!! SHREE !!

JAIN'S

SHOP NO-103-105, 1ST FLOOR, A BLOCK, SURYALOK COMPLEX,
ABIDS, HYDERABAD-500001
M.NO:- (+91) 97 00 29 11 47

AVAILABLE

ALL BRANCHES CLASS NOTES FOR GATE, IES, GENCO, TRANSCO, AEE & ALL PSU'S

ALL STATIONARY MATERIALS AVAILABLE HERE

INTERNET CAFÉ, SCANNING, COLOUR PRINT OUT, XEROX & LOT OF MORE THAT U NEED

NAME:-_________________________________

SUBJECT:_MATHS

BRANCH:-_________________________________

NO RETURN NO EXCANGE

THANK YOU VISIT AGAIN

SHREE SHANTI ENTERPRISES

SHOP NO.37-38,GROUND FLOOR, SURYALOK COMPLEX, GUNFOUNDRY, ABIDS, HYDERABAD-500 001

JAIN'S

ENGINEERING

MATHEMATICS

1. CALCULAS
2. PROBABILITY & STATISTICS
3. LINER ALGEBRA
4. THANSFORM THEORY
5. DIFFERENTIAL EQUATIONS
6. NUMERICAL METHODS
7. COMPLEX VARIABLES

CALCULUS

- Limits and Continuity
- Differentiation.
- Definite Integrals.
  Improper Integrals.
- Partial differentiation
- Multiple Integrals.
- Vector differentiation
- Fourier series.

Calculus :- is the science of acceleration, retardation and variation.

Function :- A function exists b/w $A \to B$ if $\forall x \in A$ there exists a unique $y \in B$ such that $f(x) = y$

① Explicit function :-

Eg: $y = x(x-2)$

$$\implies y = f(x)$$

General form, $z = f(x_1, x_2 \dots x_n)$

② Implicit function :-

Eg : $x^2 + xy + y^2 = 1$

$$\implies \phi(x, y) = c$$

General form $\phi(z, x_1, x_2 \dots x_n) = c$

The variables cannot be seperated and expressed in form $y = f(x)$.

③ Composite function

If $z = f(x, y)$ where $x = \phi(t)$ and $y = \varphi(t)$

Some Special Functions

① Even function :- A fn $f(x)$ is said to be even function of x if $f(x) = f(-x)$

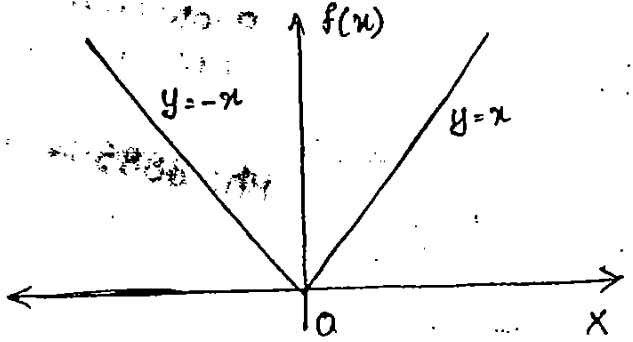
Eg: Cos $|x|$

② Odd function :- $f(-x) = -f(x)$

Eg: $\sin x$, x, . . .

③ Modulus function :-

It is defined as $f(x) = |x| = \begin{cases} x & \text{for } x > 0 \\ 0 & \text{for } x = 0 \\ -x & \text{for } x < 0 \end{cases}$



Since the curve is continuous from $-\infty$ to ∞ without any break it is continuous and differentiable.

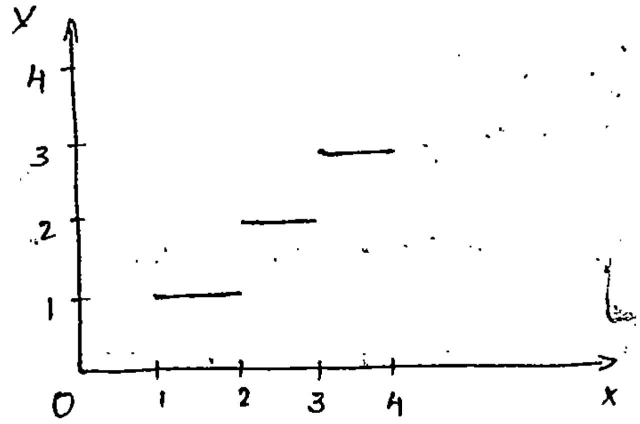
$$\frac{d}{dx}|x| = \frac{|x|}{x} \quad \text{for } x \neq 0$$

The curve is not differentiable at $x = 0$

④ Greatest Integer Function or Step or Bracket Function.

$f(x) = [x] = n \in Z$ where $n \leq x \leq n+1$

Eg: $[7.2] = 7$, $[7.999] = 7$, $[7] = 7$, $[-1.2] = -2$

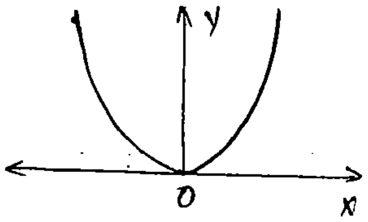


The curve is a discontinuous function at every integer points (1, 2, . . . n). So it is not differentiable

Symmetric Properties Of Curves

Let $f(x,y) = c$ be the equation of the curve.

① If $f(-x,y) = f(x,y)$, then it is symmetric about Y axis.



Eg : $x^2 - 4ay = 0$

$f(-x,y) = x^2 - 4ay$

② If $f(x,-y) = f(x,y)$, then the curve is symmetric about x axis.

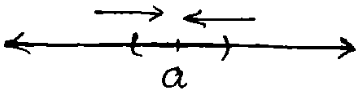
③ If $f(x,y) = f(y,x)$ then the curve is symmetric about the line $y=x$.

Eg : $x^3 + y^3 - 3axy = 0$.

Limit Of a Function

Let $f(x)$ be defined in deleted neighbourhood of $a \in R$, then $l \in R$ is said to be limit of $f(x)$ as 'x' approaches 'a' for given $\epsilon > 0$ there exists $\delta > 0$ such that

$$|f(x) - l| < \epsilon \text{ whenever } |x-a| < \delta$$



$a \in R$, $\delta > 0$

$(a-\delta, a+\delta) - \{a\}$

$$\therefore \lim_{x \to a} f(x) = l$$

Left Limit : when $x < a$ and $x \to a$ and $\boxed{\lim_{x \to a^-} f(x) = \lim_{h \to 0} f(a-h)}$

Right Limit : when $x > a$ and $x \to a$ and $\boxed{\lim_{x \to a^+} f(x) = \lim_{h \to 0} f(a+h)}$

Indeterminate Forms

$$\underbrace{\frac{0}{0}, \frac{\infty}{\infty}}, \quad 0 \times \infty, \quad \infty - \infty, \quad 0^0, \quad 1^\infty \text{ and } \infty^0.$$

$$\lim_{x \to a} \frac{f(x)}{g(x)} \left[\frac{0}{0.} \text{ or } \frac{\infty}{\infty}\right] = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

Standard Limits :-

① $\displaystyle \lim_{x \to a} \frac{x^n - a^n}{x - a} = n a^{n-1}$

② $\displaystyle \lim_{x \to 0} \frac{e^{mx} - 1}{x} = m$

③ $\displaystyle \lim_{x \to 0} \frac{a^x - 1}{x} = \log_e a$

④ $\displaystyle \lim_{x \to 0} [1 + ax]^x = e^a$

⑤ $\displaystyle \lim_{x \to \infty} \left[1 + \frac{a}{x}\right]^x = e^a$

⑥ $\displaystyle \lim_{x \to 0} \left[\frac{\sin x}{x} \text{ or } \frac{\tan x}{x}\right] = 1$

⑦ $\displaystyle \lim_{x \to 0} \left[\frac{\sin mx}{x}\right] = m$

⑧ $\displaystyle \lim_{x \to \infty} \frac{\sin x}{x} = 0$

⑨ $\displaystyle \lim_{x \to 0} \left[\frac{a^x + b^x}{2}\right] = \sqrt{ab}$

⑩ $\displaystyle \lim_{x \to 0} [\cos x + a \sin bx]^{1/x} = e^{ab}$

⑪ $\displaystyle \lim_{x \to 0} \left[\frac{1 - \cos ax}{x^2}\right] = \frac{a^2}{2}$

Problems

Q1. $\lim_{x\to 0}\left[\frac{1-\cos 3x}{x\sin 2x}\right] = \left(\frac{0}{0}\right)$ model

So L Hospital's Rule.

$$\lim_{x\to 0}\left[\frac{3\sin 3x}{\sin 2x+2x\cos 2x}\right] = \left(\frac{0}{0}\right) \text{ again}$$

$$\lim_{x\to 0}\left[\frac{+9\cos 3x}{2\cos 2x+2\cos 2x-4x\sin 2x}\right] = \frac{9(1)}{2(1)+2(1)-4(0)} = \underline{\underline{9/4}}$$

OR

Using ⑪ $\lim_{x\to 0}\left[\frac{1-\cos 3x}{x^2\left(\frac{\sin 2x}{x}\right)}\right] = \frac{9/2}{2} = \underline{\underline{9/4}}$

Q2. $\lim_{x\to 1}\frac{\cos\frac{\pi x}{2}}{1-\sqrt{x}} = \left(\frac{0}{0}\right)$

$$= \lim_{x\to 1}\frac{+\frac{\pi}{2}\sin\frac{\pi}{2}x}{0+\frac{1}{2\sqrt{x}}} = \frac{\pi/2\,\sin\pi/2}{1/2} = \underline{\underline{\pi}}$$

Q3. If $\lim_{x\to 0}\frac{\sin 2x+a\sin x}{x^3}$ = finite then a = ____ ?

ⓐ 2 b) -2 c) α d) Does not exist.

Ans $\lim_{x\to 0}\frac{\sin 2x+a\sin x}{x^3} = \left(\frac{0}{0}\right)$

$$= \lim_{x\to 0}\frac{2\cos 2x+a\cos x}{3x^2} = \frac{2+a}{0} = \text{finite (given)}$$

$\implies 2+a=0 \implies a = \underline{\underline{-2}}$

Q4. $\lim_{x\to 0}\frac{\log\sin 2x}{\log(\sin x)} = $ ____ ?

a) 0 b) 1/2 c) 1 d) 2

$$= \lim_{x\to 0}\frac{\frac{1}{\sin 2x}\times 2\cos 2x}{\frac{1}{\sin x}\times\cos x} = \lim_{x\to 0}\frac{\cancel{2}\cos 2x}{\cancel{2\sin x}\cos x}\times\frac{\cancel{\sin x}}{\cos x}$$

$= \underline{\underline{1}}$

Q5 $\lim_{x \to \pi/2} \dfrac{\tan x}{\log(\cos x)} =$

$$\lim_{x \to \pi/2} \frac{\sec^2 x}{\sec x \times (-\sin x)} = \lim_{x \to \pi/2} \frac{-1}{\sin x \cos x} = \frac{-1}{1 \times 0} = \underline{\underline{-\infty}}$$

Q6. $\lim_{x \to 0} x^2 \log x = \lim_{x \to 0} \left[\dfrac{\log x}{1/x^2}\right] = \left(\dfrac{\infty}{\infty}\right)$

$$= \lim_{x \to 0} \left(\frac{1/x}{-2/x^3}\right) = \lim_{x \to 0} \left[\frac{-x^2}{2}\right] = \underline{\underline{0}}$$

Q7. $\lim_{x \to 1} (x-1) \tan \dfrac{\pi x}{2} = \_\_\_ ?$

$$= \lim_{x \to 1} \frac{x-1}{\cot \frac{\pi x}{2}} \quad \left(\frac{0}{0}\right)$$

$$= \lim_{x \to 1} \frac{1}{-\operatorname{cosec}^2 \pi/2 x \times (\pi/2)} = \underline{\underline{-2/\pi}}$$

Q8. $\lim_{x \to 0} \left[\dfrac{1}{x} - \dfrac{1}{\tan x}\right]$

$$= \lim_{x \to 0} \left[\frac{\tan x - x}{x \tan x}\right] = \lim_{x \to 0} \frac{\tan x - x}{x^2 \left(\frac{\tan x}{x}\right)} = \frac{0}{0}$$

$$= \lim_{x \to 0} \left[\frac{\sec^2 x - 1}{2x}\right] = \lim_{x \to 0} \frac{\tan x \cdot \tan x}{2x}$$

$$= \frac{1}{2} \times 1 \times 0 = \underline{\underline{0}}$$

Q9. $\lim_{x \to 0} \left[\dfrac{1}{x} - \dfrac{1}{e^x - 1}\right]$

$$= \lim_{x \to 0} \frac{e^x - 1 - x}{x^2 \frac{(e^x - 1)}{x}} = \left(\frac{0}{0}\right)$$

$$= \lim_{x \to 0} \frac{e^x - 1}{2x} = 1/2 \times 1 = \underline{\underline{1/2}}$$

$\frac{\sin 1/x}{1/x}$ $t = 1/x$, $x \to \alpha$, $t \to 0$

$\lim_{t\to 0} \frac{\sin t}{t} = 1$

Q21. $\lim_{x\to \infty} x \sin 1/x = 1$

Q22 $\lim_{x\to a} [x]$ doesnot exist. when a is ____?

(a) Integer b) Real no. c) Rational no d) All of them.

Let $a = 2$

$L.L = \lim_{x \to 2^-} [x] = 1$

$R.L = \lim_{x\to 2^+} [x] = 2$

$LL \neq RL$

Continuity Of a Function

i) At a point

A function is said to be continuous at a point $x = a$ if $\lim_{x\to a} f(x) = f(a)$.

ii) In an interval (a, b).

A function is said to be continuous in an interval (a,b) if it satisfies the condition.

a) $f(x)$ is continuous at $\forall\, x \in (a, b)$

b) $\lim_{x\to a^+} f(x) = f(a)$

c) $\lim_{x\to b^-} f(x) = f(b)$

Eg ①: Let $f(x) = \begin{cases} \frac{x^2-4}{x-2} & \text{for } x \neq 2 \\ 0 & \text{for } x = 2 \end{cases}$

$f(2) = 0$, $\lim_{x\to 2} f(x)$

$= \lim_{x\to 2} \left[\frac{x^2-4}{x-2}\right] = 2 \times 2^{2-1} = 4 \neq f(2)$

$\therefore$ $f(x)$ is discontinuous at $x = 2$

② Let $f(x) = \begin{cases} (1+3x)^{1/x} & \text{for } x \neq 0 \\ e^3 & \text{for } x = 0 \end{cases}$

$f(0) = e^3$. $\lim_{x \to 0} (1+3x)^{1/x} = e^3 = f(0)$

∴ The function is <u>continuous</u> at $x = 0$.

③ If $f(x) = \begin{cases} (x+1)^{\cot x} & \text{for } x \neq 0 \\ K & \text{for } x = 0 \end{cases}$

is continuous at $x = 0$, then $K = ?$

<u>Ans</u> $f(0) = \lim_{x \to 0} f(x)$

$$\Longrightarrow K = \lim_{x \to 0} (x+1)^{\cot x}$$

$$= e^{\lim_{x \to 0} \cot x \, [x+1-1]}$$

$$= e^{\lim_{x \to 0} \frac{x}{\tan x}} = \underline{e^1}$$

Q4 Let $f(x) = \begin{cases} \frac{\sin [x]}{[x]} & \text{for } [x] \neq 0 \\ 0 & \text{for } [x] = 0 \end{cases}$

The reason for $f(x)$ to be discontinuous at $x = 0$ is

ⓐ $f(0)$ is not defined ⓑ $f(0)$ is defined but $\lim_{x \to 0} f(x)$ doesnot exist.

ⓒ $\lim_{x \to 0} f(x)$ exists, $f(0)$ is defined but $\lim_{x \to 0} f(0) \neq f(0)$ ⓓ $f(x)$ is continuous at $x = 0$

<u>Ans</u> $f(x) = \begin{cases} \frac{\sin(-1)}{-1} & \text{for } -1 \leq x \leq 0 \quad [x] = 1 \\ 0 & \text{for } 0 \leq x < 1 \quad [x] = 0 \end{cases}$

$f(0) = 0$

$\left.\begin{array}{l} LL = \sin 1 \\ RL = 0 \end{array}\right\}$ Limit doesnot exist.

Q5. $f(x) = \begin{cases} 0 & \text{for } x \leq 0 \\ 5x-4 & \text{for } 0<x<1 \\ 4x^2-3x & \text{for } 1<x<2 \\ 3x+4 & \text{for } x \geq 2 \end{cases}$

Then which of the following is true.

a) $f(x)$ is discontinuous at $x=0$

b) $f(x)$ is continuous at $x=1$

c) $f(x)$ is left continuous at $x=2$

d) all of the above.

Ans ⓐ $f(0)=0$, $LL = 0$

$RL = 5\times 0 - 4 = -4$

$f(x)$ is discontinuous at $x=0$

ⓑ $f(1) = 5\times 1 - 4 = 1$

$L.L = 5\times 1 - 4 = 1$

$RL = 4\times 1^2 - 3\times 1 = 1$

Since $f(1) = LL = RL$ it is continuous.

$\therefore$ Ans = ⓓ

DIFFERENTIATION

A function $f(x)$ is said to be differentiable at $x=c$ if

$f'(c) = \lim_{x\to c}\left[\frac{f(x)-f(c)}{x-c}\right]$ exists and is finite.

$$LHD = \lim_{h\to 0}\left[\frac{f(c-h)-f(c)}{-h}\right]$$

$$RHD = \lim_{h\to 0}\left[\frac{f(c+h)-f(c)}{h}\right]$$

A function $f(x)$ is said to be differentiable if the LHD, RHD exists and are finite and equal.

- $f(x) = |x|$ is not differentiable at $x=0$.

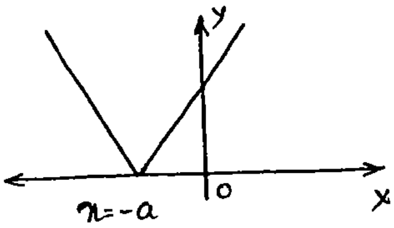
$$LHD = \lim_{h\to 0} \frac{f(0-h) - f(0)}{-h}$$

$$= \lim_{h\to 0} \frac{|-h| - |0|}{-h} = \frac{h}{-h} = \underline{\underline{-1}}$$

$$RHD = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0} \frac{|h| - |0|}{h}$$

$$= \frac{h}{h} = \underline{\underline{1}}$$

NOTE: $|x+a|$ is not differentiable at $x = -a$



Q1. Let $f(x) = x|x|$ where $x \in R$, then $f(x)$ at $x=0$ is

[✓ ⓐ] Continuous and differentiable ⓑ Continuous but not diff.

ⓒ Diff. but not continuous ⓓ Neither diff. nor continuous.

Ans

$$f(x) = \begin{cases} x^2 & \text{for } x > 0 \\ -x^2 & \text{for } x < 0 \\ 0 & \text{for } x = 0 \end{cases}$$

$f(0) = 0$, L.L $= 0$, R.L $= 0$

$$LHD = \lim_{h\to 0} \left[\frac{-(-h)^2 - 0}{-h}\right] = \underline{\underline{0}}$$

$$RHD = \lim_{h\to 0} \left[\frac{(+h)^2 - 0}{h}\right] = \lim_{h\to 0} h = \underline{\underline{0}}$$

∴ Ans = ⓐ

Q2. $f(x) = \dfrac{1}{1+|x|}$, $x = 0$

Ans $f(x) = \begin{cases} \frac{1}{1+x} & \text{for } x > 0 \\ \frac{1}{1-x} & \text{for } x < 0 \\ 1 & \text{for } x = 0 \end{cases}$

$$f(0) = 1 \, , \quad \left.\begin{array}{l} LL = \frac{1}{1+0} = \underline{\underline{1}} \\ RL = \underline{\underline{1}} \end{array}\right\} \text{ Continuous at } x = 0$$

$$LHD = \lim_{h\to 0} \left[\frac{\frac{1}{1-(-h)} - 1}{-h} \right] = \lim_{h\to 0} \frac{-h}{-h(1+h)} = 1$$

$$RHD = \lim_{h\to 0} \left[\frac{\frac{1}{(1+h)} - 1}{h} \right]$$

Continuous but $\therefore$ not diff at $x = 0$

$$= \lim_{h\to 0} \frac{-\cancel{h}1}{\cancel{h}(1+h)} = \underline{\underline{-1}}$$

Q3. Let $f(x) = \begin{cases} \frac{x^2}{|x|} & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$ then

$f(x)$ is

(a) discontinuous everywhere.

[✓ (b)] continuous everywhere.

(c) $f'(x)$ exists in $(-1, 1)$

(d) $f'(x)$ exists in $(-2, 2)$

Ans $f(x) = \begin{cases} x^2 & \text{for } x > 0 \\ -x^2 & \text{for } x < 0 \\ 0 & \text{for } x = 0 \end{cases}$

$$= |x|$$

Q4. $f(x) = |x|^3 \quad x \in R$

(a) Continuous but not differentiable

(b) Once differentiable but not twice.

(c) Twice " " " thrice.

(d) Thrice differentiable.

$$f(x) = \begin{cases} x^3 & \text{for } x > 0 \\ -x^3 & \text{for } x < 0 \\ 0 & \text{for } x = 0 \end{cases}$$

$f(0) = 0$, $LL = 0$

$RL = 0$

$$LHD = \lim_{h \to 0} \left[\frac{-(-h)^3 - 0}{-h} \right] = \lim_{h \to 0} -h^2 = 0$$

$$RHD = \lim_{h \to 0} \left[\frac{h^3 - 0}{h} \right] = \lim_{h \to 0} h^2 = 0$$

$$f'(x) = \begin{cases} 3x^2 & , x > 0 \\ -3x^2 & , x < 0 \\ 0 & , x = 0 \end{cases}$$

$$f''(x) = \begin{cases} 6x & , x > 0 \\ -6x & , x < 0 \\ 0 & , x = 0 \end{cases}$$

$$f'''(x) = \begin{cases} 6 & , x > 0 \\ -6 & , x < 0 \\ 0 & , x = 0 \end{cases}$$

Here $LHD \neq RHD$

Therefore the fn is <u>twice</u> differentiable but not thrice

Q5. $f(x) = \begin{cases} 2+x & , x \geq 0 \\ 2-x & , x < 0 \end{cases}$

ⓐ Cont. & diff. [✓ ⓑ Cont. but not diff.]

ⓒ Diff. but not cont ⓐ Neither diff not cont.

Ans $f(x) = \begin{cases} 2+x & , x > 0 \\ 2 & , x = 0 \\ 2-x & , x < 0 \end{cases}$

$f(0) = 2$

$LL = 2 - 0 = 2$

$RL = 2 + 0 = 2$

$$LHD = \lim_{h \to 0} \frac{2 - (-h) - 2}{-h} = -1$$

$$RHD = \lim_{h \to 0} \frac{2 + h - 2}{h} = 1$$

Note : Every differentiable fn is continuous but a continuous function need not be differentiable.

Mean Value Theorem

Rolle's Theorem

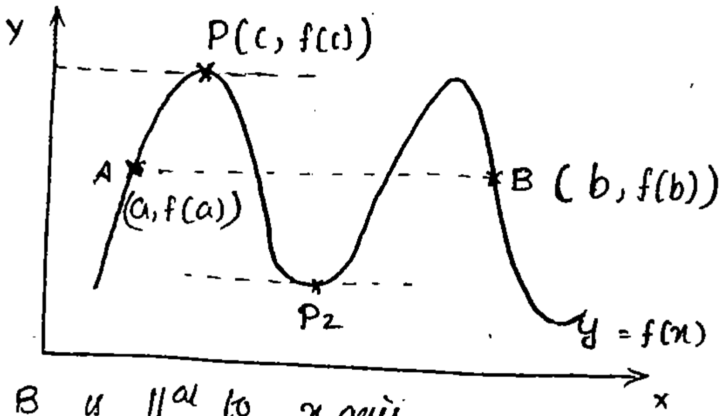
Let $f(x)$ be defined in $[a, b]$ such that.

ⓐ $f(x)$ is continuous in $[a, b]$

ⓑ $f(x)$ is differentiable in (a, b).

ⓒ $f(a) = f(b)$

Then there exists atleast one point $c \in (a, b)$ such that $f'(c) = 0$.



Line joining A & B is $||^{al}$ to x axis.

Lagrange's Mean Value Theorem

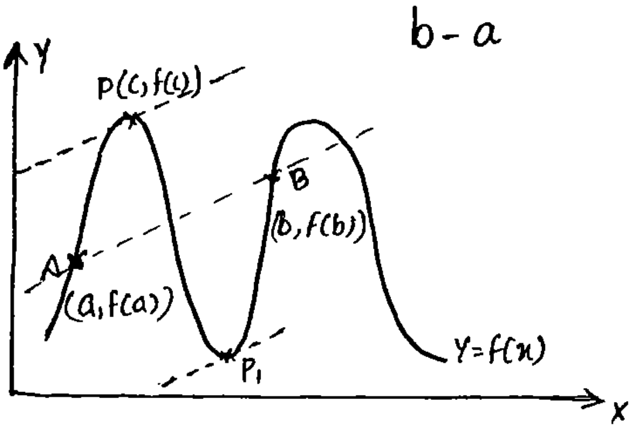
Let $f(x)$ be defined in $[a, b]$ such that:

ⓐ $f(x)$ is continuous in $[a, b]$

ⓑ $f(x)$ is differentiable in (a, b)

Then there exists atleast one point $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$



Another form

$f(x) \longrightarrow [a, a+h]$

a) cont in $[a, a+h]$

b) diff^{n} in $(a, a+h)$

then there exists $\theta \in (0,1)$ such that

$f(a+h) = f(a) + h f'(a+\theta h)$.

$c \in (a, a+h)$

$\theta \in (0,1)$

$\theta = \frac{c-a}{b-a}$

Q1. The mean value c for the function $f(x) = e^x[\sin x - \cos x]$ in $[\pi/4, 5\pi/4]$ is ______.

a) 0 b) $\pi/2$ c) $2\pi/3$ d) π

Ans $f'(x) = e^x[\sin x - \cos x] + e^x[+\cos x + \sin x]$

$= 2e^x \sin x$

$f(\pi/4) = 0$, $f(5\pi/4) = 0$

By Rolles Theorem there exists $c \in (\pi/4, 5\pi/4)$ such that $f'(c) = 0$

$2e^c \sin c = 0$

$\Longrightarrow \sin c = 0$

$\Longrightarrow c = 0, \pm\pi, \pm 2\pi, \ldots$

$c = \pi \in (\pi/4, 5\pi/4)$

② The mean value c for the function $f(x) = \sqrt{x^2-4}$ in the interval $(2,4)$ is

a) $\sqrt{6}$ b) $2\sqrt{2}$ c) $8/3$ d) none of them.

Ans $f(x) = \sqrt{x^2-4}$

$f'(x) = \frac{1}{\cancel{2}\sqrt{x^2-4}} \times \cancel{2}x = \frac{x}{\sqrt{x^2-4}}$ is finite for $\forall (2,4)$

$f(2) = \sqrt{2^2-4} = 0$ $RL = \sqrt{2^2-4} = 0$ $\Big\}$ Continuous in $[2,4]$

$f(4) = \sqrt{12}$

By Legrange's Theorem.

