Chapter 1 : Linear Algebra

Basic Operations of Matrix

1. Transpose of a Matrix :

The transpose of a matrix A written as A^T (or A'), is obtained by interchanging the rows with the corresponding columns of A.

Example : If $A=\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}_{2\times 3}$ then $A^T=\begin{bmatrix} a & d \\ b & e \\ c & f \end{bmatrix}_{3\times 2}$

2. Determinants of Matrix :

Determinant is only valid for square matrix. Determinant is the expansion or value of the matrix according to the elements position coefficient. The position coefficient $=(-1)^{i+j}$.

Let us consider matrix of order 2×2, $A=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}_{2\times 2}$

Then, the determinant of given matrix is $|A|=\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}_{2\times 2} = a_{11}a_{22}-a_{12}a_{21}$

and matrix of order 3×3, $A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ then $|A|=a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$

Similarly, we can calculate the determinants of higher order by expanding its row or column.

Properties of Determinants

(i) $|A^T|=|A|$

(ii) $|AB|=|A||B|$

(iii) $|A^n|=(|A|)^n$

(iv) $|kA|=k^n|A|$

(v) If two rows (or two columns) of a determinant are interchanged, the sign of the value of the determinant changes.

(vi) If in determinant any row or column is completely zero, the value of the determinant is zero.

(vii) If two rows (or two columns) of a determinant are identical, the value of the determinant is zero.

3. Matrix Multiplication

It is valid for both square and non-square matrix, the existence of resultant depends upon the order of the matrix.

Let $\boldsymbol{A}_{m\times n}$ and $\boldsymbol{B}_{n\times p}$ are two matrices then, the order of resultant AB is $m\times p$.

For example : $[A]=\begin{bmatrix}1 & 4\\ 2 & 5\\ 3 & 6\end{bmatrix}_{3\times 2}$ $[B]=\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}_{2\times 2}$

$$AB=\begin{bmatrix}1\times1+4\times3 & 1\times2+4\times4\\ 2\times1+5\times3 & 2\times2+5\times4\\ 3\times1+6\times3 & 3\times2+6\times4\end{bmatrix}_{3\times 2}=\begin{bmatrix}13 & 18\\ 17 & 24\\ 21 & 30\end{bmatrix}_{3\times 2}$$

Here, number of elements $=6$

Number of multiplications $=12$

Number of addition $=6$

Key Point

1. Let $A_{m\times n}$ and $B_{n\times p}$ are two matrices then, the resultant is $AB_{m\times p}$, has
   (i) Number of elements $= mp$
   (ii) Number of multiplication $=(mp)n = mnp$
   (iii) Number of addition $= mp(n-1)$
2. If A is an $m\times n$ matrix and B is an $n\times m$ matrix, then
   $\mathbf{tr}(AB)=\mathbf{tr}(BA)$, $\mathbf{tr}(AB)\neq \mathbf{tr}(A)\cdot\mathbf{tr}(B)$ and $\mathbf{tr}(BA)\neq\mathbf{tr}(B)\cdot\mathbf{tr}(A)$
   Here, **tr** represents trace of matrix i.e. sum of leading diagonal elements.

Types of Square Matrix

1. **Diagonal Matrix :** A square matrix in which all the elements except leading diagonal elements are zero is known as a diagonal matrix.

Example : $A=\begin{bmatrix}1 & 0 & 0\\ 0 & 3 & 0\\ 0 & 0 & 6\end{bmatrix}_{3\times 3}$ or $A=\text{diag}(1,3,6)$

Key Point

(i) Minimum number of zeros in a diagonal matrix of order n is $n(n-1)$.

(ii) $AB=\text{diag}(a_1, a_2, a_3)\times\text{diag}(b_1, b_2, b_3)=\text{diag}(a_1b_1, a_2b_2, a_3b_3)$

2. **Scalar Matrix :** A diagonal matrix in which all the diagonal elements are equal, is known as a scalar matrix.

Example : $A=\begin{bmatrix}3 & 0 & 0\\ 0 & 3 & 0\\ 0 & 0 & 3\end{bmatrix}_{3\times 3}$ $A=\text{diag}(3,3,3)$

3. **Unit Matrix :** A diagonal matrix in which all the diagonal elements are unity is known as unit matrix or identity matrix. The identity matrix of order n is denoted by I_n.

Example : $I_3=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}_{3\times 3}$

4. Upper Triangular Matrix : A square matrix $A=\left[a_{ij}\right]$ is said to be upper triangular matrix, if $a_{ij}=0$ whenever $i>j$.

Example : $A=\begin{bmatrix}1 & 2 & 3\\ 0 & 4 & 5\\ 0 & 0 & 6\end{bmatrix}_{3\times 3}$

5. Lower Triangular Matrix : A square matrix $A=\left[a_{ij}\right]$ is said to be lower triangular matrix, if $a_{ij}=0$ whenever $i<j$.

Example : $A=\begin{bmatrix}1 & 0 & 0\\ 2 & 3 & 0\\ 4 & 5 & 6\end{bmatrix}_{3\times 3}$

Key Point

For diagonal and triangular matrix (upper triangular or lower triangular) the determinant is equal to product of leading diagonal elements.

6. Symmetric Matrix : A square matrix is said to be symmetric, if $A^T=A$ where A^T or A' is transpose of matrix A. In transpose of matrix the rows and columns are interchanged.

Example : $A=\begin{bmatrix}1 & 2 & 3\\ 2 & 4 & 5\\ 3 & 5 & 6\end{bmatrix}_{3\times 3} \Rightarrow A^T=\begin{bmatrix}1 & 2 & 3\\ 2 & 4 & 5\\ 3 & 5 & 6\end{bmatrix}_{3\times 3}$

Properties of Symmetric Matrix :

(i) If A is a square matrix then $A+A^T$, AA^T, A^TA are symmetric matrices, while $A-A^T$, A^T-A are skew symmetric matrix.

(ii) If A is a symmetric matrix, k any real scalar, n any integer, B square matrix of order that of A, then $-A$, kA, A^T, A^n, A^{-1}, B^TAB are also symmetric matrices. All positive integral power of a symmetric matrix are symmetric.

(iii) If A, B are two symmetric matrices, then

(a) $A\pm B$, $AB+BA$ are also symmetric matrices.

(b) $AB-BA$ is a skew symmetric matrix.

(c) AB is a symmetric matrix when $AB=BA$ otherwise AB or BA may not be symmetric.

(d) A^2, A^3, A^4, B^2, B^3, B^4, $A^2\pm B^2$, $A^3\pm B^3$ are symmetric matrices.

7. Skew Symmetric Matrix : A square matrix is said to be skew symmetric matrix if $A^T=-A$

Example : $A=\begin{bmatrix}0 & -2 & -3\\ 2 & 0 & -5\\ 3 & 5 & 0\end{bmatrix}_{3\times 3} \Rightarrow A^T=\begin{bmatrix}0 & 2 & 3\\ -2 & 0 & 5\\ -3 & -5 & 0\end{bmatrix}_{3\times 3}=-A$

Properties of Skew Symmetric Matrix :

1. If A is a skew symmetric matrix, then

(i) A^{2n} is a symmetric matrix for n positive integer.

(ii) A^{2n+1} is a skew symmetric matrix for n positive integer.

(iii) kA is also skew symmetric matrix, where k is a real scalar.

(iv) $B^T AB$ is also skew symmetric where B is a square matrix of order that of A.

All positive odd integral power of a skew symmetric matrix are skew symmetric and positive even integral powers of a skew symmetric matrix are symmetric.

2. If A, B are two skew symmetric matrices, then

   (i) $A \pm B$, $AB - BA$ are skew symmetric matrices.

   (ii) $AB + BA$ is symmetric matrix.

3. If A is a skew symmetric matrix and C is a column matrix then $C^T AC$ is a zero matrix.

4. If A is any square matrix then $A + A^T$ is a symmetric matrix and $A - A^T$ is a skew symmetric matrix.

Key Point

(i) The matrix which is both symmetric and skew symmetric must be a null matrix.

(ii) If A is symmetric and B is skew-symmetric, then $\text{tr}(AB) = 0$.

(iii) Any real square matrix A may be expressed as the sum of a symmetric matrix A_S and a skew symmetric matrix A_{AS}.

$$A = \frac{1}{2}\left[A + A^T\right] + \frac{1}{2}\left[A - A^T\right] = A_S + A_{AS}$$

8. **Singular Matrix :** A singular matrix is a square matrix that is not invertible i.e. it does not have an inverse. A matrix is singular or degenerate if and only if its determinant is zero i.e. $|\mathbf{A}| = \mathbf{0}$.

9. **Non-singular Matrix or Invertible Matrix :** A square matrix is non-singular or invertible if its determinant is non-zero i.e. $|\mathbf{A}| \neq \mathbf{0}$. A non-singular matrix has a matrix inverse.

10. **Orthogonal Matrix :** A square matrix is said to be orthogonal if $A \cdot A^T = I$. In other words the transpose of orthogonal matrix is equal to the inverse of the matrix i.e. $A^T = A^{-1}$.

Example : If $A = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{bmatrix}_{3\times3}$ then $A^T = \frac{1}{3}\begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \\ 2 & -2 & -1 \end{bmatrix}_{3\times3}$

and $A \cdot A^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3\times3}$, $A^{-1} = A^T = \frac{1}{3}\begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \\ 2 & -2 & -1 \end{bmatrix}_{3\times3}$

If matrix A is orthogonal then

(i) Its inverse and transpose are also orthogonal.

(ii) Its determinant is unity i.e. $|A| = \pm 1$.

(iii) $|A||A^T| = 1$

11. **Hermitian Matrix :** A square matrix is said to be hermitian if $A = A^{\theta}$. Where A^{θ} is the transpose of conjugate of matrix A, i.e. $(\bar{A})^T$

Example : if $A=\begin{bmatrix}1 & 3-2i & 2+3i\\ 3+2i & 2 & i\\ 2-3i & -i & 3\end{bmatrix}$ then Conjugate of $A=\begin{bmatrix}1 & 3+2i & 2-3i\\ 3-2i & 2 & -i\\ 2+3i & i & 3\end{bmatrix}$

$$A^{\theta}=\begin{bmatrix}1 & 3-2i & 2+3i\\ 3+2i & 2 & i\\ 2-3i & -i & 3\end{bmatrix}=A$$

12. Skew Hermitian Matrix : A square matrix A is said to be skew hermitian if $A=-A^{\theta}$

Example : $A=\begin{bmatrix}i & 2-3i & 4+5i\\ -2-3i & 0 & 2i\\ -4+5i & 2i & -3i\end{bmatrix}_{3\times 3}$

Conjugate of $A=\begin{bmatrix}-i & 2+3i & 4-5i\\ -2+3i & 0 & -2i\\ -4-5i & -2i & 3i\end{bmatrix}_{3\times 3} \Rightarrow A^{\theta}=\begin{bmatrix}-i & -2+3i & -4-5i\\ 2+3i & 0 & -2i\\ 4-5i & -2i & 3i\end{bmatrix}_{3\times 3}=-A$

Key Point

(i) All the diagonal elements of Skew Hermitian matrix are either zero or pure imaginary.

(ii) All the diagonal elements of Hermitian matrix are real.

(iii) Upper and lower diagonal elements should be complex conjugate pair.

