2 \cong 7.39, (\text{ii})\ 0$

Q.41 **(a)** $1 + (1-z) + (1-z)^3 + ---- = \frac{1}{1-(1-z)} = \frac{1}{z}$

Q.42 **(1)** $\frac{1}{2\pi j}\oint_C \frac{z^2+1}{z^2-1}dz = \frac{1}{2\pi j}\oint_C \frac{z^2+1}{(z-1)(z+1)}dz$

Poles are at $z = 1,\ -1$

Given C is $(x-1)^2 + y^2 = 1$

Clearly 1 lies inside of C and -1 outside of C

$$\left[\text{Res } f(z)\right]_{z=1} = \lim_{z\to 1}(z-1)\frac{z^2+1}{(z-1)(z+1)} = 1$$

By Cauchy s Residue theorem

$$\frac{1}{2\pi j}\oint_C \frac{z^2+1}{z^2-1}dz = \frac{1}{2\pi j}\times 2\pi j\times 1 = 1$$

Q.43 **(b)** Using CR - Equations $a=-1, b=2$

Q.44 **(a)** $\oint_c \frac{z+1}{z^2-4}dz = 2\pi i \,\text{Res}\,[f(z)]_{z=-2} = 2\pi i \lim_{z\to -2}(z+2)\left[\frac{z+1}{(z-2)(z+2)}\right] = \frac{2\pi i}{4} = \frac{\pi i}{2}$

Q.45 **(b)** $I = \oint_c \frac{z^2}{(z-1)^2(z-2)^2}dz$; $C:|z|=4$

Finding roots of denominator,

$$(z-1)^2(z-2)^2 = 0$$
$$z = 1,1,2,2$$

[Multiple poles]

Cauchy integral formula for multiple poles

$$\oint_c \frac{z^2}{(z-1)^2(z-2)^2}dz = \frac{2\pi i}{(2-1)!}\left[\frac{d}{dz}\frac{z^2}{(z-1)^2} + \frac{d}{dz}\frac{z^2}{(z-2)^2}\right]$$

$$= \frac{2\pi i}{(2-1)!}\left[\frac{(z-1)^2 2z - z^2 2(z-1)}{(z-1)^4} + \frac{(z-2)^2 2z - z^2 2(z-2)}{(z-2)^4}\right]$$

$$= 2\pi i\left(\frac{4-8}{1} + \frac{2+2}{1}\right) = 2\pi i(-4+4) = 0$$

Q.46 **(2)**

Q.47 **(b)** $\frac{\partial z^2}{\partial \bar{z}} = 0, f_1(z) = z^2$ is analytic, $\frac{\partial \bar{z}}{\partial \bar{z}} = 1 \neq 0, f_2(z) = \bar{z}$ is not analytic.

Q.48 **(b)** $F(z) = i\,z + k\,\text{Re}(z) + i\,\text{Im}(z)$

$$F(z) = i\,z + k\left(\frac{z+\bar{z}}{2}\right) + i\left(\frac{z-\bar{z}}{2i}\right)$$

$$\frac{\partial F}{\partial \bar{z}} = 0 + \frac{k}{2} - \frac{1}{2} = 0 \Rightarrow k = 1$$

Q.49 **(a)**

ESE SOLUTIONS

Q.1 **(b)** $f(z)=\frac{1}{z^4}\left[1-\left(1+2z+\frac{(2z)^2}{2!}+\frac{(2z)^3}{3!}+....\right)\right]=\frac{-2}{z^3}-\frac{4}{2z^2}-\frac{8}{6z}-\frac{16}{24}-....$

Residue of f(z) at z = 0 = coefficient of $\frac{1}{z}=\frac{-8}{6}=\frac{-4}{3}$

Q.2 **(d)** $f(z)=\frac{1}{(z-1)(z-2)}=\frac{-1}{(z-1)}+\frac{1}{(z-2)}$

for the region $(1<|z|<2)$

$$f(z)=\frac{-1}{z\left(1-\frac{1}{z}\right)}+\frac{1}{2\left(1-\frac{z}{2}\right)}=\frac{-1}{z}\left(1-\frac{1}{z}\right)^{-1}-\frac{1}{2}\left(1-\frac{z}{2}\right)^{-1}$$

$$=\frac{-1}{z}\left[1+\frac{1}{z}+\frac{1}{z^2}+....\right]-\frac{1}{2}\left[1+\frac{z}{2}+\frac{z^2}{2^2}+...\right]$$

$$=\left[\frac{-1}{z}-\frac{1}{z^2}-\frac{1}{z^3}....\right]-\left[\frac{1}{2}+\frac{z}{2^2}+\frac{z^2}{2^3}+...\right]$$

Coefficient of $\frac{1}{z^2}=-1$

Q.3 **(b)** Both the poles z=1 and z=3 lies within given contour.

$$[\text{Residue of f(z)}]_{z=1}=\frac{1}{2!}\lim_{z\to1}\frac{d^2}{dz^2}\left[(z-1)^3 f(z)\right]=\frac{1}{2}\lim_{z\to1}\frac{d^2}{dz^2}\left[\frac{1}{(z-3)}\right]=\frac{1}{2}\lim_{z\to1}\frac{2}{(z-3)^3}=\frac{-1}{8}$$

$$[\text{Residue of f(z)}]_{z=3}=\lim_{z\to1}(z-3)f(z)=\lim_{z\to1}\frac{1}{(z-1)^3}=\frac{1}{8}$$

$$\oint_c \frac{1}{(z-1)^3(z-3)}dz=2\pi i\left(-\frac{1}{8}+\frac{1}{8}\right)=0$$

7 NUMERICAL METHODS

7.1 INTRODUCTION

Mathematical methods used to solve equations or evaluate integrals or solve differential equations can be classified broadly into two types.

1. Analytical Methods
2. Numerical Methods

Analytical methods can analyze the equation to obtain a solution directly as readymade formulae in terms of say, the coefficient present in the equations.

The limitations of analytical methods have led the engineering and scientists to evolve graphical & numerical methods. The graphical methods, though simple, give results to a low degree of accuracy. Numerical methods can, however, be derived which are more accurate. With the advent of high speed digital computers and increasing demand for numerical answers to various problems, numerical techniques have become indispensible tool in the hands of engineers.

Numerical methods are often, of a repetitive nature. These consist in repeated execution of the same process where at each step the result of the preceding step is used. This is known as iteration process and is repeated till the result us obtained to a desired degree of accuracy.

The advantage of numerical method is that usually these procedures work on a much wider range of problems as compared to analytical solutions which work only on a limited class of problems. Whereas numerical methods can be used to solve polynomial equation of any degree. Also numerical methods can be used on linear as well as nonlinear equations, whereas analytical solutions usually fail for nonlinear equations.

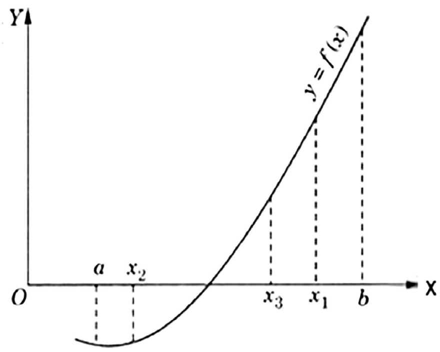
7.2 SOLUTION OF ALGEBRAIC & TRANSCENDENTAL EQUATIONS

To find the root of an equation f(x) = 0, we start with the known approximate solution and apply any of the following methods:

7.2.1 BISECTION METHOD

This method consists in locating the root of the equation f(x) =0 between a and b. If f(x) is continuous between a and b and f(a) and f(b) are of opposite signs then there is a root between a and b for definiteness. Let f(a) be negative & f(b) be positive. Then the first approximation to the root is $x_0=\frac{1}{2}(a+b)$.

If $f(x_0)=0$, then x_0 is a root of f(x)=0. Otherwise, the root lies between a and x_0 or x_0 and b according as $f(x_0)$ is positive or negative. Then we bisect the intervals as before and continue the process until the root is found to desired accuracy. Then we designate the new interval as $[a_1,b_1]$ whose length is $|b-a|/2$.



As before this is bisected at X_1 and the new interval will be exactly half the length of the previous one. We repeat this process until the latest interval is as small as desired say ε. It is clear that the interval width is reduced by a factor of one-half at each step and at the end of the n^{th} step, the new interval will be $[a_n, b_n]$ of length $|b-a|/2^n$.

Example : Find a root of the equation $x^3-4x-9=0$ in (2,3) , using the bisection method correct to two decimal places.

Solution : $f(x) = x^3 - 4x - 9 = 0$, $f(2)<0$, $f(3)>0$

First approximate to the root is $x_1 = \frac{1}{2}(2 + 3) = 2.5$, $f(x_1) < 0$

Thus second approximation to the root is $x_2 = \frac{1}{2}(2.5 + 3) = 2.75$, $f(x_2) > 0$

Thus third approximation to the root is $x_3 = \frac{1}{2}(x_1 + x_2) = 2.62$, $f(x_3) < 0$

Thus fourth approximation to the root is $x_4 = \frac{1}{2}(x_2 + x_3) = 2.68$

Repeating this process, the successive approximations are

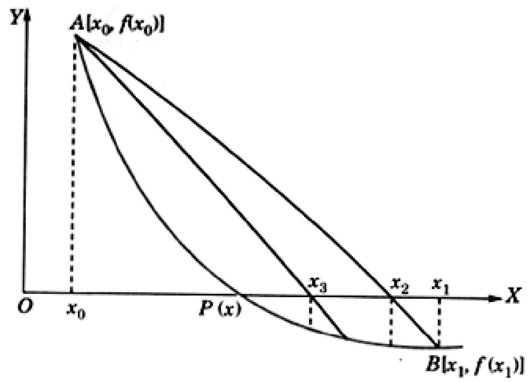
$x_5 = 2.71$, $x_6 = 2.70$, $x_7 = 2.70$, Hence the root is 2.70

Example : Find a real root of the equation $f(x) = X^3 - X - 1 = 0$.

Solution : Since f(1) is negative and f(2) is positive, a root lies between 1 and 2 and therefore we take $x_0 = 3/2$. Then $f(x_0) = \frac{27}{8} - \frac{3}{2} = \frac{15}{8}$ which is positive. Hence the root lies between 1 and 1.5 and we obtain $x_1 = \frac{(1+1.5)}{2} = 1.25$ we find $f(x_1) = -\frac{19}{64}$, which is negative. We therefore, conclude that the root lies between 1.25 and 1.5. It follows that $x_2 = \frac{(1.25+1.5)}{2} = 1.375$. The procedure is repeated and the successive approximations are $x_3=1.3125$, $x_4=1.34375$, $x_5 = 1.328125$; etc.

7.2.2 FALSE POSITION or REGULA FALSI METHOD

This is the oldest method of finding the real root of an equation f(x) = 0 and closely resembles the bisection Method. Here we choose two points x_0 and x_1 such that $f(x_0)$ and $f(x_1)$ are of opposite signs i.e., the graph of y = f(x) crosses the x-axis between these points (Fig.). This indicates that a root lies between x_0 and x_1 consequently $f(x_0)\, f(x_1) < 0$.



Equation of the chord joining the points $A[x_0, f(x_0)]$ and $B[x_1, f(x_1)]$ is

$$y - f(x_0) = \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0) \qquad \text{...(1)}$$

The method consists in replacing the curve AB by means of the chord AB and taking the point of intersection of the chord with the x- axis as an approximation to the root. So the abscissa of the point where the chord cuts the x- axis (y = 0) is given by

$$x_2 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0)$$

which is an approximation of the root.

Example : Find the real root of equation $x^3 - 2x - 5 = 0$ by the method of false position correct to three decimal places.

Solution : Let $f(x) = x^3 - 2x - 5 = 0$

So that

$f(2) = -1$ and $f(3) = 16$

i.e., A root lies between 2 and 3

Taking $x_0 = 2$, $x_1 = 3$,

$f(x_0) = -1$, $f(x_1) = 16$, in the method of false position,

We get $$x_2 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0)$$

$= 2 + 1/17 = 2.0588$ ... (i)

Now $f(x_1) = f(2.0588) = -0.3908$

i.e., the root lies between 2.0588 and 3.

Taking $x_0 = 2.0588, x_1 = 3, f(x_1) = 16$, in (i), we get

$$x_3 = 2.0588 - \frac{0.9412}{16.3908}(-0.3908) = 2.0813$$

Repeating this process, the successive approximations are

$x_4 = 2.0862$, $x_5 = 2.0915$, $x_6 = 2.0934$, $x_7 = 2.0941$, $x_8 = 2.0943$, etc.

Hence the root is 2.094 correct to three decimal places.

7.2.3 NEWTON RAPHSON METHOD

This method is generally used to improve the result obtained by one of the previous method. Let x_0 be an approximate root of the equation f(x) = 0. Then the first iteration using Newton- Raphson Method will be,

(For Single Variable)

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

And the $(n+1)$th iteration will be

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Which is known as Newton-Raphson formula of Newton's iteration formula.

This method is commonly used for its simplicity and greater speed. Here f(x) is assumed to have continuous derivative f'(x). This method fails when f'(x) = 0. It has second order of convergence or quadratic convergence, i.e. the subsequent error at each step is proportional to the square of the error at previous step.

(For Multiple Variables)

$$\begin{cases} f_1\left(x_1 \ldots\ldots, x_n\right) = f_1\left(x\right) = 0 \\ \bullet \\ \bullet \\ \bullet \\ f_n\left(x_1 \ldots\ldots, x_n\right) = f_n\left(x\right) = 0 \end{cases}$$

where $X = \left[x_1 \ldots\ldots x_n\right]^T$ is an n-dimensional vector. This equation system can be more concisely represented in vector form as f(x)=0. The Newton-Raphson formula for multi-variate problem is

$$\left[X_{n+1}\right] = \left[X_n\right] - J_n^{-1}\left[X_n\right]$$

$$J_n(x) = \begin{bmatrix} \dfrac{\partial f_1}{\partial x_1} \cdots \dfrac{\partial f_1}{\partial x_n} \\ \cdots \quad \cdots \quad \cdots \\ \dfrac{\partial f_n}{\partial x_1} \cdots \dfrac{\partial f_n}{\partial x_n} \end{bmatrix}$$

Example : Find the positive root of $x^4 - x = 10$ correct to three decimal places, using Newton-Raphson method.

Solution : Let $f(x) = x^4 - x - 10 = 0$

So that

$f(1) = -10 = -ve$,

$f(2) = 16 - 2 - 10 = 4 = +ve$

$\therefore$ a root of $f(x) = 0$ lies between 1 and 2. Let us take $x_0 = 2$

Also, $f'(x) = 4x^3 - 1$

Newton-Raphson's formula is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \qquad \ldots(i)$$

Putting $n = 0$, the first approximation x_1 is given by

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 2 - \frac{f(2)}{f'(2)} = 2 - \frac{4}{4\times 2^3 - 1} = 1.871$$

Putting $n = 1$ in (i), the second approximation is

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1.871 - \frac{f(1.871)}{f'(1.871)}$$

$$= 1.871 - \frac{4(1.871)^4 - (1.871) - 10}{4(1.871)^3 - 1} = 1.871 - \frac{0.3835}{25.199} = 1.856$$

Putting $n = 2$ in (i), the third approximation is

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 1.856 - \frac{4(1.856)^4 - (1.856) - 10}{4(1.856)^3 - 1} = 1.856 - \frac{0.010}{24.574} = 1.856$$

Here $x_3 = x_2$. Hence the desired root is 1.856 correct to three decimal places.

Example : Statement for linked answer question

Given a>0, we wish to calculate the reciprocal value $\frac{1}{a}$ by Newton Raphson method.

Example : The Newton Raphson algorithm for the function will be

(a) $x_{k+1}=\frac{1}{2}\left(x_k+\frac{a}{x_k}\right)$ (b) $x_{k+1}=x_k+\frac{a}{2}x_k^2$

(c) $x_{k+1}=2x_k-ax_k^2$ (d) $x_{k+1}=\left(x_k-\frac{a}{2}x_k^2\right)$

Solution : (c) $f(x)=\frac{1}{x}-a=0,\ f'(x)=-\frac{1}{x^2}$

For N – R Method : $x_{k+1}=x_k-\frac{f(x_k)}{f'(x_k)}$

$$x_{k+1}=x_k-\frac{\frac{1}{x_k}-a}{-\frac{1}{x_k^2}}\ ,\quad x_{k+1}=2x_k-ax_k^2$$

Example : For a = 7 and starting with x_0 = 0.2, the first two iterations will be

(a) 0.11, 0.1299 (b) 0.12, 0.1392

(c) 0.12, 0.1416 (d) 0.13, 0.1428

Solution : (b)

For a = 7 the iteration becomes

$x_{k+1}=2x_k-7x_k^2$ With x_0 = 0.2 ;

$x_1=2x_0-7x_0^2=2\times0.2-7(0.2)^2=0.12$ and

$x_2=2x_1-7x_1^2=2\times0.12-7(0.12)^2=0.1392$

7.2.3.1 USEFUL DEDUCTIONS FROM THE NEWTON-RAPHSON FORMULA

(1)Iterative formula to find 1/N is $x_{n+1}=x_n(2-Nx_n)$

(2) Iterative formula to find $\sqrt{N}$ is $x_{n+1}=\frac{1}{2}(x_n+N/x_n)$

(3) Iterative formula to find $1/\sqrt{N}$ is $x_{n+1}=\frac{1}{k}\left[(k-1)x_n+N/x_n^{k-1}\right]$

(4) Iterative formula to find $\sqrt[k]{N}$ is $x_{n+1}=\frac{1}{k}\left[(k-1)x_n+N/x_n^{k-1}\right]$

7.2.4 Secant Method

The Secant method proceeds similarly to Regula-Falsi method in the sense that it also requires two starting guess values, but the difference is that $f(x_0)\,f(x_1)$ need not be negative i.e. at any stage of iteration we do not ensure that the root is between x_0 and x_1. However, Secant method uses the same iteration equation as Regula-Falsi method.

$$x_2 = \frac{f_1 x_0 - f_0 x_1}{f_1 - f_0} \quad , \quad x_{n+1} = \frac{f_n x_{n-1} - f_{n-1} x_n}{f_n - f_{n-1}}$$

In Secant method, once the value of x_2 is obtained, to proceed to the next iteration, x_0 is always replaced by x_1 and x_1 is always replaced by x_2. This is the only and primary difference between Regula-Falsi and Secant method. Geometrically, both Regula-Falsi and Secant methods find x_2 by same way, that is by drawing the chord from (x_0, f_0) to (x_1, f_1) and intersection of this chord with x axis is. The advantage of the Secant method is that it is faster than both the Bisection and Regula-Falsi method as it has a convergence order of 1.62. However, the disadvantage is that, Secant method is not 100% reliable, since the equation

$$x_2 = \frac{f_1 x_0 - f_0 x_1}{f_1 - f_0}$$

Will fail if $f_1 = f_0$, which may happen since no effort is made to keep f_1 and f_0 to be of opposite signs as it is done in case of Regula-Falsi method, which uses the same iteration equation.