$c \in (2,4)$ such that

$$f'(c) = \frac{f(4)-f(2)}{4-2}$$

$$\frac{c}{\sqrt{c^2-4}} = \frac{\sqrt{12}-0}{2}$$

$$\frac{c}{\sqrt{c^2-4}} = \sqrt{3} \implies c^2 = 3c^2 - 12$$

$$2c^2 = 12$$

$$c^2 = \underline{\underline{\sqrt{6}}} \in (2,4)$$

5/07/12

Q3. The value of ξ of $f(b) - f(a) = (b-a)f'(\xi)$ for $f(x) = Ax^2 + Bx + C$ in $[a, b]$

ⓐ $\frac{b+4}{2}$ ⓑ $\frac{b-a}{2}$ ⓒ $\frac{b+a}{2}$ ⓓ $\frac{b-a}{4}$

Ans $f(x) = Ax^2 + Bx + C$ is continuous and differentiable in $[a, b]$.

$$f'(x) = 2Ax + B$$

$$f'(\xi) = 2A\xi + B$$

$$f'(\xi) = \frac{f(b)-f(a)}{b-a}$$

$$\implies 2A\xi B = \frac{f(b)-f(a)}{b-a} = \frac{(Ab^2+Bb+c)-(Aa^2+Ba+c)}{b-a}$$

$$2A\xi + B = (b+a)A + B$$

$$\implies \xi = \underline{\underline{\frac{b+a}{2}}}$$

Q4. The mean value c for the function $f(x) = \frac{3}{2}x^2 - 5x + \frac{11}{3}$ in the interval $[4/2, 17/2]$ is

ⓐ 6   ⓑ 6.5   ⓒ 7.5   ⓓ none of them.

Ans $f'(x) = \frac{3}{\cancel{2}} \times \cancel{2}x - 5$

$$3x - 5 = \underline{\frac{3}{2}\left(\frac{17}{2}\right)^2 - 5 \times \frac{17}{2}} \quad /$$

From above question $\varepsilon = \frac{a+b}{2}$

$$\therefore c = \frac{11/2 + 17/2}{2}$$
$$= \underline{\underline{7}}$$

Q5. If $f\ [-5, 5] \rightarrow R$ is a differentiable function and $f'(x)$ doesnot vanish anywhere in the open interval $[-5, 5]$ then.

① $f(x)$ is not continuous in $[-5, 5]$

[✓ ②] $f(-5) \neq f(5)$

③ $f(-5) = f(5)$

④ a and b

Ans Given fn is differentiable

$\therefore$ It is continuous. but $f(5)$ may or maynot be $= f(-5)$

But it is not given in question that Rolles condition are met.

Q6. If $f(x) = ax + b$ where $x \in [-1, 1]$ then the point $c \in (-1, 1)$ such that $f'(c) = \frac{f(1) - f(-1)}{2}$ is

ⓐ doesnot exist [✓ ⓑ] can be any $c \in (-1, 1)$

ⓒ $c = 1/2$ ⓓ $c = -1/2$

Ans $a = \frac{a+b+a-b}{2}$

$$\Longrightarrow \underline{\underline{a = a}}$$

Q7. The value of $\theta \in (0, 1)$ for the function $f(x) = \log x$ in the interval $[1, e]$ using a mean value theorem is —?

Ans $f'(x) = 1/x$

$f(1) = 0$, $f(e) = 1$

$\exists\ c \in (1, e)$, $f'(c) = \frac{f(e) - f(1)}{e-1}$

$$\Longrightarrow \frac{1}{c} = \frac{1}{e-1}$$

$$\underline{\underline{c = e-1}}$$

$$\Longrightarrow \theta = \frac{c-a}{b-a} = \frac{e-1-1}{e-1}$$

$$\Longrightarrow \theta = \frac{e-2}{e-1} \in (0,1)$$

Q8 Legranges MVT cannot be applied to $f(x) = 2+(x-1)^{2/3}$ in the interval $[0,2]$ because.

ⓐ $f(x)$ is not continuous in $[0,2]$

[✓ ⓑ] " " " differentiable in $(0,2)$

ⓒ $f(0) \neq f(2)$

ⓓ both a & b.

Ans $f'(x) = 0 + \frac{2}{3}(x-1)^{-1/3} = \frac{2}{3(x-1)^{1/3}}$ rh.

$f'(1) = \infty \Longrightarrow f(x)$ is not differentiable in $(0,2)$.

$f(1) = 2+0 = 2$

$$\lim_{x\to 1} f(x) = 2+(1-1)^{2/3} = \underline{\underline{2}}$$

Q9. If $f'(x) = \frac{1}{5-x^2}$ & $f(0) = 1$ then the lower and upper bounds of $f(1)$ are ___ ?

Ans We define in an interval $(0,1)$

By LMVT $\exists\, c \in (0,1)$ so that it is differentiable

$$f'(c) = \frac{f(1)-f(0)}{1-0}$$

$$\frac{1}{5-c^2} = \frac{\frac{1}{4}-\frac{1}{5}}{1}$$

$$\frac{1}{5-c^2} = \frac{1}{20}$$

$$5-c^2 = 20$$

$$c \in (0,1) \Longrightarrow 0^2 < c^2 < 1^2$$

$$\Longrightarrow 5 > 5-c^2 > 4$$

$$\frac{1}{5} < \frac{1}{5-c^2} < \frac{1}{4}$$

$$\Longrightarrow \frac{1}{5} < f(1)-1 < \frac{1}{4}$$

$$\Longrightarrow \frac{6}{5} < f(1) < \frac{5}{4}$$

Cauchy's Mean Value Theorem

Let $f(x)$ & $g(x)$ be defined in $[a,b]$ such that

ⓐ $f(x)$ and $g(x)$ are continuous in $[a,b]$.

ⓑ $f(x)$ and $g(x)$ are differentiable in (a,b)

ⓒ $g'(x) \neq 0 \ \forall \ x \in (a,b)$

Then $\exists$ atleast one point $c \in (a,b)$ such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)} .$$

Q1. The mean value 'c' for the function $f(x) = 1/x$ and $g(x) = 1/x^2$ in $[1,2]$ is

ⓐ 4/3 ⓑ 8/3 ⓒ 3/4 ⓓ none.

Ans

$$f'(x) = \frac{-1}{x^2}$$

$$g'(x) = \frac{-2}{x^3} \neq 0 \ \forall x \in (1,2).$$

$$\frac{f'(c)}{g'(c)} = \frac{f(2)-f(1)}{g(2)-g(1)}$$

$$\frac{-1/c^2}{-2/c^3} = \frac{1/2 - 1}{1/4 - 1}$$

$$c/2 = \frac{+1/2}{+3/4}$$

$$c = 4/3$$

Q2. The mean value 'c' for the functions $f(x) = \sin x, g(x) = \cos x$ in $[-\pi/2, 0]$ is

Ans $f'(x) = \cos x$ $g'(x) = -\sin x \neq 0 \ \forall x \in (-\pi/2, 0)$

$$\frac{\cos c}{-\sin c} = \frac{0+1}{1-0}$$

$$-\frac{\cos c}{\sin c} = 1$$

$$\cot c = -1$$

$$c = -\pi/4 \in (-\pi/2, 0)$$

Q3. If f and g are differentiable functions in $[0,1]$ such that $f(0) = 2$, $g(0) = 0$, $f(1) = 6$, $g(1) = 2$ and $f'(x) \neq 0 \; \forall x \in (0,1)$ then

a) $f'(c) = g'(c)$ b) $f'(c) = 2g'(c)$

c) $g'(c) = 2f'(c)$ d) none of them.

Ans
$$\frac{f'(c)}{g'(c)} = \frac{6-2}{2-0} = 2$$

$$f'(c) = 2g'(c)$$

TAYLOR'S THEOREM (Generalized MVT)

Let $f(x)$ be defined in $[a, a+h]$ such that it satisfies

i) $f(x), f'(x), f''(x), \ldots f^{n-1}(x)$ are continuous in $[a, a+h]$

ii) $f(x), f'(x), f''(x) \ldots f^{n-1}(x)$ are differentiable in $(a, a+h)$

then $\exists \; \theta \in (0,1)$ such that

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \ldots + \frac{h^{n-1}}{(n-1)!} f^{n-1}(a) + R_n$$

$$\text{where } R_n = \frac{h^n}{(n-1)!\, p} (1-\theta)^{n-p} f^n(a+\theta h)$$

Case (i): when $p = n$

$$R_n = \frac{h^n}{n!} f^n(a+\theta h)$$ is called Lagrange's Form of remainder.

Case (ii): when $p = 1$

$$R_n = \frac{h^n}{(n-1)!} (1-\theta)^{n-1} f^n(a+\theta h)$$ called Cauchy's Form of remainder.

Taylor's Series :

As $n \to \infty$, $R_n \to 0$ then

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \dots \infty$$

1) $f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!} f''(a) + \dots \infty$

is a Taylor Series expansion of $f(x)$ about $x = a$.

2) $f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \dots \infty$

is a TSE of $f(x)$ about $x = 0$. [Maclaurin's Series]

a) $e^x = 1 + x + \frac{x^2}{2!} + \dots \infty$

b) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots \infty$

c) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots \infty$

d) $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots \infty$

e) $\log(1-x) = -\left[x + \frac{x^2}{2} + \frac{x^3}{3} + \dots \infty\right]$

f) $\tan x = x + \frac{x^3}{3} + \frac{2x^2}{15} + \dots \infty$

g) $(1+x)^{-1} = 1 - x + x^2 - x^3 + \dots \infty$

Q1. The coeff. of $(x-2)^4$ in the TSE of $\log x$ about $x = 2$ is.

Ans Coeff of $(x-2)$

$f'(x) = 1/x$ $\quad f''(x) = -1/x^2$ $\quad f'''(x) = 2/x^3$

$f''''(x) = \frac{-6}{x^4}$

$f''''(2) = \frac{-6}{16} = -3/8$

$$\frac{-3/8}{4!} = \frac{-3}{8 \times 24} = \underline{\underline{\frac{-1}{64}}}$$

Q₂. The coeff. of x^3 in the power series expansion of $e^{\sin x}$ in the ascending powers of x are.

a) 0 b) $+1/3!$ c) $-1/3$ d) none of these.

Ans $f(x) = e^{\sin x}$

$f'(x) = e^{\sin x} \cos x$

$f''(x) = e^{\sin x} \times \cos^2 x + e^{\sin x} \times (-\sin x)$

$f'''(x) = e^{\sin x} \cos x (\cos^2 x - \sin x) + e^{\sin x} [-\sin 2x - \cos x]$

$f'''(0) = 1(1-0) + 1(0-1) = 0$

Q₃. The TSE of $\tan x$ about $x = \pi/4$ is

Ans $f(x) = f(\pi/4) + (x-\pi/4) f'(\pi/4) + \frac{(x-\pi/4)^2}{2!} f''(\pi/4) \ldots \infty$

$f(x) = \tan x$ $f'(x) = \sec^2 x$. $f''(x) = 2\sec^2 x \tan x$.

$f'(\pi/4) = 2$ $f''(\pi/4) = 4$

$= 1 + 2(x-\pi/4) + 2(x-\pi/4)^2 + \ldots \infty$

Q₄. The first 3 non zero term in the expansion of $e^x \tan x$ is

a) $x + x^2 + \frac{x^3}{3}$

b) $x + x^2 + \frac{5x^3}{6}$

c) $x + \frac{x^3}{3} + \frac{5x^5}{6}$

d) $x + \frac{x^2}{2!} + \frac{x^3}{3}$

Ans $f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \ldots \infty$

$f(0) = 0$

$f'(x) = e^x [\tan x + \sec^2 x] \Longrightarrow f'(0) = 1$

$f''(x) = e^x [\sec^2 x + \tan x + \sec^2 x + 2\sec^2 x \tan x]$

$\Longrightarrow f''(0) = 2$.

$f'''(x) = e^x [\tan x + 2\sec^2 x + 2\sec^2 x \tan x + \sec^2 x + 4\sec^2 x \tan x + 2\sec^4 x + 4\sec^2 x \tan^2 x]$

$$f(x) = 0 + x(1) + \frac{x^2}{2!}(2) + \frac{x^3}{3!}(5) + \cdots + \infty$$

$$= x + \frac{\cancel{2}x^2}{\cancel{2}!} + \frac{5x^3}{6} + \cdots \infty$$

(OR)

$$e^x \tan x = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)\left(x + \frac{x^3}{3} + \cdots \infty\right)$$

$$= x + \frac{x^3}{3} + x^2 + \frac{x^3}{2} + \cdots \infty$$

$$= x + x^2 + \frac{5}{6}x^3 + \cdots \infty$$

Q5. The TSE of $\tan^{-1} x$ is ...

Ans $f'(x) = \frac{1}{1+x^2} = (1+x^2)^{-1}$

$$= 1 - x^2 + x^4 - x^6 + \cdots \infty$$

On integration

$$f(x) + C = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \infty$$

Put $x = 0$

$$0 + C = 0 \implies C = 0$$

$$\therefore f(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \infty$$

Q6. The linear approximation to $x \sin x$ near $x = 2\pi$ is ?

$$f(x) = \underbrace{f(a) + (x-a)f'(a)}_{\text{linear approx.}} + \frac{(x-a)^2}{2!}f''(a) + \cdots \infty$$

$f(x) = x\sin x \implies f(2\pi) = 0$

$f'(x) = \sin x + x\cos x \implies f'(2\pi) = 2\pi$

Linear approx. $= f(2\pi) + (x - 2\pi) f'(2\pi)$

$$= (x - 2\pi)2\pi$$

DEFINITE INTEGRALS

Let $f(x)$ be continuous in $[a,b]$ and F be the antiderivative of the given fn $f(x)$

$$\text{then } \int_a^b f(x)\,dx = F(b) - F(a)$$

NOTE :
$$\frac{d}{dx}\left[\int_{u(x)}^{v(x)} f(x)\,dx\right] = f(v)\frac{dv}{dx} - f(u)\frac{du}{dx}$$

Properties

- $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$.
- If $c \in (a,b)$ then $\int_a^b f(x)\,dx = \int_a f(x)\,dx + \int_c f(x)\,dx$.
- $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$
- $\int_a^b \frac{f(x)}{f(x)+f(a+b-x)}\,dx = \frac{b-a}{2}$
- $\int_{-a}^{a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx & \text{if } f(x) \text{ is even} \\ 0 & \text{if } f(x) \text{ is odd} \end{cases}$
- $\int_0^{2a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx & \text{if } f(2a-x) = f(x) \\ 0 & \text{if } f(2a-x) = -f(x) \end{cases}$
- $\int_0^{na} f(x)\,dx = n\int_0^a f(x)\,dx \quad \text{if } f(x+a) = f(x)$
- $\int_0^a x f(x)\,dx = \frac{a}{2}\int_0^a f(x)\,dx \quad \text{if } f(a-x) = f(x)$
- $\int_0^{\pi/2} \sin^n x\,dx = \int_0^{\pi/2} \cos^n x\,dx = \left[\frac{n-1}{n} \times \frac{n-3}{n-2} \times \cdots \times \frac{2}{3} \text{ (or) } \frac{1}{2}\right] K$

$$\text{where } K = \begin{cases} 1 & \text{if } n \text{ is odd} \\ \pi/2 & \text{if } n \text{ is even} \end{cases}$$

- $$\int_0^{\pi/2} \sin^m x \cos^n x\,dx = \frac{[(m-1)(m-3)\dots 2 \text{ (or) } 1][(n-1)(n-3)\dots 2 \text{ (or) } 1] \times K}{[(m+n)(m+n-2)\dots\dots 2 \text{ (or) } 1]}$$

where $K = \begin{cases} \pi/2 & \text{when } m \,\&\, n \text{ are even} \\ 1 & \text{otherwise} \end{cases}$

Q₁ $\displaystyle\int_0^{\pi/2} \frac{\tan x}{\tan x + \cot x}\,dx = \_\_\_\_$

Let $f(x) = \tan x$

$\Longrightarrow f(0+\frac{\pi}{2}-x) = \cot x$

$$= \int_0^{\pi/2} \frac{f(x)}{f(x)+f(0+\pi/2-x)}\,dx = \frac{b-a}{2}$$

$$= \frac{\pi/2-0}{2} = \underline{\underline{\pi/4}}$$

Q₂. $\displaystyle\int_0^{\pi/2} \frac{1}{1+\sqrt{\cot x}}\,dx = \_\_\_\_$

$$\int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx = \frac{\pi/2-0}{2} = \underline{\underline{\pi/4}}$$

Q₃. $\displaystyle\int_2^5 \frac{\sqrt{x}}{\sqrt{x}+\sqrt{7-x}}\,dx = \_\_\_\_$

$$= \frac{b-a}{2} = \frac{5-2}{2} = \underline{\underline{3/2}}$$

Q₄ $\displaystyle\int_0^{\pi} |\cos x|\,dx = \_\_\_\_$

ⓐ 0 ⓑ 1 ⓒ -1 ⓓ 2

Ans
$$\int_0^{\pi/2} \cos x\,dx + \int_{\pi/2}^{\pi} -\cos x\,dx$$

$$= [\sin x]_0^{\pi/2} - [\sin x]_{\pi/2}^{\pi}$$

$$= (1-0) - (0-1)$$

$$= \underline{2}$$

Q5. $\int_0^2 |x^2+2x-3|dx = \_\_\_\_$ a) 0 b) 2 c) 4 d) None

Ans $\int_0^2 |x^2+2x-3|dx = \int_0^2 |\underbrace{(x+3)}_{+}(x-1)|dx$

$$= \int_0^1 (-x^2-2x+3)dx + \int_1^2 (x^2+2x-3)dx$$

$$= \left[-\frac{x^3}{3} - \frac{2x^2}{2} + 3x\right]_0^1 + \left[\frac{x^3}{3} + \frac{2x^2}{2} - 3x\right]_1^2$$

$$= \left(\frac{-1}{3} - 1 + 3\right) + \left(\frac{8}{3} + 4 - 6\right) - \left(\frac{1}{3} + 1 - 3\right) = \underline{\underline{4}}$$

Q6. $\int_0^n [x]dx = \_\_\_\_$. a) $\frac{n(n-1)}{2}$ b) $\frac{n(n+1)}{2}$ c) n d) $n-1$

$[x]$ = step fn.

$$\int_0^1 0dx + \int_1^2 1dx + \int_2^3 2dx + \int_3^4 3dx + \dots + \int_{n-1}^n (n-1)dx$$

$$7\ 0 + 1(2-1) + 2(3-2) \dots + (n-1)[n-(n-1)]$$

$$= 1 + 2 + 3 \dots + (n-1) = \underline{\underline{\frac{n(n-1)}{2}}}$$

Q7. $\int_0^1 x(1-x)^{99}dx = \_\_\_\_$ a) $\frac{11}{10100}$ b) $\frac{99}{10100}$ c) $\frac{1}{10100}$ d) None

Ans $\int_0^1 (1-x)x^{99}dx = \int_0^1 x^{99} - x^{100}\ du$

$$= \left[\frac{x^{100}}{100} - \frac{x^{101}}{101} \cancel{dx}\right]_0^1 = \frac{1}{100} - \frac{1}{101} = \underline{\underline{\frac{1}{10100}}}$$

Q8 $I = \int_0^{\pi/2} \log(\tan x)dx = \_\_\_\_$

$I = \int_0^{\pi/2} \log(\cot x)dx$ $\quad [\because \int_0^{\pi/2} \log(\tan x) = \int_0^{\pi/2} \log(\tan(\pi/2 - x))$

$$I + I = \int_0^{2\pi} [\log(\tan x) + \log(\cot x)]dx$$

$$= \int_0^{\pi/2} \log(1)dx = \underline{\underline{0}}$$

$$\int_0^{2a} f(x)dx = \begin{cases} 2\int_0^a f(x)dx & \text{if } f(2a-x) = f(x) \\ 0 & \text{if } f(2a-x) = -f(x) \end{cases}$$

$\therefore \quad f(\pi - x) = f(x)$

$$= 2\int_0^{\pi/2} \frac{\sec^2 x dx}{a^2+b^2\tan^2 x} dx = 2\int_0^{\infty} \frac{1}{a^2+b^2t^2} dt$$

$$= 2\times\frac{1}{a}\tan^{-1}\left[\frac{bt}{a}\right]\times\frac{1}{b}\Big]_0^{\infty}$$

$$= \frac{2}{ab}\left[\frac{\pi}{2}-0\right] = \underline{\underline{\frac{\pi}{ab}}}$$

Q. $\int_0^{\pi/2} \sin^8 x dx = \frac{7}{8}\times\frac{5}{6}\times\frac{3}{4}\times\frac{1}{2}\times\frac{\pi}{2}$ $\qquad \pi/2 \rightarrow$ even power.

Q. $\int_0^{\pi/2} \cos^7 x dx = \frac{6}{7}\times\frac{4}{5}\times\frac{2}{3}\times 1$ $\qquad 1 \rightarrow$ odd power

Q. $\int_0^{\pi/2} \sin^7 x \cos^9 x dx =$ $\qquad m = n =$ odd.

$$= \underline{\underline{\frac{(6.4.2)(8.6.4.2)}{14.12.10.8.6.4.2}}} \times 1$$

Q. $\int_0^{\pi/2} \sin^6 x \cos^3 x dx = \frac{5.3.1(2)}{7.5.3.1} = \underline{\underline{2/7}}$

Q. $\int_0^{\pi/2} \sin^8 x \cos^4 x dx = \underline{\underline{\frac{(7.5.3.1)(3.1)}{12.10.8.6.4.2}}} \times \frac{\pi}{2}$

Q. $\int_{-\pi}^{\pi} \sin^6 x dx =$ ______ $\qquad \sin x =$ odd

but $\sin^6 x =$ even

$$= 2\int_0^{\pi} \sin^6 x dx$$

$f(\pi - x) = \sin^6 x = f(x)$

$$= 2\times 2\times\int_0^{\pi/2} \sin^6 x dx = \underline{\underline{4\left[\frac{5}{6}\times\frac{3}{4}\times\frac{1}{2}\times\frac{\pi}{2}\right]}}$$

Q. $\int_0^{\pi} \sin^4 x \cos^3 x dx =$ ______

$$f(\pi - x) = \sin^4 x(-\cos x)^3 = -\underline{\underline{f(x)}}$$

Q. $\int_{-2\pi}^{2\pi} \sin^4 x \cos^8 x \, dx = \underline{\quad 30 \quad}$

$= K \int_0^{\pi/2} \sin^4 x . \cos^8 x \, dx$ where $K = \frac{b-a}{\pi/2} = \frac{2\pi-(-2\pi)}{\pi/2}$

$= \underline{8}$

$= 8 \left[\frac{(3.1)(7.5.3.1)}{12.10.8.6.4.2} \right] \frac{\pi}{2}$

Q. $\int_0^{\pi/2} \sqrt{\sin x} \, \cos^3 x \, dx = \underline{\qquad}$

Put $\sin x = t$

$\cos x \, dt = dt$

$= \int_0^1 \sqrt{t} \, dt = \left. \frac{t^{3/2}}{3/2} - \frac{t^{7/2}}{7/2} \right]_0^1 = \frac{2}{3} - \frac{2}{7} = \underline{\frac{8}{21}}$

Improper Integrals

First Kind :

$\int_a^b f(x)dx$ if $a = -\infty$ (or) $b = \infty$ (or) both.

ie $\int_{-\infty}^{b} f(x)dx$ (or) $\int_a^{\infty} f(x)dx$ (or) $\int_{-\infty}^{\infty} f(x)dx$.

Second Kind :

$\int_a^b f(x)dx$

when a and b are finite but $f(x)$ is infinite for some $x \in [a, b]$.

Eg :

① $\int_0^1 \log(1-x)dx$

② $\int_{-1}^{1} \sqrt{\frac{1-x}{1+x}} \, dx$

③ $\int_0^2 \frac{1}{1-x} dx = \int_0^1 \frac{1}{1-x} dx + \int_1^2 \frac{1}{1-x} dx$

Convergence

① If $\int_a^b f(x)dx$ = finite, then it is a convergent improper integral

② If $\int_0^b f(x)dx$ = Infinite, then it is Divergent Improper Integral.

Find the convergence of following integrals.

Q① $\int_1^\infty \frac{1}{x\sqrt{x^2-1}}dx = \sec^{-1}x \Big]_1^\infty = \pi/2 - 0 = \pi/2$

Convergent

Q2. $\int_0^\infty x\sin x\, dx =$ ______

$= \left[x\,[-\cos x] - (1)[\sin x]\right]_0^\infty = \underline{\infty}$

Divergent Improper integral

Q3. $\int_{-\infty}^0 \frac{1}{(1-3x)^2}dx$ converges to ______ a) 1/3 b) −1/3 c) ∞ d) None.

Ans $= \frac{-1}{1-3x} \times \frac{-1}{3}\Big]_{\infty}^{0} = \frac{1}{3} - 0 = \underline{1/3}$

Q4. $\int_1^\infty \log(1/x)\,dx$ = Divergent

$= \int_1^\infty -\log x\,dx$

$= -\left[x\log(1-x)\right]_0^\infty = \underline{\infty}$

Q5. $\int_{-1}^1 \sqrt{\frac{1+x}{1-x}}\,dx =$ ______

$= \int_{-1}^1 \frac{1+x}{\sqrt{1-x^2}}dx = \int_{-1}^1 \frac{1}{\sqrt{1-x^2}}dx + \int_{-1}^1 \frac{x}{\sqrt{1-x^2}}dx$

(even) (odd)

$= 2\int_0^1 \frac{1}{\sqrt{1-x^2}}dx = 2(\sin^{-1}x)_0^1$

$= 2 \times [\pi/2 - 0] = \underline{\pi}$

Q6. $\int_{-1}^{1} \frac{1}{x^2} dx = \_\_\_\_$ 32

$$= \int_{-1}^{0} \frac{1}{x^2} dx + \int_{0}^{1} \frac{1}{x^2} dx = \left[\frac{-1}{x}\right]_{-1}^{0} + \left[\frac{-1}{x}\right]_{0}^{1} = \underline{\underline{\infty}}$$

Q7. $\int_{0}^{3} \frac{1}{x^2-3x+2} dx = \_\_\_\_$

$$= \int_{0}^{3} \frac{1}{(x-1)(x-2)} dx = \int_{0}^{1} \frac{dx}{(x-1)(x-2)} + \int_{1}^{2} \frac{dx}{(x-1)(x-2)} + \int_{2}^{3} \frac{dx}{(x-1)(x-2)}$$

$$\int \frac{1}{(x-1)(x-2)} dx = \int \left[\frac{1}{x-2} - \frac{1}{x-1}\right] dx = \log\left(\frac{x-2}{x-1}\right)$$

$$= \underbrace{\log\left(\frac{x-2}{x-1}\right)\Big]_{0}^{1}}_{\Downarrow \; \infty} + \log\left(\frac{x-2}{x-1}\right)\Big]_{1}^{2} + \log\left(\frac{x-2}{x-1}\right)\Big]_{2}^{3}$$

$$= \underline{\underline{\infty}}$$

COMPARISON TEST

For First Kind of Improper Integrals :-

I Method :- Let $0 \leq f(x) \leq g(x)$ then

i) $\int_{a}^{b} f(x)dx$ converges if $\int_{a}^{b} g(x)dx$ is convergent.

ii) $\int_{a}^{b} g(x) dx$ diverges if $\int_{a}^{b} f(x) dx$ is divergent.

$x <$ finite $\Rightarrow x$ is finite

$x >$ Infinite $\Rightarrow x$ is infinite.