13. Unitary Matrix : A square matrix is said to be unitary if $A\cdot A^{\theta}=I$ where A^{θ} is transpose of conjugate of matrix A.

Example : $A=\begin{bmatrix}\frac{1+i}{2} & \frac{-1+i}{2}\\ \frac{1-i}{2} & \frac{-1-i}{2}\end{bmatrix}_{2\times 2} \Rightarrow A^{\theta}=\begin{bmatrix}\frac{1-i}{2} & \frac{1+i}{2}\\ \frac{-1-i}{2} & \frac{-1+i}{2}\end{bmatrix}_{2\times 2} \Rightarrow A\cdot A^{\theta}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}_{2\times 2}$

Key Point

If matrix A is unitary then

(i) Its inverse and transpose are also unitary.

(ii) Its determinant is unity i.e. $|A|=\pm 1$.

(iii) $|A||A^{\theta}|=1$

14. Periodic Matrix : A square matrix is said to be a periodic if $A^{K+1}=A$ where, K : Positive integer. K is known as period of the matrix.

15. Involutory Matrix : A matrix is said to be involutory if $A^2=I$.

16. Idempotent Matrix : An idempotent matrix is a square matrix which, when multiplied by itself i.e. $A^2=A$. A periodic matrix is said to be idempotent when the positive integer K is unity i.e.

$$A^{K+1}=A \quad\Rightarrow\quad A^{1+1}=A \quad\Rightarrow\quad A^2=A$$

17. Nilpotent Matrix : A square matrix is called a nilpotent matrix if there exists a positive integer K such that $A^K=0$.

The least positive value of K is called the index of nilpotent matrix A.

📖 Key Point

(i) Determinant of Idempotent matrix is either 0 or 1.

(ii) Determinant and Trace of nilpotent matrix is zero.

(iii) Inverse of nilpotent matrix does not exist.

Eigen Values and Eigen Vectors

Characteristic Roots or Eigen Values or Latent roots

Eigen value and Eigen vectors are only valid for square matrix, the roots of characteristic equation $|A-\lambda I|=0$ = 0 are called characteristic roots or Eigen values of matrix A.

Example : $A=\begin{bmatrix}2 & 4\\ 5 & 3\end{bmatrix}_{2\times 2}$

The characteristic equation of matrix is given by, $|A-\lambda I|=0$

$$\begin{vmatrix}2-\lambda & 4\\ 5 & 3-\lambda\end{vmatrix}=0$$

$$(2-\lambda)(3-\lambda)-20=0 \Rightarrow \lambda^2-5\lambda+6-20=0$$

$$\lambda^2-5\lambda-14=0 \Rightarrow \lambda^2-7\lambda+2\lambda-14=0 \Rightarrow \lambda=7,\,-2$$

Properties of Eigen Values or Characteristics Roots

(i) The sum of Eigen values of a matrix is equal to the trace of the matrix where the sum of the elements of principal diagonal of a matrix is called the **trace of matrix.**

$$\sum_i(\lambda_i)=\lambda_1+\lambda_2+\lambda_3=\text{Trace of matrix}$$

(ii) The product of Eigen values of a matrix A is equal to the determinant of matrix A.

$$\prod_i(\lambda_i)=\lambda_1\lambda_2\lambda_3=|A|$$

(iii) For Hermitian matrix every Eigen value is real.

(iv) Every Eigen value of a Unitary matrix has absolute value i.e. $|\lambda|=1$

(v) Any square matrix A and its transpose A^T have same Eigen values.

(vi) If $\lambda_1,\lambda_2.....\lambda_n$ are Eigen values of A then Eigen values of

- ✓ KA are $K\lambda_1, K\lambda_2.....K\lambda_n$.
- ✓ A^m are $\lambda_1{}^m,\lambda_2{}^m.....\lambda_n{}^m$.
- ✓ A^{-1} are $\frac{1}{\lambda_1},\frac{1}{\lambda_2},\frac{1}{\lambda_3}.....\frac{1}{\lambda_n}$.
- ✓ $A+KI$ are $\lambda_1+K,\ \lambda_2+K,\ \lambda_3+K.............\lambda_n+K$

(viii) If λ is an Eigen value of an Orthogonal matrix A then $\frac{1}{\lambda}$ is also an Eigen value of $A(A^T=A^{-1})$.

(ix) The Eigen value of a symmetric matrix are purely real.

(x) The Eigen value of skew-symmetric matrix are either purely imaginary or zero.

(xi) Zero is an Eigen value of a matrix if and only if a matrix is singular.

(xii) If all Eigen values are distinct then the corresponding Eigen vectors are independent.

(xiii) The set of Eigen values are called the spectrum of A and the largest Eigen value in magnitude is called the spectral radius of A. Where A is the given matrix.

Cayley-Hamilton Theorem :

According to Cayley-Hamilton Theorem,

" Every square matrix satisfies its own characteristic equation."

This theorem is only applicable for square matrix. This theorem is used to find the inverse of the matrix in the form of matrix polynomial.

If A be $n \times n$ matrix and its characteristic equation is,

$$a_0\lambda^n + a_1\lambda^{n-1} + ... + a_n = 0$$

Then, according to Cayley-Hamilton Theorem,

$$a_0A^n + a_1A^{n-1} + ... + a_nI_n = 0$$

Eigen Vectors

If a matrix A having characteristic root λ then we have a non-zero vector X which satisfies the equation $[A - \lambda I][X] = [0]$. Where the non-zero vector X is called characteristic vector or Eigen vector.

If there exist Eigen vector X corresponding to Eigen value λ then the relation for matrix A is given by,

$$AX = \lambda X$$

Properties of Eigen Vectors

(i) For every Eigen value there exist atleast one Eigen vector.

(ii) If λ is an Eigen value of a matrix A, then the corresponding Eigen vector X is not unique. i.e. we have infinite number of Eigen vectors corresponding to a single Eigen value.

(iii) If $\lambda_1, \lambda_2, ...\lambda_n$ be distinct Eigen values of a $n \times n$ matrix, then corresponding Eigen vectors = $X_1, X_2, ...X_n$ form a linearly independent set.

(iv) If two or more Eigen values are equal then Eigen vectors are linearly dependent.

(v) Two Eigen vectors X_1 and X_2 are called orthogonal vectors if $X_1^T X_2 = 0$.

(vi) A matrix is said to be defective if it fails to have n linearly independent Eigen vectors and therefore it is not diagonaizable. All defective matrices have fewer than n distinct Eigen values, but not all matrices having fewer than n distinct Eigen values are defective.

Normalized Eigen Vectors :

A normalized Eigen vector is an Eigen vector of length one. Consider an Eigen vector $X = \begin{bmatrix} a \\ b \end{bmatrix}_{2\times 1}$ then length of this Eigen vector is $\| X \| = \sqrt{a^2 + b^2}$

Normalized Eigen vector is $\hat{X} = \frac{X}{\| X \|} = \frac{\text{Eigen vector}}{\text{Length of Eigen vector}} = \begin{bmatrix} \frac{a}{\sqrt{a^2+b^2}} \\ \frac{b}{\sqrt{a^2+b^2}} \end{bmatrix}_{2\times 1}$

Example : $X = \begin{bmatrix} 2 \\ 7 \end{bmatrix}$, $\qquad \| X \| = \sqrt{2^2 + 7^2} = \sqrt{53}$

$$\hat{X} = \frac{X}{\|X\|} = \frac{1}{\sqrt{53}}\begin{bmatrix} 2 \\ 7 \end{bmatrix} \qquad \| \hat{X} \| = \sqrt{\left(\frac{2}{\sqrt{53}}\right)^2 + \left(\frac{7}{\sqrt{53}}\right)^2} = \sqrt{\frac{53}{53}} = 1$$

Rank of Matrix :

The rank of a matrix is a number equal to the order of the highest order non-vanishing minor, that can be formed from the matrix.

The rank of a matrix is said to be r if,

1. There is at least one non-zero minor of order r.
2. Every minor of A having order higher than r is zero.

Key Point

(i) The rank of a matrix A is the maximum number of linearly independent columns or Rows.

(ii) A matrix is **full rank**, if all the rows and columns are linearly independent. i.e. having rank as large as possible otherwise, the matrix is rank deficient

(iii) Rank of the matrix A is denoted by $\rho(A)$.

Properties of Rank of Matrix

(i) Rank of the matrix does not change by elementary transformation, we can calculate the rank by elementary transformations by changing the matrix into echelon form. In echelon form, rank of matrix is number of non-zero row of matrix.

(ii) The rank of matrix is zero, only when the matrix is a null matrix.

(iii) $\rho(A) \le \min(\text{Row, Column})$

(iv) $\rho(AB) \le \min[\rho(A), \rho(B)]$

(v) $\rho(A^T A) = \rho(AA^T) = \rho(A) = \rho(A^T)$

(vi) If A and B are matrices of same order, then $\rho(A+B) \le \rho(A) + \rho(B)$ and $\rho(A-B) \ge \rho(A) - \rho(B)$

(vii) If A^θ is the conjugate transpose of A, then $\rho(A^\theta) = \rho(A)$ and $\rho(AA^\theta) = \rho(A)$

(viii) The rank of a skew symmetric matrix cannot be one.

(ix) If A and B are two n-rowed square matrices, then $\rho(AB) \ge \rho(A) + \rho(B) - n$.

Solution of Linear Simultaneous Equations

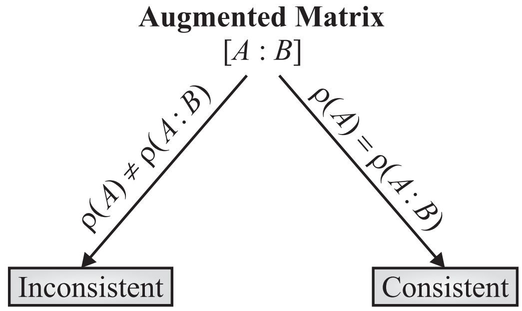
There are two types of linear simultaneous equations

(i) Linear homogeneous equation : $AX = 0$

(i) Linear non-homogeneous equation : $AX = B$

Steps to investigate the consistency of system of linear equations.

1. First represent the equations in matrix form as $AX = B$.
2. System equation $AX = B$ is checked for consistency as to make **Augmented Matrix** $[A : B]$.



$\rho(A) \neq \rho(A:B)$

Result **No solution**

$\rho(A) = \rho(A:B)$

When $\rho(A) = \rho(A:B)$
= No. of unknown variables

Result : **Unique solution**

When $\rho(A) = \rho(A:B)$
< No. of unknown variables

Result : **Infinite solution**

Chapter 2 : Differential Equations

Order and Degree of Differential Equation

Order : The order of a differential equation is maximum number of times differentiation present in the differential equation.

Degree : The degree of a differential equation is the power of the highest derivative term after removing the radical sign and fraction.

Examples of order and degree of differential equation

(1) $x\dfrac{dy}{dx} = y$ order = 1, degree = 1

(2) $\left[1+\left(\dfrac{dy}{dx}\right)^2\right]^3 = \left(\dfrac{d^2y}{dx^2}\right)^2$ order = 2, degree = 2

(3) $\left(\dfrac{d^2y}{dx^2}\right)^{\frac{3}{2}} = \left(\dfrac{dy}{dx}\right)^2$ order = 2,

To remove fraction power, squaring both sides we have,

$$\left(\frac{d^2y}{dx^2}\right)^3 = \left(\frac{dy}{dx}\right)^4 \quad \text{degree} = 3$$

Key Point

(i) Order and degree both are positive integer values.