7.3 SOLUTION of LINEAR SIMULTANEOUS EQUATIONS

Simultaneous linear equations occur in various engineering problems. The student knowthat a given system of linear equation can be solved by Cramer's rule or by Matrix Method. But these methods become tedious for large systems. However, there exist other numerical methods of solution which are well suited for computing machines. We now explain some direct and iterative methods of solution.

7.3.1 ITERATIVE METHODS OF SOLUTION

7.3.1.1 JACOBI'S ITERATION METHOD

Consider the equations

$a_1x + b_1y + c_1z = d_1$, $a_2x + b_2y + c_2z = d_2$, $a_3x + b_3y + c_3z = d_3$ ...(1)

If a_1, b_2, c_3 are large as compared to other co-efficient, then solving these x, y, z respectively, the system can be written in the form

$x = k_1 - l_1 y - m_1 z$, $y = k_2 - l_2 x - m_2 z$, $z = k_3 - l_3 x - m_3 y$ ...(2)

Let us start with the initial approximations x_0, y_0, z_0 (each = 0) for the values of x, y, z. Substituting these on the right, we get first approximation $x_1 = k_1$, $y_1 = k_2$, $z_1 = k_3$.

Substituting these on the right hand side of (2), the second approximations are given by

$x_2 = k_1 - l_1y_1 - m_1z_1$, $y_2 = k_2 - l_2x_1 - m_2z_1$, $z_2 = k_3 - l_3x_1 - m_3y_1$

This process is repeated till the difference between two consecutive approximations is negligible.

Example : Solve by Jacobi's iteration method, the equation 10x + y – z = 11.19, x + 10y + z = 28.08, –x + y + 10z = 35.61 correct up to two decimal places.

Solution: Rewriting the given equation as

$$x = \frac{1}{10}(11.19 - y + z),\ y = \frac{1}{10}(28.08 - x - z),\ z = \frac{1}{10}(35 - 61 + x - y)$$

We start from an approximation, $x_0 = y_0 = z_0 = 0$

First Iteration

$$x_1 = \frac{11.19}{10} = 1.119,\ y_1 = \frac{28.08}{10} = 2.808,\ z_1 = \frac{35.61}{10} = 3.561.$$

Second Iteration

$$x_2 = \frac{1}{10}(11.19 - y_1 + z_1) = 1.19,\ y_2 = \frac{1}{10}(28.08 - x_1 - z_1) = 2.34,\ z_2 = \frac{1}{10}(35.61 + x_1 - y_1) = 3.39$$

Third Iteration

$$x_3 = \frac{1}{10}(11.19 - y_2 + z_2) = 1.22,\ y_3 = \frac{1}{10}(28.08 - x_2 - z_2) = 2.35,\ z_3 = \frac{1}{10}(35.61 + x_2 - y_2) = 3.45$$

Fourth Iteration

$$x_4 = \frac{1}{10}(11.19 - y_3 + z_3) = 1.23,\ y_4 = \frac{1}{10}(28.08 - x_3 - z_3) = 2.34,\ z_4 = \frac{1}{10}(35.61 + x_3 - y_3) = 3.45$$

Fifth Iteration

$$x_5 = \frac{1}{10}(11.19 - y_4 + z_4) = 1.23,\ y_5 = \frac{1}{10}(28.08 - x_4 - z_4) = 2.34,\ z_5 = \frac{1}{10}(35.61 + x_4 - y_4) = 3.45$$

Hence, x = 1.23, y = 2.34, z = 3.45.

7.3.1.2 GAUSS-SEIDEL ITERATION METHOD

This is a modification of Jacobi's Iteration Method. As before, we start with the initial approximations x_0, y_0, z_0 (each=0) for the values of x, y, z respectively. Substituting y = y_0 and z = z_0 in the first equation of (2) of Jacobi's Iteration, we get $x_1 = k_1$.
Then putting x= x_1, z = z_0 in the second of the equations (2) of Jacobi's Iteration, we obtain $y_1 = k_2 - l_2x_1 - m_2z_0$

Next substituting $x = x_1$, $y = y_1$ in the third of the equations (2) of Jacobi's Iteration, we obtain $z_1 = k_3 - l_3x_1 - m_3y_1$ and so on i.e., as soon as new approximation for an unknown is found, it is immediately used in the next step. This process of iteration is continued till convergence to the desired degree of accuracy is obtained.

7.4 NUMERICAL INTEGRATION

The process of evaluating a definite integral from a set of tabulated values of the integrand f (x) is called as numerical integration. This process when applied to a function of a single variable is known as quadrature.

The problem of numerical integration like that of numerical differentiation, is solved by representing f(x)by an interpolation formula and then integrating them between given limits. In this way, we can derive quadrature formula approximate integration of a function defined by a set of numerical values only.

7.4.1 NEWTON-COTES QUADRATURE FORMULA

$$\int_{x_0}^{x_0+nh} f(x)dx = nh\left[\left(y_0 + \frac{n}{2}\Delta y_0 + \frac{n(2n-3)}{12}\Delta^2 y_0 + \frac{n(n-2)^2}{10}\Delta^3 y_0\right)\right.$$
$$+\left(\frac{n^4}{5} - \frac{3n^3}{2} + \frac{11n^2}{3} - 3n\right)\frac{\Delta^4 y_0}{4!} + \left(\frac{n^5}{6} - 2n^4 + \frac{35n^3}{4} - \frac{50n^2}{3} + 12n\right)\frac{\Delta^5 y_0}{5!}$$
$$\left. + \left(\frac{n^6}{7} - \frac{15n^5}{6} + 17n^4 - \frac{225n^3}{4} + \frac{274n^2}{3} - 60n\right)\frac{^6 y_0}{6!} + \ \ldots\right]$$

7.4.2 TRAPEZOIDAL RULE

Putting n = 1 in the above equation (Newton-Cotes Quadrature Formula) and taking polynomial of the first order so that differences of order higher than first become zero, we get

$$\int_{x_0}^{x_0+h} f(x)dx = h\left(y_0 + \frac{1}{2}\Delta y_0\right) = h\left[y_0 + \frac{1}{2}(y_1 - y_0)\right] = \frac{h}{2}(y_0 + y_1)$$

Similarly, for next interval $[x_1, x_2]$

$$\int_{x_1}^{x_2} f(x)dx = h\left(y_1 + \frac{1}{2}\Delta y_1\right) = \frac{h}{2}(y_1 + y_2)$$

For the last interval $[x_{n-1}, x_n]$

$$\int_{x_{n-1}}^{x_n} f(x)dx = \frac{h}{2}\left(y_{(n-1)} + y_n\right)$$

Adding these n integrals, we obtain

$$\int_{x_0}^{x_n} f(x)dx = \frac{h}{2}\left[(y_0 + y_n) + 2(y_1 + y_2 + \ldots + y_{n-1})\right]$$

This is known as Trapezoidal Rule.

Example : Use the trapezoidal rule to estimate the integral $\int_0^2 xe^{x^2}dx$ taking 10 intervals

Solution : Let $y = xe^{x^2}$, h = 0.2 and n = 10.

The value of x and y are as follows:

X :	0	0.2	0.4	0.6	0.8	1	1.2	1.4	1.6	1.8	2
Y :	1	1.0408	1.1735	1.4333	1.8964	2.1782	4.2206	7.0993	12.936	25.534	54.598
	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9	y_{10}

By Trapezoidal rule, we have

$$\int_0^2 xe^{x^2}dx = \frac{h}{2}\left[(y_0 + y_{10}) + 2(y_1 + y_2 + y_3 + \ldots + y_9)\right]$$

$$= \frac{0.2}{2}\left[(1 + 54.5981) + 2(1.0408 + 1.1735 + 1.4333 + 1.8964 + 2.1782 + 4.2206 + 7.0993 + 12.9358 + 25.5337 + 54.5981)\right]$$

Hence $\int_0^2 xe^{x^2} dx = 17.0621.$

Example : The definite integral $\int_1^3 \frac{1}{x} dx$ is evaluated using Trapezoidal rule with a step size of 1. The correct answer is _______

Solution : (1.1 to 1.2)

By Trapezoidal rule $\int_{x_0}^{x_n} f(x)dx = \frac{h}{2}[(y_0 + y_n) + 2(y_1 + ... + y_{n-1})]$

$$\int_1^3 \frac{1}{x} dx = \frac{1}{2}\left[\left(1 + \frac{1}{3}\right) + 2\left(\frac{1}{2}\right)\right] = \frac{1}{2}\left[\frac{4}{3} + 1\right] = \frac{1}{2} \times \frac{7}{3} = 1.1667$$

7.4.3 SIMPSON'S $\frac{1}{3}$rd RULE

Putting n=2 in the Newton-Cotes Quadrature Formula and taking polynomial of the second order so that differences of order higher than second become zero, we get

$$\int_{x_0}^{x_0+2h} f(x)dx = 2h\left(y_0 + \Delta y_0 + \frac{1}{6}\Delta^2 y_0\right), \quad \int_{x_0}^{x_2} ydx = \frac{h}{3}[y_0 + 4y_1 + y_2]$$

Similarly, $\int_{x_2}^{x_4} ydx = \frac{h}{3}[y_0 + 4y_1 + y_2]$ and finally $\int_{x_{n-2}}^{x_n} ydx = \frac{h}{3}[y_{n-2} + 4y_{n-1} + y_n]$

$$\int_{x_0}^{x_n} ydx = \frac{h}{3}[(y_0 + y_n) + 4(y_1 + y_3 + y_5 + ... + y_{n-1}) + 2(y_2 + y_4 + ... + y_{n-2})]$$

This is known as the Simpson's one-third rule.

Example : Use Simpson's 1/3rd rule to find $\int_0^{0.6} e^{-x^2} dx$ by taking seven coordinates.

Solution : Divide the intervals (0, 0.6) into 6 parts each of width h = 0.1. the value of y = f (x) = e^{-x^2} are given below :

X	0	0.1	0.2	0.3	0.4	0.5	0.6
x^2	0	0.01	0.04	0.09	0.16	0.25	0.36
Y	1	0.9900	0.9608	0.9139	0.8521	0.7788	0.6977
	Y_0	Y_1	Y_2	Y_3	Y_4	Y_5	Y_6

By Simpson's $1/3^{rd}$ rule, we have

$$\int_0^{0.6} e^{-x^2}dx = \frac{h}{3}[(y_0 + y_6) + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4)]$$

$$= \frac{0.1}{3}\left[(1+0.6977)+4(0.9900+0.9139+0.7788)+2(0.9608+8521)\right]$$

$$= \frac{0.1}{3}[1.69977 + 10.7308 + 3.6258]$$

$$= \frac{0.1}{3}[16.0543] = 0.5351$$

Example : For step-size $\Delta x = 0.4$, the value of following integral using Simpson's 1/3 rule is ____

$$\int_0^{0.8} (0.2+25x - 200x^2+675x^3 - 900x^4 + 400x^5)\,dx$$

Solution : (-3.8293) Given h = $\Delta x = 0.4$

$f(x) = 0.2 + 25x - 200x^2 + 675x^3 - 900x^4 + 400x^5$

X	0	0.4	0.8
y = f (x)	0.2	24.456	-126.7

$$\int_0^{0.8} f(x)\,dx = \frac{0.4}{3}\left[(0.2 - 126.744) + 4\,(24.456)\right] = -3.8293$$

7.4.4 SIMPSON'S $\frac{3}{8}$th RULE

Putting n = 3 in the above equation (Newton-Cotes Quadrature Formula) and taking polynomial of the third order so that differences of order higher than third become zero, we get

$$\int_{x_0}^{x_0+3h} f(x)dx = 3h\left(y_0 + \frac{3}{2}\Delta y_0 + \frac{3}{4}\Delta^2 y_0 + \frac{1}{8}\Delta^3 y_0\right) = \frac{3h}{8}(y_0 + 3y_1 + 3y_2 + y_3)$$

Similarly, $\displaystyle\int_{x_0+3h}^{x_0+6h} f(x)dx = \frac{3h}{8}(y_3 + 3y_4 + 3y_5 + y_6)$ and so on.

Adding all such expressions, we obtain

$$\int_{x_0}^{x_0+nh} f(x)dx = \frac{3h}{8}\left[(y_0 + y_n) + 3(y_1 + y_2 + y_4 + y_5 + \ldots) + 2(y_3 + y_6 + y_9 + \ldots)\right]$$

This is known as Simpson's three-eighth rule.

Example : Compute the value of $\int_{0.2}^{1.4} (\sin x - \log x + e^x)\, dx$ using Simpson's 3/8th rule.

Solution : Let y = $\sin x - \log x + e^x$ and h = 0.2, n = 6.

The value of y are as given below

X :	0.2	0.4	0.6	0.8	1	1.2	1.4
Y :	3.0295	2.7975	2.8976	2.8976	3.166	3.5597	4.4042
	y_1	y_2	y_3	y_4	y_5	y_6	y_7

By Simpson's $3/8^{th}$ rule, we have

$$\int_{0.2}^{1.4} (\sin x - \log x + e^x) dx = \frac{3h}{8}\left[\left(y_0 + y_6\right) + 2\left(y_3\right) + 3(y_1 + y_2 + y_4 + y_5)\right]$$

$$= \frac{3}{8}\left(0.2\right)\left[7.7336 + 2\left(3.1660\right) + 3\left(13.3247\right)\right] = 4.053$$

7.5 NUMERICAL SOLUTION OF ORDINARY DIFFRENTIAL EQUATIONS

Numerical solution of ordinary differential equations comprises of three method commonly used for solution of first order differential equations namely,

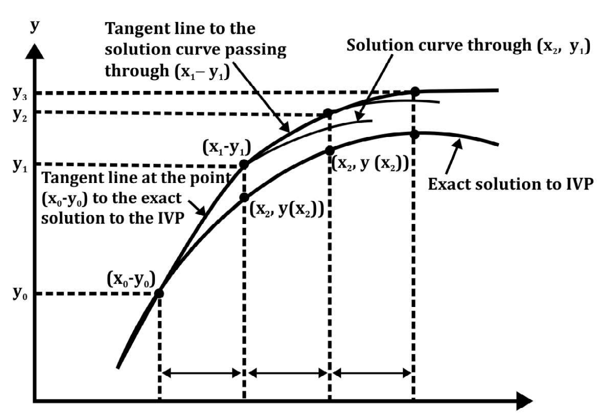
1. Euler's Method
2. Modified Euler's Method
3. Runge –Kutta Method of fourth order. (Classical Rung-Kutta Method.)

7.5.1 EULER'S METHOD

Suppose we wish to approximate the solution to the initial-value problem

$$\frac{dy}{dx} = f(x, y), y(x_0) = y_0 \qquad ...(1)$$

at $x = x_1 = x_0 + h$, where h is small. The idea behind Euler's method is to use the tangent line to the solution curve through (x_0, y_0) to obtain such an approximation. (See Figure)



Euler's method for approximating the solution to the initial-value problem
$dy/dx = f(x, y),\ y(x_0) = y_0$

The equation of the tangent line through (x_0, y_0) is $y(x) = y_0 + m(x - x_0)$,

Where m is the slope of the curve at . From Equation (1), $m = f(x, y)$, so

$y - y_0 = (x - x_0)\, f(x_0,\ y_0)$.

Setting $x = x_1$ in this equation yields the Euler approximation to the exact solution at x_1, namely, Which we write as

$$y_1 = y_0 + hf(x_0, y_0)$$

Continuing in this manner, we determine the sequence of approximations

$$y_{n+1} = y_n + hf(x_n, y_n),\ n = 0, 1, 2, \ldots$$

to the solution to the initial-value problem (1) at the points $x_{n+1} = x_n + h$.

Example : Consider a differential equation $\frac{dy(x)}{dx} - y(x) = x$ with the initial condition y (0) = 0. Using Euler's first order method with a step size of 0.1, the value of y (0.3) is

(a) 0.01 (b) 0.031

(c) 0.0631 (d) 0.1

Solution : (b) $F(x,\ y) = x + y,\ y(0) = 0,\ h = 0.1$

using Euler's first order formula is

$y_1 = y_0 + h\, f(x_0,\ y_0)$

$y_1 = 0 + 0.1f(0,0) = 0$

$y_2 = y_1 + hf(x_1, y_1)$

$y_2 = 0 + 0.1\, f(0.1, 0) = 0.01$

$y_3 = y_2 + hf(x_2, y_2)$

$y_3 = 0.01 + 0.1f(0.2, 0.01) = 0.031$

7.5.2 IMPLICIT or BACKWARD EULER'S METHOD

Let $\frac{dy}{dx} = f(x, y)$ is a first order ordinary differential equation and $y(x_0) = y_0$ is the initial condition.

$y_1 = y_0 + h\, f(x_1, y_1)$

Applying the same for n-steps, we get

$y_{n+1} = y_n + h\, f(x_{n+1}, y_{n+1})$

Example : The differential equation $(dy/dx) = 0.25y^2$ is to be solved using the backward (implicit) Euler's method with the boundary condition y = 1 at x = 0 and with a step size of 1. What would be the value of y at x =1?

(a) 1.33 (b) 1.67

(c) 2.00 (d) 2.33

Solution : (c) $f(x,y) = 0.25y^2$, $x_0 = 0,\ y_0 = 1,\ h = 1$

$$y_1 = y_0 + hf(x_1, y_1) \Rightarrow y_1 = 1 + 0.25y_1^2 \Rightarrow y_1 = 2$$

7.5.3. MODIFIED or IMPROVED EULER'S METHOD

Let $\frac{dy}{dx} = f(x,y)$ is a first order ordinary differential equation and $y(x_0) = y_0$ is initial condition. The approximation of modified Euler's Method is

$$y_{n+1} = y_n + \frac{h}{2}\left[f(x_n, y_n) + f(x_{n+1}, y^*_{n+1}) \right]$$
$$y^*_{n+1} = y_n + h f(x_n, y_n)$$

7.5.3 RUNGE - KUTTA METHOD

There is a class of methods known as Runge-Kutta methods which do not require the calculations of higher order derivatives. These methods agree with taylor's series solution up to the term in h^r, where r differs from method to method and is called the order of that method. The fourth order Runge-Kutta Method is most commonly used and is often referred as 'Runge-Kutta Method' or Classical Runge-Kutt a Method.