$x >$ finite $\Rightarrow x$ may be finite (or) infinite

$x <$ infinite $\Rightarrow x$ may be finite or Infinite.

II Method [Limit form]

Let $f(x)$ and $g(x)$ be two +ve functions such that

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = l \text{ [Non zero, finite]}$$

then $\int_a^b f(x)dx$ and $\int_a^b g(x)dx$ both diverge (or) converge together.

Q. Find the convergence of following Improper Integrals.

a) $\int_1^\infty e^{-x^2}dx = \_\_\_\_$

$e^{x^2} \geq e^{x} \quad \forall x \geq 1$

$\Longrightarrow e^{-x^2} \leq e^{-x}$

$$\int_1^\infty e^{-x}dx = \left.\frac{e^{-x}}{-1}\right]_1^\infty = \underline{\underline{1/e}}$$

$\downarrow$

Convergent.

b) $\int_2^\infty \frac{1}{\log x}dx = \underline{\text{Divergent}}$

$\log x < x \quad \forall x \geq 2$

$\Longrightarrow \frac{1}{\log x} > \frac{1}{x}$

$$\int_2^\infty \frac{1}{x}dx = \log x\Big]_2^\infty = \infty$$

$\downarrow$

Divergent

c) $\int_1^\infty \frac{1}{x^2(e^{-x}+1)}dx = \_\_\_\_$

$x^2(1+e^{-x}) > (0+1)x^2$

$\Longrightarrow \frac{1}{x^2(1+e^{-x})} \leq \frac{1}{x^2}$

Ist method.

(OR)

IInd method.

Let $g(x) = \frac{1}{x^2}$

$$\frac{f(x)}{g(x)} = \frac{1}{e^{-x}+1} \Longrightarrow \lim_{x\to\infty} \frac{f(x)}{g(x)} = \frac{1}{0+1} = \underline{1}$$

$$\int_1^\infty \frac{1}{x^2}\,dx = \left[-1/x\right]_1^\infty = \underline{\underline{1}}$$

↘ Convergent.

d) $\displaystyle\int_1^\infty \frac{x\,\mathrm{Tan}^{-1}x}{\sqrt{4+x^3}}\,dx = \_\_\_\_\_$

$$f(x) = \frac{\cancel{x}\tan^{-1}x}{\cancel{x}\sqrt{x}\sqrt{\frac{4}{x^3}+1}}$$

Let $g(x) = \frac{1}{\sqrt{x}}$

$$\frac{f(x)}{g(x)} = \frac{\mathrm{Tan}^{-1}x}{\sqrt{\frac{4}{x^3}+1}} \longrightarrow \lim_{x\to\infty} \frac{f(x)}{g(x)} = \frac{\pi/2}{\sqrt{0+1}} = \underline{\underline{\pi/2}}$$

$$\int_1^\infty \frac{1}{\sqrt{x}}\,dx = 2\sqrt{x}\Big]_1^\infty = \underline{\underline{\infty}}$$

↘ Divergent.

Comparison Test For Second Kind Of Improper Integrals.

Ist method is same.

IInd Method : [Limit Form]

Let $f(x)$ and $g(x)$ be two +ve functions such that

(i) 'a' is a point of discontinuity and $\lim_{x\to a+} \frac{f(x)}{g(x)} = l_1$ [Non zero, finite] or

(ii) 'b' is a point of discontinuity and $\lim_{x\to b-} \frac{f(x)}{g(x)} = l_2$ [Non zero, finite]

then $\int_a^b f(x)dx$ and $\int_a^b g(x)dx$ both converge (or) diverge together.

Q$_1$. $\displaystyle\int_0^{\pi/2} \frac{\sin x}{x\sqrt{x}}\,dx = \_\_\_\_\_$

$$\frac{\sin x}{x} \le 1 \Longrightarrow \frac{\sin x}{x}\cdot\frac{1}{\sqrt{x}} \le \frac{1}{\sqrt{x}}$$

$$\int_0^{\pi/2} \frac{1}{\sqrt{x}}\,dx = 2\sqrt{x}\Big]_0^{\pi/2} = 2\sqrt{\pi/2}$$

$\rightarrow$ Convergent.

$$\therefore \int_0^{\pi/2} \frac{\sin x}{x\sqrt{x}}\,dx \text{ is convergent}$$

Q1. $\displaystyle\int_1^2 \frac{\sqrt{x}}{\log x}\,dx = $ ____

$$\frac{1}{\log x} > \frac{1}{x} \implies \frac{\sqrt{x}}{\log x} > \frac{\sqrt{x}}{x} = \frac{1}{\sqrt{x}}$$

$$\int_1^2 \frac{1}{\sqrt{x}}\,dx = 2\sqrt{x}\Big]_1^2 = \underline{2\sqrt{2}-2}$$

$\searrow$ convergent. Method I fail

f(x) may be convergent or divergent.

<u>Method II</u> Let $g(x) = \frac{1}{x\log x} \implies \frac{f(x)}{g(x)} = \sqrt{x}$

$$\lim_{x\to 1^+} \frac{f(x)}{g(x)} = 1$$

$$\int_1^2 \frac{1}{\log x}\,dx \qquad \text{let } \log x = t,\quad \frac{1}{x}\,dx = dt$$

Now let $g(x) = \frac{1}{x\log x} \implies \frac{f(x)}{g(x)} = x\sqrt{x}$

$$\lim_{x\to 1^+} \frac{f(x)}{g(x)} = 1$$

$$\int_0^1 \frac{1}{x\log x}\,dx = \int_0^{\log 2} \frac{1}{t}\,dt$$

$$= \log t\Big]_0^{\log 2} = \underline{\infty}$$

Q3. How many of the following integrals is divergent.

$$I_1 = \int_0^1 \frac{1}{\sqrt{x+4x^3}}\,dx \qquad I_3 = \int_0^{\pi/2} \sec x\,dx$$

$$I_2 = \int_{e^2}^{\infty} \frac{1}{x(\log x)^3}\,dx \qquad I_4 = \int_1^{e} \frac{1}{x(\log x)^{1/3}}\,dx$$

a) One b) Two [✓ c) Three] d) None.

$$I_1 \le \int_0^1 \frac{1}{\sqrt{x}}\,dx = 2\sqrt{x}\Big]_0^1 \doteq 2 \longrightarrow \text{Convergent}$$

$$\sqrt{x+4x^3} \Longrightarrow \sqrt{x}\sqrt{1+4x^2}$$

$$I_2 = \frac{[\log x]^{-3+1}}{-3+1} = \frac{-1}{2(\log x)^2}\Big]_{e^2}^{\infty} = \underline{1/8} \longrightarrow \text{Convergent.}$$

$$I_3 = \log|\sec x + \tan x|\Big]_0^{\pi/2} = \infty$$

$$I_4 = \frac{(\log x)^{-1/3+1}}{-1/3+1} = \frac{(\log x)^{2/3}}{2/3}\Big]_1^{e} = \underline{\underline{3/2}} \longrightarrow \text{Convergent}$$

Q4. Consider the integrals $I_1 = \int_1^{\infty} \frac{x^2}{(1+x^3)^{5/2}}\,dx$ and $I_2 = \int_1^{\infty} \frac{x+1}{x\sqrt{x}}\,dx.$

ⓐ I_1 & I_2 are convergent ⓑ I_1 – conv. I_2 – diverg. ✓

ⓒ I_1 – div I_2 – conv ⓓ I_1 & I_2 div.

Ans

$$\frac{1}{x^p} = \frac{1}{x^{15/2-2}} = \int_1^{\infty} \frac{1}{x^{11/2}}\,dx = \frac{x^{-11/2+1}}{\frac{-11}{2}+1} =$$

$$-\frac{2}{9}\,\frac{1}{x^{9/2}}\Big]_1^{\infty} = \underline{\underline{2/9}} \longrightarrow \text{Convergent}$$

$$\frac{1}{x^p} = \frac{1}{x^{3/2-1}} = \int \frac{dx}{\sqrt{x}} = 2\sqrt{x}\Big]_1^{\infty} = \infty \longrightarrow \text{Divergent.}$$

Q5. $$\int_1^{\infty} \frac{1}{x^{1/3}\,[1+\sqrt{x}]}\,dx$$

$$\frac{1}{x^p} = \int_1^{\infty} \frac{1}{x^{5/6}}\,dx = \frac{x^{1/6}}{1/6}\Big]_1^{\infty} = \longrightarrow \text{Divergent}$$

Q6. $\int_0^1 x\log x\,dx =$ ______

ⓐ 1/4 ⓑ -1/4 ⓒ ∞ ⓓ None

$$= \log x\,\frac{x^2}{2} - \int \frac{1}{x}\,\frac{x^2}{2}\,dx$$

$$= \frac{x^2}{2}\log x - \frac{x^2}{4}\Big]_0^1 = \frac{-1}{4} - \lim_{x\to 0}\frac{x^2}{2}\log x$$

$$= -1/4 - \frac{\log x}{2/x^2}$$

using L.H Rule $= \dfrac{1/x}{-4/x^3}$

$$= -1/4 - \left[\lim_{x\to 0} -\frac{x^2}{4} \right] = -1/4 - 0 = \underline{\underline{-1/4}}$$

GAMMA FUNCTION

$$\Gamma n = \int_0^\infty e^{-x} x^{n-1} dx \quad (n>0)$$

Note : ① $\Gamma 1 = 1$ ② $\Gamma 1/2 = \sqrt{\pi}$

③ $\Gamma n+1 = n \Gamma n \quad \forall\ n > 0$

④ $\Gamma n+1 = n! \quad \forall\ n \in Z^+$

⑤ $\displaystyle\int_0^\infty e^{-ax} x^{n-1} dx = \frac{\Gamma n}{a^n}$

Q1. $\displaystyle\int_0^\infty e^{-x^2} dx = $ ____

Let $x^2 = t \Rightarrow 2x\,dx = dt$

$\Longrightarrow dx = \frac{1}{2} t^{-1/2} dt$

$$= \int_0^\infty e^{-t} . \frac{1}{2} t^{-1/2} dt$$

$$= \frac{1}{2} \int_0^\infty e^{-t} t^{1/2 - 1} dt = \frac{1}{2} \Gamma \frac{1}{2} = \underline{\underline{\frac{\sqrt{\pi}}{2}}}$$

Note : $\displaystyle\int_{-\infty}^{\infty} e^{-x^2} dx = 2\int_0^\infty e^{-x^2} dx = \underline{\underline{\sqrt{\pi}}}$

Q2. $\displaystyle\int_0^\infty e^{-y^3} y^{1/2} dy = $ ____ a) $\sqrt{\pi}/3$ b) $3\sqrt{\pi}/2$ c) $\frac{\sqrt{3\pi}}{2}$ d) None.

Ans : Let $y^3 = t \Rightarrow 3y^2 dy = dt$

$\Longrightarrow dy = \frac{1}{3} t^{-2/3} dt$

$$= \int_0^\infty e^{-t} t^{1/6} \frac{1}{3} t^{-2/3} dt$$

$$= \frac{1}{3} \int_0^\infty e^{-t} t^{-1/2} dt = \frac{1}{3} \Gamma 1/2 = \underline{\underline{\frac{\sqrt{\pi}}{3}}}$$

Q3. $\int_0^1 (x\log x)^4\, dx \overset{38}{=}$ ______

Let $\log x = -t \implies x = e^{-t}$

$\implies dx = e^{-t}dt$

$= \int_\infty^0 (e^{-t}(-t))^4 (-e^{-t})dt$

$= \int_0^\infty e^{-5t}\, t^{5-1}\, dt = \frac{\overline{|5}}{5^5} = \underline{\underline{\frac{4!}{5^5}}}$

Q4. $\int_0^\infty 5^{-4x^2}\, dx =$ ______

Let $5^{-4x^2} = e^{-t}$

$\implies -4x^2\log 5 = -t$

$\implies x = \frac{1}{2\sqrt{\log 5}}\sqrt{t}$

$\implies dx = \frac{1}{2\sqrt{\log 5}} \times \frac{1}{2\sqrt{t}}dt$

$= \int_0^\infty e^{-t}\frac{1}{4\sqrt{\log 5}}\, t^{-1/2}\, dt = \frac{1}{4\sqrt{\log 5}}\sqrt{1/2}$

$= \underline{\underline{\frac{\sqrt{\pi}}{4\sqrt{\log 5}}}}$

BETA FUNCTION

$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx \quad (m>0, n>0)$

NOTE: ① $\beta(m,n) = \beta(n,m)$

② $\beta(m,n) = \frac{\sqrt{m}\sqrt{n}}{\sqrt{m+n}}$

③ $\beta(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}dx = \int_0^x \frac{x^{n-1}}{(1+x)^{m+n}}dx$

④ $\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta\, \cos^{2n-1}\theta\, d\theta$

ie $\int_0^{\pi/2} \sin^p\theta\, \cos^q\theta\, d\theta = \frac{1}{2}\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$

Q1. $\int_0^2 x^7 (16-x^4)^{10}\, dx = \underline{\qquad}$

Let $x^4 = 16t \Longrightarrow 4x^3 dx = 16dt$

$$= \int_0^1 16t\,(16-16t)^{10}\, 4dt$$

$$= 4\times 16^{11} \int_0^1 t^{2-1}(1-t)^{11-1}\, dt$$

$$= 4\times 16^{11} \times \beta(2,11)$$

$$= 4\times 16^{11} \times \frac{\sqrt{2}\times\sqrt{11}}{\sqrt{13}}$$

$$= 4\times 16^{11} \times \frac{1\times 10!}{12!}$$

Q2 $\int_0^\infty \frac{x^3(1+x^5)}{(1+x)^{13}}\, dx = \underline{\qquad}$

$$= \int_0^\infty \frac{x^3}{(1+x)^{13}}\, dx + \int_0^\infty \frac{x^8}{(1+x)^{13}}\, dx$$

$$= \int_0^\infty \frac{x^{4-1}}{(1+x)^{4+9}}\, dx + \int_0^\infty \frac{x^{9-1}}{(1+x)^{9+4}}\, dx$$

$$= \beta(4,9) + \beta(9,4)$$

$$= 2\beta(4,9) = 2\left(\frac{\sqrt{4}\cdot\sqrt{9}}{\sqrt{13}}\right)$$

$$= 2\left[\frac{3!\times 8!}{12!}\right]$$

Q3. $\int_0^\infty \left[\frac{x}{1+x^2}\right]^3 dx = \underline{\qquad}$

Let $x = \tan\theta$

$dx = \sec^2\theta\, d\theta$

$$= \int_0^{\pi/2} \left[\frac{\tan\theta}{\sec^2\theta}\right]^3 \sec^2\theta\, d\theta = \int_0^{\pi/2} \frac{\tan^3\theta}{\sec^4\theta}\, d\theta$$

$$= \int_0^{\pi/2} \sin^3\theta\cos\theta\, d\theta = \frac{1}{2}\beta\left(\frac{3+1}{2}, \frac{1+1}{2}\right) = \frac{1}{2}\beta(2,1)$$

$$= \frac{1}{2}\times\frac{1}{1+1} = 1/4$$

PARTIAL DIFFERENTIATION

If $z = f(x,y)$ then

$$Z_x = \frac{\partial Z}{\partial x} = \lim_{h\to 0} \frac{f(x+h,y)-f(x,y)}{h}$$

$$Z_y = \frac{\partial Z}{\partial y} = \lim_{k\to 0} \frac{f(x,y+k)-f(x,y)}{k}$$

|||ly $\frac{\partial^2 z}{\partial x^2}, \frac{\partial^2 z}{\partial y^2}, \frac{\partial^2 z}{\partial x \partial y}, \ldots$

Homogeneous Function :

Eg : ① $2x+3y$

② $x^3z + 4x^2y^2 - 2xyz^2$

③ $\frac{x^3y + z^2y^2}{4x-3y}$, $n = 4-1 = 3$ n = order of the fn = order of Nr - Order of Dr

④ $u = \cos^{-1}\left(\frac{x^2+y^2}{2x-3y}\right)$ ⟶ Non homogeneous fn.

⑤ $z = \log(x/y)$ ⟶ Homogeneous fn.

⑥ $\sin x$ ⟶ Non homogeneous fn

If the inner function of trigonometric, exponential and logarithmic function are homogeneous with degree 0 then the whole fn will be homogeneous.

Note : ① If $f(kx, ky) = k^n f(x,y)$ then $f(x,y)$ is a homogeneous function with degree 'n'.

② If $f(x,y)$ is a homogeneous fn with degree 'n' then

$$f(x,y) = \begin{cases} x^n \phi(y/x) \\ y^n \psi(x/y) \end{cases}$$

Euler's Theorem :

If $f(x,y)$ is a homogeneous fn with deg 'n', then

ⓐ $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf.$

ⓑ $$x^2\frac{\partial^2 f}{\partial x^2} + 2xy\frac{\partial^2 f}{\partial x \partial y} + y^2\frac{\partial^2 f}{\partial y^2} = n(n-1)f$$

NOTE: If $u(x,y) = f(x,y) + g(x,y)$ where f and g are homogeneous fn with degree m & n respectively, then

ⓐ $$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = mf + ng$$

ⓑ $$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = m(m-1)f + n(n-1)g.$$

NOTE: If $f(u)$ is a homogeneous fn in two variables x&y with deg. 'n' then

ⓐ $$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\frac{f(u)}{f'(u)} = F(u)$$

ⓑ $$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = F(u)[F'(u)-1]$$

Total Differentiation

If $Z = f(x,y)$ where $x = \phi(t)$ and $y = \psi(t)$ then the total derivative of 'Z' w.r.t 't' is

$$\frac{dZ}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

The total differential of $Z = f(x,y)$ is

$$dZ = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

NOTE: If $f(x,y) = c$ is an implicit function then

$$\frac{dy}{dx} = \frac{-f_x}{f_y}$$

If $Z = f(x,y)$ where $x = \phi(u,v)$ and $y = \psi(u,v)$ then

$$\frac{\partial Z}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u}$$

$$\frac{\partial z}{\partial v} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial v}$$

Q. If $w = x^2+y^2$ where $x = \frac{t^2-1}{t}$ and $y = \frac{t}{t^2+1}$ then $\left.\frac{dw}{dt}\right|_{t=1} =$ ____ ?

a) 0 b) 1 c) -1 d) 2.

Ans

$$\frac{dw}{dt} = \frac{\partial w}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial w}{\partial y} \cdot \frac{dy}{dt}$$

$$= 2x\left[1+\frac{1}{t^2}\right] + 2y\left[\frac{(t^2+1)1 - t(2t)}{(t^2+1)^2}\right]$$

At $t = 1$

$$\frac{dw}{dt} = 2\times(0)(1+1) + 2\,(1/2)\left(\frac{2-2}{4}\right) = 0+0 = \underline{\underline{0}}$$

Q₂. If $V = x^2+y^2+z^2$ where $x = e^{2t}$, $y = e^{2t}\sin 3t$ and $z = e^{2t}\cos 3t$, then $\frac{dV}{dt} =$ ____ ?

a) e^{2t} b) $\frac{e^{4t}}{2}$ c) $2e^{4t}$ d) $8e^{4t}$

Ans

$$V = e^{4t} + e^{4t}\sin^2 3t + e^{4t}\cos^2 3t.$$

$$= e^{4t}\,[1+1]$$

$$V = 2e^{4t}$$

$$\Longrightarrow \frac{dV}{dt} = \underline{\underline{8e^{4t}}}$$

Q₃. The total derivative of x^3y^2 w.r.t x where x & y are connected by the relation $x^3+y^3-3xy = 0$ is ____ ?

Ans Let $u = x^3y^2$

$$\frac{du}{dx} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dx} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dx}$$

$$= 3x^2y^2(1) + 2x^3y\,\frac{dy}{dx}$$

$$= 3x^2y^2 + 2x^3y\left[-\frac{(3x^2-3y)}{(3y^2-3x)}\right]$$

Q4. If $u = f(2x-3y,\ 3y-4z,\ 4z-2x)$ then $6u_x + 4u_y = \_\_\_$?

a) $3u_z$ b) $4u_z$ [✓ c) $-3u_z$] d) $-4u_z$.

Ans let $r = 2x-3y$, $s = 3y-4z$, $t = 4z-2x$

$$u = f(r, s, t)$$

$$u_x = \frac{\partial f}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial f}{\partial s}\cdot\frac{\partial s}{\partial x} + \frac{\partial f}{\partial t}\frac{\partial t}{\partial x}$$

$$= f_r\cdot(2) + f_s\cdot 0 + f_t\cdot -2$$

$$\Longrightarrow 6u_x = 12f_r - 12f_t$$

$$u_y = f_r\cdot(-3) + f_s(3) + f_t(0)$$

$$\Longrightarrow 4u_y = -12f_r + 12f_s$$

$$\therefore\ 6u_x + 4u_y = 12f_s - 12f_t$$

$$u_z = f_r(0) + f_s(-4) + f_t(4)$$

$$-3u_z = \underline{\underline{12f_s - 12f_t}}$$

Q5. If $u = \log(x^3+y^3+z^3-3xyz)$ then $\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)u = \_\_\_$

a) $\frac{3}{(x+y+z)^2}$ b) $\frac{-3}{(x+y+z)^2}$ c) $\frac{9}{(x+y+z)^2}$ d) $\frac{-9}{(x+y+z)^2}$

Ans
$$\frac{\partial u}{\partial x} = \frac{1}{x^3+y^3+z^3-3xyz}(3x^2-3yz)$$

$$\frac{\partial u}{\partial y} = \frac{3y^2-3xz}{x^3+y^3+z^3-3xyz}$$

$$\frac{\partial u}{\partial z} = \frac{3z^2-3xy}{x^3+y^3+z^3-3xyz}$$

$$\therefore\ \frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z} = \frac{3(x^2+y^2+z^2-yz-xy-zx)}{x^3+y^3+z^3-3xyz}$$

$$= \frac{3}{x+y+z}$$

$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u = \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)$$

$$= \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{3}{x+y+z}\right)$$

$$= \frac{-3}{(x+y+z)^2}+\frac{-3}{(x+y+z)^2}+\frac{-3}{(x+y+z)^2} = \underline{\underline{\frac{-9}{(x+y+z)^2}}}$$

Q6. If $V = r^n$, $r = \sqrt{x^2+y^2+z^2}$ then $V_{xx}+V_{yy}+V_{zz} =$ ____

ⓐ $n r^{n-1}$ ⓑ $n(n-1)r^{n-2}$ ⓒ $n(n+1)r^{n-2}$ ⓓ None.

Ans $\frac{\partial r}{\partial x} = \frac{1}{2\sqrt{x^2+y^2+z^2}}\cdot 2x = \frac{x}{r}$

$V_x = n r^{n-1}\frac{\partial r}{\partial x} = n r^{n-1}\cdot\frac{x}{r} = n r^{n-2}\cdot x$

$V_{xx} = n\left[r^{n-2}\cdot(1)\right] + n x\,(n-2) r^{n-3}\cdot\frac{x}{r}$

$= n r^{n-2} + n(n-2) r^{n-4} x^2$

$V_{yy} = n r^{n-2} + n(n-2) r^{n-4}\cdot z^2$

∴ $V_{xx}+V_{yy}+V_{zz} = 3n r^{n-2} + n(n-2) r^{n-4}\underset{\substack{\Downarrow \\ r^2}}{(x^2+y^2+z^2)}$

$= r^{n-2}\left[3n+n^2-2n\right]$

$= \underline{\underline{n(n+1)r^{n-2}}}$

Q7. If $u = \frac{y}{z}+\frac{z}{x}$ then $xu_x + yu_y + zu_z =$ ____

a) $\frac{xy}{z^2}$ b) $\frac{yz}{x^2}$ c) $\frac{xz}{y^2}$ d) 0.

Ans By formula $= nu = 0\times u = \underline{\underline{0}}$

Q8. If $u = \frac{x^2y}{x^{5/2}+y^{5/2}}$ then $x^2u_{xx} + 2xyu_{xy} + y^2u_{yy} =$ ____

Ans a) $\frac{1}{4}u$ b) $\frac{3}{4}u$ c) $\frac{-1}{4}u$ d) $-\frac{3}{4}u$

Ans $= n(n-1)u$ $\quad n = 3-5/2 = 1/2$

$= \frac{1}{2}\left[\frac{1}{2}-1\right]u = \underline{\underline{-1/4\;u}}$

Q9. If $u = \tan^{-1}\left[\frac{x^3+3y^3}{2x-3y}\right]$ then $x u_x + y u_y =$ ____ ?

ⓐ $2u$ ⓑ $2\tan u$ ⓒ $\tan 2u$ ⓓ $\sin 2u$.

Ans

$$\tan u = \frac{x^3+3y^3}{2x-3y}$$

$$\downarrow$$

$$f(u) \longrightarrow n = 3-1 = \underline{2}$$

$$x u_x + y u_y = \frac{n f(u)}{f'(u)} = \frac{2\tan u}{\sec^2 u} = 2\sin u \cos u$$

$$= \underline{\underline{\sin 2u}}$$

Q10. If $u = \log\left(\frac{x^2+y^2}{2x+3y}\right)$ then $x^2 u_{xx} + 2xy u_{xy} + y^2 u_{yy} =$ ____

ⓐ 0 ⓑ 1 ⓒ -1 ⓓ u.

Ans

$$e^u = \frac{x^2+y^2}{2x+3y}$$

$$\downarrow$$

$$f(u) \longrightarrow n = 2-1 = 1$$

$$x u_x + y u_y = \frac{n f(u)}{f'(u)}$$

$$= 1\frac{e^u}{e^u} = 1 = \underline{\underline{F(u)}}$$

$$x^2 u_{xx} + 2xy u_{xy} + y^2 V_{yy} = F(u)[F'(u)-1]$$

$$= 1[0-1] = \underline{\underline{-1}}$$

Q11. $Z = \sin^{-1}\left[\frac{x^{1/4}+y^{1/4}}{x^{1/6}+y^{1/6}}\right]$ then $x^2 Z_{xx} + 2xy Z_{xy} + y^2 Z_{yy} =$ ____

Ans

$$\sin z = \frac{x^{1/4}+y^{1/4}}{x^{1/6}+y^{1/6}} = f(z), \quad n = 1/4 - 1/6 = 1/12$$

$$F(z) = \frac{n f(z)}{f'(z)} = \frac{1}{12}\frac{\sin z}{\cos z} = \frac{\tan z}{12}$$

$$F(z)[F'(z)-1] = \frac{\tan z}{12}\left[\frac{\sec^2 z}{12} - 1\right]$$

Q12. If $z = f(y/x) + \sqrt{x^2+y^2}$ then $x^2 Z_{xx} + 2xy Z_{xy} + y^2 Z_{yy} =$ _

a) z b) 0 c) 1 d) -1

$$= m(m-1) f(y/x) + n(n-1)\sqrt{x^2+y^2}$$

$$= 0(0-1) f(y/x) + 1(1-1)\sqrt{x^2+y^2} = 0+0 = \underline{\underline{0}}$$

Maxima & Minima :

For Functions Of Single Variable :-

① $f(x) \longrightarrow$ max $\longrightarrow x = c$ if for all $\delta > 0$ such that

$$|x - c| < \delta \implies f(x) \leq f(c)$$

② $f(x) \longrightarrow$ min $\longrightarrow x = c$ if for all $\delta > 0$ such that

$$|x - c| < \delta \implies f(x) \geq f(c)$$

Method :

① Find $f'(x)$

② Equate $f'(x)$ to zero for obtaining stationary points..