(ii) There is no relation between order and degree.

(iii) A differential equation can exists without finite degree but cannot exists without finite order.

Linear and Non-linear Differential Equations :

A differential equation in which the dependent variable and its differential coefficients (derivatives) occur only in first degree (first power) and are not multiplied together (no product of dependent variables and/or derivatives occurs) is called a **linear** differential equation.

$$P_0\frac{d^ny}{dx^n} + P_1\frac{d^{n-1}y}{dx^{n-1}} + P_2\frac{d^{n-2}y}{dx^{n-2}} + \ldots\ldots P_{n-1}\frac{dy}{dx} + P_ny = Q$$

where, $P_0, P_1, P_2 \ldots P_{n-1}, P_n$ and Q are either constants or functions of independent variable x.

A differential equation is **non-linear** differential equation if :

1. Its degree is more than one.
2. Any one of the differential coefficients has order more than one.
3. Products containing dependent variable and its differential coefficients are present.

Solution of a Differential Equation

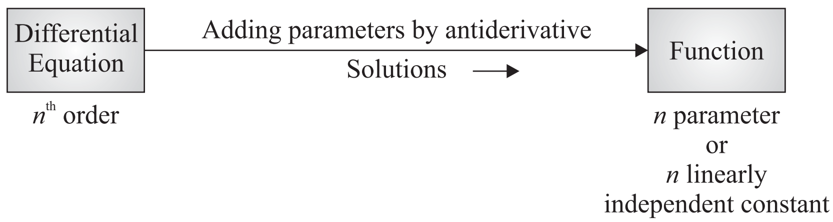


Fig. Solution of differential equation

In general there are two types of solutions that is given for ordinary differential equations.

(a) General Solution : The solution of a differential equation which contains a number of arbitrary constants equal to the order of the differential equation is called the general solution

(b) Particular Solution : A solution obtained by giving particular values to arbitrary constants (parameters) in the general solution is called a particular solution.

Basic Differential Equations and their Solutions :

Differential Equations	Solution
Separation of variables, $f_1(x)g_1(y)dx + f_2(x)g_2(y)dy = 0$	$\int \frac{f_1(x)}{f_2(x)}dx + \int \frac{g_2(y)}{g_1(y)}dy = c$
Linear first order equation $\frac{dy}{dx} + P(x)y = Q(x)$	(i) Integrating factor : $\text{I.F.} = e^{\int Pdx}$ (ii) Solution : $y \cdot \text{I.F.} = \int Q \cdot \text{I.F.}\, dx + c$
Exact equation $M(x, y)dx + N(x, y)dy = 0$ where, $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$	$\int_{y=\text{constant}} Mdx + \int (\text{term of } N, \text{not containing } x)\, dy = C$
Homogeneous equation $\frac{dy}{dx} = \phi\left(\frac{y}{x}\right)$	Put $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$ Now use separation of variables to solve the equation

Differential Equations	Solution
Linear, homogeneous second order differential equation with constant coefficients $a_0\frac{d^2y}{dx^2}+a_1\frac{dy}{dx}+a_2y=0$ a_0, a_1 and a_2 are real constants.	The solution is $y=$ C.F. (Complementary function) because for homogeneous equation P.I. is zero. **Calculation of C.F. :** Let m_1, m_2 be the roots of $a_0m^2+a_1m+a_2=0$. Then there are 4 cases. **Case 1 :** m_1, m_2 real and distinct $y=c_1e^{m_1x}+c_2e^{m_2x}$ **Case 2 :** m_1, m_2 real and equal $y=(c_1+c_2x)e^{m_1x}$ **Case 3 :** $m_1=a+bi$, $m_2=a-bi$ $y=e^{ax}(c_1\cos bx+c_2\sin bx)$ **Case 4 :** m_1 and m_2 are surds $(a\pm\sqrt{b})$, $y=e^{ax}[C_1\cosh\sqrt{b}x+C_2\sinh\sqrt{b}x]$ **Note :** ***Solve same for higher order also.***
Linear, non-homogeneous second order differential equation $a_0\frac{d^2y}{dx^2}+a_1\frac{dy}{dx}+a_2y=\phi(x)$ a_0, a_1 and a_2 are real constant	The solution is $y=\text{C.F.}+\text{P.I.}$, the calculation of C.F. is as given above. **Calculation for P.I. :** **Case - I :** **When $\phi(x)=e^{ax}$, the particular integral is as follows** $\text{P.I.}=\frac{1}{f(D)}e^{ax}=\frac{1}{f(a)}e^{ax}$ provided $f(a)\neq 0$, **Case fails** if $f(a)=0$ (where, a is a root of auxiliary equation) Then the value of P.I. $=\frac{1}{f(D)}e^{ax}=\frac{xe^{ax}}{f'(a)}$ Again **Case fails** If $f'(a)=0$ then $\text{P.I.}=\frac{1}{f(D)}e^{ax}=\frac{x^2e^{ax}}{f''(a)}$, $f''(a)\neq 0$, and so on. **Case - II :** **When $\phi(x)=\cos(ax+b)$ or $\sin(ax+b)$, the particular integration is as follows** $\text{P.I.}=\frac{1}{f(D)}\cos(ax+b)/\sin(ax+b)$ $\text{P.I.}=\frac{1}{f(-a^2)}\sin(ax+b)/\cos(ax+b)$, provide $f(-a^2)\neq 0$

Differential Equations	Solution
	Case fails if $f\left(-a^2\right)=0$ then P.I. $=\dfrac{x\cdot 1}{f'\left(-a^2\right)}\cdot \sin ax/\cos ax$ and so on. **Case - III :** **When $\phi(x)=x^m$ (m being non negative integer).** P.I. $=\dfrac{1}{f(D)}x^m=\left[f(D)\right]^{-1}x^m$ Expand $\left[f(D)\right]^{-1}$ in ascending powers of D and apply it to x^m. **Case - IV :** **When $\phi(x)=Ve^{ax}$ where, V is a function of x.** P.I. $=\dfrac{1}{f(D)}e^{ax}V=e^{ax}\dfrac{1}{f(D+a)}V$ **Note :** ***Solve same for higher order also.***
Cauchy linear differential equation : $a_0x^2\dfrac{d^2y}{dx^2}+a_1x\dfrac{dy}{dx}+a_2y=\phi(x)$	Put $x=e^t \Rightarrow x\dfrac{dy}{dx}=Dy$, $x^2\dfrac{d^2y}{dx^2}=D(D-1)y$ and so on .. then solve differential equation as above.

Partial Differential Equation

A differential equation is said to be partial differential equation if it contains partial derivatives of the dependent variable with respect to two or more independent variables.

Example : $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}=0$

General Notations :

For function $z=f(x,y)$,

$$\frac{\partial f}{\partial x}=p,\ \frac{\partial f}{\partial y}=q,\ \frac{\partial^2 f}{\partial x^2}=r,\ \frac{\partial^2 f}{\partial y^2}=t \text{ and } \frac{\partial^2 f}{\partial x\partial y}=\frac{\partial^2 f}{\partial y\partial x}=s.$$

Some standard form of partial differential equation

1. One-dimensional wave equation, $\dfrac{\partial^2 y}{\partial t^2}=c^2\dfrac{\partial^2 y}{\partial x^2}$: $y=f(x,t)$.

2. One-dimensional heat flow, $\dfrac{\partial u}{\partial t}=c^2\dfrac{\partial^2 u}{\partial x^2}$: $u=f(x,t)$.

3. Laplace equation, $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}=0$: $u=f(x,y)$.

Key Point

If the below equation represents the general form of a second order partial differential equation in two variables with constant coefficients.

$$a\frac{\partial^2 u}{\partial x^2}+b\frac{\partial^2 u}{\partial x\partial y}+c\frac{\partial^2 u}{\partial y^2}+d\frac{\partial u}{\partial x}+e\frac{\partial u}{\partial y}+fu=\phi(x,y)$$

Then properties and behaviour of its solution are largely dependent on its type, as classified below.

(i) If $b^2-4ac>0$, then the equation is called **hyperbolic.**

(ii) If $b^2-4ac=0$, then the equation is called **parabolic.**

(iii)If $b^2-4ac<0$, then the equation is called **elliptic**.

Jacobians

If u and v are function of the two independent variables x and y, then the determinant $\begin{vmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}\end{vmatrix}$ is called the Jacobian of u, v with respect to x, y and written as $\frac{\partial(u,v)}{\partial(x,y)}$ or $J\left(\frac{u,v}{x,y}\right)$

Properties of Jacobian

(i) If u and v are the functions of x and y, then $\frac{\partial(u,v)}{\partial(x,y)}\times\frac{\partial(x,y)}{\partial(u,v)}=1$

(ii) If u, v are the functions of r, s and r, s are the functions of x, y, then $\frac{\partial(u,v)}{\partial(x,y)}=\frac{\partial(u,v)}{\partial(r,s)}\times\frac{\partial(r,s)}{\partial(x,y)}$

Euler's Theorem of Homogeneous Function

Homogeneous Function

A function $f(x, y)$ is a homogeneous function of order n, if the degree of each of its terms in x and y is equal to n.

$$f(x,y)=a_0x^n+a_1x^{n-1}y+a_2x^{n-2}y^2+....+a_{n-1}xy^{n-1}+a_ny^n \qquad ...(i)$$

The function (i) which can be written as

$$f(x,y)=x^n\left[a_0+a_1\left(\frac{y}{x}\right)+a_2\left(\frac{y}{x}\right)^2+....+a_{n-1}\left(\frac{y}{x}\right)^{n-1}+a_n\left(\frac{y}{x}\right)^n\right]=x^n\phi\left(\frac{y}{x}\right)$$

$$f(x,y)=x^n\phi\left(\frac{y}{x}\right) \qquad ...(ii)$$

Equation (ii) is the general form of homogeneous function with degree n which can be any real value positive, negative or zero.

Euler's Theorem

If $u=x^n\phi\left(\frac{y}{x}\right)$ is a homogeneous function of x and y of degree n, then $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=nu$.