Working rule for finding the increment k of y corresponding to an increment h of x by Runge-Kutta method from

Runge-Kutta 1st order method :

Let $\frac{dy}{dx} = f(x,y)$ is a first order ordinary differential equation and $y(x_0) = y_0$ is the initial condition. To know the value of function y at any value of $x_n > x_0$, we proceed as below

$$y_{n+1} = y_n + hK_1$$
$$K_1 = f(x_n, y_n)$$

Runge-Kutta 2nd order method :

Let $\frac{dy}{dx} = f(x,y)$ is a first order ordinary differential equation and $y(x_0) = y_0$ is the initial condition. To know the value of function y at any value of $x_n > x_0$, we proceed as below

$$y_{n+1} = y_n + \frac{h}{2}\left(K_1 + K_2\right)$$
$$K_1 = f(x_n, y_n), \quad K_2 = f(x_n + h,\ y_n + hK_1)$$

Runge-Kutta 3rd order method :

Let $\frac{dy}{dx} = f(x, y)$ is a first order ordinary differential equation and $y(x_0) = y_0$ is the initial condition. To know the value of function y at any value of $x_n > x_0$, we proceed as below

$$y_{n+1} = y_n + \frac{h}{6}(K_1 + 4K_2 + K_3)$$
$$K_1 = f(x_n, y_n),\ K_2 = f(x_n + \frac{h}{2}, y_n + \frac{h}{2}K_1),\ K_3 = f(x_n + h, y_n - hK_1 + 2hK_2)$$

Runge-Kutta 4th order method :

Let $\frac{dy}{dx} = f(x, y)$ is a first order ordinary differential equation and $y(x_0) = y_0$ is the initial condition. To know the value of function y at any value of $x_n > x_0$, we proceed as below

$$y_{n+1} = y_n + \frac{h}{6}\left(K_1 + 2K_2 + 2K_3 + K_4\right)$$
$$K_1 = f\left(x_n,\ y_n\right),\quad K_2 = f\left(x_n + \frac{h}{2}, y_n + \frac{h}{2}K_1\right)$$
$$K_3 = f\left(x_n + \frac{h}{2}, y_n + \frac{h}{2}K_2\right),\ K_4 = f\left(x_n + h, y_n + hK_3\right)$$

Example : Apply Runge–Kutta fourth order method to find an approximate value of y when x = 0.2 given that dy/dx = x + y and y = 1 when x = 0.

Solution : Here, f(x , y) = x + y, $x_0 = 0$, $y_0 = 1$, h = 0.2

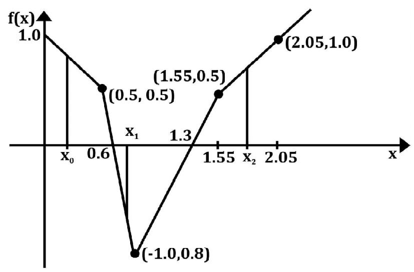
$$K_1 = hf\left(x_0, y_0\right) = 0.2$$
$$K_2 = hf\left(x_0 + \frac{h}{2},\ y_0 + \frac{K_1}{2}\right) = 0.2f\left(0.1, 1.1\right) = 0.2400$$
$$K_3 = hf\left(x_0 + \frac{h}{2},\ y_0 + \frac{K_2}{2}\right) = 0.2f\left(0.1, 1.12\right) = 0.2440$$
$$K_4 = hf\left(x_0 + h,\ y_0 + K_3\right) = 0.2f\left(0.2.1.244\right) = 0.2888$$
$$K = \frac{1}{6}\left(K_1 + 2K_2 + 2K_3 + K_4\right) = 0.2428$$

Now, $y_1 = y_0 + K = 1 + 0.2428 = 1.2428$

Hence the required approximate value of y is 1.2428.

GATE QUESTIONS

Q.1 A piecewise linear function f(x) is plotted using thick solid lines in the figure below (the plot is drawn to scale).



If we use the N-R method to find the roots of f(x) = 0 using x_0, x_1 and x_2 respectively as initial guesses, the roots obtained would be

(a) 1.3, 0.6 and 0.6 respectively (b) 0.6, 0.6 and 1.3 respectively

(c) 1.3, 1.3 and 0.6 respectively (d) 1.3, 0.6 and 1.3 respectively **[GATE-2003(CS)]**

Statement for linked answer question 2 & 3.

Given a>0, we wish to calculate the reciprocal value $\frac{1}{a}$ by Newton Raphson method for f(x) = 0.

Q.2 The Newton Raphson algorithm for the function will be

(a) $x_{k+1}=\frac{1}{2}\left(x_k+\frac{a}{x_k}\right)$ (b) $x_{k+1}=x_k+\frac{a}{2}x_k^2$

(c) $x_{k+1}=2x_k-ax_k^2$ (d) $x_{k+1}=\left(x_k-\frac{a}{2}x_k^2\right)$ **[GATE-2005 (CE)]**

Q.3 For a = 7 and starting with x_0 = 0.2, the first two iterations will be

(a) 0.11, 0.1299 (b) 0.12, 0.1392

(c) 0.12, 0.1416 (d) 0.13, 0.1428 **[GATE-2005 (CE)]**

Q.4 Starting with x_0 = 1, one step of Newton-Raphson method in solving the equation $x^3+3x-7=0$ gives the next value (x_1) as

(a) x_1 = 0.5 (b) x_1= 1.406

(c) x_1= 1.5 (d) x_1 = 2 **[GATE-2005 (ME)]**

Q.5 A 2^{nd} degree polynomial, f(x) has value of 1, 4 and 15 at x = 0, 1 and 2, respectively. The integral $\int_0^2 f(x)\,dx$ is to be estimated by applying the trapezoidal rule to this data. What is the error (defined as "true value – approximate value") in the estimate?

(a) $-\frac{4}{3}$
(b) $-\frac{2}{3}$
(c) 0
(d) $\frac{2}{3}$ **[GATE-2006 (CE)]**

Q.6 The differential equation $\frac{dy}{dx} = 0.25y^2$ is to be solved using the backward (implicit) Euler's method with the boundary condition y = 1 at x = 0 and with a step size of 1. What would be the value of y at x = 1?

(a) 1.33
(b) 1.67
(c) 2.00
(d) 2.33 **[GATE-2006(CE)]**

Q.7 Consider the series $x_{(n+1)} = \frac{x_n}{2} + \frac{9}{8x_n}$, $x_0 = 0.5$ obtained from the N-R method. The series converges to

(a) 1.5
(b) $\sqrt{2}$
(c) 1.6
(d) 1.4 **[GATE-2007(CS)]**

Q.8 The N-R iteration $x_{(n+1)} = \frac{1}{2}\left(x_n + \frac{R}{x_n}\right)$ can be used to compute the

(a) R^2
(b) $1/R$
(c) $\sqrt{R}$
(d) log R **[GATE-2007(CS)]**

Q.9 N-R Method is used to compute a root of the equation $x^2 - 13 = 0$ with 3.5 as the initial value. The approximation after first iteration is,

(a) 3.575
(b) 3.677
(c) 3.667
(d) 3.607 **[GATE-2007(CS)]**

Q.10 A calculator has accuracy up to 8 digits after decimal place. The value of $\int_0^{2\pi} \sin x\,dx$ when evaluated using this calculator by trapezoidal method with 8 equal intervals, to 5 significant digits is

(a) 0.00000
(b) 1.0000
(c) 0.00500
(d) 0.00025 **[GATE-2007 (ME)]**

Q.11 The differential equation $\left(\frac{dx}{dt}\right)=\left[\frac{(1-x)}{\tau}\right]$ is discretized using Euler's numerical integration method with a time step $\Delta T > 0$. What is the maximum permissible value of ΔT to ensure stability of the solution of the corresponding discrete time equation ?

(a) 1 (b) $\frac{\tau}{2}$

(c) τ (d) 2τ **[GATE-2007(EE)]**

Q.12 Equation $e^x - 1 = 0$ is required to be solved using N-R method with initial guess $x_0 = -1$. Then, after one step of N-R method, estimate x_1 of the solution will be given by

(a) 0.71828 (b) 0.36784

(c) 0.20587 (d) 0.00000 **[GATE-2008 (EE)]**

Q.13 The equation $x^3 - x^2 + 4x - 4 = 0$ is to be solved using the N-R method. If x = 2 is taken as the initial approximation of the solution, then the next approximation using this method will be

(a) $\frac{2}{3}$ (b) $\frac{4}{3}$

(c) 1 (d) $\frac{3}{2}$ **[GATE-2008 (EE)]**

Q.14 The minimum number of equal length subintervals needed to approximate $\int_1^2 x e^x dx$ to an accuracy of at least $\frac{1}{3}\times 10^{-6}$ using the trapezoidal rule is

(a) 1000e (b) 1000

(c) 100e (d) 100 **[GATE-2008 (CS)]**

Q.15 Torque exerted on a flywheel over a cycle is listed in the table. Flywheel energy (in J per unit cycle) using Simpson's rule is

Angle(degree)	0	60	120	180	240	300	360
Torque(Nm)	0	1066	– 323	0	323	– 355	0

(a) 542 (b) 993

(c) 1444 (d) 1986 **[GATE-2010 (ME)]**

Q.16 Consider a differential equation $\frac{dy(x)}{dx} - y(x) = x$ with the initial condition y (0) = 0. Using Euler's first order method with a step size of 0.1, the value of y (0.3) is

(a) 0.01 (b) 0.031

(c) 0.0631 (d) 0.1 **[GATE-2010(E(C)]**

Q.17 Match the CORRECT pairs :

Numerical Integration Scheme Order of Fitting Polynomial

P. Simpson's 3/8 Rule 1. First

Q. Trapezoidal Rue 2. Second

R. Simpson's 1/3 Rule 3. Third

(a) P – 2; Q – 1; R – 3 (b) P – 3; Q – 2; R – 1

(c) P – 1; Q – 2; R – 3 (d) P – 3; Q – 1; R – 2 **[ME-GATE-2012]**

Q.18 The estimate of $\int_{0.5}^{1.5} \frac{dx}{x}$ obtained using Simpson's rule with three-point function evaluation exceeds the exact value by

(a) 0.235 (b) 0.068

(c) 0.024 (d) 0.012 **[CE-GATE-2013]**

Q.19 The error in $\left.\frac{d}{dx}f(x)\right|_{x=-x_0}$ for a continuous function estimated with h = 0.03 using the central difference formula $\left.\frac{d}{dx}f(x)\right|_{x=x_0} \approx \frac{f(X_0+h)-f(X_0-h)}{2h}$, is 2×10^{-3}. The values of X_0 and f (X_0) are 19.78 and 500.01, respectively. The corresponding error in the central difference estimate for h = 0.02 is approximately

(a) 1.3×10^{-4} (b) 3×10^{-4}

(c) 4.5×10^{-4} (d) 9×10^{-4} **[CE-GATE-2013]**

Q.20 The maximum value of θ until which the approximation $\sin\theta=\theta$ holds to within 10% error is

(a) $10°$ (b) $18°$

(c) $50°$ (d) $90°$ **[EC-GATE-2013]**

Q.21 A polynomial $f(x)=a_4X^4+a_3X^3+a_2X^2+a_1X-a_0$ with all coefficients positive has

(a) no real roots

(b) no negative real root

(c) odd number of real roots

(d) at least one positive and one negative real root **[EC-GATE-2013]**

Q.22 When the Newton-Raphson method is applied to solve the equation $f(x)=x^3+2x-1=0$, the solution at the end of the first iteration with the initial guess value as $x_0=1.2$ is

(a) –0.82 (b) 0.49

(c) 0.705 (d) 1.69 **[EE-GATE-2013]**

Q.23 While numerically solving the differential equation $\frac{dy}{dx} + 2xy^2 = 0$, $y(0) = 1$ using Euler's predictor– corrector (improved Euler–Cauchy) method with a step size of 0.2, the value of y after the first step is

(a) 1.00 (b) 1.03

(c) 0.97 (d) 0.96 **[IN-GATE-2013]**

Q.24 Using the trapezoidal role, and dividing the interval of integration into the equal sub intervals, the definite integral $\int_{-1}^{1} |x|\, dx$ is _____ **[ME-GATE-2014(1)]**

Q.25 The best approximation of the minimum value attained $e^{-x}\sin(100x)$ for $x > 0$ is _______ **[ME-GATE-2014(2)]**

Q.26 The value of $\int_{2.5}^{4} \ln(x)\, dx$ calculated using the Trapezoidal rule with five sub intervals is_____ **[ME-GATE-2014(2)]**

Q.27 The definite integral $\int_{1}^{3} \frac{1}{x} dx$ is evaluated using Trapezoidal rule with a step size of 1. The correct answer is _______ **[ME-GATE-2014(3)]**

Q.28 The real root of the equation $5x - 2\cos x - 1 = 0$ (up to two decimal accuracy) is ______ **[ME-GATE-2014(3)]**

Q.29 Consider an ordinary differential equation $\frac{dx}{dt} = 4t + 4$ if $x = x_0$ at t = 0, the increment in x calculated using Runge-Kutta fourth order multi-step method with a step size of $\Delta t = 0.2$, is

(a) 0.22 (b) 0.44

(c) 0.66 (d) 0.88 **[ME-GATE-2014(4)]**

Q.30 Match the application to appropriate numerical method.

Application	Numerical Method
P1:Numerical integration	M1:Newton-Raphson Method
P2:Solution to a transcendental equation	M2:Runga-Kutta Method
P3:Solution to a system of linear equation	M3:Simoson's 1/3-rule
P4:Solution to a differential equation	M4:Gauss Elimination Method

(a) P1-M3,P2-M2,P3-M4,P4-M1 (b) P1-M3,P2-M1,P3-M4,P4-M2

(c) P1-M4,P2-M1,P3-M,P4-M2 (d) P1-M2,P2-M1,P3-M3,P4-M4 **[EC-GATE-2014(3)]**

Q.31 The function $f(x) = e^x - 1$ is to be solved using Newton-Raphson method. If the initial value x_0 of is taken as 1.0, then the absolute error observed at 2nd iteration is

[EE-GATE-2014(3)]

Q.32 The iteration step in order to solve for the cube roots of a given number N using the Newton-Raphson's method is

(a) $x_{k+1} = x_k + \frac{1}{3}\left(N - x_k^3\right)$ (b) $x_{k+1} = \frac{1}{3}\left(2x_k + \frac{N}{x_k^2}\right)$

(c) $x_{k+1} = x_k - \frac{1}{3}\left(N - x_k^3\right)$ (d) $x_{k+1} = \frac{1}{3}\left(2x_k - \frac{N}{x_k^2}\right)$ **[IN-GATE-2014]**

Q.33 Simpson's $\frac{1}{3}$ rule is used to integrate the function $f(x) = \frac{3}{5}x^2 + \frac{9}{5}$ between x = 0 and x =1 using the least number of equal sub-intervals. The value of the integral is ______

[ME-GATE-2015(1)]

Q.34 The value of function f(x) at 5 discrete point are given below :

X	0	0.1	0.2	0.3	0.4
F(x)	0	10	40	90	160

Using trapezoidal rule with step size of 0.1 the value of $\int_0^{0.4} f(x)\,dx$ is

[ME-GATE-2015(2)]

Q.35 Using a unit step, the value of integral $\int_1^2 x \ln x \, dx$ by trapezoidal rule is ___.

[ME-GATE -2015(3)]

Q.36 Newton–Raphson method is used to find the roots of the equation, $x^3 + 2x^2 + 3x - 1 = 0$ If the initial guess $x_0 = 1$ is then the value of x after 2nd iteration is ___

[ME-GATE -2015(3)]

Q.37 The integral $\int_{x_1}^{x_2} x^2\,dx$ with $x_2 > x_1 > 0$ is evaluated analytically as well as numerically using a single application of the trapezoidal rule. If I is the exact value of the integral obtained analytically and J is the approximate value obtained using the trapezoidal rule, which of the following statements is correct about their relationship?

(a) J > I (b) J < I

(c) J = I (d) Insufficient data to determine the relationship

[CE-GATE-2015(1)]

Q.38 The quadric equation $x^2 - 4x + 4 = 0$ is to be solved numerically, starting with the initial guess $x_0 = 3$. The Newton-Raphson method is applied once to get a new estimate and then the Secant method is applied once using the initial guess and this new estimate. The estimated value of the root after the application of the Secant method is ________.

[CE-GATE-2015(1)]

Q.39 In Newton-Raphson iterative method, the initial guess value (x_{ini}) is considered as zero while finding the roots of the equation: $f(x) = -2 + 6x - 4x^2 + 0.5x^3$. The correction Δx, to be added to in the first iteration is _____. **[CE-GATE-2015(2)]**

Q.40 For step-size $\Delta x = 0.4$, the value of following integral using Simpson's 1/3 rule is _____

$$\int_0^{0.8} (0.2 + 25x - 200x^2 + 675x^3 - 900x^4 + 400x^5)dx$$ **[CE-GATE-2015(2)]**

Q.41 The Newton-Raphson method is used to solve the equation $f(x) = x^3 - 5x^2 - 6x - 8 = 0$. Taking the initial guess as x = 5, the solution obtained at the end of the first iteration is ___________.

[EC-GATE-2015(3)]

Q.42 Solve the equation x = 10cos(x) using the Newton-Raphson method. The initial guess is $x = \frac{\pi}{4}$. The value of the predicted root after the first iteration, up to second decimal is_____________ **[ME-GATE -2016(1)]**

Q.43 Gauss-Seidel method is used to solve the following equations (as per the given order):

$x_1 + 2x_2 + 3x_3 = 5,\ 2x_1 + 3x_2 + x_3 = 1,\ 3x_1 + 2x_2 + x_3 = 3$

Assuming initial guess as $x_1 = x_2 = x_3 = 0$ the value of after the first iteration is ______.