③ At each stationary point find $f''(x)$

ⓐ If $f''(x) > 0 \implies$ minima

ⓑ If $f''(x) < 0 \implies$ max.

Note ; Every ~~extreme~~ stationary point is not an extreme point but every extreme point is a stationary pt.

Q1 The fn $f(x) = 3x^4 - 4x^3 + 10$, has a minimum value at $x =$ ___

a) -1 b) + 1 c) 2 a). None of them.

Ans $f'(x) = 12x^3 - 12x^2 = 0$

$$\implies 12x^2(x - 1) = 0$$

$$x = 0, 1$$

$f''(x) = 36x^2 - 24x$

$f''(0) = 0$, $f''(1) = 12 > 0 \longrightarrow$ min

Q2. $f(x) = x^2 e^{-x}$. max $x \rightarrow$ ____

Ans a) 1 b) 0 c) -1 d) 2

$f'(x) = 2xe^{-x} - e^{-x} \times x^2 = 0$

$$[2x - x^2] e^{-x} = 0$$

$$x(2x + x) = 0$$

$$x = 0, 2$$

$f''(x) = (2-2x)e^{-x} + (2x-x^2)e^{-x}$

$= (x^2 - 4x + 2)e^{-x}$

$f''(0) = 2 > 0 \longrightarrow$ min

$f''(2) = -2e^{-2} < 0 \longrightarrow$ max.

Q3. Max of $f(x) = \frac{e^{\sin x}}{e^{\cos x}}$, $x \in R$ is ______

ⓐ $e^{\sqrt{2}}$ ⓑ $e^{-\sqrt{2}}$ ⓒ $e^{1/\sqrt{2}}$ ⓓ $e^{-1/\sqrt{2}}$

Ans $f(x) = e^{\sin x - \cos x}$

Let $g(x) = \sin x - \cos x$

$g'(x) = \cos x + \sin x = 0 \Longrightarrow$

$\cos x = -\sin x$

$\Longrightarrow x = -\pi/4, 3\pi/4 \cdots$

$g''(x) = -\sin x + \cos x$

$g''(-\pi/4) = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} > 0 \rightarrow$ min

$g''(3\pi/4) = \frac{-1}{\sqrt{2}} - \frac{1}{\sqrt{2}} < 0 \rightarrow$ max

Max value $= f(3\pi/4)$

$= e^{1/\sqrt{2} - (-1/\sqrt{2})} = \underline{\underline{e^{\sqrt{2}}}}$

Q4. The function $f(x) = -x^2 \log x$, at $x = 1/\sqrt{e}$ is.

ⓐ 1 ⓑ e ⓒ max ⓓ min.

Ans $f(1/\sqrt{e}) = \frac{1}{2e}$

$f'(x) = -[x + 2x \log x] = 0$

$\Longrightarrow x = 0, 1/\sqrt{e}$

$f''(x) = -[1 + 2 + 2\log x]$

$f''(1/\sqrt{e}) = -[3 + 2 \times -1/2] = -2 < 0 \longrightarrow$ max

Q5. The max value of the fn $f(x) = x^3 - 9x^2 + 24x + 5$ in $[0, 6]$ is ____

a) 21 b) 25 c) 41 d) 46.

Ans $f'(x) = 3x^2 - 18x + 24 = 0$

$$\Longrightarrow (x-4)(x-2) = 0$$

$$x = 4, 2.$$

$f''(x) = 6x - 18$

$f''(4) = 6 > 0 \longrightarrow$ min

$f''(2) = -6 < 0 \longrightarrow$ max.

$f(2) = 25$

$f(1) = 21$

$f(6) = \underline{41}$

Q6. If $y = a \log |x| + bx^2 + x$ has extreme values at $x = +2$ and $x = -3/4$ then the values of a & b are ____.

Ans $y' = a \times \frac{1}{\cancel{|x|}} \times \frac{\cancel{|x|}}{x} + 2bx + 1 = 0$

$$\frac{a}{x} + 2bx + 1 = 0$$

$$2bx^2 + x + a = 0 \quad \text{———} ①$$

$$(x + 3/4)(x-2) = 0$$

$$(4x + 3)(x-2) = 0$$

$$4x^2 - 5x - 6 = 0$$

$$\Longrightarrow \frac{4}{5}x^2 + x + \frac{6}{5} = 0 \quad \text{———} ②$$

Equating ① & ②

$2b = \frac{4}{5}$ $a = \frac{6}{5}$

$b = \frac{2}{5}$

Maxima & Minima For Functions Of Two Variables.

Let $z = f(x,y)$, $p = \frac{\partial z}{\partial x}$, $q = \frac{\partial z}{\partial y}$, $r = \frac{\partial^2 z}{\partial x^2}$, $s = \frac{\partial^2 z}{\partial x \partial y}$, $t = \frac{\partial^2 z}{\partial y^2}$

Method

① Find p, q, r, s & t

② Equate p & q to zero for obtaining the stationary pts.

③ At each stationary pt find r, s, t

- ⓐ If $rt - s^2 > 0$ and $r > 0$ $\implies$ Min at that stationary pt.
- ⓑ If $rt - s^2 > 0$ and $r < 0$ $\implies$ Max
- ⓒ If $rt - s^2 < 0$ then $f(x,y)$ has no extreme at that st. pt. and such points are SADDLE Points.

Q. The fn $f(x,y) = 1 - x^2 - y^2$ has

ⓐ max at $(0,0)$ ⓑ min at $(0,0)$ ⓒ $(0,0)$ has a saddle pt

ⓓ None of them.

Ans $p = -2x$ $q = -2y$.

$r = -2$ $s = 0$ $t = -2$.

$$\left.\begin{matrix} p = 0 \\ q = 0 \end{matrix}\right\} \implies \begin{matrix} x = 0 \\ y = 0 \end{matrix}$$

At $(0,0)$ $rt - s^2 = 4 > 0$

$r < 0$

$\therefore$ max at $(0,0)$

Q. The min. value of the fn $x^3 - 3x^2 + 4y^2 - 10$ has at

ⓐ $(0,0)$ ⓑ $(2,0)$ ③ $(-2,0)$ ④ $(1,0)$

Ans $p = 3x^2 - 6x$

$q = 8y$

$r = 6x - 6$

$s = 0$, $t = 8$

$p = q = 0 \implies (0,0)$ & $(2,0)$ are the stationary pt.

At (0,0), $r = -6$, $s = 0$, $t = 8$.

$$rt - s^2 = -48 < 0 \longrightarrow \text{Saddle pt.}$$

At (2,0), $r = 6$, $s = 0$, $t = 8$

$$rt - s^2 = 48 > 0$$

$$r = 6 > 0 \quad \text{min at } (2,0)$$

Q. $f(x,y) = x^3 + y^3 - 3axy$ has

ⓐ max at (a,a) if $a > 0$

ⓑ max at (a,a) if $a < 0$

ⓒ min at (a,a) $\forall\, a \in B$.

ⓓ None.

Ans

$$\left.\begin{array}{l} p = 3x^2 - 3ay = 0 \implies x^2 = ay \\ q = 3y^2 - 3ax = 0 \implies y^2 = ax \end{array}\right\} \frac{y^2}{a^2} = ay \text{ ; } ay = y^4 - a^3y$$

$$\implies y = 0, y = a$$
$$x = 0, x = a$$

$r = 6x$ $\qquad (0,0)$ & (a,a)

$s = -3a$

$t = 6y$ $\qquad$ At (a,a), $r = 6a$, $s = -3a$, $t = 6a$

$$rt - s^2 = 27a^2 > 0$$

max when $r < 0$

ie $a < 0$

$\therefore$ Ans : ⓑ

Q. A rectangular box open at the top is to have a volume of 32 cubic feet. Then the dimensions of the box requiring least material for its construction are ___?

a) (16, 1, 2) b) (4, 4, 2) c) (2, 2, 8) d) (8, 8, 1/2)

Ans Let x, y, z be the dimensions

$$S = xy + 2yz + 2xz$$

$$V = xyz = 32$$

$$f = xy + \frac{64}{x} + \frac{64}{y}$$

$$\left.\begin{aligned} p &= y - \frac{64}{x^2} = 0 \implies y = \frac{64}{x^2} \\ q &= x - \frac{64}{y^2} = 0 \implies x = \frac{64}{y^2} \end{aligned}\right\} \implies y = \frac{64 \times y^4}{(64)^2}$$

$$\implies 64y = y^4$$

$$\implies \underset{\substack{\downarrow}}{y=0}, \quad \underset{x=4}{y=4}$$

N.A as dimension can't be $=0$

$\therefore$ Ans : $\underline{\underline{(4,4,2)}}$

Q. The distance b/w origin and a point nearest to it on the surface $z^2 = (1+xy)$ is ____?

A a) 1 b) $\sqrt{3}$ c) $\sqrt{3/2}$ d) None.

Ans Let $P(x,y,z)$ be a pt. on the surface $z^2 = (1+xy)$

$$d = OP = \sqrt{x^2+y^2+z^2} = \sqrt{x^2+y^2+1+xy}$$

A Comp

Let $f = x^2+y^2+1+xy$

$$\left.\begin{aligned} p &= 2x+y = 0 \implies x = -y/2 \\ q &= 2y+x = 0 \implies y = -x/2 \end{aligned}\right\} \implies \begin{aligned} x&=0 \\ y&=0 \end{aligned} \quad (0,0) \text{ is a st pt.}$$

$r = 2$

$s = 1$

$t = 2$

At $(0,0)$ $rt - s^2 = 3 > 0$

$r = 2 > 0 \longrightarrow$ min

Min distance $= \sqrt{0+0+1+0} = \underline{\underline{1}}$

Constrained Maxima & Minima

Lagrange's Method of Undetermined multipliers :

$f(x,y,z)$ where $\phi(x,y,z) = c \longrightarrow$ ①

Let $F(x,y,z) = f(x,y,z) + \underset{\substack{\downarrow \\ \text{Lagrange's Multipliers.}}}{\lambda}\,\phi(x,y,z)$

$F_x = 0, \quad F_y = 0, \quad F_z = 0$

ie

$$\left.\begin{aligned} \frac{\partial F}{\partial x} + \lambda \frac{\partial \phi}{\partial x} &= 0 \quad \text{——} ② \\ \frac{\partial F}{\partial y} + \lambda \frac{\partial \phi}{\partial y} &= 0 \quad \text{——} ③ \\ \frac{\partial F}{\partial z} + \lambda \frac{\partial \phi}{\partial z} &= 0 \quad \text{——} ④ \end{aligned}\right\} \text{Legranges eq.}$$

Solving eqs ① – ④ we obtain x, y, z & λ

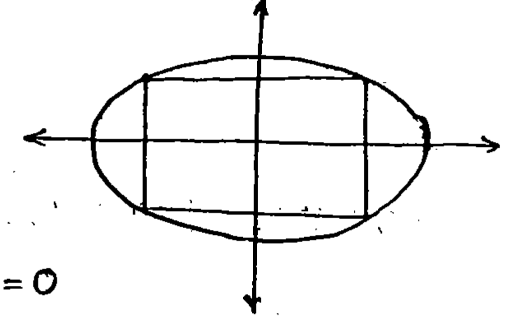
$x, y, z \longrightarrow$ Stationary pt.

$f(x,y,z) \longrightarrow$ extreme value.

Q. The volume of max. value of parallelopiped that can be inscribed in an ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

ⓐ $\frac{8abc}{3\sqrt{3}}$ ⓑ $\frac{9abc}{\sqrt{3}}$ ⓒ $\frac{abc}{\sqrt{3}}$ ⓓ $\frac{abc}{3\sqrt{3}}$

Ans $2x, 2y, 2z$



$V = 8xyz$

$f = 8xyz, \; \phi = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0$

$8yz + \lambda\left(\frac{2x}{a^2}\right) = 0 \implies \frac{-\lambda}{4} = \frac{a^2yz}{x}$

$8xz + \lambda\left(\frac{2y}{b^2}\right) = 0 \implies \frac{-\lambda}{4} = \frac{b^2xz}{y}$

$8xy + \lambda\left(\frac{2z}{c^2}\right) = 0 \implies \frac{-\lambda}{4} = \frac{c^2xy}{z}$

$\frac{a^2yz}{x} = \frac{b^2xz}{y} \implies \frac{y^2}{b^2} = \frac{x^2}{a^2}$

$\frac{b^2xz}{y} = \frac{c^2xy}{z} \implies \frac{z^2}{c^2} = \frac{y^2}{b^2}$

$\therefore \frac{x^2}{a^2} = \frac{y^2}{b^2} = \frac{z^2}{c^2} \implies \frac{3x^2}{a^2} = 1$

$\implies x = a/\sqrt{3}$ |||ly $y = b/\sqrt{3}$ $z = c/\sqrt{3}$

$\left(\frac{a}{\sqrt{3}}, \frac{b}{\sqrt{3}}, \frac{c}{\sqrt{3}}\right) \longrightarrow$ St pt.

Extreme value $= f\left(a/\sqrt{3}, b/\sqrt{3}, c/\sqrt{3}\right) = \underline{\underline{\frac{8abc}{3\sqrt{3}}}}$

Q. The minimum value of the function $f = x^2+y^2+z^2$ such that $x+y+z = 1$ is

a) 1　　b) 1/3　　c) 1/9　　d) 1/27.

Ans $f = x^2+y^2+z^2$, $\phi = x+y+z-1 = 0$

$$\left.\begin{aligned} 2x + \lambda(1) &= 0 \\ 2y + \lambda(1) &= 0 \\ 2z + \lambda(1) &= 0 \end{aligned}\right\} \quad x = y = z = 1/3$$

$(1/3, 1/3, 1/3) \longrightarrow$ St pt.

Extreme value $= f(1/3, 1/3, 1/3)$

$= 1/9 + 1/9 + 1/9 = \underline{\underline{1/3}}$

MULTIPLE INTEGRALS

Double Integral :

$f(x,y) \longrightarrow R$

$\delta R_1, \delta R_2, \delta R_3 \ldots \delta R_n$

$(x_j, y_j) \longrightarrow \delta R_j$

$$\lim_{n\to\infty}\left[\sum_{j=1}^{n}\left[f(x_j,y_j)\,\delta R_j\right]\right] = \iint_R f(x,y)\,dx\,dy.$$

Let $f(x,y)$ be defined at each point in a region R. Divide the region R into n sub regions each of area $\delta R_1, \delta R_2, \ldots \delta R_n$. Let (x_i, y_i) be an arbitrary point in a subregion with area δR_i.

Then $\lim_{n\to\infty}\left[\sum_{i=1}^{n}\left[f(x_i,y_i)\,\delta R_i\right]\right]$

$$= \iint_R f(x,y)\,dx\,dy$$

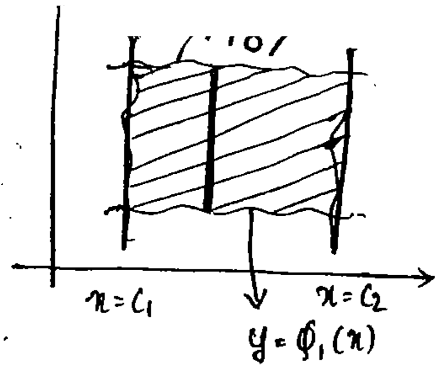
$$\therefore A = \iint_R 1\,dx\,dy$$

Methods of Evaluation

Case (i): When the limits are

$y = \phi_1(x)$ to $y = \phi_2(x)$

$x = c_1$ to $x = c_2$



$$\therefore \iint_R f(x,y)\,dx\,dy = \int_{c_1}^{c_2}\int_{\phi_1(y)}^{\phi_2(y)}\left[f(x,y)\,dy\right]dx$$

Vertical Strip

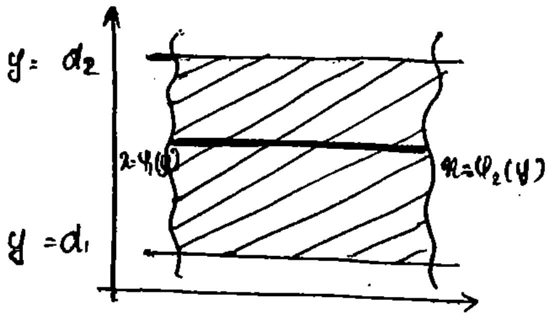
Vertical Strip

Case (ii): when the limits are $x = \psi_1(y)$ to $x = \psi_2(y)$

$y = d_1$ to $y = d_2$

$$\iint_R f(x,y)\,dx\,dy$$

$$= \int_{y=d_1}^{d_2}\left[\int_{\psi_1(y)}^{\psi_2(y)} f(x,y)\,dx\right]dy$$



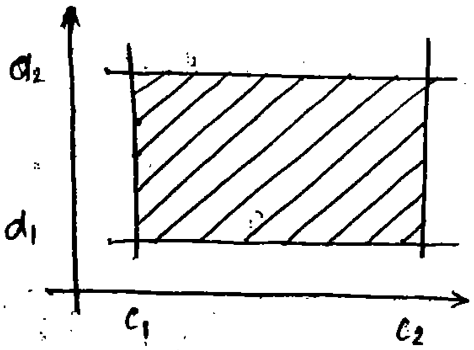
Horizontal Strip.

Case (iii): when the limits are $x = c_1$ to $x = c_2$

$y = d_1$ to $y = d_2$

$$\iint_R f(x,y)\,dx\,dy =$$

$$= \int_{x=c_1}^{c_2}\left[\int_{y=d_1}^{d_2} f(x,y)\,dy\right]dx$$



$$= \int_{y=d_1}^{d_2}\left[\int_{x=c_1}^{c_2} f(x,y)\,dx\right]dy$$

Q. Evaluate the following

① $\int_1^2 \int_3^4 \frac{1}{(x+y)^2}\, dx\, dy$

$$= \int_1^2 \left[\frac{-1}{x+y} \right]_3^4 dy = \int_1^2 \left[\frac{1}{3+y} - \frac{1}{4+y} \right] dy$$

$$= \left[\log\left(\frac{3+y}{4+y} \right) \right]_1^2 = \log\frac{5}{6} - \log\frac{4}{5}$$

$$= \underline{\underline{\log\frac{25}{24}}}$$

② $\int_0^3 \int_0^x (6-x-y)\, dy\, dx$

$$= \int_0^3 \left(6y - \cancel{\frac{x^2}{2}}\, xy - \frac{y^2}{2} \right)_0^x dx$$

$$= \int_0^3 \left(6x - x^2 - \frac{x^2}{2} \right) dx = \left[\frac{\cancel{6}x^2}{\cancel{2}} - \frac{3x^3}{\cancel{6}2} \right]_0^3$$

$$= 27 - \frac{27}{2} = \underline{\underline{\frac{27}{2}}}$$

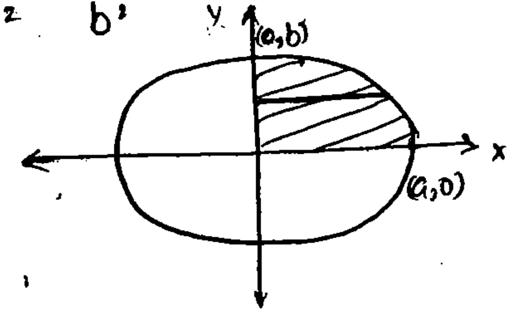
③ $\int_0^4 \int_0^{y^2} e^{x/y}\, dx\, dy = \int_0^4 \left[\frac{e^{x/y}}{1/y} \right]_0^{y^2} dy$

$$= \int_0^4 y\left[e^y - 1 \right] dy = e^y \times (y - 1) - \frac{y^2}{2} \Big]_0^4$$

$$= (3e^4 - 8) + 1$$

$$= \underline{\underline{3e^4 - 7}}$$

④ The value of $\iint_R xy\, dx\, dy$ where R is the region of the Ist quadrant of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is —



Ans $\int_0^b \int_0^{\frac{a}{b}\sqrt{b^2-y^2}} xy\, dx\, dy$

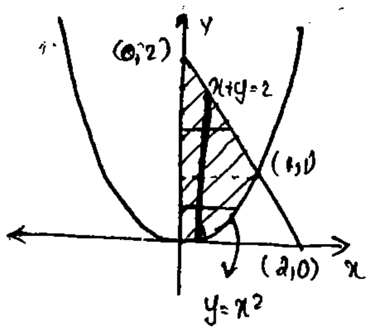
$$= \int_0^b \left[\frac{y x^2}{2} \right]_0^{\frac{a}{b}\sqrt{b^2-y^2}} dy$$

$x = 0$ to $x = \frac{a}{b}\sqrt{b^2-y^2}$

$$= \int_0^b \left[\frac{y}{2} \frac{a^2}{b^2} (b^2 - y^2) \right] dy = \frac{a^2}{2b^2} \left[\frac{b^2 y^2}{2} - \frac{y^4}{4} \right]_0^b$$

$$= \frac{a^2}{2b^2}\left[\frac{b^4}{2} - \frac{b^4}{4}\right] = \frac{a^2b^2}{8}$$

⑤ The value of $\iint_R y\,dx\,dy$ where R is the area bounded by $x = 0$, $y = x^2$, $x+y = 2$ in the 1st quad is.



Ans $x^2 + x - 2 = 0$

$(x+2)(x-1) = 0$

$x = -2, 1$

$y = 4, 1$

$(-2,4), (1,1)$

Consider the vertical strip

$$\int_0^1\int_{x^2}^{2-x} y\,dx\,dy = \int_0^1 \left[\frac{y^2}{2}\right]_{x^2}^{2-x} dx$$

$$= \int_0^1 \left[\frac{(2-x)^2}{2} - \frac{x^4}{2}\right] dx$$

$$= \frac{1}{2}\int_0^1 \left[4 + x^2 - 4x - x^4\right] dx$$

$$= \cancel{2\int_0^1 (1-x)\,dx} = 2\left[x - \frac{x^2}{2}\right]_0^1$$

$$= 2\left[1 - \frac{1}{2}\right] = 1$$

$$= \frac{1}{2}\left[4x + \frac{x^3}{3} - \frac{\overset{2}{\cancel{4}}x^2}{\cancel{2}} - \frac{x^5}{5}\right]_0^1$$

$$= \frac{1}{2}\left[4 + \frac{1}{3} - 2 - \frac{1}{5}\right]$$

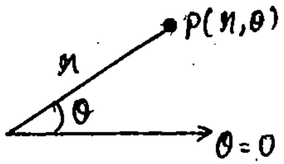
$\frac{1}{3} - \frac{1}{5}$

$\frac{5-3}{15}$

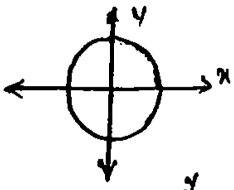
$$= \frac{1}{2}\left[2 + \frac{2}{15}\right] = \frac{1}{2}\left[\frac{32}{15}\right]$$

$$= 16/15$$

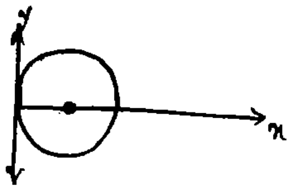
⑥ The value of $\iint_R r^2 \sin\theta\,dr\,d\theta$ where R is the semicircle $r = 2a\cos\theta$ above the initial line.



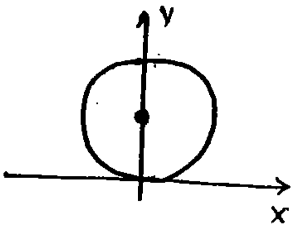
Ans ① $r = a$



② $r = a\cos\theta$

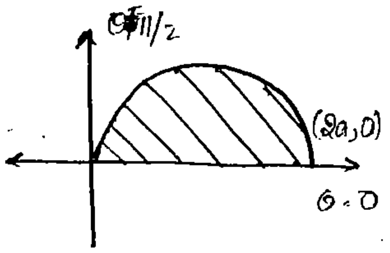


③ $r = a\sin\theta$



$r = 0$ to $2a\cos\theta$

$\theta = 0$ to $\pi/2$



$$\int_0^{\pi/2}\int_0^{2a\cos\theta} r^2\sin\theta\,dr\,d\theta = \int_0^{\pi/2}\sin\theta\left[\frac{r^3}{3}\right]_0^{2a\cos\theta}d\theta$$

$$= \frac{1}{3}\int_0^{\pi/2} 8a^3\cos^3\theta\,\sin\theta\,d\theta$$

$$= \frac{8a^3}{3}\left[-\frac{\cos^4\theta}{4}\right]_0^{\pi/4} = \frac{2a^3}{3}$$

Area of the Region

① The area of the region bounded by $y = f(x)$ & $y = g(x)$ b/w $x = c_1$ and $x = c_2$ is

$$A = \int_{x=c_1}^{c_2}\left[g(x) - f(x)\right]dx$$

OR

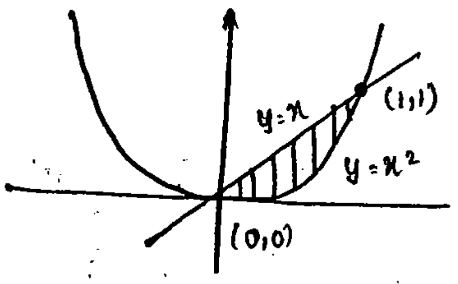
$$\int_{c_1}^{c_2}\int_{f(x)}^{g(x)} 1\,dy\,dx.$$

In Polar Form

$$\text{Area} = \int_{\theta_1}^{\theta_2}\int_{r=f_1(\theta)}^{f_2(\theta)} r\,dr\,d\theta$$

Q1. The area bounded by the curves $y = x^2$ & $y = x$ is ____

ⓐ 1/2 ⓑ 1/4 ⓒ 1/6 ⓓ None.



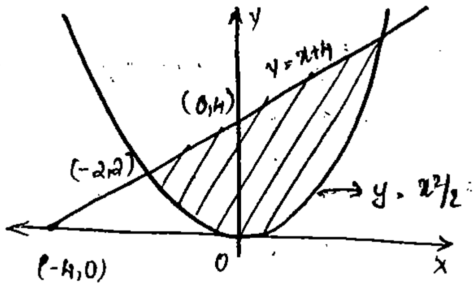
Ans
$$A = \int_0^1 \left[x - x^2\right]dx$$

$$= \frac{1}{2} - \frac{1}{3} = 1/6$$

Q1. The area bounded by $2y = x^2$ and $x = y - 4$ is ____

ⓐ 6 ⓑ 18 ⓒ 8 ⓓ None.

Ans



$x = \frac{x^2}{2} - 4$

$x^2 - 2x - 8 = 0$

$(x \mp 4)(x+2) = 0$

$x = 4, -2$

$y : 8, 2$

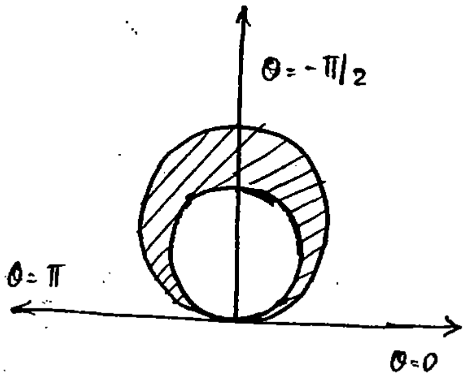
$$A = \int_{-2}^{4}\left(x+4-\frac{x^2}{2}\right)dx = \frac{x^2}{2} + 4x - \frac{x^3}{6}\Big]_{-2}^{4}$$

$$= \left(8+16-\frac{64}{6}\right) - \left(2-8+\frac{8}{5}\right)$$

$$= \underline{\underline{18}}$$

Q3. Area in b/w the circle $r = 2\sin\theta$ and $r = 4\sin\theta$ is ___

ⓐ π ⓑ 2π ⓒ 3π ⓓ None.