Deductions from Euler's theorem

If u is not a homogeneous function of x and y but $f(u)$ is homogenous function,

Then $$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=n\frac{f(u)}{f'(u)}$$

Euler's theorem for 2nd derivative

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=g(u)[g'(u)-1]\left[\text{Here}\quad g(u)=n\frac{f(u)}{f'(u)}\right]$$

Chapter 3 : Integral & Differential Calculus

Definite Integration

Some useful properties of Definite Integrals

1. $\int_a^b f(x)dx=[F(x)]_a^b=F(b)-F(a)$

2. $\int_a^b f(x)dx=\int_a^b f(t)dt$

3. $\int_a^b f(x)dx=-\int_b^a f(x)dx$

4. $\int_a^b f(x)dx=\int_a^c f(x)dx+\int_c^b f(x)dx$ where $a<c<b$

5. $\int_0^a f(x)dx=\int_0^a f(a-x)dx$

6. $\int_{-a}^a f(x)dx=\begin{cases}2\int_0^a f(x)\,dx, & \text{if } f(-x)=f(x); \quad \text{Even function}\\ 0, & \text{if } f(-x)=-f(x); \quad \text{Odd function}\end{cases}$

7. $\int_0^{2a} f(x)dx=\begin{cases}2\int_0^a f(x)dx, & \text{if } f(2a-x)=f(x)\\ 0, & \text{if } f(2a-x)=-f(x)\end{cases}$

8. $\int_0^{na} f(x)dx=n\int_0^a f(x)dx$ if $f(x)=f(x+a)$

9. $\int_a^b f(x)dx=\int_a^b f(a+b-x)dx$

10. $\int_a^b x\,f(x)dx=\frac{b-a}{2}\int_a^b f(x)dx$ if $f(a+b-x)=f(x)$

11. $\int f^n.gdx=f^{n-1}g-f^{n-2}g'+f^{n-3}g''-......(-1)^n\int fg^n dx$

    [Generalized form of integration by parts]

Special Functions (Gamma and Beta Functions)

Gamma Function

Gamma function denoted by $\Gamma(n)$ is defined by the improper integral which is dependent on the parameter n,

$$\Gamma(n)=\int_0^\infty e^{-t}t^{n-1}dt,\quad (n>0)$$

Key Point

Standard results of Gamma function :

(i) $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

(ii) $\Gamma(n) = (n-1)!$

(iii) $\Gamma(n+1) = \begin{cases} n\Gamma(n), & \text{if } n \text{ is any fraction} \\ n!, & \text{if } n \text{ is an integer} \end{cases}$

(iv) $\Gamma\left(-\frac{1}{2}\right) = -2\sqrt{\pi}$

Beta Function

Beta function $\beta(m,n)$ defined by

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx,\ (m>0, n>0)$$

Key Point

Standard results of Gamma function :

(i) $\beta(m,n) = \dfrac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$

(ii) $\int_0^{\frac{\pi}{2}} \sin^m \cos^n x.dx = \dfrac{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{2\Gamma\left(\frac{m+n+2}{2}\right)}$

(iii) $\int_0^{\frac{\pi}{2}} \sin^n x\,dx = \frac{1}{2}\beta\left(\frac{n+1}{2}, \frac{1}{2}\right) = \dfrac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n+2}{2}\right)} \cdot \dfrac{\sqrt{\pi}}{2}$

(iv) $\int_0^{\frac{\pi}{2}} \cos^n x\,dx = \frac{1}{2}\beta\left(\frac{1}{2}, \frac{n+1}{2}\right) = \dfrac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n+2}{2}\right)} \dfrac{\sqrt{\pi}}{2}$

Application of Definite Integral (Area, Length and Volume)

Applications		Formula
Area or Quadrature	Cartesian form	$\int_{x_1}^{x_2}\int_{y_1}^{y_2} dydx$
	Polar form $r = f(\theta)$	$\int_\alpha^\beta \frac{r^2}{2} d\theta$ or $\iint_S r dr d\theta$
Length of Curve	Cartesian form $y = f(x)$	$\int_a^b \sqrt{\left[1+\left(\frac{dy}{dx}\right)^2\right]}\,dx$
	Polar form $r = f(\theta)$	$\int_\alpha^\beta \sqrt{\left[r^2+\left(\frac{dr}{d\theta}\right)^2\right]}\,d\theta$

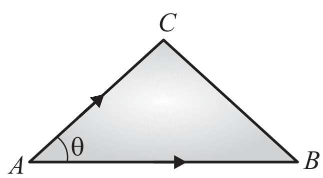
Volume of Revolution	Volume revolved by x - axis	$\int_a^b \pi y^2 dx$
	Volume revolved by y - axis,	$\int_a^b \pi x^2 dy$
	About the initial line ($\theta = 0$)	$\int_\alpha^\beta \frac{2\pi}{3} r^3 \sin\theta \, d\theta$
	About the line $\left(\theta = \frac{\pi}{2}\right)$	$\int_\alpha^\beta \frac{2\pi}{3} r^3 \cos\theta \, d\theta$
Volume as Double and Triple Integral	Triple integral	Cartesian: $\iiint_V dx\,dy\,dz$ Cylindrical: $\iiint r dr d\phi dz$ Spherical : $\iiint r^2 \sin\theta dr d\theta d\phi$
	Double integral	$\iint_s f(x,y) dx\, dy$

Chapter 4 : Vector Calculus

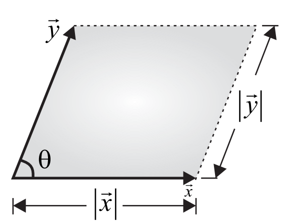
Applications of Vector Analysis

Area of the Triangle : If $\vec{a}, \vec{b}, \vec{c}$ are the position vectors of the vertices A, B, C, of a triangle, then the area of the triangle is given by

$$\text{Area } (\Delta ABC) = \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right|$$



Area of Parallelogram



$$\text{Area } = |\vec{x} \times \vec{y}| = |\vec{x}||\vec{y}|\sin\theta$$

Let $\vec{x} = x_1\hat{i} + x_2\hat{j} + x_3\hat{k}$ and $\vec{y} = y_1\hat{i} + y_2\hat{j} + y_3\hat{k}$

$$\vec{x} \times \vec{y} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}$$

Orthogonal and Orthonormal Vectors

(i) Two vectors $\vec{A}$ and $\vec{B}$ are said to be orthogonal if their dot product is equal to zero.

(ii) Two vectors $\vec{A}$ and $\vec{B}$ are said to be orthonormal if their dot product is equal to zero and magnitude of both vectors are unity.

Vector (Differential and Integral) Calculus

Del Operator or Nabla Operator

Del operator is three dimension vector operator used for the derivative in 3D vector space. When Del operator is applied to a field (scalar or vector) then it gives the gradient of a scalar field, the divergence of a vector field, or the curl of a vector field depending on the way it is applied.

Del operator, $\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$

Application	Formula	Key points
Gradient	$\text{grad}(\phi) = \nabla\phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\phi$ $\text{grad}(\phi) = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$	(i) It is only valid for scalar point function. (ii) Gradient of scalar point function is a vector quantity. (iii) It gives the maximum rate of change.
Divergence	$\text{div}(\vec{F}) = \nabla\cdot\vec{F}$ $\text{div}(\vec{F}) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\left(\hat{i}F_1 + \hat{j}F_2 + \hat{k}F_3\right)$ $\text{div}(\vec{F}) = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$	(i) Divergence is only valid for vector point function. (ii) It is calculated by dot product of del operator with given vector quantity. (iii) It is used to calculate the net flow of vector quantity. (iv) If $\nabla\cdot\vec{F} = 0$ then $\vec{F}$ is called a solenoidal vector/ incompressible vector. $\text{div}\vec{F} = \begin{cases} -\text{ve}, & \text{inward flow} \\ 0, & \text{no flow (zero flow)} \\ +\text{ve}, & \text{outward flow} \end{cases}$
Curl	$\text{curl}(\vec{F}) = \nabla\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$ where, $\vec{F} = \left(F_1\hat{i} + F_2\hat{j} + F_3\hat{k}\right)$	(i) It is only valid for vector point function. (ii) Curl is basically used to find the rotation of vector quantity. (iii) If curl $\vec{F} = 0$, the field $\vec{F}$ is termed as irrotational.

Directional Derivative	(i) Find the gradient of scalar potential function i.e. $\nabla\phi$ (ii) Find the unit vector in the given direction $\vec{d}$ i.e. $\hat{d} = \dfrac{\vec{d}}{\lvert\vec{d}\rvert}$ (iii) Calculate dot product of gradient and unit vector that gives directional derivative as $\nabla\phi.\,\hat{d}$	(i) It gives the rate of change of a scalar point function in particular direction (ii) Maximum magnitude of directional derivative is the magnitude of gradient.

Key Point

(i) Divergence of curl of $\vec{A}$ is zero i.e. div (curl $\vec{A}$) = 0, $\nabla\cdot(\nabla\times\vec{A}) = 0$

(ii) Curl of gradient of ϕ is zero i.e. curl (grad ϕ) = 0, $\nabla\times(\nabla\phi) = 0$

(iii)Divergence of gradient of ϕ is i.e. div (grad ϕ) , $\nabla\cdot(\nabla\phi) = \nabla^2\phi$

Vector Integral Theorems

Green's Theorem

(i) It is used to simplify the vector integration.

(ii) It gives the relation between closed line and open surface integration.

Statement : If $\phi(x,y), \psi(x,y), \dfrac{\partial\phi}{\partial y}$ and $\dfrac{\partial\psi}{\partial x}$ be continuous functions over a region R bounded by simple closed curve c in x-y plane, then according to this theorem

$$\oint_c (\phi\, dx + \psi\, dy) = \iint_R \left(\frac{\partial\psi}{\partial x} - \frac{\partial\phi}{\partial y}\right) dx\, dy$$

Stoke's Theorem

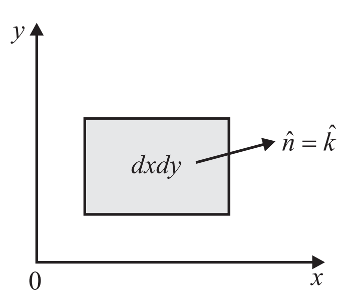
(i) It is used to simplify the vector integration.

(ii) It gives the relation between closed line and open surface integration.

Statement : Surface integral of curl of $\vec{F}$ along the normal to the surface S ,bounded by curve c is equal to the line integral of the vector point function $\vec{F}$ taken along the closed curve c.

Mathematically, $\oint_c \vec{F}.\vec{dr} = \iint_s \text{curl}\,\vec{F}.\hat{n}\,ds$

or $$\oint_c \vec{F}.\vec{dr} = \iint_s (\nabla\times\vec{F})\cdot ds\cdot(\hat{n})$$



where, $\hat{n}$ is the direction of the surface S and this direction is normal or perpendicular outward to the surface.

Example : Let the surface is xy plane, $ds = dxdy$ and $\hat{n} =$ direction perpendicular to surface i.e. z-axis, so the direction is $\hat{k}$.

$\therefore \quad \hat{n} = \hat{k}$

Gauss Theorem or Divergence Theorem

(i) This theorem is used to simplify the vector integration.

(ii) It gives the relationship between closed surface and open volume integration

Statement : The surface integral of the normal component of a vector function $\vec{F}$ taken around a closed surfaced S is equal to the integral of the divergence of F taken over the volume V enclosed by the surface S.

Mathematically, $\oiint_s \vec{F}.\hat{n}\, ds = \iiint_v div\, \vec{F}\, dv$

Or $\oiint_s \vec{F}.\overrightarrow{ds} = \iiint_v \nabla.\vec{F}\, dv$

Chapter 5 : Maxima & Minima

Maxima and Minima for Function of One Independent Variable

To find maxima and minima of a function $y = f(x)$, follow these steps

Step 1 : Find $\frac{dy}{dx}$, and put $\frac{dy}{dx} = 0$ find the value of x and this value is said to be the stationary point, this is the necessary condition to find extremum value of function.

Step 2 : Find $\frac{d^2y}{dx^2}$ and check the value at the stationary point obtained in step 1.

(i) A function $f(x)$ has a maxima at $x = a$ if $f'(a) = 0$ and $f''(a) < 0$.

(ii) A function $f(x)$ has a minima at $x = a$ if $f'(a) = 0$ and $f''(a) > 0$.

(iii) A function $f(x)$ has no maxima and minima at $x = a$ if $f'(a) = 0$ and $f''(a) = 0$.