[ME-GATE -2016(1)]

Q.44 Numerical integration using trapezoidal rule gives the best for single variable function, which is

(a) linear (b) parabolic

(c) logarithmic (d) hyperbolic **[ME-GATE -2016(2)]**

Q.45 The error in numerically computing the integral $\int_0^{\pi}(\sin x + \cos x)dx$ using the trapezoidal rule with three intervals of equal length between 0 and π is _____. **[ME-GATE -2016(2)]**

Q.46 The root of the function obtained after first iteration on application of Newton-Raphson scheme using an initial guess of

(a) 0.682 (b) 0.686

(c) 0.750 (d) 1.000 **[ME-GATE -2016(3)]**

Q.47 Newton-Raphson method is to be used to find root of equation $3x - e^x + \sin x = 0$. If the initial trial value for the root is taken as 0.333, the next approximation for the root would be ______. **(note :** answer up to three decimal**)** **[CE-GATE-2016(1)]**

Q.48 Consider the first order initial value problem $y' = y + 2x - x^3$, $y(0) = 1$, $(0 \le x < \infty)$ with exact solution $y(x)=x^2 + e^x$ For x = 0.1, the percentage difference between the exact solution & the solution obtained using a single iteration of the second order Runge-Kutta method with step-size h = 0.1 is **[EC-GATE-2016(3)]**

Q.49 P(0, 3), Q(0.5, 4) and R (1, 5) are three points on the curve defined by f(x). Numerical integration is carried out using both Trapezoidal rule and Simpson's rule within limits x = 0 and x = 1 for the curve. The difference between the two results will be

(a) 0 (b) 0.25

(c) 0.5 (d) 1 **[GATE -2017]**

Q.50 Variation of water depth (y) in a gradually varied open channel flow is given by the first order differential equation $\dfrac{dy}{dx} = \dfrac{1 - e^{-\frac{10}{3}\ln(y)}}{250 - 45e^{-3\ln(y)}}$ Given initial condition: y(x = 0) = 0.8m. The depth (in m, up to three decimal places) of flow at a downstream section at x = 1m from one calculation step of Single step Euler Method is ______. **[CE-GATE-2018(1)]**

Q.51 The quadratic equation $2x^2 - 3x + 3 = 0$ is to be solved numerically starting with an initial guess as $x_0 = 2$. The new estimate of x after the first iteration using Newton-Raphson method is ______. **[CE-GATE-2018(2)]**

Q.52 An explicit forward Euler method is used to numerically integrate the differential equation $\dfrac{dy}{dt} = y$ using a time step of 0.1. With the initial condition y(0) = 1, the value of y(1) computed by this method is __________ (correct to two decimal places).

[ME-GATE-2018(1)]

ANSWER KEY:

1	2	3	4	5	6	7	8	9	10	11	12	13	14
(d)	(c)	(b)	(c)	(a)	(c)	(a)	(c)	(d)	(a)	(d)	(a)	(b)	(a)
15	16	17	18	19	20	21	22	23	24	25	26	27	28
(b)	(b)	(d)	(d)	(a)	(b)	(d)	(c)	(d)	1.12	0.94	1.76	1.2	0.56
29	30	31	(b)	33	34	35	36	37	38	39	40	41	42
(d)	(b)	0.06		3	22	0.69	0.3	(a)	2.33	0.33	-3.82	4.3	1.56
43	44	45	46	47	48	49	50	51	52				
-6	(a)	0.17	(c)	0.36	0.06	(a)	0.793	1	2.5937				

ESE QUESTIONS

Q.1 What is the form of the function f(x) for the following data ?

x	0	1	2	3
f(x)	3	6	11	18

(a) x^2+2x+3 (b) x^2-2x+3

(c) x^2+2x-3 (d) x^2-2x-3

Q.2 What is the cube root of 1468 to 3 decimal places?

(a) 11.340 (b) 11.353

(c) 11.365 (d) 11.382

Q.3 What is the value of $(1525)^{0.2}$ to 2 decimals places ?

ANSWER KEY :

1. (a)

2. (c)

3. (c)

GATE SOLUTIONS

Q.1 **(d)**

Q.2 **(c)** To calculate $\frac{1}{a}$ using N-R Method, Set up the equation as $x = \frac{1}{a}$;

$$f(x) = \frac{1}{x} - a = 0, \quad f'(x) = -\frac{1}{x^2}$$

For N-R Method : $x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} = x_k - \frac{\frac{1}{x_k} - a}{-\frac{1}{x_k^2}}$

Simplifying which we get, $x_{k+1} = 2x_k - ax_k^2$

Q.3 **(b)** For a = 7 the iteration becomes $x_{k+1} = 2x_k - 7x_k^2$, With $x_0 = 0.2$;

$x_1 = 2x_0 - 7x_0^2 = 2 \times 0.2 - 7(0.2)^2 = 0.12$ & $x_2 = 2x_1 - 7x_1^2 = 2 \times 0.12 - 7(0.12)^2 = 0.1392$

Q.4 **(c)** From Newton-Raphson method $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$

Given function is $f(x) = x^3 + 3x - 7,\ f'(x) = 3x^2 + 3$

Putting $x_0 = 1$

$f(x_0) = f(1) = (1)^3 + 3 \times (1) - 7 = -3$, $\quad f'(x_0) = f'(1) = 3 \times (1)^2 + 3 = 6$

Substituting x_0, $f(x_0)$, $f'(x_0)$ values into (i) we get, $x_1 = 1 - \left(\frac{-3}{6}\right) \times 1 = 1.5$

Q.5 **(a)** f(x) = 1,4,15 at x = 0, 1 and 2 respectively

$$\int_0^2 f(x)\,dx = \frac{h}{2}(f_1 + 2f_2 + f_3) \text{ (3 point trapezoidal rule)}$$

Here h = 1, Therefore $\int_0^2 f(x)\,dx = \frac{1}{2}(1 + 2 \times 4 + 15) = 12$

Therefore approximate value by Trapezoidal Rule = 12

Since f(x) is second degree polynomial, let f(x) = $a_0 + a_1x + a_2x^2$

a_0+ 0 + 0 = 1; a_0 = 1 , f(1) = 4 ;

$a_0 + a_1 + a_2 = 4$; $a_1 + a_2 = 3$ ...(i)

f(2) = 15 ; $a_0 + 2a_1 + 4a_2 = 15$; $1 + 2a_1 + 4a_2 = 15$; $2a_1 + 4a_2 = 14$ ...(ii)

Solving (i) and (ii) $a_1 = -1$ and $a_2 = 4$

f(x) = 1 – x + 4x^2

Now, exact value of $\int_0^2 f(x)dx = \int_0^2 (1 - x + 4x^2)dx \quad = \left[x - \frac{x^2}{2} + \frac{4x^3}{3}\right]_0^2 = \frac{32}{3}$

Error = exact value – approximate value $= \frac{32}{3} - 12 = -\frac{4}{3}$

Q.6 **(c)** $\frac{dy}{dx} = 0.25y^2$ (y = 1 at x = 0) $\Rightarrow$ h = 1

Iterative equation for backward (implicit) Euler's methods for the above equation would be

$$y_{k+1} = y_k + h\ f(x_{k+1}, y_{k+1}) \rightarrow y_{k+1} = y_k + h \times 0.25y_{k+1}^2 \rightarrow 0.25y_{k+1}^2 - y_{k+1} + y_k = 0$$

Putting k = 0 in above equation

$0.25h\,y_1^2 - y_1 + y_0 = 0$, Since, y_0= 1 and h = 1, $0.25y_1^2 - y_1 + 1 = 0 \Rightarrow y_1 = 2$

Q.7 **(a)** Given, $x_{(n+1)} = \frac{x_n}{2} + \frac{9}{8x_n}$, $x_0 = 0.5$

As $n \to \infty$, when series converges, $x_{(n+1)} = x_n = \alpha$ = root of equation

$$\alpha = \frac{\alpha}{2} + \frac{9}{8\alpha} \Rightarrow \alpha = 1.5$$

Q.8 **(c)** $x_{(n+1)} = \frac{1}{2}\left(x_n + \frac{R}{x_n}\right)$

At convergence $x_{(n+1)} = x_n = \alpha$

$$\alpha = \frac{1}{2}\left(\alpha + \frac{R}{\alpha}\right) \Rightarrow 2\alpha = \frac{\alpha^2 + R}{\alpha} \Rightarrow 2\alpha^2 = \alpha^2 + R \Rightarrow \alpha^2 = R, \alpha = \sqrt{R}$$

So, this iteration will compute the square root of R

Q.9 **(d)** The equation is $f(x) = x^2 - 13 = 0$

N-R iteration equation is $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$

$f(x_0) = x_0^2 - 13$, $f'(x_0) = 2x_0$

$x_1 = x_0 - \frac{x_0^2 - 13}{2x_0} = \frac{x_0^2 + 13}{2x_0}$, put $x_0 = 3.5$ (as given) $x_1 = \frac{3.5^2 + 13}{2 \times 3.5} = 3.607$

$\therefore$ the approximation after 1st iteration is 3.607

Q.10 **(a)** $h = \frac{2\pi - 0}{8} = \frac{\pi}{4}$

$y_1 = \sin\left(\frac{\pi}{4}\right) = 0.70710$, $y_2 = \sin\left(\frac{\pi}{2}\right) = 1$, $y_3 = \sin\left(\frac{3\pi}{4}\right) = 0.7010$, $y_4 = \sin(\pi) = 0$

$y_5 = \sin\left(\frac{5\pi}{4}\right) = 0.7010$, $y_6 = \sin\left(\frac{6\pi}{4}\right) = -1$, $y_7 = \sin\left(\frac{7\pi}{4}\right) = -0.70710$, $y_8 = \sin\left(\frac{8\pi}{4}\right) = 0$

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{h}{2}\left[(y_0 + y_n) + 2(y_1 + y_2 + \dots + y_{n-1})\right]$$

$\int_0^{2\pi} \sin x\, dx = \frac{h}{2}$ × [(0+0) + 2(0.70710 + 1 + 0.70710 + 0 – 0.70710 – 0.70710)] = 0.000

Q.11 **(d)** Here, $\frac{dx}{dt} = \frac{1-x}{\tau}$; Here, f(x, y) = $\frac{1-x}{\tau}$

Euler's Method Equation is $x_{i+1} = x_i + h\, f(x_i, y_i)$

$x_{i+1} = x_i + h\left(\frac{1-x}{\tau}\right)$, $x_{i+1} = \left(1 - \frac{h}{\tau}\right)x_i + \frac{h}{\tau}$

For Stability $\left|1 - \frac{h}{\tau}\right| < 1$, $-1 \le 1 - \frac{h}{\tau} \le 1$

Since, $h = \Delta T$ here, $-1 \le 1 - \frac{\Delta T}{\tau} < 1$

$\Delta T < 2\tau$, So , maximum permissible value of ΔT is 2τ

Q.12 **(a)** Here f(x) = $e^x - 1$ and f '(x) = e^x

From Newton-Raphson method , $x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$

$$= x_i - \frac{e^{x_i}-1}{e^{x_i}} = \frac{x_i e^{x_i}-(e^{x_i}-1)}{e^{x_i}} = \frac{e^{x_i}\left(x_i-1\right)+1}{e^{x_i}}$$

Now put i = 0 ; $x_1 = \frac{e^{x_0}\left(x_0-1\right)+1}{e^{x_0}}$

put x_0 = – 1 as given, $x_1 = e^{-1}\left(-2\right)+1/e^{-1} = 0.71828$

Q.13 **(b)** Here, $x_0 = 2$

$f(x) = x^3 - x^2 + 4x - 4$, $f(x_0) = f(2) = 8$

$f'(x) = 3x^3 - 2x + 4$ $f'(x_0) = f'(2) = 12$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 2 - \frac{8}{12} = \frac{4}{3}$$

Q.14 **(a)** Here, the function being integrated is $f(x) = xe^x$

$f'(x) = xe^x + e^x = e^x(x+1)$, $f''(x) = xe^x + e^x + e^x = e^x(x+2)$

Since, both e^x and x are increasing functions of x, maximum value of $f''\left(\xi\right)$ in interval $1 \le \xi \le 2$, occurs at $\xi = 2$.

So, max $|f''(\xi)| = e^2(2+2) = 4e^2$

$\therefore$ Truncation error for trapezoidal rule = TE(bound)

$= \frac{h^3}{12} \max |f''(\xi)| * N_i$, Where N_i is number of subintervals $N_1 = \frac{b-a}{h}$

$$T_{\epsilon(\text{bound})} = \frac{h^3}{12} \max |f''(\xi)| * \frac{b-a}{h}$$

$$= \frac{h^2}{12}(b-a)\max |f''(\xi)|, \; 1 \le \xi \le 2 = \frac{h^2}{12}(2-1)(4e^2) = \frac{h^2}{3}e^2$$

Now putting $T_{\epsilon(\text{bound})} = \frac{1}{3}\times 10^{-6}$ We get $\frac{h^2}{3}e^2 = \frac{1}{3}\times 10^{-6} \rightarrow h^2 = \frac{10^{-6}}{e^2}$, $\rightarrow h = \frac{10^{-3}}{e}$

Now, the number of Intervals $= \frac{b-a}{h} = \frac{2-1}{\frac{10^{-3}}{e}} = 1000e$

Q.15 **(b)**

Flywheel energy = $\int_0^{2\pi} T(\theta)\,dx$, where $T\left(\theta\right)$ is torque exerted.

The integral by using Simpson's rule is $\Rightarrow \frac{h}{3}\left(f_0 + 4f_1 + 2f_2 + 4f_3 + 2f_4 + 4f_5 + f_6\right)$

h = 60 degrees $= \frac{\pi}{3}$ radius

$$I = \frac{1}{3} \times \frac{\pi}{3} \times [0 + 4 \times 1066 + 2(-323) + 4(0) + 2(323) + 4(-355) + 0] = 993$$

Q.16 **(b)**

Q.17 **(d)**

Q.18 **(d)** Exact value of $\int_{0.5}^{1.5} \frac{1}{x}\,dx = 1.0986$

Using simpson's rule, in three point form, $h = \frac{b-a}{2} = \frac{1.5-0.5}{2} = 0.5$

x	0.5	1	1.5
y	2	1	0.67

$\int \frac{1}{x} dx = \frac{h}{3}[y_0 + y_n + 4y_1] = \frac{0.5}{3}$

[2 + 0.67 + 4 × 3] = 1.1116

It exceeds the exacts value by = 0.012 (approx)

Q.19 **(a)**

Q.20 **(b)**

Q.21 **(d)** Use Routh Hurwitz Criteria to get the condition.

Q.22 **(c)**

Q.23 **(d)** $\frac{dy}{dx} + 2xy^2 = 0, \quad \therefore \frac{dy}{dx} = -2xy^2$

After one iteration

$$y_1^* = y_0 + h\left[-2x_0y_0^2\right] = 1 + 0.2\left[-2 \times 0 \times 1^2\right] = 1 + 0 = 1$$

$$y_1 = y_0 + \frac{1}{2} \times (0.2)\left[-2x_0y_0^2 - 2x_1y_1^*\right] = 1 + 0.1\left[-\left(2 \times 0 \times 1^2\right) - \left(2 \times 0.2 \times 1\right)\right]$$

$$= 1 + 0.1\left[-0 - 0.4\right] = 1 - 0.04 = 0.96$$

Q.24 **(1.10 to 1.12)**

$\int_{-1}^{+1} |x|\,dx$, Let $y = |x|$, N = no of subintervals =3

$$h = \frac{x_n - n_o}{n} = \frac{1-(-1)}{3} = \frac{2}{3}$$

X	**-1**	$-\frac{1}{3}$	$\frac{1}{3}$	**1**
y = |x|	**1**	$\frac{1}{3}$	$\frac{1}{3}$	**1**

Trape zodial rule

$$= \int_{x_0}^{x^n} f(x)dx = \frac{h}{2}\left[\left(y_0 + y_n\right) + 2\left(y_1 + ... + y_{n-1}\right)\right] = \int_{-1}^{1} |x|\,dx = \frac{1}{3}\left[(1+1) + 2\left(\frac{1}{3} + \frac{1}{3}\right)\right]$$

$$= \frac{1}{3}\left[2 + \frac{4}{3}\right] = \frac{1}{3} \times \frac{10}{3} = \frac{10}{9} = 1.1111$$

Q.25 **(1.00 to - 0.94)**

Q.26 **(1.74 to 1.76)**

Given $\int_{2.5}^{4} \ln x \; dx$, $n = 5$, Let $y = \ln x$ $n = \frac{4-2.5}{5} = \frac{1.5}{5} = 0.30$

x	2.5	2.8	3.1	3.4	3.7	4.0
y = lnx	0.9163	1.0296	1.1314	1.2238	1.3083	1.3863

Trapezoidal Rule is $\int_{X_0}^{X_n} f(x)dx = \frac{h}{2}\left[\left(y_0 + y_n\right) + 2y_1 + ...y_{n-1}\right]$

$$\int_{2.5}^{4} \ln\,x\;dx = \frac{0.3}{2}[(0.9163 + 1.3863) + (1.0296 + 1.314 + 1.2238 + 1.3083)]$$

$$= 0.15(2.3026 + 9.3862) = 1.7533$$

Q.27 **(1.1 to 1.2)**

Q.28 **(0.53 to 0.56)**

Q.29 **(d)**

Q.30 **(b)** P1-M3, P2-M1, P3-M4, P4-M2

Q.31 **(0.06)**

Q.32 **(b)** Newtons's formula for finding $\sqrt[m]{N}$ is

$$x_{k+1}=\frac{1}{m}\left[(m-1)x_k+\frac{N}{X_k^{m-1}}\right] \quad \text{For } m = 3\,; \quad x_{k+1}=\frac{1}{3}\left[2x_k+\frac{N}{X_k^2}\right]$$

Q.33 **(3)**

Q.34 **(22)**

X	0	0.1	0.2	0.3	0.4
Y=f(x)	0	10	40	90	160
	y_0	y_1	y_2	y_3	y_4

$$\int_0^{0.4} f(x)\,dx=\frac{h}{2}\left[(y_0+y_4)+2(y_1+y_2+y_3)\right]=\frac{0.1}{2}\left[(0+160)+2(10+40+90)\right]=22$$

Q.35 **(0.69)** By Trapezoidal Rule,

$$\int_1^2 x\ln x\,dx=\frac{1}{2}\left[0+2\ln 2\right]=\ln 2=0.69$$

Q.36 **(0.30)** By Newton –Raphson Method

$$1\text{st iteration } x_1=x_0-\frac{f(x_0)}{f'(x_0)}=1-\frac{f(1)}{f'(1)}=1-\frac{5}{10}=\frac{1}{2}$$

$$2\text{nd iteration },\ x_2=x_1-\frac{f(x_1)}{f'(x_1)}=0.5-\frac{f(0.5)}{f'(0.5)}=0.3043$$

Q.37 **(a)**

Q.38 (2.333) $f(x) = x^2 - 4x + 4$, $x_0 = 3$, $f'(x) = 2x - 4$

By Newton Raphson method $x_1=x_0-\dfrac{f(x_0)}{f'(X_0)}=3-\dfrac{1}{2}=2.5$

For secant method let $x_0 = 2.5$ and $x_1 = 3$

By secant method $x_2=x_1-\dfrac{X_1-X_0}{f(X_1)-f(x_0)}f(x_1)=3-\dfrac{(3-2.5)}{f(3)-f(2.5)}f(3)=2.333$

Q.39 **(0.3333)** $f(x) = -2 + 6x - 4x^2 + (0.5)x^3$, $x_0 = 0$

$f'(x) = 6 - 8x + 1.5x^2$, $f(0) = -2$, $f'(0) = 6$

By Newton-Raphson method $\Delta x=x_1-x_0=-\dfrac{f(x_0)}{f'(x_0)}=-\dfrac{(-2)}{6}=0.3333$

Q.40 **(–3.8293)**

Given h = Δx = 0.4

$$f(x) = 0.2 + 25x - 200x^2 + 675x^3 - 900x^4 + 400x^5$$

X	0	0.4	0.8
y = f(x)	0.2	24.456	-126.7

$x_0 = 0$, $x_n = 0.8$, $n = \dfrac{0.8-0}{0.4} = 2$

$$\int_0^{0.8} f(x)\,dx = \frac{0.4}{3}[(0.2 - 126.744) + 4\,(24.456)] = -3.8293$$

Q.41 **(4.2903)**

$f(x) = x^3 - 5x^2 + 6x - 8$, $\quad x_0 = 5$

f’ (x) = $3x^2 - 10x + 6$

By Newton-Raphson method $x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)} = 5 - \dfrac{f(5)}{f'(5)} = 5 - \dfrac{22}{31} = 5 - 0.7097 = 4.2903$

Q.42 **(1.56)** By Newton-Raphson method; the iterative formula for finding approximate root at $(n+1)^{th}$ iteration

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}\text{ ; where } x = 0,1,2 \ldots$$

Putting n = 0 ; then $x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)}$

Let $f(x) = x - 10\cos x \Rightarrow f(x_0) = f\left(\dfrac{\pi}{4}\right) = \dfrac{\pi}{4} - \dfrac{10}{\sqrt{2}}$

$$f'(x) = 1 + 10\sin x$$

$$\Rightarrow f'(x_0) = f'\left(\frac{\pi}{4}\right) = 1 + \frac{10}{\sqrt{2}} \quad \Rightarrow \quad x_1 = \frac{\pi}{4} - \left[\frac{\dfrac{\pi}{4} - \dfrac{10}{\sqrt{2}}}{1 + \dfrac{10}{\sqrt{2}}}\right] \cong 1.56$$

Q.43 **(–6)**

$x_1^{(1)} - 0 - 0 = 5$ ... (1) ; $\quad 2x_1^{(1)} + 3x_2^{(1)} + 0 = 1$ .... (2) ; $\quad 3x_1^{(1)} + 2x_2^{(1)} + x_3^{(1)} = 3$ .... (3)

$\therefore$ From equation (1) $x_1^{(1)} = 5$

$\therefore$ From equation (2) $2x_1^{(1)} + 3x_2^{(1)} = 1$

$= 3x_2^{(1)} = 1 - 2x_1^{(1)} \quad = 1 - 2(5) \quad \Rightarrow \quad 3x_2^{(1)} = -9$

From equation (3), $x_3^{(1)} = 3 - 3x_1^{(1)} - 2x_2^{(1)} = 3 - 3(5) - 2(-3) = -6$

After the first iteration, the value of is $x_3 = -6$.