Ans $r = 2\sin\theta$ to $4\sin\theta$

$\theta = 0$ to π

$$A = \int_{0}^{\pi}\int_{2\sin\theta}^{4\sin\theta} r\,dr\,d\theta$$

$$= \int_{0}^{\pi}\left[\frac{r^2}{2}\right]_{2\sin\theta}^{4\sin\theta} d\theta = \frac{1}{2}\int_{0}^{\pi} 12\sin^2\theta\,d\theta$$

$$= 3\left[\int_{0}^{\pi}(1-\cos 2\theta)\,d\theta\right] = 3\left[\theta - \frac{\sin 2\theta}{2}\right]_{0}^{\pi}$$

$$= \underline{\underline{3\pi}}$$

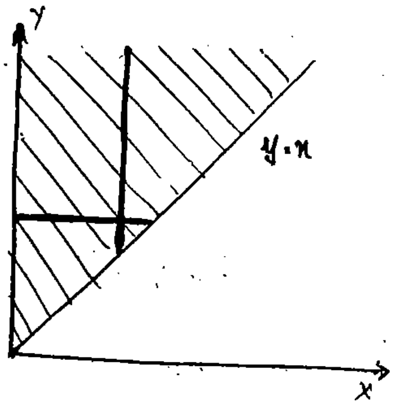
OR $r_1 = 1 \quad r_2 = 2$

$A_1 = \pi, \; A_2 = 4\pi \quad \therefore A_2 - A_1 = \underline{\underline{3\pi}}$

Change Of Order Of Integration :

Vertical Strip Problem ⟺ Horizontal Strip Problem.

Q1. $\int_{0}^{\infty}\int_{0}^{\infty}\frac{e^{-y}}{y}\,dy\,dx.$



$y = x$ to $y = \infty$

$x = 0$ to $x = \infty$

Horizontal strip : $x = 0$ to $x = y$

$y = 0$ to $y = \infty$

$$= \int_{0}^{\infty}\int_{0}^{y}\frac{e^{-y}}{y}\,dx\,dy = \int_{0}^{\infty}\frac{e^{-y}}{y}[x]_{0}^{y}\,dy = \int_{0}^{\infty}\frac{e^{-y}}{y}\,y\,dy = \frac{e^{-y}}{-1}\Big]_{0}^{\infty} = \underline{\underline{1}}$$

Q2. By reversing order of integral ~~evaluate~~ $\int_0^2 \int_{y^3}^{4\sqrt{2y}} f(x,y)\,dx\,dy$.

Ans $= \int_P^q \int_H^S f(x,y)\,dy\,dx$. Then $q \times H =$ ____

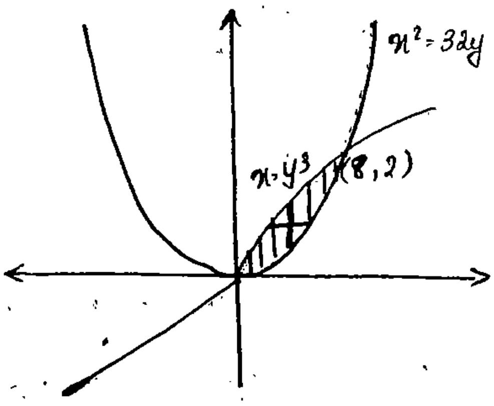
ⓐ $\frac{x^2}{2}$ [✓ ⓑ $\frac{x^2}{4}$] ⓒ $3\sqrt{x}$ ⓓ 0

Ans Given limits are.

$x = y^3$ to $x = 4\sqrt{2y}$

i.e. $x^2 = 32y$

$y = 0$ to $y = 2$



$y^6 = 32y$

$\Longrightarrow y = 0,\ y = 2$

$x = 0,\ x = 8$

Vertical Strip

$y = \frac{x^2}{\sqrt{2}}$ to $y = 3\sqrt{x}$

$x = 0$ to $x = 8$

$$= \int_0^8 \int_{x^2/32}^{3\sqrt{x}} f(x,y)\,dy\,dx.$$

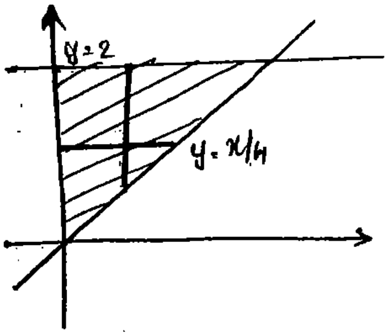
$\therefore\ q \times H = \underline{\underline{x^2/4}}$

Q3. By reversing the order of integration $I = \int_0^8 \int_{x/4}^2 f(x,y)\,dy\,dx$ changes to $I = \int_H^S \int_P^q f(x,y)\,dx\,dy$ then $q =$ ____

ⓐ $4y$ ⓑ $16y^2$ ⓒ x ⓓ 8.

Ans Given limits are $y = \frac{x}{4}$ to $y = 2$

$x = 0$ to $x = 8$



Horizontal Strip

$x = 0$ to $x = 4y$

$y = 0$ to $y = 2$

$$I = \int_0^2 \int_0^{4y} f(x,y)\,dx\,dy.$$

Triple Integral

$$\lim_{n\to\infty}\left[\sum_{j=1}^{n}\phi(x_j,y_j,z_j)\,\delta V_j\right]$$

$$= \iiint_R \phi(x,y,z)\,dx\,dy\,dz$$

Let $z = f_1(x,y)$ to $z = f_2(x,y)$

$y = g_1(x,y)$ to $y = g_2(x,y)$

$x = c_1$ to c_2

$$\iiint_R \phi(x,y,z)\,dx\,dy\,dz = \int_{x=c_1}^{c_2}\left[\int_{y=g_1(x)}^{g_1(x)}\left[\int_{z=f_1(x,y)}^{f_2(x,y)}\phi(x,y,z)\,dz\right]dy\right]dz$$

Q1. $$\int_0^a\int_0^x\int_0^{x+y} e^{x+y+z}\,dz\,dy\,dx$$

$$= \int_0^a\int_0^x \left(e^{x+y+z}\right)_0^{x+y} dy\,dx$$

$$= \int_0^a\int_0^x \left[e^{2(x+y)} - e^{(x+y)}\right] dy\,dx$$

$$= \int_0^a \left[\frac{e^{2(x+y)}}{2} - e^{x+y}\right]_0^x dx$$

$$= \int_0^a \left[\frac{e^{4x}}{2} - e^{2x} - \frac{e^{2x}}{2} + e^x\right] dx$$

$$= \int_0^a \left[\frac{e^{4x}}{8} - 3\frac{e^{2x}}{4} + e^x\right]_0^a dx$$

$$= \frac{e^{4a}}{8} - \frac{3e^{2a}}{4} + e^a - \left(\frac{1}{8} - \frac{3}{4} + 1\right)$$

Q2. $\iiint_R y\,dx\,dy\,dz$, $x=0$, $y=0$, $z=0$, $x+y+z=1$

$$= \int_0^1\int_0^{1-x}\int_0^{1-x-y} y\,dx\,dy\,dz = \int_0^1\int_0^{1-x} y\,[z]_0^{1-x-y}\,dx\,dy$$

$$= \int_0^1 \int_0^{1-x} y(1-x-y)\,dy\,dx = \int_0^1 \left[(1-x)\frac{y^2}{2} - \frac{y^3}{3}\right]_0^{1-x} dx$$

$$= \int_0^1 \left[\frac{(1-x)^3}{2} - \frac{(1-x)^3}{3}\right] dx = \frac{1}{6}\int_0^1 (1-x)^3\,dx$$

$$= \frac{1}{6}\left[\frac{(1-x)^4}{-4}\right]_0^1 = \frac{1}{6}\left[0+\frac{1}{4}\right]$$

$$= \underline{\underline{1/24}}$$

Change Of Variables:

- In double integral.

Let $x = f(u,v)$, $y = g(u,v)$

$$\iint_R \phi(x,y)\,dx\,dy = \iint_{R'} \phi(f,g)\,|J|\,du\,dv.$$

$$= \iint_{R'} \psi(u,v)\,|J|\,du\,dv$$

$|J|$ = Jacobian of Transformation

$$|J| = J\left[\frac{(x,y)}{u,v}\right] = \begin{bmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{bmatrix}$$

① Cartesian form $\longrightarrow$ polar form

$$(x,y) \longrightarrow (r,\theta)$$

$x = r\cos\theta$, $y = r\sin\theta$

$x^2+y^2 = r^2$

$$|J| = J\left[\frac{x,y}{r,\theta}\right] = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$$

$$\Longrightarrow |J| = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = 0$$

$$\therefore \iint_R \phi(x,y)\,dx\,dy = \iint_R \phi(r\cos\theta, r\sin\theta)\,r\,dr.d\theta.$$

In tripple integral.

① Cartesian ⟶ cylindrical

$(x, y, z) \longrightarrow (r, \theta, z)$

$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z.$

$x^2 + y^2 = r^2, \quad z = z.$

$$|J| = J\left[\frac{x, y, z}{r, \theta, z}\right] = r.$$

② Cartesian ⟶ Spherical polar form.

$(x, y, z) \longrightarrow (r, \theta, \phi)$

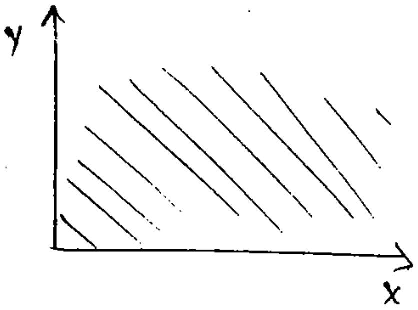
$x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\theta$

$\Longrightarrow x^2 + y^2 + z^2 = r^2$

$$|J| = J\left[\frac{x, y, z}{r, \theta, \phi}\right] = \underline{\underline{r^2\sin\theta}}$$

<u>Cylinder</u>	<u>Sphere</u>
$r : 0$ to r.	$r : 0$ to r.
$\theta : 0$ to 2π	$\phi : 0$ to 2π
$z : z_1$ to z_2.	$\theta : 0$ to π

Q1 $$\int_0^\infty\int_0^\infty e^{-(x^2+y^2)}\,dx\,dy.$$



$x = r\cos\theta$

$y = r\sin\theta$

$|J| = r$

Limits : $r : 0$ to ∞

$\theta : 0$ to $\pi/2$

$$= \int_0^{\pi/2}\int_0^\infty e^{-r^2}\, r\,dr\,d\theta$$

Let $r^2 = t \Longrightarrow r\,dr = dt/2$

$$= \int_0^{\pi/2}\int_0^{\infty} e^{-t}\,\frac{dt}{2}\,d\theta = \frac{1}{2}\int_0^{\pi/2}\left[\frac{e^{-t}}{-1}\right]_0^{\infty}d\theta = 1/2\times 1\times \pi/2 = \underline{\underline{\pi/4}}$$

2) By the change of variables to spherical polar coordinates.

$$\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}} \frac{dz\,dy\,dx}{\sqrt{1-x^2-y^2-y^2}}$$ may be represented as.

Ans Let $y = r\sin\theta\sin\phi$, $x = r\sin\theta\cos\phi$, $z = r\cos\theta$

$\Longrightarrow x^2+y^2+z^2 = r^2$ & $|J| = r^2\sin\theta$

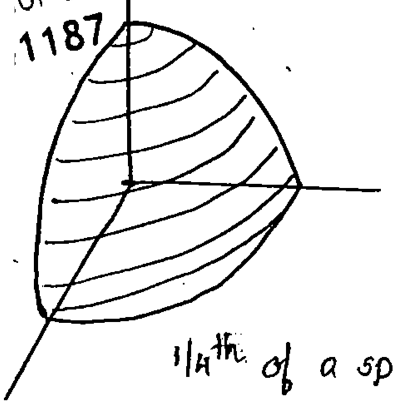
$z = 0$ to $z = \sqrt{1-x^2-y^2} \Longrightarrow x^2+y^2+z^2 = 1$

r : 0 to 1

ϕ : 0 to $\pi/2$

θ : 0 to $\pi/2$

$$\int_0^{\pi/2}\int_0^{\pi/2}\int_0^{1} \frac{r^2\sin\theta}{\sqrt{1-r^2}}\,dr\,d\theta\,d\phi.$$



1/4th of a sphere

③ $x(u,v) = uv$, $y(u,v) = v/u$

In a double integral $f(x,y)$ changes to $f(uv, v/u)\,\phi(u,v)$ then $\phi(u,v) = \_\_\_\_$?

a) $\frac{2v}{u}$ b) $\frac{v}{u}$ c) 1 d) $2v$.

Ans $$\phi(u,v) = |J| = J\left[\frac{x,y}{u,v}\right] = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} v & u \\ \frac{-v}{u^2} & \frac{1}{u} \end{vmatrix}$$

$$= \frac{v}{u} + \frac{v}{u} = \underline{\underline{\frac{2v}{u}}}$$

Length of a curve :-

① Length of an arc of a curve.

$y = f(x)$ b/w $x = x_1$ and $x = x_2$ is

$$L = \int_{x_1}^{x_2} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx.$$

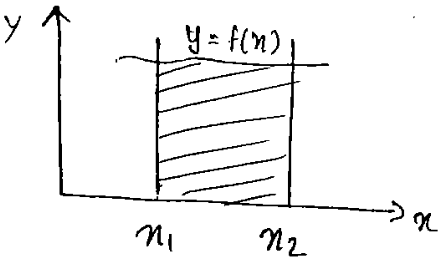
In polar form

$$L = \int_{\theta_1}^{\theta_2} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta.$$

Volume Of Solid of Revolution

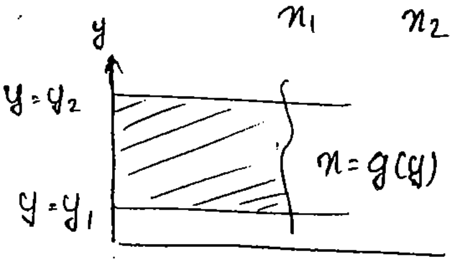
1) Volume generated by revolving the area enclosed by $y = f(x)$ b/w $x = x_1$ & $x = x_2$ about x axis

is $$V = \int^{x_2} \pi y^2\, dx$$



2) ||aly about

$$V = \int_{y_1}^{y_2} \pi x^2\, dy.$$



In Polar Form

③ about initial line, $V = \int_{\theta_1}^{\theta_2} \frac{2\pi}{3} r^3 \sin\theta\, d\theta.$

④ About the line $\theta = \pi/2$, $V = \int_{\theta_1}^{\theta_2} \frac{2\pi}{3} r^3 \cos\theta\, d\theta.$

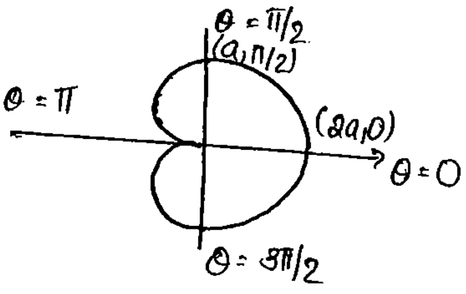
Q₁. The length of the curve $y = \frac{2}{3} x^{3/2}$ b/w $x = 0$ & $x = 1$ is ____

a) 0.27 b) 0.67 c) 1 d) 1.22.

Ans $\frac{dy}{dx} = \frac{2}{3} \times \frac{3}{2} x^{1/2} = \sqrt{x}$

$$L = \int_0^1 \sqrt{1+x}\, dx = \left.\frac{(1+x)^{3/2}}{3/2}\right]_0^1 = \underline{\underline{2/3(2\sqrt{2} - 1)}}$$

Q₂. Cardiod $r = a(1+\cos\theta)$

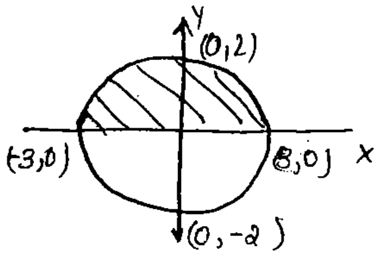


$$L = \int_0^{\pi} \sqrt{r^2 + (-a\sin^2\theta)}\, d\theta$$

$$= \int_0^{\pi} \sqrt{a^2(1+\cos^2\theta + 2\cos\theta) + a^2\sin^2\theta}\, d\theta = a\int_0^{\pi} \sqrt{2(1+\cos\theta)}\, d\theta$$

$$= a\int_0^{\pi} 2\cos\theta/2\, d\theta = 2a\left[\frac{\sin\theta/2}{1/2}\right]_0^{\pi} = 4a[1-0] = \underline{\underline{4a}}$$

Q3. $\frac{x^2}{9}+\frac{y^2}{4}=1$ about x axis

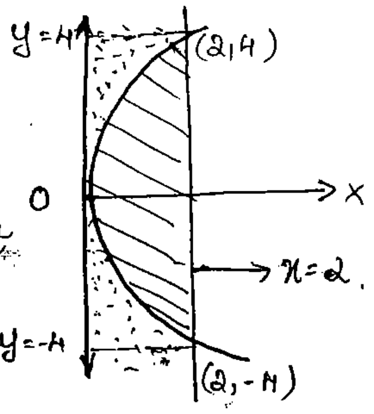


Ans $V=\int_{x_1}^{x_2}\pi y^2dx = \int_{-3}^{\pi}\pi\frac{4}{9}(9-x^2)dx = 2\pi\times\frac{4}{9}\left[9x-\frac{x^3}{3}\right]_0^3$

$=\frac{8\pi}{9}\left[27-9\right]=\underline{16\pi}$

Q4. The volume generated by revolving the area bounded by the parabola $y^2=8x$ and $x=2$ about y axis is ______

Ans a) $\frac{128\pi}{5}$ b) $\frac{127\pi}{5}$ c) $\frac{128}{5\pi}$ d) None



- The volume generated by revolving the straight line $x=2$ b/w $y=-4$ to $+4$ about y axis is

$$V_1=\int_{-4}^{4}\pi x^2dy = \int_{-4}^{4}\pi(2)^2dy = 4\pi\left[y\right]_{-4}^{4}=32\pi$$

- The volume generated by revolving the parabola $x=y^2/8$ b/w $x=-4$ to 4 about Y axis is.

$$V_2=\int_{-4}^{4}\pi x^2dy = \int_{-4}^{4}\pi\left(\frac{y^4}{64}\right)dy = 2\times\frac{\pi}{64}\left[\frac{y^5}{5}\right]_0^4 = \frac{\pi}{32}\left[\frac{16\times16\times4}{5}\right]$$

$$=\frac{32\pi}{5}$$

∴ Req. volume $=V_1-V_2=32\pi-\frac{32\pi}{5}$

$=\underline{\frac{128\pi}{5}}$

A Cor

Q5. The volume generated by revolving the cardoid $r=a(1-\cos\theta)$ about the initial line is ______?

Ans $V=\int_{\theta_1}^{\theta_2}\frac{2\pi}{3}r^3\sin\theta\, d\theta$

$=\int_{0}^{\pi}\frac{2\pi}{3}a^3(1-\cos\theta)^3\sin\theta\, d\theta = \frac{2\pi a^3}{3}\left[\frac{(1-\cos\theta)^4}{4}\right]_0^{\pi}$

$=\underline{\frac{8\pi a^3}{3}}$

Vecton Calculus

<u>Scalar</u> : Time, distance, temperature.

<u>Vecton</u> : displacement, velocity, - - - - .

<u>Position vecton</u> : The P.V of $P(x,y,z)$ in the space is $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

and $r = |\bar{r}| = \sqrt{x^2+y^2+z^2}$

In parametric form, $\bar{r}(t) = x(t)\bar{i} + y(t)\bar{j} + z(t)\bar{k}$

Let $\bar{a} = a_1\bar{i} + a_2\bar{j} + a_3\bar{k}$

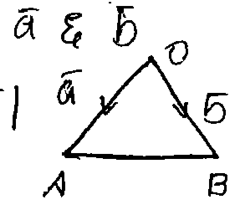
$\bar{b} = b_1\bar{i} + b_2\bar{j} + b_3\bar{k}$

① $\bar{a}\cdot\bar{b} = |\bar{a}||\bar{b}|\cos(\bar{a},\bar{b}) = a_1b_1 + a_2b_2 + a_3b_3$

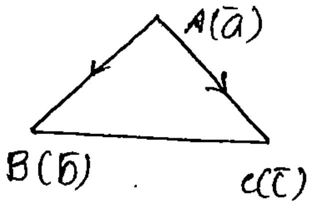
② $\bar{a}\times\bar{b} = |\bar{a}||\bar{b}|\sin(\bar{a},\bar{b})\,\hat{n} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

where $\hat{n} \longrightarrow$ unit vecton $\perp^r$ to the plane containing $\bar{a}$ & $\bar{b}$

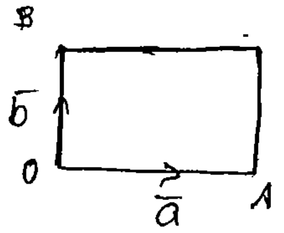
<u>Note</u> : ① Area of Δ^{le} OAB $= \frac{1}{2}|\overline{OA}\times\overline{OB}| = \frac{1}{2}|\bar{a}\times\bar{b}|$



② Area of $\Delta ABC = \frac{1}{2}|\overline{AB}\times\overline{AC}| = \frac{1}{2}|(\bar{b}-\bar{a})\times(\bar{c}-\bar{a})|$



③ Area of parallelogram $= |\bar{a}\times\bar{b}|$



<u>Scalar Triple Product</u>

$(\bar{a}\times\bar{b})\bar{c} = \bar{a}\cdot(\bar{b}\times\bar{c}) = [a\ b\ c] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

<u>Note</u> : If $\bar{a}, \bar{b}, \bar{c}$ are edge vectors, then

① Volume of parallelopiped $= |[\bar{a}\ \bar{b}\ \bar{c}]|$

② Volume of Tetrahedron $= \frac{1}{6}|[\bar{a}\ \bar{b}\ \bar{c}]|$

<u>Vecton Triple Product</u>

① $(\bar{a}\times\bar{b})\times\bar{c} = (\bar{c}\cdot\bar{a})\bar{b} - (\bar{c}\cdot\bar{b})\bar{a}$

② $\bar{a}\times(\bar{b}\times\bar{c}) = (\bar{c}\cdot\bar{a})\bar{b} - (\bar{b}\cdot\bar{a})\bar{c}$

Vector Differentiation

Derivative of a fn :

Let $\bar{r}(t) = \bar{f}(t)$ be the vector eq. of the curve. Then

$$\frac{d\bar{r}}{dt} = \lim_{\delta t \to 0} \left[\frac{\bar{f}(t+\delta t) - \bar{f}(t)}{\delta t} \right]$$

Directional Derivative

$D.D = \nabla\phi \cdot \frac{\bar{a}}{|\bar{a}|}$

Let $\bar{a} = \bar{i}$

$D.D = \nabla\phi \cdot \bar{i}$

$= \frac{\partial\phi}{\partial x}$

Note : Let $\hat{b} = \frac{\bar{a}}{|\bar{a}|}$

$D.D = \nabla\phi \cdot \hat{b}$

$= |\nabla\phi||\hat{b}| \cos\theta$

$= |\nabla\phi| \cos\theta$

The max value of $\cos\theta = 1$ ie when $\theta = 0°$.

$\Longrightarrow \hat{b} \parallel \nabla\phi$.

$\Longrightarrow \hat{b}$ is normal to ϕ.

$\therefore$ The value of directional derivative is maximum in the dirn of normal to the surface ϕ.

$\therefore$ Max. value of $D.D = |\nabla\phi|$

(OR)

Greatest rate of increase $= |\nabla\phi|$

$\angle$le b/w two surfaces

Let $\phi_1(x,y,z) = c_1$ and $\phi_2(x,y,z) = c_2$ be two surfaces and θ be the angle b/w them. Then $\cos\theta = \frac{\nabla\phi_1 \cdot \nabla\phi_2}{|\nabla\phi_1||\nabla\phi_2|}$

Q. If $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$ and $r = |\bar{r}|$ then $\nabla r =$ ____

a) $\frac{\bar{r}}{r}$ b) $r\bar{r}$ c) $-\frac{\bar{r}}{r}$ d) $-r\bar{r}$

Ans $r = |\bar{r}| = \sqrt{x^2+y^2+z^2}$

$\frac{\partial r}{\partial x} = \frac{1}{2\sqrt{x^2+y^2+z^2}} \times 2x = \frac{x}{r}$

III^{ly} $\frac{dr}{dy} = \frac{y}{r}$, $\frac{dr}{dz} = \frac{z}{r}$

$$\therefore \quad \nabla r = i\frac{\partial r}{\partial x} + j\frac{\partial r}{\partial y} + k\frac{\partial r}{\partial z}$$

$$= \frac{xi + yj + zk}{r} = \underline{\underline{\frac{\bar{r}}{r}}}$$

Note 3 $\nabla[f(r)] = f'(r)\frac{\bar{r}}{r}$

② $\nabla[r^n] = n r^{n-1}\frac{\bar{r}}{r} = \underline{\underline{n r^{n-2}\bar{r}}}$

③ $\nabla[\sin(\log r)] = \cos(\log r) \times \frac{1}{r}\frac{\bar{r}}{r}$.

$$= \underline{\underline{\frac{\cos(\log r)}{r^2} \times \bar{r}}}$$

Q4. The unit vector normal to the surface $xy^3z^2 = 4$ at $(-1,-1,2)$ is ____.

Ans Let $\phi = xy^3z^2$

$\Longrightarrow \nabla\phi = \hat{i}(y^3z^2) + \hat{j}[3xy^2z^2] + \hat{k}[2xy^3z]$

At $(-1,-1,2)$

$$\nabla\phi = \hat{i}(-1^3 \times 4) + \hat{j}[3\times -1 \times 1 \times 4] + \hat{k}[2 \times -1 \times -1^3 \times 2]$$

$$= -\hat{i}4 + -12\hat{j} + 4\hat{k}$$

$$= -4i - 12\hat{j} + 4\hat{k}$$

$$\frac{\nabla\phi}{|\nabla\phi|} = \frac{\hat{i} - 3\hat{j} + \hat{k}}{\sqrt{1+9+1}} = \underline{\underline{\frac{-i-3j+k}{\sqrt{11}}}}$$

Q5. A sphere of unit radius is centred at origin. Then a unique vector normal to the surface of the sphere at any point $P(x,y,z)$ is the vector

a) $\left(\frac{x}{\sqrt{3}}, \frac{y}{\sqrt{3}}, \frac{z}{\sqrt{3}}\right)$ b) $\left(\frac{x}{\sqrt{2}}, \frac{y}{\sqrt{2}}, \frac{z}{\sqrt{2}}\right)$ c) $(x, y, 3)$ d) $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$

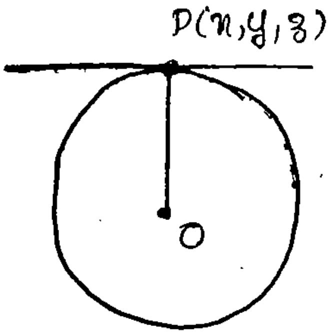
Ans $x^2 + y^2 + z^2 = 1$

Let $\phi = x^2 + y^2 + z^2$

$$\implies \nabla\phi = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$$

$$\implies \frac{\nabla\phi}{|\nabla\phi|} = \frac{x\hat{i}+y\hat{j}+z\hat{k}}{\sqrt{x^2+y^2+z^2}}$$

$$\implies \frac{\nabla\phi}{|\nabla\phi|} = \underline{\underline{\frac{x\hat{i}+y\hat{j}+z\hat{k}}{1}}}$$



Note : Normal to the sphere with centre at origin at any point $P(x,y,z)$ is the position vector of the point itself.