Key Point

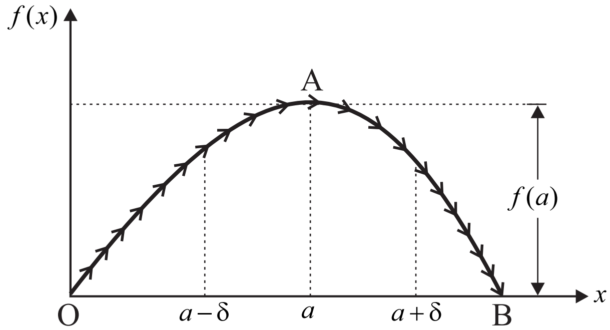
Stationary points : For a continuous and differentiable function $f(x)$, the values of x for which the slope of the function $f'(x) = 0$ are called stationary points or turning points or critical points. These are the points of x in the domain where $\boldsymbol{f'(x) = 0}$.

Saddle point : A point where function is neither maximum nor minimum is said to be a **saddle point**. At such point function is maximum in one direction while minimum in another direction.

Local or Relative Maxima and Minima

Local Maxima

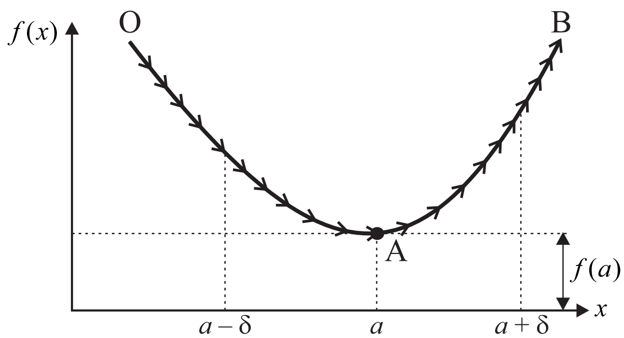
A function $f(x)$ has a maximum at $x = a$ if there exists some interval $(a-\delta,\, a+\delta)$ around 'a' such that $f(a) \geq f(x)$ for all x in $(a-\delta,\, a+\delta)$.



OA = Increasing Function, A = Stationary Point, AB = Decreasing Function

Local Minima

A function $f(x)$ has a minimum at $x = a$ if there exists some interval $(a-\delta,\ a+\delta)$ around 'a' such that $f(a) \le f(x)$ for all values of x in $(a-\delta,\ a+\delta)$.



OA = Decreasing function, A = Stationary Point, AB = Increasing Function

Steps to find Absolute Maximum and Minimum value of the Function in the Interval [*a*, *b*]

To find the absolute value or maximum and minimum value of the function follow the steps given below,

Steps 1 : Find stationary points by putting $f'(x) = 0$.

Step 2 : Find the value of $f(x)$ at stationary points.

Step 3 : Also find $f(a)$ and $f(b)$.

Then the maximum of the value is the absolute maximum of the given function $f(x)$ and minimum of these values is the absolute minimum of the given function $f(x)$.

Key Point

Convexity and Concavity of a Curve

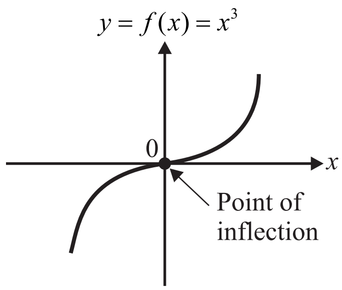
(i) If $f''(x) < 0$, $x \in (a,b)$ then the curve $y = f(x)$ is convex upward or concave downward in (a,b).

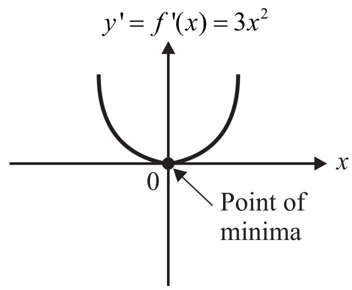
(ii) If $f''(x) > 0$, $x \in (a,b)$ then the curve $y = f(x)$ is concave upward or convex downward in (a,b).

Point of inflection :

An inflection point is a point on a curve at which the sign of the curvature (i.e. the concavity) changes. An inflection point does not have to be a stationary point, but if it is, then it would also be a saddle point.

A function f (or the curve $y = f(x)$) has a point of inflection at $x = c$ if $f'(c) = 0$, $f''(c) = 0$ and $f'''(c) \ne 0$.





Maxima and Minima for Function of Two Independent Variables

To find maxima and minima of a function $z = f(x, y)$, follow these steps

Step 1 : Find $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$

Step 2 : Solve $\frac{df}{dx} = 0$ and $\frac{df}{dy} = 0$ to get stationary points.

Step 3 : Find the values of

$$r = \frac{\partial^2 f}{\partial x^2}, s = \frac{\partial^2 f}{\partial x \partial y}, t = \frac{\partial^2 f}{\partial y^2}$$

Step 4 : Check the conditions

If	$rt - s^2 > 0, r > 0$	then it gives minima
	$rt - s^2 > 0, r < 0$	then it gives maxima
	$rt - s^2 = 0$	then we need further investigation required
	$rt - s^2 < 0$	then neither maxima nor minima

Chapter 6 : Mean Value Theorem

Concept of Continuity and Differentiability

Continuity

The word continuous means without any break or gap. A function is continuous when its graph is a single unbroken curve.

Example : $\sin x,\ x,\ \cos x,\ e^x$ etc.

Continuity of a Function at a Point :

A function $f(x)$ is continuous at $x = a$ if the following three conditions are satisfied :

(i) $f(a)$ is defined

(ii) $\lim_{x \to a} f(x)$ exists i.e. $\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x)$ or R.H.L. = L.H.L.

(iii) $\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = f(a)$

$$\lim_{x \to a^-} f(x) = \lim_{h \to 0} f(a - h) = \text{Left hand limit}$$

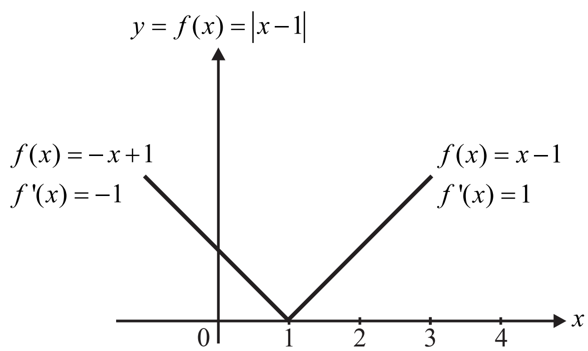
$$\lim_{x \to a^+} f(x) = \lim_{h \to 0} f(a + h) = \text{Right hand limit}$$

If the above conditions are not satisfied then it is referred as **Discontinuous Function**.

Differentiability

The function $f(x)$ is differentiable at point P, if there exists a unique tangent at point P or if the curve does not have P as a corner point i.e. the function is not differentiable at those points on which function has jumps and sharp edges.

Consider the function $f(x)=|x-1|$, which can be graphically shown as,



which shows that $f(x)$ is not differentiable at $x=1$. $f(x)$ has sharp edge at $x=1$.

Differentiability of a Function at a Point

A function $f(x)$ is said to be differentiable (finitely) at $x=a$ if $f'(a^+)=f'(a^-)=$ finite, i.e., $\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}=\lim_{h\to 0}\frac{f(a-h)-f(a)}{-h}=$ finite and the common limit is called the derivative of $f(x)$ at $x=a$, denoted by $f'(a)$.

First derivative of $f(x)$ at $x=a$,

$$f'(a)=\lim_{x\to a}\frac{f(x)-f(a)}{x-a} \qquad \{x\to a \text{ from the left as well as from the right}\}$$

Mean Value Theorem :

Rolle's Mean Value Theorem

If $f(x)$ is real valued function such that

(i) $f(x)$ is continuous in the closed interval $[a,b]$

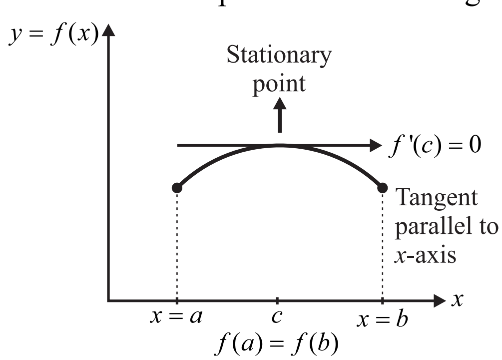
(ii) $f(x)$ is differentiable in the open interval (a,b)

(iii) $f(a)=f(b)$

Then there exists atleast one value of x, $c\in(a,b)$ such that $f'(c)=0$.

Geometrical Interpretation :

This theorem states that between two points with equal coordinates on the graph of the function $f(x)$, there exists at least one point where the tangent is parallel to x-axis.



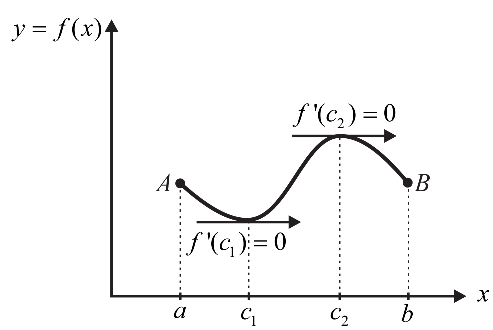


Fig. Geometric interpretation of Rolle's mean value theorem

Key Point

(i) There may be more than one stationary points where $f'(x)$ vanishes i.e. becomes zero.

(ii) Rolle's theorem fails even if any one of the three conditions is not satisfied by the function.

(iii) $f'(x)$ may be zero at a point in (a,b) without satisfying all the three conditions of Rolle's theorem. Hence, the converse of Rolle's theorem is not true.

Lagrange's Mean Value Theorem

If $f(x)$ is real valued function such that,

(i) $f(x)$ is continuous in the closed interval $[a,b]$

(ii) $f(x)$ is differentiable in the open interval (a,b)

(iii) $f(a) \neq f(b)$

Then there exists atleast one value x, $c \in (a,b)$ such that $f'(c) = \dfrac{f(b)-f(a)}{b-a}$

Geometrical Interpretation :

This theorem states that, between two points *a* and *b*, $f(a) \neq f(b)$ of the graph of $f(x)$ then there exists atleast one point where the tangent is parallel to the chord or link *AB*.

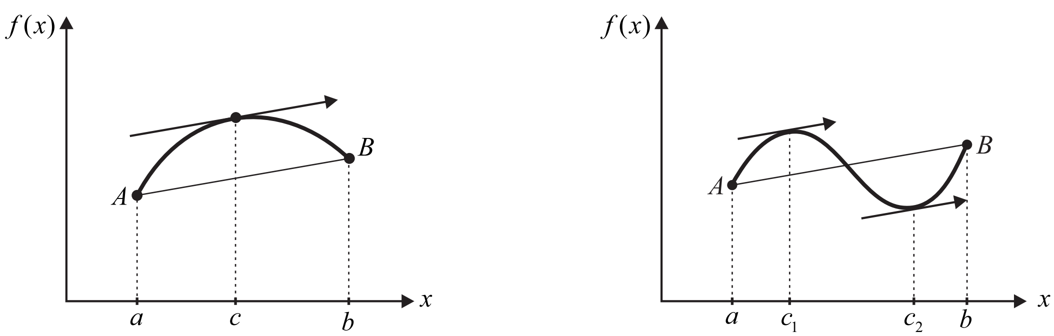


Fig. Geometric interpretation of Lagrange's mean value theorem

Key Point

(i) Lagrange's mean value theorem fails if the function does not satisfy even one of the three conditions.