Q.44 **(a)**

Q.45 **(0.178)**

Q.46 **(c)** We have $x_{n+1} = X_n - \dfrac{f(X_n)}{f'(X_n)}$

For $n = 0,\ x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)}$; $f(x) = x^3 + x - 1 \Rightarrow f'(x) = 3x^2 + 1$

given $x_0 = 1 \quad \Rightarrow \ f(x_0) = f(1) = 1,\ f'(x_0) = f'(1) = 4 \ \Rightarrow x_1 = 1 - \dfrac{1}{4} = \dfrac{3}{4} = 0.75$

Q.47 **(0.36)** Let f(x) = 3x – e^x + sin x and $x_0 = 0.333 = \dfrac{1}{3}$

f ’(x) = 3 – e^x + cosx

$\text{for } x_0 = 0.333 \Rightarrow \text{for } f(x_0) = -0.069 \text{ and } f'(x_0) = 2.55$

(Using Newton–Rapshon method)

$= 0.333 + \dfrac{0.069}{2.55} = 0.360$ is the required next approximation

Q.48 **(0.06 to 0.063)**

Q.49 **(a)** By Trapezoidal Rule

$$\int_0^1 f(x)dx = \frac{h}{2}[(y_0 + y_2) + 2(y_1)] \quad = \frac{0.5}{2}[(3+5) + 2\times 4] = 4$$

By simpson $\left(\dfrac{1}{3}\right)^{rd}$ rule

$$\int_0^1 f(x)\,dx = \frac{h}{3}[(y_o + y_2) + 4(y_1)] \ = \frac{0.5}{3}[(3+5) + 4\times 4] = 4$$

Q.50 **0.793**

From Euler method; we have

$$\frac{dy}{dx} = \frac{1 - e^{-\frac{10}{3}\ln(y)}}{250 - 45e^{-3\ln(y)}} \ ; \ y_1 = y_0 + hf(x_0, y_0)$$

Given D.E is $\dfrac{dy}{dx} = \dfrac{1 - e^{-\frac{10}{3}\ln(y)}}{250 - 45e^{-3\ln(y)}}$ & intial condition $x_0 = 0, \quad y_0 = 0.8$

The above D.E is of the form $\dfrac{dy}{dx} = f(x,y)$

where $f(x, y) = \dfrac{1 - e^{-\frac{10}{3}\ln(y)}}{250 - 45e^{-3\ln(y)}}$ & $y \rightarrow$ depth of flow

$\therefore$ The depth of flow at $x = 1m$ is

$y(1) = 0.8 + 1.f(0,0.8) \quad [\because \text{form (1) } h = 1,\ x_0 = 0 \ ; y_0 = 0.8]$

Q.51 **(1)** $f(x) = 2x^2 - 3x + 3$

$f'(x) = 4x - 3$; Given $x = 2$

New estimate of 'x' after 1st iteration $x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)} = 2 - \dfrac{f(2)}{f'(2)} = 2 - \dfrac{5}{5} = 1$

Q.52 **(2.5937)** $f(x, y) = y, \quad h = 0.1, \Rightarrow y_1 = y_0 + 0.1y_0 = (1 + 0.1)y_0$

$\Rightarrow y_{10} = (1 + 0.1)^{10} y_0$, Given $y_0 = 1 \Rightarrow y_{10} = (1 + 0.1)^{10} = 2.5937$

ESE SOLUTIONS

Q.1 (a) Forward difference table for the given data

$$\begin{array}{ccccc}
x & f(x) & D & D^2 & D^3 \\
0 & 3 & & & \\
 & & 3 & & \\
1 & 6 & & 2 & \\
 & & 5 & & 0 \\
2 & 11 & & 2 & \\
 & & 7 & & \\
3 & 18 & & &
\end{array}$$

By using Newton's forward interpolation

$$f(x) = f(0) + x\Delta f(0) + \frac{x(x-1)^2}{2!}\Delta^2 f(0) = 3 + 3x + \frac{x(x-1)}{2} \times 2 = x^2 + 2x + 3$$

$$f(x) = x^2 + 2x + 3$$

Given data satisfies only first option

i.e $f(x) = x^2 + 2x + 3 = f(0) = 3$; $f(1) = 6$; $f(2) = 11$; $f(3) = 18$

Q.2 **(c)** Let $\sqrt[3]{1468} = x \Rightarrow x^3 - 1468 = 0 \Rightarrow f'(x) = 3x^2$

By Newton Raphson method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{(x_n^3 - 1468)}{3x_n^2} = x_{n+1} = \frac{2x_n^3 + 1468}{3x_n^2}$$

Let $x_0 = 11$

$$x_1 = \frac{2(11)^3 + 1468}{3(11)^2} = \frac{2(11)^3 + 11^3 + 137}{3(11)^2} = \frac{3(11)^3}{3(11)^2} + \frac{137}{3(11)^2} = 11 + 0.36 = 11.36$$

Q.3 Let $(1525)^{0.2} = x$, i.e, $x^5 = 1525 \Rightarrow f(x) = x^5 - 1525 = f'(x) = 5x^4$

$$x_{n+1} = x_n - \left(\frac{f(x_n)}{f'(x_n)}\right) = x_n - \left(\frac{x_n^5 - 1525}{5x_n^4}\right) = \frac{4}{5}x_n + \frac{305}{x_n^4}$$

Let $x_0 = 4.3$, $x_1 = \frac{4}{5}x_0 + \frac{305}{x_0^4} = \frac{4}{5}(4.3) + \frac{305}{(4.3)^4} = 4.33$

8 LAPLACE TRANSFORM

8.1 TRANSFORM THEORY

The transform of a function $f(t)$ is defined as

$$T\{f(t)\} = \int_{-\infty}^{\infty} k(s, t) f(t)\, dt$$

Where, $k(s, t)$, is known as kernel of the transform.

8.1.1 Laplace Transform

For the Laplace transform, the kernel $k(s, t)$ is e^{-st} and Laplace transform of the function $f(t)$ is given as

$$L(f(t)) = F(s) = \int_{-\infty}^{\infty} e^{-st} f(t)\, dt$$

Provided that the integral exists. S is a parameter which may be a real or complex number. $L(f(t)) = F(S)$ being clearly a function of s is briefly written as f(s).

Unilateral Transform:

The Unilateral Laplace transform of the function $f(t)$ is given by

$$L(f(t)) = F(s) = \int_{0}^{\infty} e^{-st} f(t)\, dt,\ 0 < t < \infty$$

Bilateral Transform :

The Bilateral Laplace transform of the function $f(t)$ is given by

$$L(f(t)) = F(s) = \int_{-\infty}^{\infty} e^{-st} f(t)\, dt\ ,\ -\infty < t < \infty$$

Existence of Laplace transform :

1. The Laplace transform of f(t) i.e. $\int_0^{\infty} e^{-st} f(t)\, dt$ exists if $\int_0^{\lambda} e^{-st} f(t) dt$ can be evaluated as $\lambda \to \infty$.

2. If f(t) is continuous and $\lim_{t\to\infty}[e^{-at} f(t)]$ is finite. Then the Laplace transform of f(t) i.e. $\int_0^{\infty} e^{-st} f(t)\, dt$ exists for S > a.

8.3 TRANSFORMS OF ELEMENTARY FUNCTIONS

(1) $L(1)=\frac{1}{s}$ $(s>0)$

(2) $L(t^n)=\frac{n!}{s^{n+1}}$, When n=0, 1, 2, 3,... $\left[\text{Otherwise } L(t^n)=\frac{\overline{|(n+1)}}{s^{n+1}}\right]$; $n>-1$

(3) $L(e^{at})=\frac{1}{(s-a)}$ $(s>a)$

(4) $L(\sin at)=\frac{a}{s^2+a^2}$ $(s>0)$

(5) $L(\cos at)=\frac{s}{s^2+a^2}$ $(s>0)$

(6) $L(\sinh at)=\frac{a}{s^2-a^2}$ $(s>|a|)$

(7) $L(\cosh at)=\frac{s}{s^2-a^2}$ $(s>|a|)$

Proof:

(1) $L(1)=\frac{1}{s}$

Proof. $L(1)=\int_0^\infty 1.e^{-st}dt=\left[\frac{e^{-st}}{-s}\right]_0^\infty=\frac{1}{s}$

Hence, $L(1)=\frac{1}{s}$

(2) $L(t^n)=\frac{n!}{s^{n+1}}$

Proof. $L(t^n)=\int_0^\infty e^{-st}\,t^n\,dt$

Putting $st=x$ or $t=\frac{x}{s}$ or $dt=\frac{dx}{s}$

Thus, we have, $L(t^n)=\int_0^\infty e^{-x}\left(\frac{x}{s}\right)^n\frac{dx}{s}$

$$L(t^n)=\frac{1}{s^{n+1}}\int_0^\infty e^{-x}x^n dx$$

$$L(t^n)=\frac{n!}{s^{n+1}}\qquad\left[\begin{array}{l}\overline{|n+1}=\int_0^\infty e^{-x}x^n dx\\ \overline{|n+1}=n!\end{array}\right]$$

(3) $L(e^{at})=\frac{1}{s-a}$ where $s>a$

Proof. $L\left(e^{at}\right) = \int_0^\infty e^{-st}.\, e^{at}\ dt = \int_0^\infty e^{-st+at}.\, dt$

$= \int_0^\infty e^{(-s+a)t}\ dt = \int_0^\infty e^{-(s-a)t}.\, dt = \left[\dfrac{e^{-(s-a)t}}{-(s-a)}\right]_0^\infty = -\dfrac{1}{s-a}\left[\dfrac{1}{e^{(s-a)t}}\right]_0^\infty$

$= \dfrac{-1}{(s-a)}(0-1) = \dfrac{1}{s-a}$

(4) $L\,(\cosh at) = \dfrac{s}{s^2-a^2}$

Proof. $L(\cosh\ at) = L\left[\dfrac{e^{at}-e^{-at}}{2}\right]$ $\quad\left(\cosh\ at = \dfrac{e^{at}+e^{-at}}{2}\right)$

$= \dfrac{1}{2}L\left(e^{at}\right) + \dfrac{1}{2}L\left(e^{-at}\right)$

$= \dfrac{1}{2}\left[\dfrac{1}{s-a}+\dfrac{1}{s+a}\right]$ $\quad\left[L\left(e^{at}\right) = \dfrac{1}{s-a}\right]$

$= \dfrac{1}{2}\left[\dfrac{s+a+s-a}{s^2-a^2}\right] = \dfrac{s}{s^2-a^2}$

(5) $L\,(\sinh\ at) = \dfrac{a}{s^2-a^2}$

Proof. $L(\sinh\ at) = L\left(\dfrac{e^{at}-e^{-at}}{2}\right)$

$= \dfrac{1}{2}\left[L\left(e^{at}\right) - L\left(e^{-at}\right)\right]$

$= \dfrac{1}{2}\left[\dfrac{1}{s-a}-\dfrac{1}{s+a}\right] = \dfrac{1}{2}\left[\dfrac{s+a-s+a}{s^2-a^2}\right]$

$= \dfrac{a}{s^2-a^2}$

(6) $L\,(\sin at) = \dfrac{a}{s^2+a^2}$

Proof. $L\,(\sin at) = L\left[\dfrac{e^{at}-e^{-at}}{2i}\right]$ $\quad\left[\sin at = \dfrac{e^{iat}-e^{-iat}}{2i}\right]$

$= \dfrac{1}{2i}\left[L\left(e^{iat}-e^{-iat}\right)\right] = \dfrac{1}{2i}\left[L\left(e^{iat}\right) - L\left(e^{-iat}\right)\right]$

$= \dfrac{1}{2i}\left[\dfrac{1}{s-ia}-\dfrac{1}{s+ia}\right] = \dfrac{1}{2i}\ \dfrac{s+ia-s+ia}{s^2+a^2}$

$= \dfrac{1}{2i}\ \dfrac{2ia}{s^2+a^2} = \dfrac{a}{s^2+a^2}$

(7) $L(\cos at) = \dfrac{s}{s^2 + a^2}$

Proof. $L(\cos at) = L\left(\dfrac{e^{iat} + e^{-iat}}{2}\right)$ $\qquad \left[\cos at = \dfrac{e^{iat} + e^{-iat}}{2}\right]$

$$= \frac{1}{2}\left[L\left(e^{iat} + e^{-iat}\right)\right] = \frac{1}{2}\left[L\left(e^{iat}\right) + L\left(e^{-iat}\right)\right]$$

$$= \frac{1}{2}\left[\frac{1}{s - ia} + \frac{1}{s + ia}\right] = \frac{1}{2}\,\frac{s + ia + s - ia}{s^2 + a^2}$$

$$= \frac{s}{s^2 + a^2}$$

8.4 PROPERTIES OF LAPLACE TRANSFORMS

(1) LINEARITY PROPERTY

If a,b,c be any constants and f,g,h any functions of t, then

L[af(t) + bg(t) + ch(t)]= aL{f(t)} + bL{g(t)} + cL{h(t)}

For by definition,

$$\text{L.H.S} = \int_0^\infty e^{-st}\left[af(t) + bg(t) + ch(t)\right] dt = a\int_0^\infty e^{-st} f(t)\, dt + b\int_0^\infty e^{-st} g(t)\, dt + c\int_0^\infty e^{-st} h(t)\, dt$$

This result can easily be generalized.

Because of the above property of Laplace transform, L is called a linear operator.

(2) FIRST SHIFTING PROPERTY

If L[f(t)]= F(s), then

$$L\left\{e^{at} f(t)\right\} = F(s - a).$$

Proof : $L\left\{e^{at} f(t)\right\} = \int_0^\infty e^{at} e^{-st} f(t)\, dt = \int_0^\infty e^{-(s-a)t} f(t)\, dt = F(s - a)$

(1) $L(e^{at}) = \dfrac{1}{(s - a)}$ $\qquad \left[\because L(1) = \dfrac{1}{S}\right]$

(2) $L(e^{at} t^n) = \dfrac{n!}{(s - a)^{n+1}}$ $\qquad \left[\because L(t^n) = \dfrac{n!}{s^{n+1}}\right]$

(3) $L(e^{at} \sin bt) = \dfrac{b}{(s - a)^2 + b^2}$ $\qquad \left[\because L(\sin bt) = \dfrac{b}{s^2 + b^2}\right]$

(4) $L(e^{at} \cos bt) = \dfrac{s - a}{(s - a)^2 + b^2}$ $\qquad \left[\because L(\cos bt) = \dfrac{s}{s^2 + b^2}\right]$

(5) $L(e^{at}\sinh bt) = \dfrac{b}{(s-a)^2 - b^2}$ $\left[\because L(\sinh bt) = \dfrac{b}{s^2 - b^2}\right]$

(6) $L(e^{at}\cosh bt) = \dfrac{s-a}{(s-a)^2 - b^2}$ $\left[\because L(\cosh bt) = \dfrac{s}{s^2 - b^2}\right]$

Where in each case s > a.

Example : Find the laplace transforms of

(i) sin 2t sin 3t

(ii) $\cos^2 2t$

Solution : **(i)** Since $\sin 2t \sin 3t = \dfrac{1}{2}[\cos t - \cos 5t]$

Therefore

$$L(\sin 2t \sin 3t) = \frac{1}{2}[L(\cos t) - L(\cos 5t)] = \frac{1}{2}\left[\frac{S}{S^2+1^2} - \frac{S}{S^2+5^2}\right] = \frac{12S}{(S^2+1)(S^2+25)}$$

(ii) Since $\cos^2 2t = \dfrac{1}{2}(1 + \cos 4t)$

Therefore

$$L(\cos^2 2t) = \frac{1}{2}[L(1) + L(\cos 4t)] = \frac{1}{2}\left(\frac{1}{s} + \frac{s}{s^2+16}\right)$$

(3) CHANGE OF SCALE PROPERTY

If $L[f(t)] = F(s)$, Then $L[f(at)] = \dfrac{1}{a}F\left(\dfrac{s}{a}\right)$

Proof : $L\{f(at)\} = \int_0^\infty e^{-st} f(at)dt$ Put $at = u$

$dt = du/a$

$$= \int_0^\infty e^{-su/a} f(u)du/a$$

$$= \frac{1}{a}\int_0^\infty e^{-su/a} f(u)du = \frac{1}{a}F\left(\frac{s}{a}\right)$$

Example : Find the Laplace transform of f(t)=cos4t

Solution : $L\{f(at)\} = \dfrac{1}{a}F\left(\dfrac{s}{a}\right)$

$L(\cos t) = \dfrac{s}{s^2+1}$

Thus, $L(\cos 4t) = \left[\dfrac{s}{s^2+16}\right]$

(4) TRANSFORMS OF DERIVATIVES

- If f ' (t) be continuous and L{f(t)} = F(s), then

L{f '(t)} = s F(s) – f(0).