Q6. The directional derivative of $f = xy^2z$ at $(1,-1,1)$ in the dirn of the vector $\bar{a} = \bar{i} + \bar{j} - 2\bar{k}$ is ___

a) $\frac{3}{\sqrt{6}}$ b) $\frac{-3}{\sqrt{6}}$ c) $\frac{5}{\sqrt{6}}$ d) $-\frac{5}{\sqrt{6}}$

Ans $\nabla f = \hat{i}(y^2z) + \hat{j}(2xyz) + \hat{k}(xy^2)$

$$\nabla f\Big]_{(1,-1,1)} = \hat{i} - 2\hat{j} + \hat{k}$$

$$\therefore D\cdot D = \nabla f \cdot \frac{\bar{a}}{|\bar{a}|} = (\hat{i}-2\hat{j}+\hat{k}) \cdot \frac{(i+j-2k)}{\sqrt{1+1+4}}$$

$$= \frac{1-2-2}{\sqrt{6}} = \underline{\underline{-3/\sqrt{6}}}$$

Q7. The directional derivative of $\phi = xy^2 + yz^2 + zx^2$ at $(1,1,1)$ along the direction of tangent to the curve $x = t, y = t^2, z = t^3$ is ___

Ans $\bar{r}(t) = t\hat{i} + t^2\hat{j} + t^3\hat{k}$

$$\frac{dr}{dt} = \hat{i} + 2t\hat{j} + 3t^2\hat{k}$$

At $(1,1,1)$ it $t = 1$

$$\frac{dr}{dt} = \hat{i} + 2\hat{j} + 3\hat{k}$$

$$\nabla\phi = \hat{i}(y^2+2xz) + \hat{j}(z^2+2xy) + \hat{k}(x^2+2yz)$$

$$\nabla\phi\Big]_{(1,1,1)} = 3\hat{i} + 3\hat{j} + 3\hat{k}$$

$$\therefore \quad D.D = \frac{\nabla\phi \cdot \frac{d\bar{r}}{dt}}{\left|\frac{d\bar{r}}{dt}\right|} = \frac{3+6+9}{\sqrt{1+4+9}} = \frac{18}{\sqrt{14}}$$

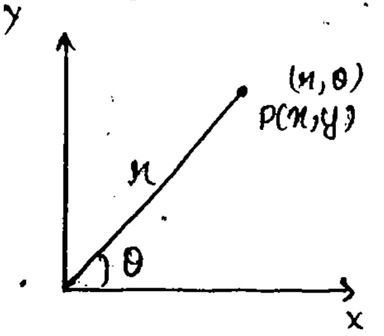
Q8. Let $f = x^{2/3} + y^{2/3}$ be a scalar point fn. Then the derivative of f along the line $y = x$ directed away from the origin at the pt $(8,8)$ is ___ a) $\frac{2}{\sqrt{3}}$ b) $\frac{\sqrt{2}}{3}$ c) $\frac{3}{\sqrt{2}}$ d) $\frac{\sqrt{3}}{2}$

Ans $f = x^{2/3} + y^{2/3}$

~~df~~ $\bar{r} = x\hat{i} + y\hat{j}$

$= (r\cos\theta)\hat{i} + (r\sin\theta)\hat{j}$

$\frac{\bar{r}}{|r|} = \cos\theta\,\hat{i} + \sin\theta\,\hat{j}$



Here $\theta = \pi/4$

$\Longrightarrow \hat{e} = \frac{1}{\sqrt{2}}\hat{i} + \frac{1}{\sqrt{2}}\hat{j}$

(or)

$\hat{r} = 8\hat{i} + 8\hat{j}$

$\frac{\hat{r}}{|r|} = \frac{\hat{i}+\hat{j}}{\sqrt{2}}$

$\nabla f = \hat{i}\left(\frac{2}{3}x^{-1/3}\right) + \hat{j}\left(\frac{2}{3}y^{-1/3}\right)$

$\nabla f\big]_{(8,8)} = \frac{1}{3}\hat{i} + \frac{1}{3}\hat{j}$

$\therefore \; D.D = \nabla f \cdot \hat{e} = \frac{1}{3\sqrt{2}} + \frac{1}{3\sqrt{2}} = \frac{2}{3\sqrt{2}} = \frac{\sqrt{2}}{3}$

Q9. The max. value of directional derivative of $\phi = e^{3x}\sin(yz^4)$ at $(0, \pi/2, 1)$ is ___

a) 3 b) 4 c) 5 d) 6

Ans $\nabla\phi = \hat{i}[3e^{x}\sin(yz^4)] + \hat{j}[e^{3x}\cos(yz^4) \times z^4] + \hat{k}[e^{3x}\cos(yz^4)\,4yz^3]$

At $(0, \pi/2, 1)$

$\Longrightarrow \nabla\phi = 3\hat{i}$. Max value of D.D $= |\nabla\phi| = 3$

Q10. The directional derivative of $\phi = ax^2 + by^2 + cz^2$ at $(1,1,2)$ has max magnitude $|\nabla\phi| = 4$ in the dirn ||al to y axis. Then the values of a, b, c are ___

a) (0, 1, 0) b) (0, 2, 0) c) (0, 4, 0) d) (1, 0, 1)

Ans

$$\nabla\phi = 2ax\,\hat{i} + 2by\,\hat{j} + 2cz\,\hat{k}$$

$$\nabla\phi\Big]_{(1,1,2)} = 2a\,\hat{i} + 2b\,\hat{j} + 4c\,\hat{k}$$

Given $\nabla\phi$ ||al to Y axis.

$$\Longrightarrow a = 0 \,, c = 0$$

$$\therefore \nabla\phi = 2b\,j$$

$$|\nabla\phi| = 4 \Longrightarrow 2b = 4$$

$$b = \underline{\underline{2}}$$

$$\therefore \underline{(0,2,0)}$$

Q11. The angle b/w the surfaces $x^2+y^2+z^2=9$ and $x^2+y^2-z=3$ at $(2,-1,2)$.

Ans Let $\phi_1 = x^2+y^2+z^2$

$$\Longrightarrow \nabla\phi_1 = 2x\,\hat{i} + 2y\,\hat{j} + 2z\,\hat{k}$$

$$\Longrightarrow \nabla\phi_1\Big]_{(2,-1,2)} = 4\hat{i} - 2\hat{j} + 4\hat{k}$$

Let $\phi_2 = x^2+y^2-z$

$$\Longrightarrow \nabla\phi_2 = 2x\,\hat{i} + 2y\,\hat{j} - \hat{k}$$

$$\Longrightarrow \nabla\phi_2\Big]_{(2,-1,2)} = 4\hat{i} - 2\hat{j} - \hat{k}$$

$$\cos\theta = \frac{\nabla\phi_1 \cdot \nabla\phi_2}{|\nabla\phi_1|\,|\nabla\phi_2|} = \frac{16+4-4}{\sqrt{16+4+16}\,\sqrt{16+4+1}}$$

$$\Longrightarrow \theta = \cos^{-1}\left(\frac{8}{3\sqrt{21}}\right)$$

Divergence Of a Vector Fn

Let $\bar F(x,y,z) = F_1\hat i + F_2\hat j + F_2\hat k$ be a differential vector point fn

then $\text{div}\,\bar F = \nabla\cdot\bar F$

$$= \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

Note: If $\text{div}\,\bar F = 0$ then $\bar F$ is Solenoidal vector

Curl Of a Vector Function

$$\text{curl}\,\bar F = \nabla\times\bar F$$

$$= \begin{vmatrix} \hat i & \hat j & \hat k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

Note: If $\bar v$ → linear velocity.

$\bar\omega$ → Angular velocity

then $\bar v = \bar\omega\times\bar r$

$$\Longrightarrow \text{curl}\,\bar v = \nabla\times(\bar\omega\times\bar r)$$

$$= 2\bar\omega$$

$$\Longrightarrow \boxed{\bar\omega = \frac{1}{2}\,\text{curl}\,\bar v}$$

Note: If $\text{curl}\,\bar F = 0$ then $\bar F$ is said to be irrotational vector.

Scalar Potential Vector

If $\bar F$ is a irrotational P.V then there exist a scalar potential fn ϕ such that $\boxed{\bar F = \nabla\phi}$

$$\phi(x,y,z) = \int_a^x F_1(x,y,z)\,dx + \int_b^y F_2(a,y,z) + \int_c^z F_3(a,b,z)\,dz$$

NOTE: ① $\text{Curl}\,(\text{grad}\,\phi) = \bar 0$

② $\text{Div}\,(\text{Curl}\,\bar F) = 0$

③ $\text{Div}\,(\text{grad}\,\phi) = \nabla\cdot(\nabla\phi)$

Laplacian operator $\longleftarrow = \nabla^2 \phi$

$$= \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}$$

④ Curl (curl F̄) $= \nabla \times (\nabla \times \bar{F})$

$$= \nabla(\nabla \cdot \bar{F}) - (\nabla \cdot \nabla)\bar{F}$$

$$= \text{grad}(\text{div}\,\bar{F}) - \nabla^2 \bar{F}$$

⑤ $\text{div}(\bar{A} \times \bar{B})$

$$= \bar{B} \cdot \text{curl}\,\bar{A} - \bar{A} \cdot \text{curl}\,\bar{B}$$

Q1. $\bar{F} = 4x^2 z\,\hat{i} - (7y^2 + 2xz)\,\hat{j} + 3yz^2\,\hat{k}$ represents a velocity vector

(i) Then Div of $\bar{F}$ at $(1,-1,2)$ is ___

(ii) Corresp. angular velocity at $(2,1,-2)$ is ___.

Ans i) $\text{div}\,\bar{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$

$$= 8xz + (-14y) + 6yz$$

At $(1,-1,2)$

$$\text{div}\,\bar{F} = 16 + 14 - 12 = \underline{\underline{18}}$$

ii) $\bar{\omega} = \frac{1}{2}\,\text{curl}\,\bar{F}$

$$\text{curl}\,\bar{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 4x^2 z & -(7y^2+2xz) & 3yz^2 \end{vmatrix} = i[3z^2 + 2x] - j[0 - 4x^2] + \bar{k}[2z - 0]$$

At $(2,1,-2)$

$$\text{Curl}\,\bar{F} = 16\hat{i} + 16\hat{j} + 4\hat{k}$$

$$\therefore \omega = \frac{1}{2}\,\text{curl}\,\bar{F} = \underline{\underline{8\hat{i} + 8\hat{j} + 2\hat{k}}}$$

Q2. The value of a such that the vector fn $\bar{F} = (ax^2y - yz)\hat{i} + (xy^2 - xz^2)\hat{j} + (2xyz + y^2x^2)\hat{k}$ is solenoid is ___

a) 0 b) 1 c) 2 d) −2

Ans $\Longrightarrow \text{div}\,\bar{F} = 0$

$$\Longrightarrow (2\lambda xy - 0) + (2xy - 0) + (2xy + 0) = 0$$

$$\Longrightarrow 2xy[\lambda + 1 + 1] = 0 \quad \Longrightarrow \lambda = \underline{\underline{-2}}$$

Q③ If $\bar{F} = (x - 2y + az)\hat{i} + (bx - 3y + 4z)\hat{j} + (2x + cy - 5z)\hat{k}$ is irrotational then a, b, c is ____

Ans Curl $\bar{F} = 0$

$$\text{Curl } \bar{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x - 2y + az & bx - 3y + 4z & 2x + cy - 5z \end{vmatrix} = 0$$

$$\Longrightarrow i[c - 4] - \hat{j}[2 - a] + k[b + 2] = 0.$$

$$\Longrightarrow \underline{\underline{c = 4, \; a = 2, \; b = -2}}$$

Q4. $\phi = xyz$ then $\nabla\phi$ is

ⓐ Solenoidal ⓑ irrotational [✓ ⓒ a & b] ⓓ None.

Ans $\nabla\phi = i(yz) + j(xz) + k(xy)$

$\text{div}(\nabla\phi) = 0 + 0 + 0 = \underline{\underline{0}}$

$\text{Curl}(\nabla\phi) = \text{Curl}(\text{Grad } \phi) = \underline{\underline{\bar{0}}}$

Q5. If $\bar{F} = 3xy^2\hat{i} - 4yz\hat{j} + xz^2\hat{k}$ then the value of $\nabla \cdot (\nabla \times F)$ at $(3, -5, 1)$ is ____

Ans $\text{div}(\text{Curl } \bar{F}) = \underline{\underline{0}}$ [By property]

Q6. The directional derivative of div $\bar{F}$ in the dirn of outer normal to the surface of the sphere with centre at origin and radius = 3 where $\bar{F} = x^4\hat{i} + y^4\hat{j} + z^4\hat{k}$ is

a) 67 b) 68 c) 69 d) None.

Ans div $\bar{F} = 4x^3 + 4y^3 + 4z^3 = f$

$\nabla f = 12x^2\hat{i} + 12y^2\hat{j} + 12z^2\hat{k}$

$\nabla f\Big]_{(1,2,2)} = 12\hat{i} + 48\hat{j} + 48\hat{k}$

$\Longrightarrow \nabla f = 12\,(\hat{i}+4\hat{j}+4\hat{k})$

Given sphere is $x^2+y^2+z^2=9$

Normal to the sphere at $(1,2,2)$ is

$\hat{a} = \hat{i}+2\hat{j}+2\hat{k}$

$\therefore$ D.D = $\nabla f \cdot \frac{\bar{a}}{|\bar{a}|} = 12(\hat{i}+4\hat{j}+4k)\,\frac{(\hat{i}+2\hat{j}+2\hat{k})}{\sqrt{1+4+4}}$

$= 4(1+8+8) = \underline{\underline{68}}$

Q7. $\bar{r} = x\hat{i}+y\hat{j}+z\hat{k}$, & $r = |\bar{r}|$ then $\mathrm{div}(r^n\bar{r}) =$ ___

a) $(n+3)r^n$ b) $(n-3)r^n$ c) $n\,r^{n-3}$ d) None.

Ans $r^n\bar{r} = \underbrace{r^n x}_{F_1}\hat{i} + \underbrace{r^n y}_{F_2}\hat{j} + \underbrace{r^n z}_{F_3}\hat{j}$

$\mathrm{div}(r^n\bar{r}) = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$

$\frac{\partial F_1}{\partial x} = \frac{\partial}{\partial x}(r^n x) = r^n(1) + x\,n\,r^{n-1}\frac{\partial r}{\partial x}$

$= r^n + x\,n\,r^{n-1} \times \frac{x}{r}$

$\frac{\partial F_1}{\partial x} = r^n + n r^{n-2} x^2$

III^{ly}. $\frac{\partial F_2}{\partial y} = r^n + n r^{n-2} y^2$

$\frac{\partial F_3}{\partial z} = r^n + n r^{n-2} z^2$

$\therefore \mathrm{div}(r^n\bar{r}) = 3r^n + n r^{n-2}\underbrace{(x^2+y^2+z^2)}_{r^2}$

$= 3r^n + n r^n = (n+3)r^n$

If $n=-3$ $\mathrm{div}(r^n\bar{r}) = 0$

$\Longrightarrow \frac{\bar{r}}{r^3}$ is solenoidal vector

VECTOR INTEGRATION

Line Integral :

Let $\bar{F}(x,y,z) = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$ be a diff. vector fn defined along the curve 'c' then its line integral is

$$\int_c \bar{F}\cdot d\bar{s}$$

In Cartesian form,

$$\int_c \bar{F}\cdot d\bar{s} = \int_c F_1\,dx + F_2\,dy + F_3\,dz.$$

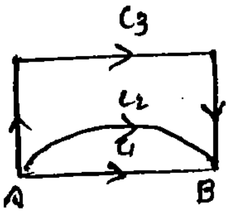
Note : If c is a closed curve then the line integral of $\bar{F}$ along C is called circulation of $\bar{F}$ denoted by

$$\oint_c \bar{F}\cdot d\bar{s}$$

Work done by a Force :

The total work done by force $\bar{F}$ in moving some particle along the curve C is

$$W.D = \oint_c \bar{F}\cdot d\bar{r}$$



Note :

The line integral of an irrotational vector fn is independent of the path of the curve.

If $\bar{F}$ is irrotational then $\bar{F} = \nabla\phi$. where ϕ is a scalar potential fn. The $\int_A^B \bar{F}\cdot d\bar{r} = \phi_B - \phi_A$

Q1. The value of $\int_c \bar{F}\cdot d\bar{r}$ where $\bar{F} = 2x^2y\,\hat{i} - y^2x\,\hat{j}$ and c is the curve $y = 2x^2$ joining the points (0,0) & (1,2) is _____ ?

Ans

$$\int_C \bar{F}\cdot d\bar{r} = \int_C 2x^2y\,dx - y^2x\,dy$$

$$y = 2x^2 \implies dy = 4x\,dx$$

$$= \int_0^1 2x^2(2x^2)\,dx - (4x^4)x\cdot 4x\,dx$$

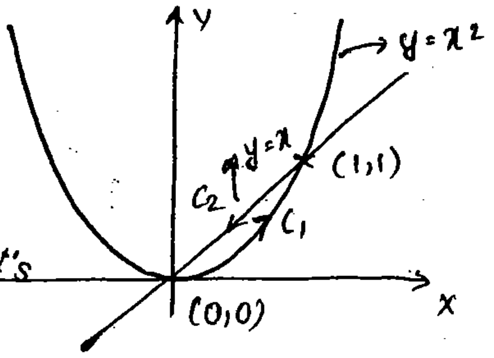
$$= \int_0^1 [4x^4 - 16x^6]\,dx = \frac{4}{5} - \frac{16}{7} = \frac{28-80}{35}$$

$$= \underline{\underline{\frac{-52}{35}}}$$

Q2. The value of $\int_C \bar{F}\cdot d\bar{r}$ where $\bar{F} = 3xy\,\hat{i} - y^2\hat{j}$ and C is the curve bounded by $y = x^2$ and $y = x$ is ___

Ans

$$\int_C \bar{F}\cdot d\bar{r} = \int_{C_1} (\;) + \int_{C_2} (\;)$$



i) Along C_1 : $y = x^2$

$$dy = 2x\,dx.$$

$$\int_{C_1} \bar{F}\,d\bar{r} = \int_C 3xy\,dx - y^2\,dy = \int_0^1 3x\times x^2\,dx - x^4\times 2x\,dx$$

$$= \frac{3x^4}{4} - \frac{2x^6}{6}\Big]_0^1 = \frac{3}{4} - \frac{2}{6} = \underline{\underline{5/12}}$$

ii) Along C_2 : $y = x$

$$dy = dx$$

Here $0 \to 1$
dirⁿ is important

$$\int_{C_2} \bar{F}\cdot d\bar{r} = \int_C 3xy\,dx - y^2\,dy = \int_1^0 3x^2\,dx - x^2\,dx$$

$$= \frac{3x^3}{3} - \frac{x^3}{3}\Big]_1^0 = \underline{\underline{-2/3}}$$

$$\therefore \int_C \bar{F}\cdot d\bar{r} = \frac{5}{12} - \frac{2}{3} = \frac{5-8}{12} = \underline{\underline{\frac{-1}{4}}}$$

Note : The line integral of an irrotational vector fn over any closed curve is zero.

Q3. The value of $\int_C (3x+4y)dx + (2x-3y)dy$ where C is a circle with centre at the origin and radius as 2 in x-y plane is ——.

a) 8π b) 4π c) − 8π d) None .

Ans Circle ⟶ $x^2+y^2=4$.

Let $x = 2\cos t$ & $y = 2\sin t$

$dx = -2\sin t\,dt$ & $dy = 2\cos t\,dt$

$$= \int_0^{2\pi} (6\cos t + 8\sin t)(-2\sin t\,dt) + (4\cos t - 6\sin t)(2\cos t\,dt)$$

$$= \int_0^{2\pi} (-24\sin t\cos t - 16\sin^2 t + 8\cos^2 t\,\cancel{dt})\,dt$$

$$= \int_0^{2\pi} (-12\sin 2t - 8[1-\cos 2t) + 4[1+\cos 2t])dt$$

$$= \int_0^{2\pi} [-12\sin 2t - 4 + 12\cos 2t]dt$$

$$= \int_0^{2\pi} \underbrace{\frac{12\cancel{\cos}2t}{\cancel{2}}}_{0} - 4t + \underbrace{\frac{12\sin\cancel{2t}}{\cancel{2}}}_{0}\Bigg]_0^{2\pi}$$

$$= -4\times 2\pi = \underline{-8\pi}$$

Q4. The value of $\int_C \bar{F}\cdot d\bar{r}$, $\bar{F} = (2y+3)\hat{i} + xz\hat{j} - 3y^2z\hat{k}$ and C is the line joining the points

i) (0,0,1) to (0,1,1)

ii) (0,1,1) to (2,1,1)

Ans $x=0$, $z=1$

$\Longrightarrow dx=0$, $dz=0$

y : 0 to 1

$$\int_C \bar{F}\cdot d\bar{r} = \int F_2\,dy = \int^1 xz\,dy = \int^1 0\,dy = 0$$

ii) $y = 1$, $z = 1$

$\Longrightarrow dy = 0$, $dz = 0$

x : 0 to 2.

$$\therefore \int_C \bar{F} \cdot d\bar{r} = \int_0^2 F_1 dx = \int_0^2 (2y+3)dx = \int_0^2 5dx = \underline{\underline{10}}$$

Q5. The total work done by the force $\bar{F} = (3x^2 + 6y)\hat{i} - 14yz\hat{j} + 20xz^2\hat{k}$ in moving a particle along the straight line joining the set of points $(0,0,0.)$ & $(1,1,2)$ is ——

Ans
$$\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1} = t$$

$$\frac{x-0}{1-0} = \frac{y-0}{1-0} = \frac{z-0}{2-0} = t$$

$x = t$, $y = t$, $z = 2t$

$dx = dt$, $dy = dt$, $dz = 2dt$.

$$W.D = \int_C \bar{F} \cdot d\bar{s} = \int_C (3x^2+6y)dx - 14yz\,dy + 20xz^2dz$$

$$= \int_0^1 (3t^2+6t)\,dt - 28t^2\,dt + 160t^3\,dt.$$

$$= \frac{3t^3}{3} + \frac{6t^2}{2} - \frac{28t^3}{3} + 160\frac{t^4}{4} \Big]_0^1$$

$$= \frac{-25}{3} + \frac{\cancel{6}^{3}}{\cancel{2}} + \frac{\overset{4}{\cancel{160}}}{\cancel{4}} = 43 - \frac{25}{3}$$

$$= \underline{\underline{\frac{-98}{3}}}$$

Q6. Find $\int \bar{v} \cdot d\bar{r}$ where $\bar{V} = yz\hat{i} + (x+z+1)\hat{j} + xy\hat{k}$ from $(0,1,0)$ to $(2,1,4)$.

a) 7 b) 8 c) 9 d) Cannot be determined without specifying the path.

Here path is not specified. Check whether the fn is irrotational. If it is irrotational proceed to line integral. If not irrotational

$$\text{curl } \bar{V} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ yz & xz+1 & xy \end{vmatrix}$$

$$= i(x-x) - j(y-y) + k(z-z) = 0$$

$\Longrightarrow \bar{V}$ is irrotational.

$$\Longrightarrow \bar{V} = \nabla\phi$$

$$\phi = \int_a^x F_1(x,y,z)\,dx + \int_b^y F_2(a,y,z)\,dy + \int_c^z F_3(a,b,z)\,dz$$

$$= \int_a^x yz\,dx + \int_b^y (az+1)\,dy + \int_c^z ab\,dz.$$

$$= yz[x]_a^x + (az+1)y\Big]_b^y + abz\Big]_c^z$$

$$= xyz \cancel{-ayz} + ayz + y - abz - b + abz - abc$$

$$\Longrightarrow \phi(x,y,z) = xyz + y + k$$

$$\therefore \int \bar{V}\cdot d\bar{r} = \phi(2,1,4) - \phi(0,1,0)$$

$$= (8+1+k) - (0+1+k)$$

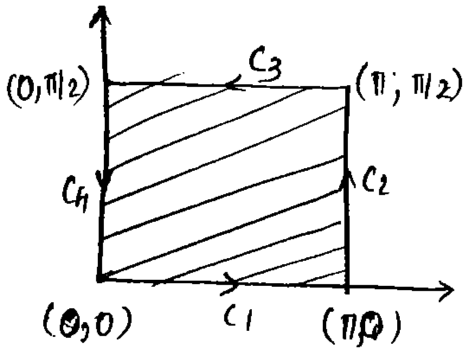
$$= \underline{8}$$

Green's Theorem in a Plane

Let $M(x,y)$ and $N(x,y)$ be the continuous fn having const. first order ~~potential~~ part derivatives defined in a closed region 'R' bounded by closed curve 'C' then

$$\oint M\,dx + N\,dy = \iint_R \left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right] dx\,dy$$

Q1. $\oint_c e^{-x}\sin y\,dx + e^{-x}\cos y\,dy$ where C is a rectangle with vertices $(0,0)$, $(\pi,0)$, $(\pi,\pi/2)$, $(0,\pi/2)$.



Ans $M = e^{-x}\sin y \implies \dfrac{\partial M}{\partial y} = e^{-x}\cos y$

$N = e^{-x}\cos y \implies \dfrac{\partial N}{\partial x} = -e^{-x}\cos y$

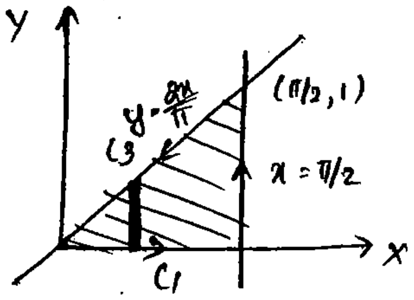
$\therefore \dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y} = -2e^{-x}\cos y$.

$\therefore \oint M dx + N dy = \iint_R \left. \dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y} \right] dx\,dy$

$$= \int_0^{\pi/2}\int_0^{\pi} -2e^{-x}\cos y\,dx\,dy = \int_0^{\pi/2} -2\cos y\left[\frac{e^{-x}}{-1}\right]_0^{\pi} dy$$

$$= 2(e^{-\pi}-1)\left[\sin y\right]_0^{\pi/2} = \underline{2[e^{-\pi}-1]}$$

Q2. Find $\oint_c (y-\sin x)\,dx + \cos x\,dy$ where C is the curve bounded by $y=0,\ x=\pi/2,\ y=\dfrac{2x}{\pi}$.



Ans Considering the vertical strip

$y=0$ to $y=\dfrac{2x}{\pi}$

$x=0$ to $x=\pi/2$.