(ii) The converse of Lagrange's mean value theorem may not be true for, $f'(c) = \dfrac{f(b)-f(a)}{b-a}$ at a point c in (a, b) without satisfying both the conditions of Lagrange's mean value theorem.

Cauchy's Mean Value Theorem

If two function $f(x)$ and $g(x)$ are,

(i) Continuous in a closed interval $[a,b]$

(ii) Differentiable in the open interval (a,b)

(iii) $g'(x) \neq 0$ for any point of the open interval (a,b)

Then there exist atleast one value *c* in (a,b) such that,

$$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$$

Chapter 7 : Complex Variables

Complex Number

The numbers in the form of $z = x + iy$ is a complex number where $x \in R$, $y \in R$, $i = \sqrt{-1}$, i.e., $i^2 = -1$; real part of $z = \text{Re}(z) = x$, imaginary part of $z = \text{Im}(z) = y$.

If $z = x + iy$ then

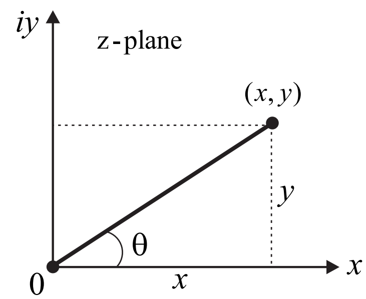
(i) Modulus of $z = |z| = \sqrt{x^2 + y^2}$

(ii) Argument of z (arg z) $= \theta = \tan^{-1}\left(\frac{y}{x}\right)$.

(iii) Conjugate of a complex number $\bar{z} = x - iy$

(iv) $z = re^{i\theta} = r(\cos\theta + i\sin\theta)$

(v) $z\bar{z} = \bar{z}z = |z|^2$



$$|z_1 + z_2| \le |z_1| + |z_2| \text{ and } |z_1 - z_2| \ge |z_1| - |z_2|.$$

Analytic Function

A single valued function $f(z)$ which is differentiable at $z = z_0$ is said to be analytic at point $z = z_0$.

The point at which function is not differentiable is called **singular point** of the function.

Cauchy Riemann Equation (Condition for function to be analytic)

If $f(z) = u(x, y) + iv(x, y)$ is differentiable at $z = z_0$ then at this point the first order partial derivatives of u and v exist and satisfy the **Cauchy-Riemann** equations. The necessary conditions for a function $f(z) = u + iv$ to be analytic at all points in a region R are :

(i) $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ (ii) $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

(i) and (ii) both are called **C-R equations.**

For a function to be analytic

(i) The C-R equations should be satisfied.

(ii) The partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ should be continuous.

C-R Equations in Polar Form

For the complex function $f(z) = u(r, \theta) + iv(r, \theta)$, to be analytic following equations should be satisfied

(i) $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$ (ii) $\frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}$

Entire Function

A function $f(z)$ which is analytic at every point of the finite complex plane is known as entire function. E.g. Polynomial and exponential functions are entire functions.

Complex Line Integral

Let $f(z)$ be a continuous function of the complex variable $z = x + iy$ defined at all points of the curve C having end points a and b.

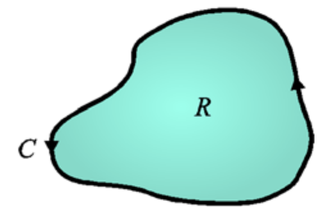
$$\int_c f(z)dz = \int_c (u+iv)(dx+idy)$$

$$\int_c f(z)dz = \int_c (udx - vdy) + i(vdx + udy)$$

Complex Integration

Cauchy's Theorem

If $f(z)$ is single valued and an analytic function of z and $f'(z)$ is continuous at each point within and on the closed curve c, then according to the theorem,



$$\oint_c f(z)dz = 0$$

Cauchy's Integral Formula

(i) For Simple Pole : If $f(z)$ is analytic within and on a closed curve c and if a (simple pole) is any point within c, then

$$\oint_c \frac{f(z)}{z-a}dz = 2\pi i . f(a)$$

(ii) For Multiple Poles : If $f(z)$ is analytic within and on a closed curve c, and if a (multiple poles) are points within c, then

$$\oint_c \frac{f(z)}{(z-a)^n}.dz = \frac{2\pi i}{(n-1)!}\left(\frac{d^{n-1}f(z)}{dz^{n-1}}\right)_{z=a}$$

Residues and Residues Theorem

Residue

The coefficient of $(z-a)^{-1}$ in the expansion of $f(z)$ around an isolated singularity is called the residue of $f(z)$ at that point.

Method of Finding Residues

(a) Residue at simple pole

If $f(z)$ has a simple pole at $z = a$ then

$$\text{Res } f(a) = \lim_{z\to a}(z-a)f(z)$$

(b) Residue at a pole of order *n*

If $f(z)$ has a pole of order n at $z = a$, then

$$\text{Res }(\text{at } z = a) = \frac{1}{(n-1)!}\left\{\frac{d^{n-1}}{dz^{n-1}}\left[(z-a)^n f(z)\right]\right\}_{z=a}$$

Residue Theorem

If $f(z)$ is analytic in a closed curve C, except at a finite number of poles within C, then

$$\oint_c f(z)dz = 2\pi i \times (\text{Sum of residues at the poles inside or on C})$$

Complex Function Series Expantion

1. **Taylor's Series :** If a function $f(z)$ is analytic at all points inside a circle c, with its centre at the point a and radius R, then at each point z inside c,

$$f(z)=f(a)+(z-a)f'(a)+(z-a)^2\frac{f''(a)}{2!}+.........+(z-a)^n\frac{f''(a)}{n!}+...$$

2. **Laurent's Theorem :**

$$f(z)=a_0+a_1(z-z_0)+a_2(z-z_0)^2+....+\frac{b_1}{z-z_0}+\frac{b_2}{(z-z_0)^2}+...$$

$$f(z)=\sum_{n=0}^{\infty}a_n(z-z_0)^n+\sum_{n=1}^{\infty}b_n(z-z_0)^{-n}$$

Chapter 8 : Limit & Series Expansion

Standard Result of limits

(i) $\lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1$ (θ is in radian)

(ii) $\lim_{\theta\to 0}\cos\theta=1$ (θ is in radian)

(iii) $\lim_{\theta\to 0}\frac{\tan\theta}{\theta}=1$ (θ is in radian)

(iv) $\lim_{x\to a}\frac{x^n-a^n}{x-a}=na^{n-1}$

(v) $\lim_{x\to\infty}\left(\frac{1}{x}\right)=0$

(vi) $\lim_{x\to 0}(1+x)^{1/x}=\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x=e$

(vii) $\lim_{x\to 0}\frac{\log(1+x)}{x}=1$

(viii) $\lim_{x\to 0}\frac{a^x-1}{x}=\ln(a)$

(ix) $\lim_{x\to\infty}\frac{\sin x}{x}=\lim_{x\to\infty}\frac{\cos x}{x}=\frac{\text{any number between }-1\text{ and }1}{\infty}=0$

(x) If $\lim_{x\to a}f(x)=1$ and $\lim_{x\to a}g(x)=\infty$

then $\lim_{x\to a}\{f(x)\}^{g(x)}=\lim_{x\to a}\ e^{g(x)\{f(x)-1\}}$

(xi) If $\lim_{x\to a}f(x)=\lim_{x\to a}g(x)=0$, then $\lim_{x\to a}\{1+f(x)\}^{\frac{1}{g(x)}}=e^{\lim_{x\to a}\frac{f(x)}{g(x)}}$

(xii) If $\lim_{x\to a}f(x)=A>0$ and $\lim_{x\to a}g(x)=B$, then $\lim_{x\to a}\{f(x)\}^{g(x)}=A^B$

L-Hospital's Rule for Indeterminate form

Indeterminate forms : Algebraic expressions sometime become indeterminate for particular values of the variable on which they depend but its limit can be evaluate. Intermediate forms are,

$$\frac{0}{0},\ \frac{\infty}{\infty},\ \infty\times\infty,\ 0^0,\ 1^\infty,\ \infty^0,\ \infty-\infty,\ \frac{0}{\infty},\ \frac{\infty}{0},\ \infty^\infty,\ 0\times\infty,\ 0^\infty$$

L-Hospital's rule : L-Hospital's rule is a general method for evaluating the basic indeterminants forms $\frac{0}{0}$ and $\frac{\infty}{\infty}$ all the other forms can be converted to these two basic forms.

This states that if $\lim_{x\to a}\frac{f(x)}{g(x)}$ reduces to $\frac{0}{0}$ or $\frac{\infty}{\infty}$, then, differentiate numerator and denominator until and unless this form is eliminated

i.e. $$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}$$

But if again it comes in the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

Then, $$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}=\lim_{x\to a}\frac{f''(x)}{g''(x)}$$

And this process is continued till $\frac{0}{0}$ or $\frac{\infty}{\infty}$ form is eliminated.

Important Transformation of Calculation of Indeterminate form

Indeterminate form	Conditions	Transformation to $\frac{0}{0}$	Transformation to $\frac{\infty}{\infty}$
$\frac{0}{0}$	$\lim_{x\to a}f(x)=0$ $\lim_{x\to a}g(x)=0$	–	$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{\frac{1}{g(x)}}{\frac{1}{f(x)}}$
$\frac{\infty}{\infty}$	$\lim_{x\to a}f(x)=\infty$ $\lim_{x\to a}g(x)=\infty$	$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{\frac{1}{g(x)}}{\frac{1}{f(x)}}$	–
$0\times\infty$	$\lim_{x\to a}f(x)=0$ $\lim_{x\to a}g(x)=\infty$	$\lim_{x\to a}f(x)\cdot g(x)=\lim_{x\to a}\frac{f(x)}{\frac{1}{g(x)}}$	$\lim_{x\to a}f(x)\cdot g(x)=\lim_{x\to a}\frac{g(x)}{\frac{1}{f(x)}}$
$\infty-\infty$	$\lim_{x\to a}f(x)=\infty$ $\lim_{x\to a}g(x)=\infty$	$\lim_{x\to a}[f(x)-g(x)]=\lim_{x\to a}\frac{\frac{1}{g(x)}-\frac{1}{f(x)}}{\frac{1}{[f(x)g(x)]}}$	$\lim_{x\to a}[f(x)-g(x)]=\ln\lim_{x\to a}\frac{e^{f(x)}}{e^{g(x)}}$
0^0	$\lim_{x\to a}f(x)=0$ $\lim_{x\to a}g(x)=0$	$\lim_{x\to a}f(x)^{g(x)}=\exp\lim_{x\to a}\frac{g(x)}{\left(\frac{1}{\ln f(x)}\right)}$	$\lim_{x\to a}f(x)^{g(x)}=\exp\lim_{x\to a}\frac{\ln f(x)}{\left(\frac{1}{g(x)}\right)}$
1^∞	$\lim_{x\to a}f(x)=1$ $\lim_{x\to a}g(x)=\infty$	$\lim_{x\to a}f(x)^{g(x)}=\exp\lim_{x\to a}\frac{\ln f(x)}{\left(\frac{1}{g(x)}\right)}$	$\lim_{x\to a}f(x)^{g(x)}=\exp\lim_{x\to a}\frac{g(x)}{\left(\frac{1}{\ln f(x)}\right)}$
∞^0	$\lim_{x\to a}f(x)=\infty$ $\lim_{x\to a}g(x)=0$	$\lim_{x\to a}f(x)^{g(x)}=\exp\lim_{x\to a}\frac{g(x)}{\left(\frac{1}{\ln f(x)}\right)}$	$\lim_{x\to a}f(x)^{g(x)}=\exp\lim_{x\to a}\frac{\ln f(x)}{\left(\frac{1}{g(x)}\right)}$
Here, **exp** represents Exponential function and **ln** represents Logarithmic function.			