- If f' (t) and its first (n – 1) derivatives be continuous, then

$$L\{f^n(t)\} = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \dots - f^{n-1}(0)$$

(5) TRANSFORMS OF INTEGRALS

$$\text{If } L\{f(t)\} = F(s), \text{ Then } L\left\{\int_0^t f(u)\,du\right\} = \frac{1}{s}F(s)$$

Example : Find the Laplace transform of f(t)=1 when t>0

Solution : $L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$

$$L\{1\} = \int_0^\infty e^{-st}\cdot 1\,dt = \left.\frac{e^{-st}}{-s}\right|_0^\infty = \frac{1}{s}$$

(6) MULTIPLICATION BY t^n

$$\text{If } L\{f(t)\} = F(s), \text{ then}$$

$$L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}\left[F(s)\right], \quad \text{where } n = 1, 2, 3\dots.$$

Example : Find the Laplace transform of t^2 cos at

Solution :

$$L\left(t^2 \cos at\right) = (-1)^2 \frac{d^2}{ds^2} = \left[\frac{s}{s^2 + a^2}\right]$$

$$= \frac{d}{ds}\,\frac{\left(s^2 + a^2\right) \times -s(2s)}{\left(s^2 + a^2\right)^2} = \frac{d}{ds}\,\frac{a^2 - s^2}{\left(s^2 + a^2\right)}$$

$$= \frac{\left(s^2 + a^2\right)^2(-2s) - \left(a^2 - s^2\right)\times 2\left(s^2 + a^2\right)^2(2s)}{\left(s^2 + a^2\right)^4}$$

$$= \frac{-2s^3 - 2a^2 s - 4a^2 s + 4s^3}{\left(s^2 + a^2\right)^3} = \frac{2s\left(s^2 - 3a^2\right)}{\left(s^2 + a^2\right)^3}$$

Example : Obtain the Laplace transform of $t^2 e^t \sin 4t$.

Solution : $L(\sin 4t) = \dfrac{4}{s^2+16}$ $L(e^t \sin 4t) = \dfrac{4}{(s-1)^2+16}$

$$L(t\,e^t \sin 4t) = -\frac{d}{ds}\,\frac{4}{s^2-2s+17} = \frac{4(2s-2)}{(s^2-2s+17)^2}$$

$$L(t^2\,e^t \sin 4t) = -4\frac{d}{ds}\frac{2s-2}{(s^2-2s+17)^2}$$

$$= 4\,\frac{(s^2-2s+17)^2\,2-(2s-2)\,2(s^2-2s+17)(2s-2)}{(s^2-2s+17)^4}$$

$$= \frac{-4(2s^2-4s+34-8s^2+16s-8)}{(s^2-2s+17)^2}$$

$$= \frac{-4(-6s^2+12s+26)}{(s^2-2s+17)^3} = \frac{8(3s^2-6s-13)}{(s^2-2s+17)^3}$$

(7) DIVISION BY t

If $L\{f(t)\} = F(s)$, then $L\left\{\dfrac{1}{t}f(t)\right\} = \displaystyle\int_s^\infty F(s)\,ds$

provided the integral exists.

Example : Laplace transform of function $\left\{\dfrac{1}{t}(1-e^{-t})\right\}$ is

(a) $\log_e(s(s+1))$ (b) $\log_e\left(\dfrac{s+1}{s}\right)$

(c) $\log_{10}\left(\dfrac{s}{s+1}\right)$ (d) $\log_{10}(s(s+1))$

Solution (B) $= L\{(1-e^{-t})\} = \dfrac{1}{s} - \dfrac{1}{s+1} = \dfrac{1}{s(s+1)}$

$$= \left\{\frac{1}{t}(1-e^{-t})\right\} = \int_s^\infty L\{1-e^{-t}\}\,ds$$

$$= \int_s^\infty \left(\frac{1}{s} - \frac{1}{(s+1)}\right)ds$$

$$= \left[\log_e s - \log_e(s+1)\right]_s^\infty$$

$$= \left[\log_e\left(\frac{s}{s+1}\right)\right]_s^\infty \quad = \log_e\left(\frac{s+1}{s}\right)$$

EVALUATION OF IMPROPER INTEGRALS : We can evaluate improper integral of type-I with the help of Laplace transform.

Example : Evaluate $\int_0^\infty t\,e^{-3t}\sin t\,dt$

Solution : $\int_0^\infty t\,e^{-3t}\sin t\,dt = \int_0^\infty t\,e^{-st}\sin t\,dt \qquad (s=3)$

$$L(t\sin t) = -\frac{d}{ds}\left(\frac{1}{s^2+1}\right) = \frac{2s}{\left(s^2+1\right)^2}$$

$$= \frac{2\times 3}{\left(3^2+1\right)^2} = \frac{6}{100} = \frac{3}{50}$$

Example : Evaluate $\int_0^\infty \frac{e^{-t}\sin t}{t}\,dt$ and $\int_0^\infty \frac{\sin t}{t}dt$

Solution : $\int_0^\infty \frac{e^{-t}\sin t}{t}\,dt = \int_0^\infty e^{-st}\frac{\sin t}{t}dt \qquad (s=1)$

$$L\left[\frac{\sin t}{t}\right] = \int_s^\infty \frac{1}{s^2+1}ds = \left[\tan^{-1}s\right]_s^\infty$$

$$= \frac{\pi}{2} - \tan^{-1}s \qquad ....(1)$$

Putting $s = 1$ in (1), we get

$$\int_0^\infty \frac{e^{-t}\sin t}{t}\,dt = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$

on substituting $s = 0$ in (1), we get

$$\int_0^\infty \frac{\sin t}{t}dt = \frac{\pi}{2} - \tan^{-1}(0) = \frac{\pi}{2}$$

(8) MODULATION PROPERTY

$$L\{f(t)\ \sin at\} = \frac{1}{2}\left[F(s-ia) - F(s+ia)\right]$$

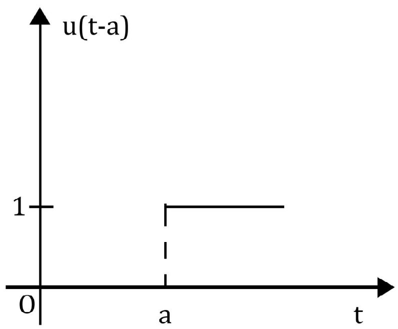
$$L\{f(t)\ \cos at\} = \frac{1}{2}\left[F(s-ia) + F(s+ia)\right]$$

8.5 STANDARD FUNCTIONS

UNIT STEP FUNCTION :

At times, we come across such fractions of which the inverse transform cannot be determined from the formulae so far derived. In order to cover such cases, we introduce the unit step function (or Heaviside's unit function which is named after the British Engineer Oliver Heaviside)

Def: The unit step function $u(t-a)$ is defined as follows:



$$U(t-a)=\begin{cases}0 \text{ for } t<a\\ 1 \text{ for } t\geq a\end{cases}$$

Where, a is always positive. It is also denoted as $H(t-a)$.

LAPLACE TRANSFORM OF UNIT STEP FUNCTION

$$L[u(t-a)]=\frac{e^{-as}}{s}$$

$$L[u(t-a)]=\int_0^\infty e^{-st}u(t-a)dt$$

$$=\int_0^a e^{-st}\cdot 0\,dt+\int_a^\infty e^{-st}\times 1dt = 0+\left[\frac{e^{-st}}{-s}\right]_a^\infty$$

$$L[u(t-a)]=\frac{e^{-as}}{s}$$

The product $f(t-a)u(t-a)=\begin{cases}0 & \text{for } t<a\\ f(t) & \text{for } t\geq a\end{cases}$

The function $f(t-a)u(t-a)$ represents the graph of f(t) shifted through a distance a to the right and is of special importance.

Example : Express the following function in terms of unit step function :

$$f(t)=\begin{bmatrix}t-1 & 1<t<2\\ 3-t & 2<t<3\end{bmatrix} \quad \text{and find its Laplace transform.}$$

Solution : $f(t)=\begin{bmatrix}t-1 & 1<t<2\\ 3-t & 2<t<3\end{bmatrix}$

$$=(t-1)[u(t-1)-u(t-2)]+(3-t)[u(t-2)-u(t-3)]$$

$$=(t-1)u(t-1)-(t-1)u(t-2)+(3-t)u(t-3)u(t-3)$$

$$=(t-1)u(t-1)-2(t-2)u\,(t-2)+(3-t)u(3-t)$$

$$L\{f(t)\}=\frac{e^{-s}}{s^2}-2\frac{e^{-2s}}{s^2}+\frac{e^{-3s}}{s^2}$$

SECOND SHIFTING PROPERTY

If $L\{f(t)\} = F(s)$, then

$L\{f(t-a)u(t-a)\} = e^{-as}F(s)$

Proof : $L\{f(t-a)u(t-a)\} = \int_0^\infty e^{-st} f(t-a)\; u(t-a)\,dt$

$= \int_0^\infty e^{-st} f(t-a)(0)\,dt + \int_0^\infty e^{-st} f(t-a)\,dt$ [Put $t - a = u$]

$= \int_0^\infty e^{-s(u+a)} f(u)\,du = e^{-as}\int_0^\infty e^{-su} f(u)\,du = e^{-as}F(s).$

Example : Find Laplace Transform of $t^2 u(t-3)$

Solution : $t^2 u(t-3) = \left[(t-3)^2 + 6(t-3) + 9\right] u(t-3)$

$= (t-3)^2 u(t-3) + 6(t-3)\,u(t-3) + 9u(t-3)$

$L\{t^2 u(t-3)\} = L\{(t-3)^2 u(t-3)\} + 6L\{(t-3)\,u(t-3)\} + 9L\{u(t-3)\}$

$= e^{-3s}\left[\frac{2}{s^3} + \frac{6}{s^2} + \frac{9}{s}\right]$

UNIT IMPULSE FUNCTION

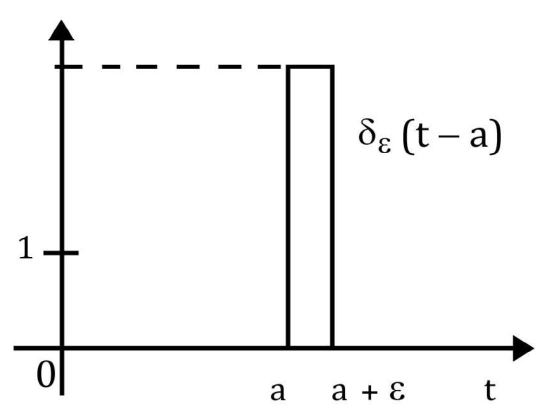
The idea of very large force acting for a very short time is of frequent occurrence in mechanics. To deal with such and similar ideas, we introduce the unit impulse function (also called Dirac delta function, after the English physicist Paul Dirac who has awarded the Nobel prize in 1933 for his work in Quantum mechanics.)

Thus unit impulse function is considered as the limiting form of the function:

$\delta_\varepsilon(t-a) = 1/\varepsilon,$ $\quad a \le t \le a+\varepsilon$

$= 0,$ $\quad$ otherwise

As $\varepsilon \to 0$. It is clear from figure that as $\varepsilon \to 0$, the height of the strip increases indefinitely and the width decreases in such a way that its area is always unity.



The Unit Impulse function is defined as follows :

$$\delta(t-a) = \begin{cases} \infty & \text{for } t = a \\ 0 & \text{for } t \neq a \end{cases}$$

$$\int_{-\infty}^{\infty} \delta(t-a)\, dt = 1 \qquad [\text{Area of strip} = 1]$$

LAPLACE TRANSFORM OF IMPULSE FUNCTION

$$\int_{-\infty}^{\infty} f(t)\ \delta(t-a)\ dt = \int_{a}^{a+\varepsilon} f(t)\frac{1}{\varepsilon}dt \qquad \begin{cases} \text{Mean value Theorem} \\ \int_a^b f(t)\, dt = (b-a) f(\eta) \end{cases}$$

$$= (a + \varepsilon - a) f(\eta)\frac{1}{\varepsilon} \qquad \textit{where } a < \eta < a + \varepsilon$$

$$= f(\eta)$$

Property I : $\int_{-\infty}^{\infty} f(t)\ \delta(t-a)dt = f(a)$

Example : Evaluate $\int_0^{\infty} e^{-5t}\delta(t-2)dt$

Solution : $\int_0^{\infty} e^{-5t}\delta(t-2)dt = e^{-5\times 2} = e^{-10}$

Property II : $\int_{-\infty}^{\infty} f(t)\delta'(t-a)\, dt = -f'(a)$

Proof : $\int_{-\infty}^{\infty} f(t)\ \delta'(t-a)dt = \left[f(t)\ \delta(t-a)\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f'(t)\delta(t-a)dt$

$= 0 - 0 - f'(a) = -f'(a)$

Example : Find the Laplace transform of $t^3\,\delta(t-4)$.

Solution : $L\{t^3\,\delta(t-4)\} = \int_0^{\infty} e^{-st}t^3\,\delta(t-4)dt$

$= 4^3 e^{-4s}$

CONVOLUTION THEOREM

If $L\{f(t)\} = F(s)$, and $L\{g(t)\} = G(s)$,

Then $L\{f(t) * g(t)\} = L\left\{\int_0^t f(u)g(t-u)du\right\} = F(s)G(s)$

[* is represent the **convolution** or **falting** of f and g.]

Example : The value of $\frac{Y(s)}{U(s)}$ is __________, where input–output relation of a system is given as,

$$y(t) = \int_{-\infty}^{+\infty} u(J)\sin\omega(t-J)dJ$$

(a) $\frac{s}{(s^2+\omega^2)}$ (b) $\frac{s}{\sin\omega}$ (c) $\frac{\omega}{(s^2+\omega^2)}$ (d) $\frac{2}{\sin\omega}$

Solution : **(c)**

$$y(t)=\int_{-\infty}^{+\infty}u(J)\sin\omega(t-J)\,dJ$$

$$y(t)=u(t)*\sin\omega t$$

on taking Laplace transform

$$Y(s)=U(s)L\{\sin\omega t\}$$

$$\frac{Y(s)}{U(s)}=\frac{\omega}{(s^2+\omega^2)}$$

LAPLACE TRANSFORMS OF PERIODIC FUNCTIONS

Let $f(t)$ *be* a periodic function with Period T, then

$$L[f(t)]=\frac{\int_0^T e^{-st}f(t)\,dt}{1-e^{-sT}}$$

Example : Find the Laplace transform of the waveform

$$f(t)=\left(\frac{2t}{3}\right),\ 3n\le t\le 3(n+1),\ n=0,1,2,.......$$

Solution : $L[f(t)]=\frac{1}{1-e^{-st}}\int_0^T e^{-st}f(t)\,dt$

$$L[f(t)]=\frac{1}{1-e^{-3s}}\int_0^3 e^{-st}\left(\frac{2}{3}t\right)dt=\frac{1}{1-e^{-3s}}\frac{2}{3}\left[\frac{te^{-st}}{-s}-(1)\frac{e^{-st}}{s^2}\right]_0^3$$

$$=\frac{2}{3}\frac{1}{1-e^{-3s}}\left[\frac{3e^{-3s}}{-s}-\frac{e^{-3s}}{s^2}+\frac{1}{s^2}\right]=\frac{2}{3}\cdot\frac{1}{1-e^{-3s}}\left[\frac{3e^{-3s}}{-s}-\frac{1-e^{-3s}}{s^2}\right]$$

$$=\frac{2e^{-3s}}{-s(1-e^{-3s})}+\frac{2}{3s^2}.$$

APPLICATION OF LAPLACE TRANSFORM

1. Initial value theorem
2. final value theorem
3. to solve differential equations

INITIAL VALUE THEOREM

Let f(t) and its derivative are Laplace Transformable, then

$$\lim_{t\to 0}f(t)=\lim_{s\to\infty}sF(s)$$

Example : If $L[f(t)]=\frac{2s+13}{s(s^2+8)}$, then $\frac{df(0^+)}{dt}$ is

a) 1 b) 2 c) 13 d) 8

Solution : (b)

$$f(0^+) = \lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s)$$

$$= \lim_{s\to\infty} s\frac{(2S+13)}{S(S^2+8)} = 0$$

$$L(f'(t)) = SF(s) - f(0^+) = \frac{(2S+13)}{(S^2+8)}$$

$$f'(0^+) = \lim_{t\to 0} f'(t) = \lim_{s\to\infty} SL\{f'(t)\}$$

$$= \lim_{s\to\infty} s\frac{(2s+13)}{(s^2+8)} = 2$$

FINAL VALUE THEOREM

Let f(t) and its derivative are Laplace Transformable, then

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$$

Note : **i)** For applying final value theorem, it is required that all the poles of sF(s) strictly be in the left half of s-plane i.e. poles at $j\omega$ axis is also not allowed.

ii) F(s) must not have any poles on imaginary axis, except at most one pole at s=0.

TO SOLVE DIFFERENTIAL EQUATIONS

The Laplace transform method of solving differential equations yields particular solutions without the necessity of first finding the general solution and then evaluating the arbitrary constants. This method is, in general, shorter than our earlier methods and is especially useful for solving linear differential equations with constant coefficients.

To solve a linear differential equation with constant coefficients by Laplace transform method :

(1) Take the Laplace transform of both sides of the differential equation using the formula of Transform of Derivates, and the given initial conditions.

(2) Transpose the terms with minus signs to the right.

(3) Divide by the coefficient of y, getting y as a known function of s.

(4) Resolve this function of s into partial fraction and take the inverse transform of both sides. This gives y as a function of t which is the desired solution satisfying the given conditions.

Example : Solve by the method of transforms, the equation y'''+2y''-y'-2y=0 given y(0)= y'(0) = 0 and y''(0) = 6.