$\dfrac{\partial M}{\partial y} = 1$, $\dfrac{\partial N}{\partial x} = -\sin x$

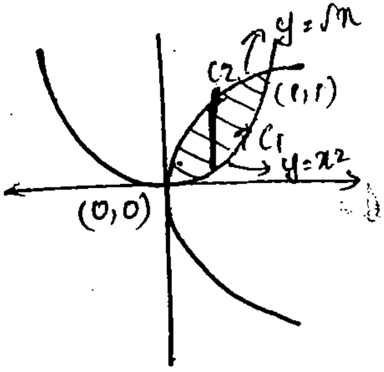
$\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y} = -\sin x - 1$

$$= \int_0^{\pi/2}\int_0^{2x/\pi} -(1+\sin x)\,dy\,dx = \int_0^{\pi/2} -(1+\sin x)\frac{2x}{\pi}\,dx$$

$$= \frac{-2}{\pi}\left[\frac{x^2}{2} + x(-\cos x) - (1)(-\sin x)\right]_0^{\pi/2}$$

$$= \underline{\frac{-2}{\pi}\left[\frac{\pi^2}{8} + 1\right]}$$

Q3. $\oint_C (3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy$ bounded by $y = \sqrt{x}$ and $y = x^2$



Ans Consider the vertical strip.

$y = x^2$ to $y = \sqrt{x}$

$x = 0$ to $x = 1$

$$\frac{\partial M}{\partial y} = -16y, \quad \frac{\partial N}{\partial x} = -6y.$$

$$\Longrightarrow \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 10y$$

$$= \int_0^1 \int_{x^2}^{\sqrt{x}} 10y\,dy\,dx = 10\int_0^1 \left[\frac{y^2}{2}\right]_{x^2}^{\sqrt{x}} dx$$

$$= 5\int_0^1 (x - x^4)\,dx$$

$$= 5\left[1/2 - 1/5\right]$$

$$= \underline{3/2}$$

Surface Integral.

Let $\bar{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$ be a differentiable vector fn defined over a surface S, then its surface integral is

$$\int_S \bar{F}\cdot d\bar{s} \quad (\text{or}) \quad \int_S \bar{F}\cdot\bar{N}\,ds$$

$\bar{N} \longrightarrow$ Unit outward dracon normal to the surface S.

In cartesian form,

$$\int_S \bar{F}\cdot\bar{N}\,ds = \int_S F_1\,dy\,dz + F_2\,dx\,dz + F_3\,dx\,dy.$$

Methods of Evaluation

① If R_1 is the projection of 'S' onto XY plane then

$$\int_S \bar{F}\cdot\bar{N}\,ds = \iint_{R_1} \bar{F}\cdot\bar{N}\,\frac{dx\,dy}{|\bar{N}\cdot\bar{k}|}$$

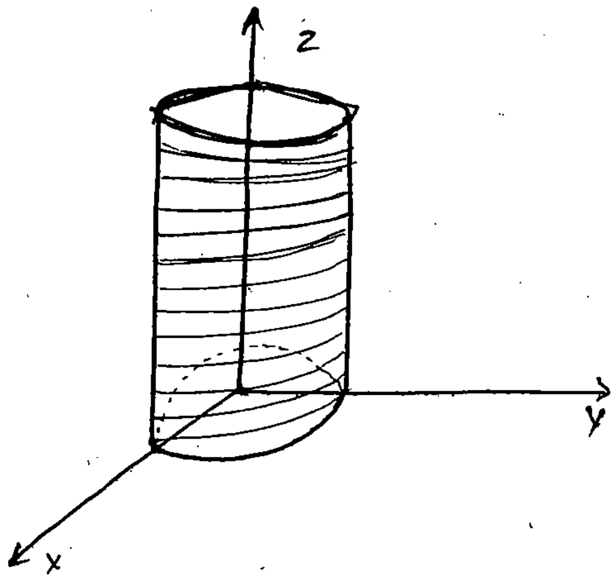
② $B_2 \longrightarrow$ YZ plane

$$\int_S \bar{F}\cdot\bar{N}\,ds = \iint_{B_2} \bar{F}\cdot\bar{N}\,\frac{dy\,dz}{|\bar{N}\cdot\bar{i}|}$$

③ $B_3 \longrightarrow$ XZ plane.

$$\int_S \bar{F}\cdot\bar{N}\,ds = \iint_{B_3} \bar{F}\cdot\bar{N}\,\frac{dx\,dz}{|\bar{N}\cdot\bar{j}|}$$

Q1. The value of $\int_S \bar{F}\cdot\bar{N}\,ds$, where $\bar{F} = z\hat{i} + x\hat{j} - 3y^2z\hat{k}$ and S = surface of the cylinder $x^2+y^2=16$ included in the first octant b/w $z=0$ and $z=5$ is ____



Ans Let $\phi = x^2+y^2$

$\nabla\phi = 2x\hat{i} + 2y\hat{j}$

$\Longrightarrow \bar{N} = \dfrac{\nabla\phi}{|\nabla\phi|}$

$\Longrightarrow \bar{N} = \dfrac{x\hat{i}+y\hat{j}}{\sqrt{x^2+y^2}} = \dfrac{x\hat{i}+y\hat{j}}{4}$

$\bar{F}\cdot\bar{N} = \dfrac{xz}{4} + \dfrac{xy}{4}$

$= \dfrac{x}{y}(z+y)$

Let B be the projection of S onto YZ plane.

$$\int_S \bar{F}\cdot\bar{N}\,ds = \iint_B \bar{F}\cdot\bar{N}\,\frac{dy\,dz}{|\bar{N}\cdot\bar{i}|} = \iint_B \cancel{\frac{x}{4}}(y+z)\,\frac{dy\,dz}{\cancel{x/4}}$$

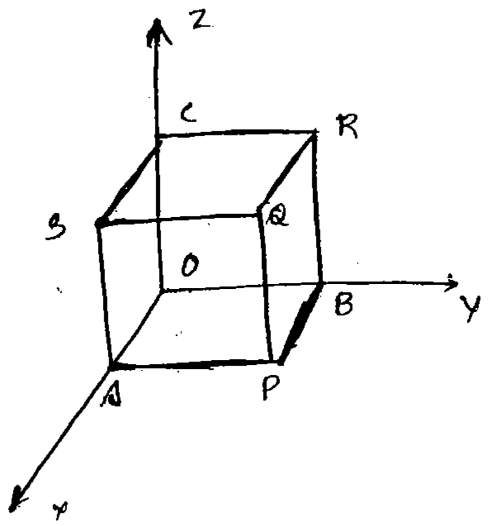
$$= \int_{z=0}^{5}\int_{y=0}^{4}(y+z)\,dy\,dz$$

$$= \int_{z=0}^{5}\left[\frac{y^2}{2} + zy\right]_0^4 dz$$

$$= \int_0^5 [8+4z]\,dz = 8z + \frac{\cancel{4}^{2}z^2}{\cancel{2}}\Big]_0^5 = 40 + 50$$

$$= \underline{\underline{90}}$$

Q. The value of $\int_S \bar{F}\cdot\bar{N}\,ds$ where $\bar{F} = 4xz\,\hat{i} - y^2\hat{j} + yz\,\hat{k}$ and S is a surface of the cube bounded by $x=0, x=1, y=0, y=1$ and $z=0$, $z=1$ is ——.



An $\int_S \bar{F}\cdot\bar{N}\,ds = \int_{S_1} + \int_{S_2} \cdots\cdots + \int_{S_6}$

i) Over S_1 : In XY plane (OAPB)

$z=0$, $\bar{N} = -\bar{k}$

$\bar{F}\cdot\bar{N} = -yz = 0$

$\int_{S_1} \bar{F}\cdot\bar{N}\,ds = 0$

ii) Over S_2 : ||al to XY plane (SQRC)

$z=1$, $\bar{N} = \bar{k}$, $\bar{F}\cdot\bar{N} = yz = y$

$\int_S \bar{F}\cdot\bar{N}\,ds = \int_S y\,ds = \iint_R y \frac{dx\,dy}{|\bar{N}\cdot\bar{k}|}$

$= \int_0^1\int_0^1 y\,dy\,dx = \underline{\underline{1/2}}$

iii) Over S_3 : In YZ plane (OBRC)

$x=0$, $\bar{N} = -\bar{i}$ $\bar{F}\cdot\bar{N} = -4xz = 0$.

$\therefore \int_{S_3} \bar{F}\cdot\bar{N}\,ds = 0$

iv) Over S_4 : ||al to y-z plane (APQS)

$x=1$, $\bar{N} = \bar{i}$, $\bar{F}\cdot\bar{N} = 4xy = 4z$

$\int_{S_4} \bar{F}\cdot\bar{N}\,ds = \int_0^1\int_0^1 4z\,dz\,dy = \underline{\underline{2}}$

v) Over S_5 : In XZ plane (OCSA)

$y=0$, $\bar{N} = -\bar{j}$, $\bar{F}\cdot\bar{N} = y^2 = 0$

$\int_{S_5} \bar{F}\cdot\bar{N}\,ds = 0$

vi) Over S_6 : ||al to XZ plane (BRQP)

$y=1$, $\bar{N} = \bar{j} = -y^2 = -1$

$$\int_{S_6} \bar{F}\cdot\bar{N}\,ds = \int_{S_6} -1\,ds = -1$$

$$\therefore \int_{S} \bar{F}\cdot\bar{N}\,ds = 0 + 1/2 + 0 + 2 + 0 - 1 = \underline{\underline{3/2}}$$

(OR)

$$\int_{S} \bar{F}\cdot\bar{N}\,ds = \int_{V} \operatorname{div}\bar{F}\,dV = \int_{V} (4z - 2y + y)\,dV$$

$$= \int_0^1\int_0^1\int_0^1 (4z - y)\,dz\,dy\,dx$$

$$= \underline{\underline{3/2}}$$

AG

Volume Integral :

Let $\phi(x,y,z)$ be a differentiable ~~vector~~ scalar fn and let $\bar{F}(x,y,z)$ be a differentiable vector fn defined over a region whose volume bounded is V.

Then the volume integrals are ~~$\oint$~~ $\int_V \phi(x,y,z)\,dV$ and

$$\int_V \bar{F}(x,y,z)\,dV.$$

$$= i\int_V F_1\,dV + j\int_V F_2\,dV + k\int_V F_3\,dV$$

Q1. Find $\int_V (2x+y)\,dV$ where V is the volume bounded by $x = 0, y = 0, y = 2, z = x^2, z = 4$.

Ans

$$\int_V (2x+y)\,dV = \int_{y=0}^{2}\int_{x=0}^{2}\int_{z=x^2}^{4} (2x+y)\,dz\,dx\,dy$$

$$= \int_0^2\int_0^2 (2x+y)(4-x^2)\,dx\,dy$$

$$= \int_0^2\int_0^2 [8x - 2x^3 + 4y - yx^2]\,dx\,dy$$

$$= \int_0^2 \left[8\cdot\frac{x^2}{2} - \frac{2x^4}{4} + 4yx - y\cdot\frac{x^3}{3} \right]_0^2 dy$$

$$= \int_0^2 \left[16 - 8 + 8y - \frac{8y}{3} \right] dy = \int_0^2 \left[8 + \frac{16y}{3} \right] dy = 8y + \frac{\overset{8}{\cancel{16}}y^2}{3\times\cancel{2}} \Big]_0^2$$

$$= 16 + \frac{32}{3} = \underline{\underline{\frac{80}{3}}}$$

Q2. The volume of an object expressed in spherical coordinates is given by $V = \int_0^{2\pi}\int_0^{\pi/3}\int_0^1 r^2 \sin\theta\, dr\, d\phi\, d\theta$ then the value of V is ______

a) $\pi/6$ b) $\pi/3$ c) $\pi/2$ d) None.

Ans
$$= \int_0^{2\pi}\int_0^{\pi/3} \left[\sin\phi \frac{r^3}{3} \right]_0^1 d\phi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi/3} \frac{1}{3} \sin\phi\, d\phi\, d\theta$$

$$= \frac{1}{3}\int_0^{2\pi} \left[-\cos\phi \right]_0^{\pi/3} d\theta = \frac{1}{3}\int_0^{2\pi} \left(-\frac{1}{2} + 1 \right) d\theta$$

$$= \frac{1}{3} \times \frac{1}{2} \times 2\pi = \underline{\underline{\pi/3}}$$

Gauss Divergence Theorem

[Surface Integral $\Longleftrightarrow$ Volume Integral]

Let S be a closed surface enclosing a volume 'V' and $\bar{F}(x,y,z)$ be a differentiable vector fn defined over the surface S.

Then $\int_S \bar{F}\cdot\bar{N}\, ds = \int_V \operatorname{div} \bar{F}\, dV.$

In Cartesian form,

$$\int_S F_1\, dy\, dz + F_2\, d\cancel{y}x\, dz + F_3\, dx\, dy$$

$$= \iint \left[\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \right] dv$$

Q1. The value of $\int_S \bar{r}\cdot\bar{N}\,ds$ where $\bar{r} = x\hat{i}+y\hat{j}+z\hat{k}$ and S is the closed surface enclosing a volume V is. ____

a) V b) 2V c) 3V d) 4V.

Ans

$$\text{div } \bar{r} = 1+1+1 = 3$$

$$\int_S \bar{r}\cdot\bar{N}\,ds = \int_V \text{div } \bar{r}\,dV = \int_V 3\,dV = \underline{\underline{3V}}$$

Q2. The value of $\int_S x\,dy\,dz + y\,dx\,dz + z\,dx\,dy$ where S is the surface of

i) cylinder $x^2+y^2 = 9$, $y=0$, $y=4$.

ii) Sphere $x^2+y^2+z^2 = 16$.

Ans

$$\int_V (1+1+1)\,dV = 3V$$

i) $3V = 3\times \pi r^2 h$

$$= 3\times\pi\times 3^2\times 4 = \underline{\underline{108\pi}}$$

ii) $3V = \cancel{3}\times\frac{4}{\cancel{3}}\times\pi r^3$

$$= 4\pi\times(4)^3 = \underline{\underline{252\pi}}$$

Q3. The value of $\int_S (x^2+2y^2+3z^2)\,ds$ where S is the surface of a unit sphere with centre at the origin.

Ans $x^2+y^2+z^2 = 1$

$$\bar{N} = \frac{\nabla\phi}{|\nabla\phi|} = x\hat{i}+y\hat{j}+z\hat{k}$$

Let $\bar{F} = F_1\hat{i}+F_2\hat{j}+F_3\hat{k}$

$$\therefore \bar{F}\cdot\bar{N} = F_1x+F_2y+F_3z = x^2+2y^2+3z^2$$

$\Longrightarrow F_1 = x$, $F_2 = 2y$, $F_3 = 3z$.

$$\therefore \bar{F} = x\hat{i}+2y\hat{j}+3z\hat{k}$$

$$\text{div } \bar{F} = 1+2+3 = 6$$

$$\therefore \int_S (x^2 + 2y^2 + 3z^2)\, ds = \int_V \operatorname{div} \bar{F}\, dV = \int_V 6\, dV$$

$$= 6V = 6 \times \frac{4}{3} \pi r^3 = 8\pi (1)^3 = \underline{\underline{8\pi}}$$

Q4. Evaluate $\int_S \text{Curl}\, \bar{F} . \bar{N} . ds$ where $\bar{F} = 4xz\, \hat{i} - 2y^2x\, \hat{j} + 8xz^2\, \hat{k}$ and S is the surface of a cylinder bounded by $y^2 + z^2 = 36$, $x = 0$ and $x = 1$.

Ans $$\int_S \text{Curl}\, \bar{F} \cdot \bar{N}\, ds = \int_V \operatorname{div} (\text{curl}\, \bar{F})\, dV = \int_V 0\, dv = \underline{0}$$

Q5. ~~Solution~~ Value of $\int_S \bar{F} \cdot \bar{N}\, ds$ where $\bar{F} = 8x^2z\, \hat{i} - y^2\, \hat{j} + 2xz^2\, \hat{k}$ and S is the surface bounded by $0 \le x \le 1$, $0 \le y \le 2$, $0 \le z \le 3$. is ______

Ans $$\operatorname{div} \bar{F} = 16xz - 2y + 4xz.$$

$$= 20xz - 2y$$

$$\int_S \bar{F} \cdot \bar{N}\, dS = \int_V \operatorname{div} \bar{F}\, dV = \int_V (20xz - 2y)\, dV$$

$$= \int_{x=0}^{1} \int_{y=0}^{2} \int_{z=0}^{3} (20xz - 2y)\, dz\, dy\, dx = \int_0^1 \int_0^2 \left[20x \cdot \frac{9}{2} - 2y \times 3\right] dy\, dx$$

$$= \int_0^1 [90x \times 2 - 6 \cdot 2]\, dx = \int_0^1 (180x - 12)\, dx$$

$$= 90 - 12 = \underline{\underline{78}}$$

Q6. The value of $\int_S \bar{F} \cdot \bar{N}\, ds$ where $\bar{F} = 4x\, \hat{i} - 2y^2\, \hat{j} + z^2\, \hat{k}$ where S is the surface bounded by $x^2 + y^2 = 4$, $z = 0$ and $z = 3$ is ______.

a) 64π b) 84π c) 104π d) None.

Ans $$\operatorname{div} \bar{F} = 4 - 4y + 2z$$

$$\int_S \bar{F} \cdot \bar{N}\, ds = \int_V \operatorname{div} \bar{F}\, dV = \int_V (4 - 4y + 2z)\, dV$$

$$= \int_{z=0}^{3}\int_{x=-2}^{2}\int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (4-4y+2z)\,dy\,dx\,dz$$

Let $x = r\cos\theta$, $y = r\sin\theta$, $z = z$.

$|J| = r$, $r = 0$ to 2

$\theta = 0$ to 2π

$z = 0$ to 3

$$= \int_0^3\int_0^{2\pi}\int_0^2 (4-4r\sin\theta+2z)\,r\,dr\,d\theta\,dz$$

$$= \int_0^3\int_0^{2\pi}\left[4\times\frac{r^2}{2} - 4\sin\theta\cdot\frac{r^3}{3} + 2z\cdot\frac{r^2}{2}\right]_0^2 d\theta\,dz$$

$$= \int_0^3\int_0^{2\pi} 8 - \frac{32}{3}\sin\theta + 4z\,\Big]\,d\theta\,dz$$

$$= \int_0^3 (8+4z)\theta - \frac{32}{3}(-\cos\theta)\Big]_0^{\pi} dz$$

$$= \int_0^3 (8+4z)\,2\pi\,dz = 2\pi\left[8z+2z^2\right]_0^3$$

$$= 2\pi\left[24+18\right] = \underline{84\pi}$$

STOKE'S THEOREM

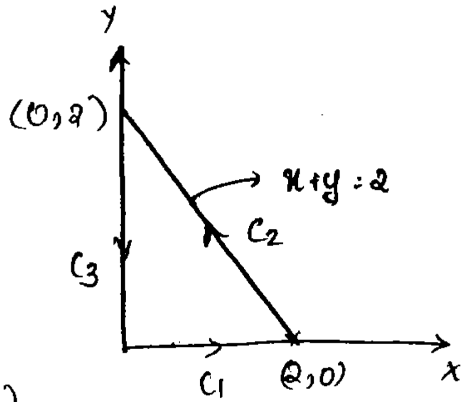
[Line Integral $\Longrightarrow$ Surface Integral]

Let S be an open surface bounded by a closed curve 'C' and $\bar{F}(x,y,z)$ be a differentiable vector function defined along the curve 'C' then $\oint_C \bar{F}\cdot d\bar{r} = \int_S \text{Curl}\,\bar{F}\cdot\bar{N}\,ds$

ie $\oint_C F_1dx + F_2dy + F_3dz = \int_S (\nabla\times\bar{F})\cdot\bar{N}\,ds$

$$\text{Curl}\,\bar{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_1 & F_2 & 0 \end{vmatrix} = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}$$

Q1. The value of $\int_{C} \bar{F}\cdot d\bar{r}$ where $\bar{F} = yz\hat{i} + xz\hat{j} + xy\hat{k}$ and C is the curve bounded by $x+y=2$, $x=0, y=0$ in xy plane is ______



Ans curl $\bar{F}$ = $\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ yz & xz & xy \end{vmatrix}$

$$= \hat{i}(x-x) - \hat{j}(y-y) + \hat{k}(z-z)$$

$\Longrightarrow$ curl $\bar{F} = 0 \Longrightarrow \bar{F}$ is irrotational.

$$\oint_{C} \bar{F}\cdot d\bar{s} = \int_{S} \text{curl}\,\bar{F}\cdot\bar{N}\,ds = \int_{S} \bar{0}\cdot\bar{N}\,ds = \underline{\underline{0}}$$

Q2 The value of $\oint_{C} \bar{F}\cdot d\bar{r}$, where $\bar{F} = -y^3\hat{i} + x^3\hat{j}$ and C is the boundary of circular disc $x^2+y^2 \leq 1$, $z=0$.

Ans curl $\bar{F}$ = $\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -y^3 & x^3 & 0 \end{vmatrix}$ A Cc

$$= \hat{i}(0-0) - \hat{j}(0-0) + \hat{k}(3x^2+3y^2)$$

$\therefore$ curl $\bar{F} = 3(x^2+y^2)\bar{k}$

$\bar{N} = \bar{k}$

$\Longrightarrow$ curl $\bar{F}\cdot\bar{N} = 3(x^2+y^2)$

$$\oint_{C} \bar{F}\cdot d\bar{s} = \int_{S} \text{curl}\,\bar{F}\cdot\bar{N}\,ds = \int_{S} 3(x^2+y^2)\,ds$$

$$= \iint_{R} 3(x^2+y^2)\,dx\,dy$$

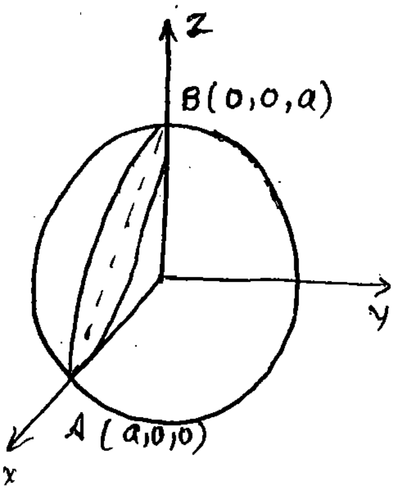
$x = r\cos\theta$, $y = r\sin\theta$

$|J| = r$, $x^2+y^2 = r^2$

$$= \int_{\theta=0}^{2\pi}\int_{r=0}^{1} 3r^2\cdot r\,dr\,d\theta = 3\times\frac{1}{4}\times 2\pi = \underline{\underline{\frac{3\pi}{2}}}$$

Q3. Evaluate $\oint y\,dx + z\,dy + x\,dz$ where C is the curve of intersection of $x^2+y^2+z^2=a^2$ and $x+z=0$

Ans The intersection of the sphere $x^2+y^2+z^2=a^2$ with the plane $x+z=a$ is a circle in the plane $x+z=a$ with 'AB' as diameter.



$$AB = \sqrt{2a^2} = \sqrt{2}a$$

$$\Longrightarrow \quad r = \frac{a}{\sqrt{2}}$$

$$\text{curl } \bar{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y & z & x \end{vmatrix} = i(0-1) - \hat{j}(1-0) + k(0-1)$$

$$\therefore \quad \text{Curl } \bar{F} = -(i+j+k)$$

$$\text{Let } \phi = x+z =$$

$$\nabla\phi = i+k$$

$$\bar{N} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{i+k}{\sqrt{2}}$$

$$\text{Curl } \bar{F}\cdot\bar{N} = \frac{-1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = -\sqrt{2}$$

$$\oint_C y\,dx + z\,dy + x\,dx = \int_S \text{Curl } \bar{F}\cdot\bar{N}\,ds = \int_S -\sqrt{2}\,ds = \underline{\underline{-\sqrt{2}S}}$$

$$= -\sqrt{2}\pi r^2 = -\sqrt{2}\pi\left(\frac{a}{\sqrt{2}}\right)^2 = \underline{\underline{\frac{-\pi a^2}{\sqrt{2}}}}$$

$$V = \nabla\times\bar{A}$$

$$\oint_C \bar{A}\cdot d\bar{r} = \int_{S_c} \bar{V}\cdot d\bar{s}$$

FOURIER SERIES

Periodic Function:

$f(x) = \{f(x+T) = f(x+2T) = \dots$

Period $= T$.

Trignometric Series:

A functional series of the form $a_0 + a_1\cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x + \dots \infty$ is called a Trignometric series:

Fourier Series

Let $f(x)$ be a periodic function defined in $[c, c+2l]$ with period $2l$, then the Fourier Series of $f(x)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{l}\right) + b_n \sin\left(\frac{n\pi x}{l}\right)\right]$$

where a_0, a_n and b_n are the Fourier coeff. given by

$$a_0 = \frac{1}{l}\int_{c}^{c+2l} f(x)\,dx$$

$$a_n = \frac{1}{l}\int_{c}^{c+2l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$b_n = \frac{1}{l}\int_{c}^{c+2l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

NOTE:

$$\begin{array}{cccc} [-l, l], & [0, 2l], & [-\pi, \pi], & [0, 2\pi] \\ \downarrow & \downarrow & \downarrow & \downarrow \\ c = -l & c = 0 & c = -\pi & c = 0 \\ & & l = \pi & l = \pi \end{array}$$

Dirichlet's Conditions:

A function is said to satisfy Dirichlet's conditions if

i) $f(x)$ and its integrals are finite and single valued.

2) $f(x)$ has finite number of finite discontinuities.

3) $f(x)$ has finite number of maxima and minima.

Convergence .

- If $f(x)$ is continuous at $x = c \in (a,b)$ then the Fourier series of $f(x)$ at $x = c$ converges to $f(c)$.
- If $f(x)$ is discontinuous at $x = c \in (a,b)$ then the Fourier Series of $f(x)$ at $x = c$ converges to $\frac{1}{2}\left[\lim_{x\to c^-} f(x) + \lim_{x\to c^+} f(x)\right]$
- The Fourier Series of $f(x)$ at the end points i.e at $x = a$ or b converges to $\frac{1}{2}\left[\lim_{x\to a^+} f(x) + \lim_{x\to b^-} f(x)\right]$

Fourier Series Of Even & Odd Functions in $[-l, l]$ or $[-\pi, \pi]$:-

① Fourier Series of an odd function $f(x)$ in $[-l, l]$ is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

where $$b_n = \frac{2}{l}\int_0^l f(x) \sin\left(\frac{n\pi x}{l}\right) dx$$

② Fourier Series of an Even function $f(x)$ in $[-l, l]$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right)$$

where $$a_0 = \frac{2}{l}\int_0^l f(x) dx$$

$$a_n = \frac{2}{l}\int_0^1 f(x) \cos\left(\frac{n\pi x}{l}\right) dx$$

Half Range Series : $[0, l]$, $[0, \pi]$

a) Half Range Cosine Series of $f(x)$ in $[0, l]$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right)$$

where $a_0 = \frac{2}{l}\int^l f(x) dx$, $a_n = \frac{2}{l}\int^l f(x) \cos\left(\frac{n\pi x}{l}\right) dx$

③ Half Range Sine Series of $f(x)$ in $[0, l]$ is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

where $b_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx$

Q① The coeff of $\sin x$ in the Fourier series expansion of $f(x) = x^2$ in the interval $(-\pi, \pi)$ is ——

a) $\sum \frac{(-1)^n}{n^2}$ b) $\sum \frac{1}{n^2}$ c) $\frac{-1}{n}$ d) 0

Ans $f(x) = x^2 \longrightarrow$ odd fn.

$\therefore$ Coeff of $\sin x = 0$

Q2. If $f(x) = \begin{cases} 0, & -2 < x < 0 \\ 1, & 0 < x < 2 \end{cases}$ then the term independent of x in the Fourier series of $f(x)$ is ——

a) 0   b) 1   c) 1/2   d) 2.