Series Expansion of Functions

Taylor's Series

If $f(x)$ is differentiable at point $x=a$ then it can be expanded as an infinite series as follows

$$f(x)=f(a)+(x-a)f'(a)+\frac{(x-a)^2}{2!}f''(a)+\frac{(x-a)^3}{3!}f'''(a)+.....\infty$$

When $a=0$,then the series is called as Maclaurin series.

If any function $f(a),\ f'(a),\ f''(a),\ ......$ becomes infinite or does not exist for any value of a in the interval under considerations then Taylor's series fails to expand.

Maclaurin's Series

If $f(x)$ is differentiable at point $x=0$ then it can be expanded as an infinite series as follows

$$f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f'''(0)+.......\infty$$

Useful Series Expansion (Mainly Derived from Maclaurin's Series)

1.	$e^x=1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+.......................................\infty$	
2.	$e^{-x}=1-\frac{x}{1!}+\frac{x^2}{2!}-\frac{x^3}{3!}+.......................................\infty$	
3.	$\frac{e^x+e^{-x}}{2}=\cos h(x)=1+\frac{x^2}{2!}+\frac{x^4}{4!}+\frac{x^6}{6!}+..............\infty$	
4.	$\frac{e^x-e^{-x}}{2}=\sin h(x)=x+\frac{x^3}{3!}+\frac{x^5}{5!}+\frac{x^7}{7!}+.............\infty$	
5.	$\log(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+..........................\infty$	where $\|x\|<1$
6.	$\log(1-x)=-\left[x+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{4}+.......................\infty\right]$	where $\|x\|<1$
7.	$\frac{1}{2}\log\frac{1+x}{1-x}=x+\frac{x^3}{3}+\frac{x^5}{5}+\frac{x^7}{7}+.........................\infty$	where $\|x\|<1$
8.	$\sin x=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+................=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}x^{2n+1}$	
9.	$\cos x=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+...................=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}x^{2n}$	
10.	$\tan x=x+\frac{1}{3}x^3+\frac{2}{15}x^5+\frac{17}{315}x^7+..........$	for $\|x\|<\frac{\pi}{2}$
11.	$\cot x=x^{-1}-\frac{1}{3}x-\frac{1}{45}x^3-\frac{2}{945}x^5-.....$	for $0<\|x\|<\pi$
12.	$\sec x=1+\frac{1}{2}x^2+\frac{5}{24}x^4+\frac{61}{720}x^6+.....$	for $\|x\|<\frac{\pi}{2}$

13.	$\text{cosec}\, x = x^{-1} + \frac{1}{6}x + \frac{7}{360}x^3 + \frac{31}{15120}x^5 + \ldots\ldots\ldots\ldots$	for $0<\lvert x\rvert<\pi$
14.	$\sin^{-1} x = x + \frac{1^2.x^3}{3!} + \frac{1^2.3^2x^5}{5!} + \frac{1^2.3^2.5^2x^7}{7!} + \ldots\ldots\ldots\ldots\infty$	$\lvert x\rvert<1$
15.	$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots\ldots\ldots\ldots\infty$	$\lvert x\rvert<1$
16.	$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots$	
17.	$(1+x)^{-1} = 1 - x + x^2 - x^3 + x^4 - x^5 + \ldots\ldots\ldots\ldots\infty$	
18.	$(1-x)^{-1} = 1 + x + x^2 + x^3 + x^4 + x^5 + \ldots\ldots\ldots\ldots\infty$	
19.	$(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + 5x^4 - \ldots\ldots\ldots\ldots\infty$	
20.	$(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \ldots\ldots\ldots\ldots\infty$	

Chapter 9 : Probability & Statistics

Random Experiment :

A random experiment is an experiment or a process for which the outcomes cannot be predicted with certainty.

Example : In an experiment of throwing a dice and getting a number 1, 2, 3, 4, 5 or 6 are different events.

Events :

A set of one or more outcome of an random experiment is called event.

Types of Events :

1. **Equally Likely Events :** Events are said to be equally likely when the chances are same for occurrence of all events.
   Example : When a dice is thrown any one number from 1 to 6 may occur. In this trial, the six events are equally likely.
2. **Mutually Exhaustive Events :** Two or more events in any trial are known as exhaustive events. If one of them is necessarily (must) occurs.
   Example : When a dice is thrown, there are six exhaustive events.
3. **Mutually Exclusive Events :** If the occurrence of anyone of the events in a trial prevents the occurrence of the other events, then the events are said to be mutually exclusive events.
   Example : When a dice is thrown the event of getting faces numbered 1 to 6 are mutually exclusive.

Key Point

If A and B both are mutually exclusive events in same sample space then $P(A \cap B) = 0$.

4. **Independent Events :** If there are two or more event such that the occurrence of any one does not depend on occurrence of other, they are said to be independent event.
   Example : Throwing two dice, event A is face 4 in first dice and event B face 3 in second dice, these both events are independent, and also not mutually exclusive because it can happen simultaneously.

Key Point

If A and B both are independent events in different sample space then $P(A \cap B) = P(A)P(B)$.

5. **Dependent Events :** When two events are dependent, the occurrence of one event influences the probability of another event.

   Example : Throwing two dice, event A is face 4 in first dice and event B is sum total value 7. The value 4 in first dice gives the information that for total sum 7 it should be 3 in second dice.

Classical Definition of Probability

In a random experiment, probability is the ratio of favourable events to corresponding sample space. The probability of event E in the sample space S is defined as,

$$P(E) = \frac{\text{Favourable events}}{\text{Total number of events}} = \frac{n(E)}{n(S)}$$

Examples :

(i) The typical example of classical probability would be fair dice roll because it is equally probable that you will land on any of the six numbers on the die : 1, 2, 3, 4, 5 or 6

(ii) Throwing a dice, the event A that even number on the dice. Sample space for a dice $S = \{1, 2, 3, 4, 5, 6\}$ elements of event $A = \{2, 4, 6\}$.

The required probability, $P(A) = \dfrac{n(A)}{n(S)} = \dfrac{3}{6} = \dfrac{1}{2} = 0.5$

Conditional Probability

It gives the probability of happening of any event if the another is already occurred.

$P\left(\dfrac{E_1}{E_2}\right) =$ Probability of getting the event E_1 when event E_2 is already occurred.

$$P\left(\frac{E_1}{E_2}\right) = \frac{P(E_1 \cap E_2)}{P(E_2)}$$

Key Point

Results of Probability

(i) $0 \le P(E) \le 1$

(ii) $P(\bar{E}) = 1 - P(E)$

(iii) $P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$ If mutually exclusive then $P(E_1 \cap E_2) = 0$

Therefore, $P(E_1 \cup E_2) = P(E_1) + P(E_2)$

(iv) $P(E_1 \cup E_2 \cup E_3) = P(E_1) + P(E_2) + P(E_3) - P(E_2 \cap E_3) - P(E_1 \cap E_2) - P(E_1 \cap E_3) + P(E_1 \cap E_2 \cap E_3)$

(v) $P(\bar{E}_1 \cap \bar{E}_2) = P(\overline{E_1 \cup E_2}) = 1 - P(E_1 \cup E_2)$

Probability Density function

A continuous function $f(x)$ is called a probability density function of a random variable, if it satisfies the following conditions,

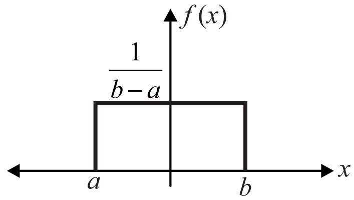
(i) $f(x) \ge 0$ (ii) $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx = 1$

Probability of Distribution

Uniform Distribution

The probability density function of a uniform random variable on the interval (a, b) is given by,

$$f(x) = \begin{cases} \dfrac{1}{b-a} & \text{if } a < x < b \\ 0 & \text{otherewise} \end{cases}$$



Key Point

(i) Mean $\bar{x} = \dfrac{a+b}{2}$

(ii) Variable $\sigma^2 = \dfrac{(b-a)^2}{12}$

(iii)Standard deviation $\sigma = \sqrt{\text{Variance}}$

Binomial Distribution

Binomial Distribution gives the probability of happening of event 'r' times exactly in 'n' trials

$$P(r) = {}^nC_r\, p^r q^{n-r}$$

where, n = Number of trials

r = Number of favourable events

p = Probability of happening of event

q = Probability of not happening of event $= 1 - p$

Key Point

(i) Mean $= np$ (ii) Variance $= npq$ (iii) Standard Deviation $= \sqrt{npq}$

Poisson Distribution

Poisson Distribution is a particular limiting form of Binomial distribution when p or q is very small and 'n' is large enough.

$$P(r) = \frac{m^r e^{-m}}{r!}, \; r = 0, 1, 2, ...$$

where, $m = np$, n = Number of trials

p = Success case probability

r = Number of the success trial

where, m is mean of distribution

Key Point

The expected value (mean) and variance of a Poisson distributed random variable are approximately equal. i.e.

Mean $\approx$ Variance $= m$

Normal Distribution

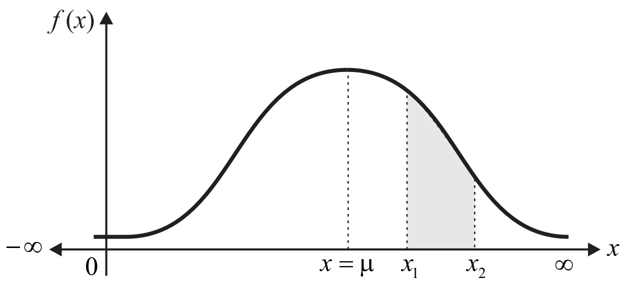
Normal Distribution is a continuous distribution and it is derived as the limiting form of Binomial Distribution for large values of 'n' and when neither 'p' nor 'q' is very small.