Solution : Taking the Laplace transform of both sides, we get

$$\left[s^3\bar{y} - s^2y(0) - sy'(0) - y''(0)\right] + 2\left[s^2\bar{y} - sy(0) - y'(0)\right] - \left[s\bar{y} - y(0)\right] - 2\bar{y} = 0$$

Using the given conditions, it reduces to

$$(s^3 + 2s^2 - s - 2)\bar{y} = 6$$

Therefore, $\bar{y} = \dfrac{6}{(s-1)(s+1)(s+2)} = \dfrac{6}{(s-1)(6)} + \dfrac{6}{(-2)(s+1)} + \dfrac{6}{3(s+2)}$

On taking inverse Laplace Transform

$y(t) = L^{-1}\dfrac{1}{(s-1)} - 3L^{-1}\left(\dfrac{1}{s+1}\right) + 2L^{-1}\left(\dfrac{1}{s+2}\right)$ or $y = e^{t} - 3e^{-t} + 2e^{-2t}$ which is the desired result.

Remarks : We will learn inverse Laplace Transform below

INVERSE LAPLACE TRANSFORM

If the Laplace transform of f(t) is F(s), then the inverse laplace transform of F(s) is f(t).

(1) $L^{-1}\left[\dfrac{1}{s}\right] = 1.$

(2) $L^{-1}\left[\dfrac{1}{s-a}\right] = e^{at}$

(3) $L^{-1}\left[\dfrac{1}{s^n}\right] = \dfrac{t^{n-1}}{(n-1)!}, n = 1, 2, 3, ..$

(4) $L^{-1}\left[\dfrac{1}{(s-a)^n}\right] = \dfrac{e^{at}t^{n-1}}{(n-1)!}.$

(5) $L^{-1}\left[\dfrac{1}{s^2+a^2}\right] = \dfrac{1}{a}\sin at.$

(6) $L^{-1}\left[\dfrac{s}{s^2+a^2}\right] = \cos at.$

(7) $L^{-1}\left[\dfrac{1}{s^2-a^2}\right] = \dfrac{1}{a}\sinh at.$

(8) $L^{-1}\left[\dfrac{s}{s^2-a^2}\right] = \cosh\ at$

(9) $L^{-1}\left[\dfrac{1}{(s-a)^2+b^2}\right] = \dfrac{1}{b}e^{at}\sin bt.$

(10) $L^{-1}\left[\dfrac{s-a}{(s-a)^2+b^2}\right] = \dfrac{1}{b}e^{at}\cos bt.$

(11) $L^{-1}\left[\dfrac{s}{\left(s^2+a^2\right)^2}\right] = \dfrac{1}{2a}t\sin at$

(12) $L^{-1}\left[\dfrac{1}{\left(s^2+a^2\right)^2}\right] = \dfrac{1}{2a^2}(\sin at - at\cos at)$

(13) $L^{-1}(1) = \delta(t)$

Example : Find the inverse transform of

$$\text{(i)}\ \frac{s^2-3s+4}{s^3} \qquad \text{(ii)}\ \frac{s+2}{s^2-4s+13}$$

Solution : (i) $L^{-1}\left(\frac{s^2-3s+4}{s^3}\right) = L^{-1}\left(\frac{1}{s}\right) - 3L^{-1}\left(\frac{1}{s^2}\right) + 4L^{-1}\left(\frac{1}{s^3}\right)$

$$= 1-3t+4\left(t^2/2!\right) = 1-3t+2t^2$$

(ii) $L^{-1}\left(\frac{s+2}{s^2-4s+13}\right) = L^{-1}\left[\frac{s+2}{(s-2)^2+9}\right]$

$$= L^{-1}\left[\frac{s-2+4}{(s-2)^2+3^2}\right] = L^{-1}\left[\frac{s-2}{(s-2)^2+3^2}\right] + 4\,L^{-1}\left[\frac{1}{(s-2)^2+3^2}\right]$$

$$= e^{2t}\cos 3t + \frac{4}{3}e^{2t}\sin 3t$$

Example : Inverse Laplace transform of $\frac{2s+1}{s^2+s}$ is

a) $1+e^{-t}$ b) $1-e^{-t}$ c) te^{-t} d) e^{-t}

Solution : (a)

$$\frac{2s+1}{s(s+1)} = \frac{1}{s} + \frac{1}{s+1}$$

$$L^{-1}\left\{\frac{2s+1}{s(s+1)}\right\} = 1+e^{-t}$$

(1) Multiplication by S

$$L^{-1}\left[s\,F(s)\right] = \frac{d}{dt}f(t) + f(0)\,\delta(t)$$

Example : Find the inverse Laplace transform of

$$(i)\frac{s}{4s^2-25} \qquad (ii)\frac{3s}{2s+9}$$

Solution : (i) $L^{-1}\left(\frac{1}{4s^2-25}\right) = \frac{1}{4}L^{-1}\left(\frac{1}{s^2-\frac{25}{4}}\right) = \frac{1}{4}.\frac{2}{5}L^{-1}\left(\frac{\frac{5}{2}}{s^2-\left(\frac{5}{2}\right)^2}\right) = \frac{1}{10}\sinh\frac{5}{2}t$

$$L^{-1}\frac{1}{4s^2-25} = \frac{1}{10}\frac{d}{dt}\sinh\frac{5}{2}t + \frac{1}{10}\sinh\frac{5}{2}(0)\,\delta(t)$$

$$= \frac{1}{10}\left(\frac{5}{2}\right)\cosh\frac{5}{2}t = \frac{1}{4}\cosh\frac{5}{2}t$$

$$(ii)\quad L^{-1}\left(\frac{3}{2s+9}\right)=\frac{3}{2}L^{-1}\left(\frac{1}{s+\frac{9}{2}}\right)=\frac{3}{2}e^{-(9/2)t}$$

$$L^{-1}\left(\frac{3s}{2s+9}\right)=\frac{3}{2}\frac{d}{dt}\left(e^{-(9/2)t}\right)+\frac{3}{2}e^{-(9/2)(0)}=\frac{3}{2}\left(-\frac{9}{2}\right)e^{-\left(\frac{11}{2}\right)t}+\frac{3}{2}$$

$$=-\frac{27}{4}e^{-(11/2)t}+\frac{3}{2}$$

(2) Division by S

$$L^{-1}\left[\frac{F(s)}{s}\right]=\int_0^t L^{-1}\left[F(s)\right]dt=\int_0^t f(t)\,dt$$

Example : Find the inverse Laplace transform of

$$(i)\ \frac{1}{s(s+a)}\quad (ii)\ \frac{1}{s(s^2+1)}\quad (iii)\ \frac{s^2+3}{s(s^2+9)}$$

Solution. $(i)\ L^{-1}\left[\frac{1}{s(s+a)}\right]=\int_0^t L^{-1}\left(\frac{1}{s+a}\right)dt=\int_0^t e^{-at}dt=\left[\frac{e^{-at}}{-a}\right]_0^t$

$$=\frac{e^{-at}}{-a}+\frac{1}{a}=\frac{1}{a}\left[1-e^{-at}\right]$$

$$(ii)\quad L^{-1}\frac{1}{s^2+1}=\sin t$$

$$L^{-1}\frac{1}{s}\left(\frac{1}{s^2+1}\right)=\int_0^t L^{-1}\left(\frac{1}{s^2+1}\right)dt=\int_0^t \sin t\,dt=\left[-\cos t\right]_0^t=-\cos t+1$$

$$(iii)\ L^{-1}\frac{s^2+3}{s(s^2+9)}=L^{-1}\left[\frac{s^2+9-6}{s(s^2+9)}\right]=L^{-1}\left[\frac{1}{s}-\frac{6}{s(s^2+9)}\right]$$

$$=1-2\int_0^t \sin 3t\,dt=1-\int_0^t L^{-1}\left(\frac{6}{s^2+9}\right)ds=1+2\times\frac{1}{3}\left[\cos 3t\right]_0^t=1+\frac{2}{3}\cos 3t-\frac{2}{3}$$

$$=\frac{2}{3}\cos 3t+\frac{1}{3}=\frac{1}{3}\left[2\cos 3t+1\right]$$

(3) FIRST SHIFTING PROPERTY

If $L^{-1}F(s)=f(t)$, then $L^{-1}F(s+a)=e^{-at}L^{-1}\left[F(s)\right]$

Example : Find the inverse Laplace transform of

$$\textbf{(i)}\ \frac{1}{(s+2)^5}\quad \textbf{(ii)}\ \frac{s}{s^2+4s+13}\quad \textbf{(iii)}\ \frac{1}{9s^2+6s+1}$$

Solution : $(i)\ L^{-1}\frac{1}{s^5}=\frac{t^4}{4\,!}$

$$(\mathbf{ii})\quad L^{-1}\left(\frac{s}{s^2+4s+13}\right)=L^{-1}\frac{s+2-2}{(s+2)^2+3^2}=L^{-1}\frac{s+2}{(s+2)^2+3^2}-L^{-1}\frac{2}{(s+2)^2+3^2}$$

$$=e^{-2t}\left(L^{-1}\frac{s}{s^2+3^2}-L^{-1}\frac{2}{s^2+3^2}\right)$$

$$=e^{-2t}\left(\cos 3t-\frac{2}{3}\sin 3t\right)$$

$$(\mathbf{iii})\quad L^{-1}\frac{1}{9s^2+6s+1}=L^{-1}\frac{1}{(3s+1)^2}=\frac{1}{9}L^{-1}\frac{1}{\left(s+\frac{1}{3}\right)^2}=\frac{1}{9}e^{-\frac{1}{3}t}L^{-1}\left(\frac{1}{s^2}\right)$$

$$=\frac{1}{9}e^{-\frac{1}{3}t}L^{-1}\left(\frac{1}{s^2}\right)=\frac{1}{9}te^{-\frac{1}{3}t}$$

(4) SECOND SHIFTING PROPERTY

$$\mathrm{L}^{-1}\left[e^{-as}F(s)\right]=f(t-a)\,U(t-a)$$

Example : Obtain inverse Laplace transform of

$$(\mathbf{i})\ \frac{e^{-\pi s}}{(s+3)}\qquad(\mathbf{ii})\ \frac{e^{-s}}{(s+1)^3}$$

Solution : $(i)\ \mathrm{L}^{-1}\dfrac{1}{s+3}=e^{-3t}$

$$\mathrm{L}^{-1}\frac{e^{-\pi s}}{s+3}=e^{-3(t-\pi)}\mathrm{u}(t-\pi)$$

$$(ii)\ \mathrm{L}^{-1}\frac{1}{s^3}=\frac{t^2}{2!}$$

$$\mathrm{L}^{-1}\frac{1}{(s+1)^3}=e^{-t}\frac{t^2}{2!}$$

$$\mathrm{L}^{-1}\frac{e^{-s}}{(s+1)^3}=e^{-(t-1)}\frac{(t-1)^2}{2!}u(t-1)$$

(5) INVERSE LAPLACE TRANSFORMS OF DERIVATIVES

$$\mathrm{L}^{-1}\left[\frac{d}{ds}F(s)\right]=-t\,\mathrm{L}^{-1}\left[F(s)\right]=-t\,f(t)$$

or $$\mathrm{L}^{-1}\left[F(s)\right]=-\frac{1}{t}\mathrm{L}^{-1}\left[\frac{d}{ds}F(s)\right]$$

Example : Find inverse Laplace transform of $\tan^{-1}\dfrac{1}{s}$.

$$L^{-1}\left(\tan^{-1}\frac{1}{s}\right)=-\frac{1}{t}L^{-1}\left[\frac{d}{ds}\tan^{-1}\frac{1}{s}\right]$$

$$= -\frac{1}{t} L^{-1}\left[\frac{1}{1+\frac{1}{s^2}}\left(-\frac{1}{s^2}\right)\right] = \frac{1}{t} L^{-1}\left[\frac{1}{1+s^2}\right]$$

$$= \frac{\sin t}{t}$$

(6) INVERSE LAPLACE TRANSFORM OF INTEGRALS

$$L^{-1}\left[\int_s^{\infty} F(s)ds\right] = \frac{f(t)}{t} = \frac{1}{t} L^{-1}\left[F(s)\right]$$

$$\text{or} \quad L^{-1}\left[F(s)\right] = t\, L^{-1}\left[\int_s^{\infty} F(s)\, ds\right]$$

Example : Obtain $L^{-1} \dfrac{2s}{\left(s^2+1\right)^2}$

Solution : $L^{-1} \dfrac{2s}{\left(s^2+1\right)^2} = t\, L^{-1} \displaystyle\int_s^{\infty} \dfrac{2s\ ds}{\left(s^2+1\right)^2} = t\, L^{-1}\left[\dfrac{1}{s^2+1}\right] = t\, L^{-1}\left[-0 + \dfrac{1}{s^2+1}\right]$

$$= t \sin t$$

(7) PARTIAL FRACTIONS METHOD

Example : Find the inverse transforms of $\dfrac{1}{s^2-5s+6}$

Solution : Let us convert the given function into partial fractions.

$$L^{-1}\left[\frac{1}{s^2-5s+6}\right] = L^{-1}\left[\frac{1}{s-3} - \frac{1}{s-2}\right]$$

$$= L^{-1}\left(\frac{1}{s-3}\right) - L^{-1}\left(\frac{1}{s-2}\right) = e^{3t} - e^{2t}$$

Example : Find the inverse Laplace transforms of $\dfrac{s-1}{s^2-6s+25}$

Solution : $L^{-1}\left(\dfrac{s-1}{s^2-6s+25}\right) = L^{-1}\left[\dfrac{s-1}{(s-3)^2+(4)^2}\right] = L^{-1}\left[\dfrac{s-3+2}{(s-3)^2+(4)^2}\right]$

$$= L^{-1}\left[\frac{s-3}{(s-3)^2+(4)^2}\right] + \frac{1}{2} L^{-1}\left[\frac{4}{(s-3)^2+(4)^2}\right]$$

$$= e^{3t} \cos 4t + \frac{1}{2} e^{3t} \sin 4t$$

(8) INVERSE LAPLACE TRANSFORM BY CONVOLUTION

$$L\left\{\int_0^t f_1(x) * f_2(t-x)\,dx\right\} = F_1(s)\,F_2(s)$$

$$\text{or} \quad \int_0^t f_1(x)\, f_2(t-x)\,dx = L^{-1}\left(F_1(s)F_2(s)\right)$$

Example : Obtain $L^{-1}\dfrac{1}{s\left(s^2+a^2\right)}$.

Solution : $L^{-1}\dfrac{1}{s} = 1$ and $L^{-1}\dfrac{1}{s^2+a^2} = \dfrac{\sin at}{a}$.

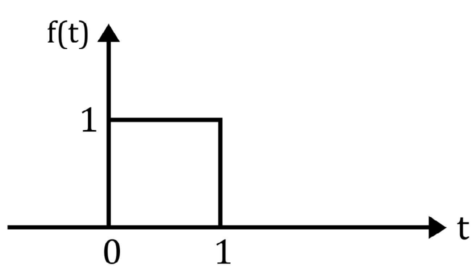
Hence by the convolution theorem

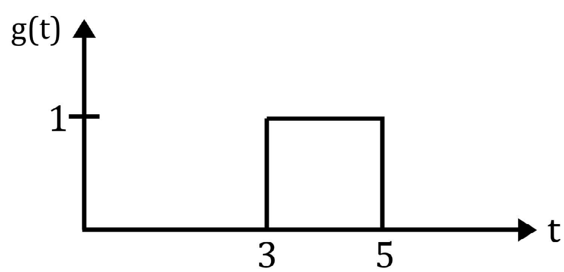
$$L\int_0^t \left\{\frac{\sin a(t-x)}{a}dx\right\} = \left(\frac{1}{s}\right)\left(\frac{1}{s^2+a^2}\right)$$

$$L^{-1}\left\{\frac{1}{s\left(s^2+a^2\right)}\right\} = \int_0^t \frac{\sin a\left(t-x\right)}{a}\,dx = \left[\frac{-\cos\left(at-ax\right)}{-a^2}\right]_0^t = \frac{1}{a^2}\left[1-\cos\,at\right]$$

Common Data Questions 1 & 2

Given f(t) and g(t) as shown below.





Q.1 g(t) can be expressed as

(a) $g(t) = f(2t-3)$ (b) $g(t) = f\left(\dfrac{t}{2}-3\right)$

(c) $g(t) = f\left(2t-\dfrac{3}{2}\right)$ (d) $g(t) = f\left(\dfrac{t}{2}-\dfrac{3}{2}\right)$

Solution: (d)

We need $g(3) = f(0)$ and $g(5) = f(1)$

Only choice (d) satisfies both these conditions as seen below.

$$g(t) = f\left(\frac{t}{2}-\frac{3}{2}\right)$$

Q.2 The Laplace transform of $g(t)$ is

(a) $\frac{1}{S}\left(e^{3s}-e^{5s}\right)$ (b) $\frac{1}{S}\left(e^{-5s}-e^{-3s}\right)$

(c) $\frac{e^{-3s}}{s}\left(1-e^{-2s}\right)$ (d) $\frac{1}{s}\left(e^{5s}-e^{3s}\right)$

Solution: **(C)** By definition of Laplace transform

$$L\{g(t)\}=\int_0^\infty e^{-st}g(t)\,dt$$

$$L\{g(t)\}=\int_0^3 e^{-st}g(t)\,dt+\int_3^5 e^{-st}g(t)\,dt+\int_5^\infty e^{-st}g(t)\,dt$$

$$=\int_0^3 e^{-st}(0)\,dt+\int_3^5 e^{-st}(1)\,dt+\int_5^\infty e^{-st}(0)\,dt$$

$$=\left[-\frac{e^{-st}}{S}\right]_3^5=-\left[\frac{e^{-5s}-e^{-3s}}{s}\right]$$

$$=\frac{e^{-3s}-e^{-5s}}{S}=\frac{e^{-3s}}{S}[1-e^{-2s}]$$

GATE QUESTIONS

Q.1 If L defines the Laplace Transform of a function, L [sin (at)] will be equal to

(a) $\frac{a}{s^2-a^2}$ (b) $\frac{a}{s^2+a^2}$

(c) $\frac{s}{s^2+a^2}$ (d) $\frac{s}{s^2-a^2}$ **[GATE-2003(CE)]**

Q.2 Laplace transform of the function $\sin\omega t$ is

(a) $\frac{s}{s^2+\omega^2}$ (b) $\frac{\omega}{s^2+\omega^2}$

(c) $\frac{s}{s^2-\omega^2}$ (d) $\frac{\omega}{s^2-\omega^2}$ **[GATE-2003(ME)]**

Q.3 A delayed unit step function is defined as $u(t-a)=\begin{cases}0, & \text{for } t<a\\ 1, & \text{for } t\geq a\end{cases}$ Its Laplace transform is

(a) ae^{-as} (b) $\frac{e^{-as}}{S}$

(c) $\frac{e^{as}}{S}$ (d) $\frac{e^{as}}{a}$ **[GATE-2004(ME)]**

Q.4 In what range should Re(s) remain so that the Laplace transform of the function $e^{(a+2)t+5}$ exits.

(a) $\text{Re}(s)>a+2$ (b) $\text{Re}(s)>a+7$

(c) $\text{Re}(s)<2$ (d) $\text{Re}(s)>a+5$ **[GATE-2005(E(C)]**

Q.5 A solution for the differential equation $x'(t)+2x(t)=\delta(t)$ with initial condition $x(0)=0$ is

(a) $e^{-2t}u(t)$ (b) $e^{2t}u(t)$

(c) $e^{-t}u(t)$ (d) $e^{t}u(t)$ **[GATE-2006(E(C)]**

Q.6 If F(s) is the Laplace transform of function f(t) , then Laplace transform of $\int_0^t f(\tau)\,d\tau$ is

(a) $\frac{1}{s}F(s)$ (b) $\frac{1}{s}F(s)-f(0)$

(c) $sF(s)-f(0)$ (d) $\int F(s)\,ds$ **[GATE-2007(ME)]**

Q.7 Evaluate $\int_0^\infty \frac{\sin t}{t}dt$

(a) π (b) $\frac{\pi}{2}$

(c) $\frac{\pi}{4}$ (d) $\frac{\pi}{8}$ **[GATE-2007(CE)]**

Q.8 Laplace transform for the function $f(t)=\cosh(at)$ is

(a) $\frac{a}{s^2-a^2}$ (b) $\frac{s}{s^2-a^2}$

(c) $\frac{a}{s^2+a^2}$ (d) $\frac{s}{s^2+a^2}$ **[GATE-2009(CE)]**

Q.9 The inverse Laplace transform of $\frac{1}{(s^2+s)}$ is

(a) $1+e^t$ (b) $1-e^t$

(c) $1-e^{-t}$ (d) $1+e^{-t}$ **[GATE-2009(ME)]**

Q.10 The Laplace transform of a function $f(t)=\frac{1}{s^2(s+1)}$. The function $f(t)$ is

(a) $t-1+e^{-t}$ (b) $t+1+e^{-t}$

(c) $-1+e^{-t}$ (d) $2t+e^t$ **[GATE-2010(ME)]**

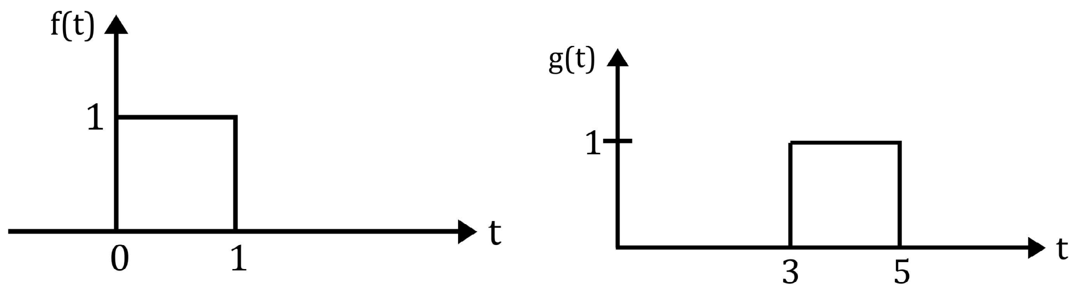
Q.11 Given $L^{-1}\left[\frac{3s+1}{s^3+4s^2+(K-3)s}\right]$. If $\lim_{t\to\infty} f(t)=1$, then the value of K is

(a)1 (b)2

(c)3 (d)4 **[GATE-2010(E(C)]**

Common Data Questions 12 and 13

Given f(t) and g(t) as shown below.



Q.12 g(t) can be expressed as

(a) $g(t) = f(2t-3)$ (b) $g(t) = f\left(\frac{t}{2}-3\right)$

(c) $g(t) = f\left(2t-\frac{3}{2}\right)$ (d) $g(t) = f\left(\frac{t}{2}-\frac{3}{2}\right)$ **[GATE-2010(EE)]**

Q.13 The Laplace transform of $g(t)$ is

(a) $\frac{1}{s}\left(e^{3s}-e^{5s}\right)$ (b) $\frac{1}{s}\left(e^{-5s}-e^{-3s}\right)$

(c) $\frac{e^{-3s}}{s}\left(1-e^{-2s}\right)$ (d) $\frac{1}{s}\left(e^{5s}-e^{3s}\right)$ **[GATE-2010(EE)]**

Q.14 Let $X(s) = \frac{3s+5}{s^2+10s+21}$ be the Laplace Transform of a signal x(t). Then, x (0^+) is

(a) 0 (b) 3

(c) 5 (d) 21 **[EE-GATE-2013]**

Q.15 The Laplace Transform of $f(t) = e^{2t}\sin 5t\, u(t)$ is

(a) $\frac{5}{s^2-4s+29}$ (b) $\frac{5}{s^2+5}$

(c) $\frac{s-2}{s^2-4s+29}$ (d) $\frac{5}{s+5}$ **[EE-GATE-2016(1)]**

Q.16 The Laplace transform F(s) of the exponential function, f(t) = e^{at} when $t > 0$, where a is a constant and $(s-a) > 0$, is

(a) $\frac{1}{s+a}$ (b) $\frac{1}{s-a}$

(c) $\frac{1}{a-s}$ (d) ∞ **[CE-GATE-2018(2)]**

Q.17 F(s) is the Laplace transform of the function $f(t)=2t^2e^{-t}$. F(1) is ____ (correct to two decimal places). **[ME-GATE-2018(1)]**

ANSWER KEY:

1	2	3	4	5	6	7	8	9	10	11	12	13	14
(b)	(b)	(b)	(a)	(a)	(a)	(b)	(b)	(c)	(a)	(d)	(d)	(c)	(c)
15	**16**	**17**											
(b)	(a)	0.5											

GATE SOLUTIONS

Q.1 **(b)** $L[f(t)] = \int_0^\infty e^{-st} f(t)dt, \ \therefore L[\sin(at)] = \int_0^\infty e^{-st}\sin(at)dt = \frac{a}{s^2+a^2}$

Q.2 **(b)** $L[\sin\omega t] = \frac{\omega}{s^2+\omega^2}$

Q.3 **(b)** $L[U(t-a)] = \int_0^\infty e^{-st}U(t-a)dt = \int_0^a e^{-st}.0.dt + \int_a^\infty e^{-st}.1.dt = 0 + \int_a^\infty e^{-st}dt = \left|\frac{e^{-st}}{-s}\right|_a^\infty = \frac{e^{-as}}{s}$

Q.4 **(a)** $f(t) = e^{(a+2)t+5} = e^5 e^{(a+2)t}, \ F(s) = \left[\frac{1}{s-(a+2)}\right]e^5$

$\therefore$ For L.T. to exist, $\text{Re}(s) > a+2$

Q.5 **(a)** $x'(t) + 2x(t) = \delta(t)$, $x(0) = 0$

Taking L.T on both sides

$$sX(s) - X(0) + 2X(s) = 1 \Rightarrow X(s)[s+2] = 1 \Rightarrow X(s) = \frac{1}{S+2}$$

$$x(t) = e^{-2t}u(t)$$

Q.6 **(a)** Using Integration in time domain property $L\left[\int_0^t f(\tau)d\tau\right] = \frac{1}{s}F(s)$

Q.7 **(b)** $L(\sin t) = F(s) = \frac{1}{(s^2+1)}$

$$\therefore L\left(\frac{\sin t}{t}\right) = \int_s^\infty F(s)ds = \int_s^\infty \frac{ds}{(s^2+1)} = \left|\tan^{-1}s\right|_s^\infty = \frac{\pi}{2} - \tan^{-1}(s)$$

$$L\left(\int_0^t \frac{\sin t}{t}dt\right) = \frac{1}{s}\left(\frac{\pi}{2} - \tan^{-1}(s)\right)$$

using final value theorem

$$\lim_{t\to\infty} L\left(\int_0^t \frac{\sin t}{t}dt\right) = \lim_{s\to 0} s\,\frac{1}{s}\left(\frac{\pi}{2} - \tan^{-1}(s)\right) = \frac{\pi}{2}$$

Q.8 **(b)** It is a standard result that $L(\cosh at) = \dfrac{s}{s^2 - a^2}$

Q.9 **(c)** $\dfrac{1}{s^2+s} = \dfrac{1}{s} - \dfrac{1}{s+1} \Rightarrow L^{-1}\left(\dfrac{1}{s^2+s}\right) = L^{-1}\left(\dfrac{1}{s}\right) - L^{-1}\left(\dfrac{1}{s+1}\right) = 1 - e^{-t}$

Q.10 **(a)** $L^{-1}\left(\dfrac{1}{s+1}\right) = e^{-t},\ L^{-1}\left(\dfrac{1}{s^2}\left(\dfrac{1}{s+1}\right)\right) = \int_0^t\int_0^t \left(e^{-t}\right) dt\, dt = e^{-t} + t - 1$

Q.11 **(d)** $\lim\limits_{t\to\infty} f(t) = \lim\limits_{s\to 0} sF(s)$

Given that $F(s) = \left[\dfrac{3s+1}{s^3 + 4s^2 + (K-3)s}\right],\ \lim\limits_{t\to\infty} f(t) = 1$

$\lim\limits_{s\to 0} s\left[\dfrac{3s+1}{s^3 + 4s^2 + (K-3)s}\right] = 1,\ \Rightarrow\ \dfrac{1}{K-3} = 1 \Rightarrow K = 4$

Q.12 **(d)** We need $g(3) = f(0)$ and $g(5) = f(1)$

Only choice (d) satisfies both these conditions as seen below.

Choice (d) is $g(t) = f\left(\dfrac{t}{2} - \dfrac{3}{2}\right)$

Q.13 **(c)** By definition of Laplace transform

$f(t) = \begin{cases} 1, & 3 \le t \le 5 \\ 0, & \text{otherwise} \end{cases},\ L\{f(t)\} = \int_3^5 e^{-st}.1.dt = \dfrac{e^{-3s} - e^{-5s}}{s}$

Q.14 **(b)** $x\left[0^+\right] = \lim\limits_{s\to\infty} \dfrac{s(3s+5)}{(s^2 + 10s + 21)} = \lim\limits_{s\to\infty} \dfrac{s^2\left[3 + \frac{5}{s}\right]}{s^2\left[1 + \frac{10}{s} + \frac{21}{s^2}\right]} = 3$

Q.15 **(a)** $f(t) = e^{2t}\sin 5t\, u(t),\ L(\sin 5t) = \dfrac{5}{s^2+25},\ L\left(e^{2t}\sin 5t\, u(t)\right) = \dfrac{5}{(s-2)^2 + 25} = \dfrac{5}{s^2 - 4s + 29}$

Q.16 **(b)** $L[e^{at}] = \dfrac{1}{s-a}$

Q.17 **(0.5)** $f(t) = 2t^2e^{-t}\ \because\ L\{t^n\} = \dfrac{\lceil n+1}{s^{n+1}} \Rightarrow L\{t^2\} = \dfrac{\lceil 2+1}{s^3} = \dfrac{2}{s^3}$ then , $L\{2t^2\} = \dfrac{4}{s^3}$

Hence , $F(s) = L\{e^{-t}(2t^2)\} = \dfrac{4}{(s+1)^3} \Rightarrow F(1) = \dfrac{4}{(1+1)^3} = \dfrac{1}{2} = 0.5$

8 FOURIER SERIES

Periodic Functions

A function f(x) is said to be periodic function of period T>0, if f (x) = f(x+T) for all x.

Trigonometric System

The set of functions 1, cos x, cos 2x, cos 3x,…, cos mx,…, sin x, sin 2x, sin 3x,…, sin nx,…

Orthogonal Functions

If $\int_a^b f(x)g(x)dx = 0$ then f and g are said to be orthogonal over the interval [a, b].

Results :

The trigonometric system is orthogonal over any interval of length 2π.

(a) $\int_{-\pi}^{\pi} \cos nx dx = \int_{-\pi}^{\pi} \sin nx dx = 0$

(b) $\int_{-\pi}^{\pi} \cos mx \ \cos nx \ dx = 0 \text{ for } m \neq n$

(c) $\int_{-\pi}^{\pi} \sin mx \ \sin nx \ dx = 0 \text{ for } m \neq n$

(d) $\int_{-\pi}^{\pi} \cos mx \ \sin nx \ dx = 0 \text{ for all m and n}$

(e) $\int_{-\pi}^{\pi} \cos^2 mx \ dx = \int_{-\pi}^{\pi} \sin^2 mx \ dx = \pi$

Fourier Series

Drichlet's Conditions

The fourier series of f(x) converges if

(i) f(x) is periodic, single valued and finite.

(ii) The function f(x) has a finite number of finite discontinuities in any one period.

(iii) The function f(x) must have finite number of maxima and minima.

The **Fourier series** of periodic function f(x) of period 2L on the interval (-L, L) is

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi}{L}x+b_n\sin\frac{n\pi}{L}x\right)$$

Where,

$$a_0=\frac{1}{L}\int_{-L}^{L}f(x)dx,\ a_n=\frac{1}{L}\int_{-L}^{L}f(x)\cos\left(\frac{n\pi}{L}x\right)dx,\ b_n=\frac{1}{L}\int_{-L}^{L}f(x)\sin\left(\frac{n\pi}{L}x\right)dx$$

The **Fourier series** of periodic function f(x) of period 2π on the interval (-π, π) is

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)$$

Where,

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)dx,\ a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nx\,dx,\ b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nx\,dx$$

The **Parseval's identity** for Fourier series in the interval (c, c + 2L) is

$$\frac{2}{L}\int_{c}^{c+2L}[f(x)]dx=\frac{a_0^2}{2}+\sum_{n=1}^{\infty}\left(a_n^2+b_n^2\right)$$

Fourier Cosine Series: If f is an even periodic function, then its Fourier series contains only cosine (include, possibly, the constant term) terms. The Fourier series of even periodic function f (x) of period 2π on the interval $(-\pi,\pi)$ is **Fourier Cosine Series.**

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx$$

Where,

$$a_0=\frac{2}{\pi}\int_{0}^{\pi}f(x)dx,\ a_n=\frac{2}{\pi}\int_{0}^{\pi}f(x)\cos nx\,dx$$

Fourier Sine Series : If f is an odd periodic function, then its Fourier series contains only sine terms.

The Fourier series of odd periodic function f(x) of period 2 on the interval is **Fourier Sine Series**.

$$f(x)=\sum_{n=1}^{\infty}b_n\sin nx\ ,\quad \text{Where}\quad b_n=\frac{2}{\pi}\int_{0}^{\pi}f(x)\sin nx dx$$

Half Range Fourier Series

A Fourier series for f(x), valid on [0, L], may be constructed by extension of the domain to [– L, L]. An even extension leads to a **Fourier cosine series**. An odd extension leads to a **Fourier Sine Series.**

Half Range Fourier Cosine Series

The Half Range Fourier Cosine Series of f(x) defined on the interval is

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx$$

$$\text{Where } a_0=\frac{2}{\pi}\int_0^{\pi}f(x)\,dx,\ a_n=\frac{2}{\pi}\int_0^{\pi}f(x)\cos nx\,dx$$

Half Range Fourier Sine Series

The Half Range Fourier Sine Series of f(x) defined on the interval $(0,\pi)$ is

$$f(x)=\sum_{n=1}^{\infty}b_n\sin nxdx\ ,\quad \text{Where}\quad b_n=\frac{2}{\pi}\int_0^{\pi}f(x)\sin nxdx$$

Example : Expand f(x) = x, 0<x<2 in half range sine series

Solution : Here L = 2

$$b_n=\frac{2}{L}\int_0^{L}f(t)\sin\left(\frac{n\pi}{L}t\right)dt\ =\int_0^{2}t\sin\left(\frac{n\pi}{2}t\right)dt=-\frac{4}{n\pi}(-1)^n$$

$$f(t)=\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\left(\frac{n\pi}{2}t\right)$$

$$f(1)=1=\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\left(\frac{n\pi}{2}t\right)$$

GATE QUESTIONS

Q.1 The Fourier cosine series for an even function f(x) is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(nx)$$

The value of the coefficient a_2 for the function $f(x) = \cos^2(x)$ in [0, À] is

(a) –0.5 (b) 0.0

(c) 0.5 (d) 1.0 **[ME-GATE-2018(2)]**

ANSWER KEY :

1. (C)

ESE QUESTIONS

Q.1 Let $f(x) = \begin{cases} -\pi, & \text{if } -\pi < x < 0 \\ \pi, & \text{if } 0 < x < \pi \end{cases}$ is a periodic function of period 2π. The coefficient of $\sin 5x$ in the Fourier series expansion of f(x) in the interval $[-\pi, \pi]$ is

(a) $\frac{4}{5}$

(b) $\frac{5}{4}$

(c) $\frac{4}{3}$

(d) $\frac{3}{4}$

ANSWER KEY :

1. (a)

GATE SOLUTIONS

Q.1 $f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(nx)$

Given that f (x) = $\cos^2(x)$ in [0, π]= [0, L]

The fourier cosine series for an even function f(x) is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(nx)$$

where, $a_n = \frac{2}{L}\int_0^1 f(x).\cos\left(\frac{n\pi x}{L}\right)dx$

Now, $a_2 = \frac{2}{\pi}\int_0^{\pi} \cos^2 x \cdot \cos 2x \; dx$

$$= \frac{2}{\pi}\int_0^{\pi}\left(\frac{1+\cos 2x}{2}\right)\cos 2x \; dx$$

$$= \frac{2}{\pi}\int_0^{\pi}\left(\frac{\cos 2x}{2} + \frac{1}{2}\cos^2(2x)\right) dx$$

$$= \frac{2}{\pi}\int_0^{\pi}\left(\frac{\cos 2x}{2} + \frac{1+\cos 4x}{4}\right) dx$$

$$= \frac{2}{\pi}\left[\left(\frac{\sin(2x)}{2} + \frac{x}{4} + \frac{\sin(4x)}{16}\right)\right]_0^{\pi}$$

$$a_2 = \frac{2}{\pi}\left[\left(0 + \frac{\pi}{4} + 0\right) - (0+0+0)\right] = \frac{1}{2} = 0.5$$

ESE SOLUTIONS

Q.1 f(x) is an odd function

$$b_n = \frac{2}{\pi}\int_0^\pi f(x)\sin nx\,dx$$

$$b_5 = \frac{2}{\pi}\int_0^\pi \pi\sin 5x\,dx$$

$$= \frac{2}{\pi}\times\left[\frac{-\cos 5x}{5}\right]_0^\pi = -\frac{2}{\pi}\times\left[\cos 5\pi - 1\right] = \frac{4}{5}$$