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$$

where, $\mu =$ mean, $\sigma =$ Standard deviation

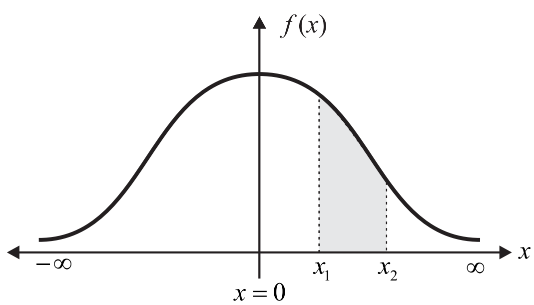
Probability of x lying between x_1 and x_2 is given by the area under normal curve from x_1 and x_2 i.e.

$$P(x_1 \le x \le x_2) = \frac{1}{\sigma\sqrt{2\pi}} \int_{x_1}^{x_2} e^{\frac{-(x-\mu)^2}{2\sigma^2}} dx$$



Probability when mean = 0,

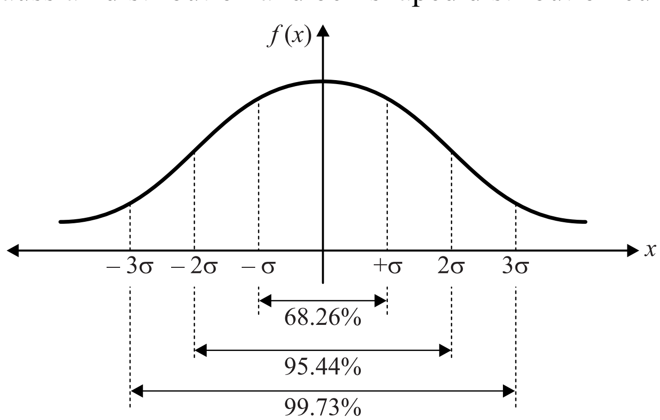
$$P(x_1 \le x \le x_2) = \frac{1}{\sigma\sqrt{2\pi}} \int_{x_1}^{x_2} e^{\frac{-x^2}{2\sigma^2}} dx$$



🕮 Key Point

(i) Normal distribution is symmetric about its mean

(ii) It is also referred as Gaussian distribution and bell shaped distribution curve.



Exponential Distribution

A continuous random variable x assuming non negative values is said to have exponential distribution with parameter $\lambda > 0$, if its probability density function is given by,

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{for } x \geq 0 \\ 0 & \text{for } x < 0 \end{cases}$$

where, λ is a parameter.

Key Point

(i) Mean $= \frac{1}{\lambda}$ (ii) Variance $= \frac{1}{\lambda^2}$ (iii) Standard derivation $= \frac{1}{\lambda}$

Random Variables and Statistics

Random Variable

A random variable is defined as a real number x connected with the outcomes of a random experiment.

1. **Discrete random variable :** A real valued function defined on a discrete sample space is called a discrete random variable.
2. **Continuous random variable :** A random variable is said to be continuous, if it can take all possible values between certain limits.

Values and Parameters	Types of Random Variables	Formula
Expected Value	discrete random variable	$E(X) = \sum_{i=1}^{n} x_i P(x_i)$
	continuous random variable	$E(X) = \int_{-\infty}^{\infty} x \,.\, f(x) dx$
Mean Square Value	discrete random variable	$E(X^2) = \sum_{i=1}^{n} x_i^2 P(x_i)$
	continuous random variable	$E(X^2) = \int_{-\infty}^{\infty} x^2 \,.\, f(x) dx$
Variance		$V(X) = \sigma^2 = E\left[(X-\mu)^2\right] = E(X^2) - [E(X)]^2$

Statistics, Correlation and Regression

1. Mean, median and mode all together referred as central tendency.
2. Mean i.e. average value or expected value.
3. Median is nothing but central item of the data or observation after arrangement.
4. Mode is referred as highest occurred item in the given observation.
5. The relationship among mean, median and mode is given by,

Mode = 3 median – 2 mean.

Standard Deviation		$\sigma_x = \sqrt{\text{Var}(X)} = \sqrt{\dfrac{\sum(x-\bar{x})^2 f_i}{n}}$
Mean	For ungrouped data	$\bar{x} = \dfrac{\Sigma x_i}{n}$
	For grouped data	$\bar{x} = \dfrac{\sum f_i x_i}{\sum f_i}$
Median	For ungrouped data	Median $= \begin{cases} x_{k+1} & \text{When } n \text{ is odd and } n = 2k+1 \\ \dfrac{x_k + x_{k+1}}{2} & \text{When } n \text{ is even and } n = 2k \end{cases}$
	For grouped data	Median $= L + \left(\dfrac{\frac{F}{2} - C}{f}\right)K$ where L = Lower limit of the median class F = Total frequency f = Frequency of median class K = Width of median class C = Cumulative frequency up to the class preceding the median class
Mode	For ungrouped data	The mode is the value of the variables which occurs most often.
	For grouped data	Mode $= l + \left(\dfrac{F - F_{-1}}{2F - F_1 - F_{-1}}\right)K$ where, l = Lower limit of class containing mode K = Size of model class or common width of the class F_1 = Frequency after modal class F = Frequency of modal class F_{-1} = Frequency of before modal class
Karl Pearson's Co-efficient of Correlation		$r = \dfrac{\sum xy}{\sqrt{\left(\sum x^2 \sum y^2\right)}}$ where, $x = X - M_x$ = deviation of variable X measured from its mean M_x. $y = Y - M_y$ = deviation of variable Y measured from its mean M_y.

Spearman's formula for Rank Correlation Coefficient		$r = 1 - \frac{6\sum d^2}{n(n^2-1)}$ where, d_i = difference in rank of i^{th} individual value n = number of individuals.
Regression	The regression line of x on y	$x - \bar{x} = r\frac{\sigma_x}{\sigma_y}(y - \bar{y})$
	The regression line of y on x	$y - \bar{y} = r \cdot \frac{\sigma_y}{\sigma_x}[x - \bar{x}]$
	Angle between two line of regression	$\tan\theta = \left(\frac{1-r^2}{r}\right)\left(\frac{\sigma_x \cdot \sigma_y}{\sigma_x^2 + \sigma_y^2}\right)$

Properties of Expectation

1. $E(c) = c$, c is a constant.
2. $E(cX) = cE(X)$, c is a constant.
3. $E(aX + b) = aE(X) + b$, a and b are constants.
4. $E(X + Y) = E(X) + E(Y)$
5. $E(X - Y) = E(X) - E(Y)$.

Properties of Variance

1. $V(c) = 0$, c is a constant.
2. $V(aX) = a^2V(X)$
3. $V(aX \pm b) = V(aX) + V(b) = a^2V(X)$, a and b are constants.
4. $V(aX + bY) = a^2V(X) + b^2V(Y) + 2ab\,\text{Cov}(X, Y)$

Covariance

The measures of the simultaneous variation between the random variable X and Y is called the covariance and written as $\text{Cov}(X, Y)$.

If X and Y are two random variables with respective expected values $E(X)$ and $E(Y)$, then

$$\text{Cov}(X, Y) = E(XY) - E(X)E(Y)$$

Key Point

(i) Rank of correlation is always less than equal to 1 ($r \le 1$).

(ii) Mode = 3 median – 2 mean.

(iii)The covariance of two independent variable is equal to zero.

Chapter 10 : Numerical Methods

Numerical Integration (Quadrature)

Name of Method	Formula	Key Points
Trapezoidal rule (2 point quadrature)	$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{h}{2}\left[(y_0+y_n)+2(y_1+y_2+......y_{n-1})\right]$	In Trapezoidal rule we used **Straight Lines** to model the curve.
Simpson's $\frac{1}{3}$ rule (3 point quadrature)	$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{h}{3}\left[(y_0+y_n) + 4(y_1+y_3+....+y_{n-1}) + 2(y_2+y_4+...+y_{n-2})\right]$ **Or** $\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{h}{3}\left[(y_0+y_n)+4\times \text{O}+2\times \text{E}\right]$	(i) In Simpson's rule we used **parabolas** to approximate each part of the curve. (ii) In Simpson's $\frac{1}{3}$ rule the given interval must be divided into **even number of equal sub intervals**.
Simpson's $\frac{3}{8}$ rule	$\int_{x_0}^{x_0+nh} f(x)dx = \frac{3h}{8}\left[(y_0+y_n) + 3(y_1+y_2+y_4+y_5+....+y_{n-1}) + 2(y_3+y_6+...y_{n-3})\right]$ **Or** $\int_{x_0}^{x_0+nh} f(x)dx = \frac{3h}{8}\left[(y_0+y_n) + 2\times(\text{Multiple of } 3)+3\times \text{rest}\right]$	In Simpson's $\frac{3}{8}$ rule the number of sub intervals **should be multiple of 3.**
Weddle's rule	$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{3h}{10}\left[y_0+5y_1+y_2+6y_3 + y_4+5y_5+2y_6+5y_7+y_8+....\right]$	(i) In Weddle's rule the number of sub intervals **should be multiple of 6.**

Numerical Solution of Linear and Non-Linear Equations :

Name of Method	Iterative Formula	Key Points
Bisection Method	$\text{I.F.} = \frac{a+b}{2}$ where, $f(a)$ and $f(b)$ are of opposite sign.	(i) The order of convergence is linear (1st order) (ii) $\frac{b-a}{2^n} \le \in$ (for error analysis)
Regula Falsi Method or False Position Method	$x_2 = x_0 - \left(\frac{x_1-x_0}{f(x_1)-f(x_0)}\right) f(x_0)$ $x_2 \in (x_0, x_1)$ Here, $f(x_0) = -\text{ve}$, $f(x_1) = +\text{ve}$ i.e., opposite sign.	(i) This method is also slower but faster than bisection method because in this method we reach the final value at only one side of polynomial. (ii) The order of convergence is linear (1st order)

Secant Method	$x_{n+1} = x_n - \left(\dfrac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}\right) f(x_n)$	The order of convergence is 1.62.
Newton-Raphson's Method (Tangent Method)	$x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$ For $n = 0$; First iteration, $x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)}$ **Note :** Iterative formula to find 1. $f(x) = \sqrt{N}$ then, $x_{n+1} = \dfrac{1}{2}\left(x_n + \dfrac{N}{x_n}\right)$ 2. $f(x) = \dfrac{1}{\sqrt{N}}$ then, $x_{n+1} = \dfrac{1}{2}\left(x_n + \dfrac{1}{Nx_n}\right)$	(i) This method has a quadrature convergence i.e. order of convergence is two. (ii) Number of function to be evaluated per iteration is 2. (iii) This method is more sensitive at starting point or initial value. (iv) Geometrically, this method is also known as **tangent method.** **Drawback :** This method is not applicable when $f'(x) = 0$, in this case we apply False position method.

Numerical Solution of Ordinary Differential Equation

Consider the first order differential equation. $\dfrac{dy}{dx} = f(x, y)$ with initial value (x_0, y_0) and step size h

Name of Method	Iterative Formula
Picard's Methods	$y_{n+1} = y_n + \int_{x_0}^{x} f(x, y_n)\, dx$
Euler's Methods (Runge-Kutta first order)	(i) Euler's Forward Method : $y_{n+1} = y_n + hf(x_n, y_n)$ (ii) Euler's Backwards Method : $y_{n+1} = y_n + hf(x_{n+1}, y_{n+1})$
Modified Euler's Methods	$y_{n+1} = y_n + \dfrac{h}{2}\left[f(x_n, y_n) + f(x_{n+1}, y_{n+1})\right]$
Runge - Kutta Methods (or Runge- Kutta 4th order method)	$y_1 = y_0 + k$, where, $k = \dfrac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$ to calculate the value of k_1, k_2, k_3 and k_4 use $k_1 = hf(x_0, y_0)$, $k_2 = hf\left(x_0 + \dfrac{1}{2}h, y_0 + \dfrac{1}{2}k_1\right)$ $k_3 = hf\left(x_0 + \dfrac{1}{2}h, y + \dfrac{1}{2}k_2\right)$, $k_4 = hf(x_0 + h, y_0 + k_3)$

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