Ans $a_0 = \frac{1}{l}\int_{-2}^{2} f(x)dx = \frac{1}{2}\int_0^2 1\,dx = 1$

Independent Term $= \frac{a_0}{2} = \underline{\underline{\frac{1}{2}}}$

Q3. The Fourier series of $f(x) = \begin{cases} -x+1, & -\pi \le x \le 0 \\ x+1, & 0 \le x \le \pi \end{cases}$ has the following terms in its expansion.

[✓ a) cosine terms only]   b) sine terms only   c) both cosine & sine terms

d) Cannot be applied.

Ans $f(-x) = \begin{cases} x+1, & -\pi \le -x \le 0 \\ -x+1, & 0 \le -x \le \pi \end{cases}$

$= \begin{cases} x+1, & \pi \ge x \ge 0 \\ -x+1, & 0 \ge x \ge -\pi. \end{cases} = f(x) \longrightarrow$ Even fn

Q4. If $f(x)=\begin{cases}0 & \text{for } -2<x<0\\ 1 & \text{for } 0<x<2\end{cases}$. Then the coeff. of $\cos\left(\frac{n\pi x}{2}\right)$ in the Fourier Series expansion of $f(x)$ is ______?

~~Ans~~ a) 0 b) $1/n$ c) $-1/n$ d) $1/n^2$.

Ans $(-2,2) \longrightarrow l=2$

$$f(x) = \frac{a_0}{2} + \left\{ a_n \cos\left(\frac{n\pi x}{2}\right) + b_n \sin\left(\frac{n\pi x}{2}\right)\right\}$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$a_n = \frac{1}{2}\int_0^2 1.\cos\frac{n\pi x}{2}dx = \frac{1}{2}\left[\sin\frac{n\pi x}{2}\right]_0^2$$

$$= \underline{\underline{0}}$$

Q5. If $f(x)=|x|$ in $[-\pi,\pi]$ is expanded as a Fourier series, then which of the following is true.?

a) $a_0=\frac{1}{\pi}$ b) $a_n=(-1)^{n+1}\times 2$ [✓ c) $b_n=0$] d) All of them.

Ans $$a_0 = \frac{2}{\pi}\int_0^\pi f(x)dx = \frac{2}{\pi}\int_0^\pi x\,dx = \frac{2}{\pi}\left[\frac{x^2}{2}\right]_0^\pi$$

$$= \frac{2}{\pi}\times\frac{\pi^2}{2} = \underline{\pi}$$

$a_0 = 1/\pi$ $\therefore$ $\underline{\underline{Ⓒ}}$

Q6. The F.S of a periodic fn of period 2π, where $f(x)=\begin{cases}0 & ,-\pi<x<0\\ x & ,0<x<\pi\end{cases}$ is given by $\frac{\pi}{4}+\sum\frac{1}{\pi n^2}\left[(\cos n\pi-1)\cos nx\right]-\frac{1}{n}\cos n\pi\sin nx]$ then the value of $1/1^2+1/3^2+1/5^2+\ldots\infty=$ ____

ⓐ $\frac{\pi^2}{2}$ ⓑ $\frac{\pi^2}{4}$ ⓒ π^2 [✓ ⓓ $\frac{\pi^2}{8}$]

Ans At $x=\pi$

$$\frac{\pi}{4}+\sum\left[\frac{1}{\pi n^2}(\cos n\pi-1)\cos n\pi-\frac{1}{n}(0)\right]$$

$$= \frac{\pi}{4}\frac{1}{2}\left[\lim_{x\to-\pi^+}f(x)+\lim_{x\to\pi^-}f(x)\right] = \frac{1}{2}[0+\pi] = \pi/2$$

$$\Longrightarrow \frac{\pi}{4} + \frac{1}{\pi}\left[\frac{2}{1^2} + 0 + \frac{2}{3^2} + 0 + \frac{2}{5^2} + 0 + \cdots\cdots \infty\right] = \pi/2$$

$$\Longrightarrow \frac{2}{\pi}\left[\frac{1}{1^2} + \frac{1}{3^2} + \cdots\cdots \infty\right] = \frac{\pi}{2} - \frac{\pi}{4} = \underline{\underline{\pi/4}}$$

$$\Longrightarrow \left[1/1^2 + 1/3^2 + 1/5^2 + \cdots \infty\right] = \underline{\underline{\pi^2/8}}$$

Q7. The F.S of $f(x) = x^2$ in $(-\pi, \pi)$ is given by $\frac{\pi^2}{3} - 4\left[\frac{\cos x}{1^2} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} + \cdots\cdots\right]$ then the value of $1/1^2 + 1/2^2 + 1/3^2 + \cdots\cdots \infty =$ ____.

a) π^2 b) $\pi^2/3$ c) $\pi^2/6$ d) $\pi^2/12$

Ans At $x = \pi$ (or) $-\pi$

$$\frac{\pi^2}{3} - 4\left[\frac{-1}{1^2} - \frac{1}{2^2} + \frac{-1}{3^2} + \cdots\cdots \infty\right] = \frac{1}{2}\left[\lim_{x\to-\pi^+} f(x) + \lim_{x\to\pi^-} f(x)\right]$$

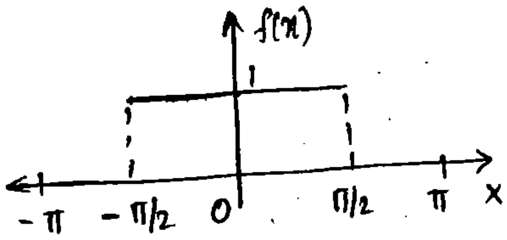
$$\Longrightarrow \frac{\pi^2}{3} + 4\left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\cdots \infty\right]$$

$$= \frac{1}{2}\left[(-\pi)^2 + \pi^2\right] = \pi^2$$

$$\Longrightarrow \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\cdots \infty = \frac{1}{4}\left[\pi^2 - \frac{\pi^2}{3}\right] = \frac{1}{4}\,\frac{2\pi^2}{3}$$

$$= \underline{\underline{\pi^2/6}}$$

Q8. A fn with period 2π is shown below.



The F.S of $f(x)$ is ____

ⓐ $\frac{1}{2} + \Sigma \frac{4}{n^2\pi^2}\left(\sin\frac{n\pi}{2}\right) \sin nx$ ⓑ $\frac{1}{2} + \Sigma \frac{2}{n\pi} \sin\frac{n\pi}{2} \cos nx$

ⓒ $\Sigma \frac{2}{n\pi} \sin\frac{n\pi}{2} \cos nx$ ⓓ $\Sigma \frac{4}{n^2\pi^2} \sin\frac{n\pi}{2} \sin nx$

$$f(x) = \begin{cases} 0 & , -\pi < x < -\pi/2 \\ 1 & , -\pi/2 < x < \pi/2 \\ 0 & , \pi/2 < x < \pi \end{cases}$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)dx = \frac{2}{\pi}\int_0^{\pi/2} 1\,dx = \frac{2}{\pi}\times\frac{\pi}{2} = \underline{\underline{1}}$$

Q9. The F.S of a symmetric even fn $f(x) = \begin{cases} 1+\frac{2x}{\pi}, & -\pi \le x < 0 \\ 1-\frac{2x}{\pi}, & 0 \le x \le \pi \end{cases}$

[✓ ⓐ] $\Sigma \frac{4}{\pi^2 n^2}(1-\cos n\pi)\cos nx$.

ⓑ $\Sigma \frac{4}{\pi^2 n^2}(1-\sin n\pi)\sin nx$

ⓒ $\Sigma \frac{4}{\pi^2 n^2}(\cos n\pi - 1)\cos nx$

ⓓ $\Sigma \frac{4}{\pi^2 n^2}(\sin n\pi - 1)\sin nx$

Ans $$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi}\left(1-\frac{2x}{\pi}\right)\cos nx\,dx$$

$$= \frac{2}{\pi}\left[\left(1-\frac{2x}{\pi}\right)\left[\frac{\sin nx}{n}\right] - \left(\frac{-2}{\pi}\right)\left[-\frac{\cos nx}{n^2}\right]\right]_0^{\pi}$$

$$= \frac{-4}{n^2\pi^2}\left[\cos n\pi - 1\right] = \underline{\underline{\frac{4}{n^2\pi^2}(1-\cos n\pi)}}$$

Q10. The constant term in the cosine series of $f(x) = x^2+2x$ in $(0,\pi)$ is ___?

Ans $$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)dx$$

$$= \frac{2}{\pi}\int_0^{\pi}(x^2+2x)dx$$

$$= \frac{2}{\pi}\left[\frac{x^3}{3}+x^2\right]_0^{\pi} = \frac{2}{\pi}\left[\frac{\pi^3}{3}+\pi^2\right] = 2\pi\left[\pi/3+1\right]$$

Const term $= \frac{a_0}{2} = \underline{\underline{\pi(\pi/3+1)}}$

Q11. $f(x) = x$ in $(0,2)$ is expressed as half range cosine series. Then coeff of $\cos \pi x$ is ___

ⓐ 0 ⓑ $1/\pi^2$ ⓒ $-1/\pi^2$ ⓓ $1/\pi$

Ans $$f(x) = \frac{a_0}{2} + \Sigma \underset{\underset{a_n}{\downarrow}}{a_n} \cos\left(\frac{n\pi x}{2}\right)$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$a_2 = \frac{2}{2}\int_0^2 x\cos(\pi x)\,dx = x\left[\frac{\sin\pi x}{\pi}\right] - (1)\left[-\frac{\cos\pi x}{\pi^2}\right]\Bigg]_0^2$$

$$= \frac{1}{\pi^2}(1-1) = \underline{\underline{0}}$$

Q12. If a constant c is expressed as a sine series in $(0,\pi)$ then the coeff of $\sin 5x$ is ______ a) $\frac{4c}{5\pi}$ b) $\frac{2c}{5\pi}$ c) $\frac{c}{5\pi}$ d) 0

Ans $f(x) = c$, $(0,\pi)$

$$f(x) = \sum \underset{\substack{\downarrow \\ b_5}}{b_n} \sin nx$$

$$b_n = \frac{2}{\pi}\int_0^\pi f(x)\sin nx\,dx = \frac{2}{\pi}\int_0^\pi c\sin 5x\,dx$$

$$= \frac{2c}{\pi}\left[-\frac{\cos 5x}{5}\right]_0^\pi$$

$$= \frac{-2c}{5\pi}[-1-1] = \underline{\underline{\frac{4c}{5\pi}}}$$

Solve 100

Q1 $\frac{d^3y}{dx^3} - \frac{3dy}{dx} + 2y = 0$

Auxillary Eqn is $(D^3 - 3D + 2)y = 0$

$(D^3 - D - 2D + 2)y = 0$

$D(D^2-1) - 2(D-1)\ y = 0$

ie $(D-1)[D(D+1) - 2]y = 0$

$(D-1)(D^2+D-2)y = 0$

$(D-1)(D+2)(D-1)y = 0$

$1 \quad -2 \quad 1$

$\therefore y = (c_1 + c_2 x)e^{x} + c_3 e^{-2x}$

Q2. $(D^4 + 8D^2 + 16)y = 0$

$D^4 + 4D^2 + 4D^2 + 16 = 0$

$D^2(D^2+4) + 4(D^2+4) = 0$

$(D^2+4)(D^2+4) = 0$

$(D^2+4)^2 = 0$

$D^2 = -4, -4$

$D = \pm 2i, \pm 2i$

$\therefore y = (c_1 + c_2 x)\cos 2x + (c_3 + c_4 x)\sin 2x$

Q3. $(D^4 - 2D^3 - 3D^2 + 4D + 4)y = 0$

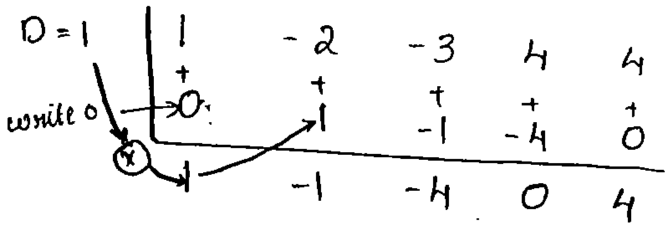
$(D+1)(D^3 - 3D^2 + 4)y = 0$

$(D+1)(D+1)(D^2 - 4D + 4) = 0$

$(D+1)^2(D-2)^2 = 0$

$D = -1, -1, 2, 2$

$\therefore y = (c_1 + c_2 x)e^{-x}$
$+ (c_3 + c_4 x)e^{2x}$



D = -1	1	-2	-3	4	4
	0	-1	3	0	-4
D = -1	+1	-3	-0	4	0
	0	-1	4	-4	
	1	-4	4	0	

$\therefore$ (D+1) is a factor

Q4. The sol^n of $y'' - 2y' + 10y = 0$ satisfying $y(0) = 4$ & $y'(0) = 1$

Soln

$$(D^2 - 2D + 10)y = 0$$

$$D^2 - 2D + 10 = 0$$

$$D = \frac{+2 \pm \sqrt{4-40}}{2} = \frac{2 \pm 6i}{2} = 1 \pm 3i$$

$$y = e^x (c_1 \cos 3x + c_2 \sin 3x)$$

$y(0) = 4$

$\therefore \underline{\underline{4 = c_1}}$

$y'(0) = 1$

$$\frac{dy}{dx} = e^x(-4\,3\sin 3x + 3C_2 \cos 3x) + (4_1 \cos 3x + C_2 \sin 3x)e^x$$

$y'(0) = 1$

$1 = 3C_2 + 4$

$\therefore \underline{\underline{C_2 = 1}}$

$\therefore \underline{\underline{y = e^x(4\cos 3x - \sin 3x)}}$

Q5. The solution of $\frac{d^2y}{dx^2} + \frac{2dy}{dx} + 17y = 0$ satisfying the conditions $y(0) = 1$, $\frac{dy}{dx}(\pi/4) = 0$

Ans

$$(D^2 + 2D + 17)y = 0$$

$$D^2 + 2D + 17 = 0$$

ie $D = \frac{-2 \pm \sqrt{4-68}}{2} = \frac{-2 \pm 8i}{2} = \underline{\underline{-1 \pm 4i}}$

$$y = e^{-x}(C_1 \cos 4x + C_2 \sin 4x)$$

$y(0) = 1$

$C_1 = 1$

$$\frac{dy}{dx} = -4\bar{e}^x(-4C_1 \sin 4x + 4C_2 \cos 4x) + -e^{-x}(C_1 \cos 4x + C_2 \sin 4x)$$

$$= e^x(-4\sin 4x + 4C_2 \cos 4x) + -e^{-x}(\cos 4x + C_2 \sin 4x)$$

$$\frac{dy}{dx}(\pi/4) = 0$$

$$0 = e^{\pi/4}\left(\cancel{-}4 \times \overset{0}{\cancel{1/\sqrt{2}}} + \cancel{4}C_2 \times \overset{1}{\cancel{1/\sqrt{2}}}\right) - \cancel{e}^{-\pi/4}\cancel{x}$$

$$= \underline{\underline{1/4}}$$

$$\underline{\underline{y = e^{-x}\left(\cos 4x + \frac{1}{4}\sin 4x\right)}}$$

Q6. $$\frac{d^4y}{dx^4} + 4y = 0$$

$$(D^4 + 4)y = 0$$

$$[(D^2)^2 + (2)^2]y = [(D^2+2)^2 - 2D^2 \times 2]y = 0$$

$$(D^2+2)^2 - (2D)^2 = (D^2+2+2D)(D^2+2-2D)$$

$$D = \frac{2 \pm \sqrt{4-8}}{2} \qquad D = \frac{-2 \pm \sqrt{4-8}}{2}$$

$$D = 1 \pm i \qquad D = -1 \pm i$$

$$\underline{\underline{y = e^{-x}(c_1\cos x + c_2\sin x) + e^{x}(c_3\cos x + c_4\sin x)}}$$

Q7. $$\frac{d^3y}{dx^3} + \frac{d^2y}{dx^2} + 100\frac{dy}{dx} + 100y = 0$$

$$(D^3 + D^2 + 100D + 100)y = 0$$

$$D^2(D+1) + 100(D+1) = 0$$

$$(D^2+100)(D+1) = 0$$

$$D = -1, \quad D^2 = -100$$

$$D = \pm 10i$$

$$\underline{\underline{y = c_1e^{-x} + c_2\cos 10x + c_3\sin 10x}}$$

Q8. (i) The solution of $\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = 0$ is given as

$y = c_1 e^{-x} + c_2 e^{-3x}$, then $(a, b) = (\_, \_)$?

ⓐ $(-4, 3)$ ⓑ $(4, -3)$ ⓒ $(-4, -3)$ ⓓ $(4, 3)$

(ii) which of the following is a solution of

$\frac{d^2y}{dx^2} + a\frac{dy}{dx} + (b+1)y = 0$

ⓐ xe^{-4x} ⓑ xe^{-3x} ⓒ xe^{-2x} ⓓ x^2e^{-2x}

Ans $(D^2 + aD + b)y = 0$

$$D = \frac{-a \pm \sqrt{a^2 - 4b}}{2}$$

$y = c_1e^{-x} + c_2e^{-3x}$

$(D+1)(D+3) = 0$

$D^2 + 4D + 3 = 0$

Given $D^2 + aD + b = 0$

$\therefore \underline{\underline{a = 4, b = 3}}$

(ii) $(D^2 + Da + (b+1))y = 0$

$a = 4, b = 3$

$(D^2 + 4D + 4)y = 0$

$(D+2)^2 = 0$

$D = -2, -2.$

$y_1 = \underline{\underline{(c_1 + c_2x)}}e^{-2x}$

Solution of $f(D)y = X$ form ——①

From ① we can observe that $y = \frac{X}{f(D)}$ is also a solⁿ to it which is called particular integral or particular solution of ①. and is denoted by y_p.

1) $\frac{X}{(D-a)} = e^{ax}\int X e^{-ax} dx$.

2) $\frac{X}{(D+a)} = e^{-ax}\int X e^{ax} dx$

3) $\frac{X}{D} = \int X dx$.

Note :- Now by selecting $X = e^{ax+b}$ or $\sin(ax+b)$ or $\cos(ax+b)$ or x^m - - - - on the RHS of ①, we can analyze various PI in various cases.

Case I :

Consider the eqn $f(D)y = e^{ax+b}$, then

$$y_p = \frac{e^{ax+b}}{f(D)} = \frac{e^{ax+b}}{f(a)}$$

$$D(e^{ax+b}) = a(e^{ax+b})$$

$$D^2(e^{ax+b}) = a^2(e^{ax+b})$$

$$\vdots$$

$$D^n(e^{ax+b}) = a^n(e^{ax+b})$$

$$f(D)(e^{ax+b}) = f(a)\, e^{(ax+b)}$$

ie $$\frac{e^{(ax+b)}}{f(a)} = \frac{e^{(ax+b)}}{f(D)}$$

Note :-

(i) $$\frac{e^{2-3x}}{(D+1)(D+2)} = \frac{e^{2-3x}}{(-3+1)(-3+2)}$$

$$= \frac{e^{2-3x}}{2}$$

(ii) $$\frac{e^{ax+b}}{(D-a)} = e^{ax} \int e^{ax+b} \times e^{-ax}\, dx$$

$$= x e^{ax+b}$$

(iii) $$\frac{e^{ax+b}}{(D-a)^2} = \frac{1}{(D-a)} x \cdot e^{ax+b}$$

$$= e^{ax} \int x e^{ax+b} \times e^{-ax} dx$$

$$= \frac{x^2}{2} e^{ax+b}$$

(iv) $$\frac{e^{ax+b}}{(D-a)^3} = \frac{1}{(D-a)} \frac{x^2}{2} e^{ax+b}$$

$$= e^{ax} \int \frac{x^2}{2} e^{ax+b} \times e^{-ax} dx$$

$$= \frac{x^3}{3!} e^{ax+b}$$

(v) $$\frac{e^{ax+b}}{(D-a)^K} = \frac{x^k}{K!} e^{ax+b}$$

(vi) $$\frac{e^{ax+b}}{(D-a)^m (D-n)(D+p)^r} = \frac{x^m}{m!} \frac{e^{ax+b}}{(a-n)(a+p)^r}$$

Case II

$f(D)y = \sin(ax+b)$ or $\cos(ax+b)$

$$y_p = \frac{\sin(ax+b)}{f(D)} = ?$$

$$D[\sin(ax+b)] = a\cos(ax+b)$$

$$D^2[\sin(ax+b)] = -a^2\sin(ax+b)$$

$$\underline{D^2 = -a^2}$$

Let $f(D) = \phi(D^2)$ then

$$y_p = \frac{\sin(ax+b)}{\phi(D^2)}$$

$$= \frac{\sin(ax+b)}{\phi(-a^2)} \quad [\phi(-a^2) \neq 0]$$

Similarly $\dfrac{\cos(ax+b)}{\phi(D^2)} = \dfrac{\cos(ax+b)}{\phi(-a^2)} \quad [\phi(-a^2) \neq 0]$

Note 1:

(i) $\dfrac{\cos(3x+4)}{(D^2+4)} = \dfrac{\cos(3x+4)}{\underset{-5}{(-9+4)}} = \dfrac{\cos(3x+4)}{-5}$

(ii) $\dfrac{\sin(2-x)}{D^3+1} = \dfrac{\sin(2-x)}{D \times D^2 + 1}$

$$= \frac{\sin(2-x)}{-D+1}$$

$$= \frac{(1+D)\sin(2-x)}{(1-D^2)}$$

$$= \frac{(1+D)\sin(2-x)}{1-(-1)}$$

$$= \frac{1}{2}[\sin(2-x) - \cos(2-x)]$$

(iii) $\dfrac{\cos(ax+b)}{D^2+a^2} = \dfrac{x}{2a}\sin(ax+b)$

(iv) $\dfrac{\sin(ax+b)}{D^2+a^2} = \dfrac{-x}{2a}\cos(ax+b)$

$$\frac{e^{i(ax+b)}}{D^2+a^2} = \frac{a\, e^{iax+ib}}{(D+ia)(D-ia)}$$

ie $$\frac{\cos(ax+b) + i\sin(ax+b)}{(D+ia)(D-ia)} = \frac{e^{iax+ib}}{(D+ia)(D-ia)}$$

$$\frac{\cos(ax+b)}{D^2+a^2} + \frac{i\sin(ax+b)}{D^2+a^2} = \frac{ix}{-2a}\left[\cos(ax+b) + i\sin(ax+b)\right]$$

Comparing we get the results

(v) $$\frac{\sin(3x+4)}{D^2+9} = \frac{-x}{2.3}\cos(3x+4)$$

$$= \frac{-x}{6}\cos(3x+4)$$

(vi) $$\frac{\cos(2x-3)}{D^2+4} = \frac{x}{2}\,\frac{\sin(2x-3)}{2}$$

$$= \frac{x}{4}\sin(2x-3)$$

<u>Remarks</u> : By solving the given eqn

$$f(D)\,y = X \quad \text{—— ①}$$

we can observe that the soln of $f(D)y = 0$ —— ② as a part of solution of ① which is called complementary function (CF) and is denoted by y_c.

We know that $y_p = \frac{X}{f(D)}$ —— ③ as a particular solution of ①,

$$\therefore y = y_c + y_p \quad \text{—— ④}$$

gives the complete solution of ①.

Q1. $(D^3 - 5D^2 + 8D - 4)y = e^{2x}$

Aux eqn $D^3 - 5D^2 + 8D - 4 = 0$

D=1	1	-5	8	-4
	0	1	-4	4
	1	-4	4	0

$(D+1)(D^2 - 4D + 4) = 0$

ii $(D-1)(D-2)^2 = 0$

$D = 1, 2, 2$

C.F, $y_c = c_1 e^{x} + (c_2 + c_3 x) e^{2x}$

P.I, $y_p = \frac{X}{f(D)} = \frac{e^{2x}}{\underbrace{(D-1)}_{2-1=1}(D-2)^2}$ $a = 2$

$= \frac{x^2}{2!} e^{2x}$

Q2. $y'' - 8y' + 16y = 3e^{4x}$ satisfying $y = 0$, at $x = 0$ & $x = 2$

$(D^2 - 8D + 16) = 0$ $y_c = (c_1 + c_2 x) e^{4x}$

$(D-4)^2 = 0$ $y_p = \frac{X}{f(D)} = \frac{3e^{4x}}{(D-4)^2} = 3 \times \frac{x^2}{2!} e^{4x}$

$D = 4, 4.$

$\therefore y = (c_1 + c_2 x) e^{4x} + 3 \frac{x^2}{2!} \times e^{4x}$

$0 = c_1 + 0$ $0 = 2c_2 e^8 + 6e^8$

$c_1 = 0$ $= e^8(2c_2 + 1)$

$c_2 = -3.$

$\therefore y = -3x e^{4x} + \frac{3x^2}{2} e^{4x}$

Q3. $\frac{d^4 y}{dx^4} - y = 15 \cos 2x$

$(D4 - 1) = 0$

$(D^2)^2 - 1 = 0$

$(D^2 + 1)(D^2 - 1) = 0$

$D = \pm 1, \pm i$

$y = (c_1 + c_2 x) e^{x} + c_3 \cos x + c_4 \sin x$

$y_p = 15 \times \frac{\cos 2x}{(D^2)^2 - 1} = \frac{15 \cos 2x}{(-4)^2 - 1} = \cos 2x$

Q_4 $(D^2-4)y = \cosh(2x-1) + 3^x$

Ans

$$(D^2-4)y = 0$$

$$(D+2)(D-2) = 0$$

$$D = -2, +2$$

$$\therefore y_c = c_1e^{-2x} + c_2e^{2x}$$

$$\cosh(2x-1) + 3^x = \frac{e^{2x-1}+e^{-2x+1}}{2} + e^{(\log 3)x}$$

$$y_p = \frac{\frac{1}{2}\left[e^{2x-1}+e^{-2x+1}\right] + e^{x\log 3}}{(D+2)(D-2)}$$

$$= \frac{1}{2}\left[\frac{x}{4}e^{2x-1} + \frac{x}{-4}e^{-2x+1}\right] + \frac{3^x}{(\log 3+2)(\log 3-2)}$$

$$= \frac{x}{4}\sinh(2x-1) + \frac{3^x}{(\log 3)^2-4}$$

Q_5. $(D^3+1)y = \cos(2x-1)$

$$(D^3+1) = 0$$

$$(D^3+1^3) = (D+1)(D^2-D+1)$$

$$D = -1, \frac{1\pm\sqrt{3}i}{2}$$

$$y_c = c_1e^{-x} + e^{x/2}\left(c_2\cos\frac{\sqrt{3}}{2} + c_3\sin\frac{\sqrt{3}}{2}\right)$$

$$y_p = \frac{\cos 2x-1}{D^2\cdot D+1}$$

$$= \frac{\cos 2x-1}{-4D+1}$$

$$= \frac{(1+4D)(\cos 2x-1)}{(1+4D)(1-4D)}$$

$$= \frac{(1+4D)\,(\cos 2x - 1)}{1 - 16D^2} = \frac{(1+4D)\,(\cos 2x - 1)}{1+16\times 4^2}$$

$$= \frac{1}{65}\left[\cos(2x-1) - 8\sin(2x-1)\right]$$

Q6. $(D^4 + 2D^3 - 3D^2)y = 2e^{3x} + 4\sin x$

$D^2(D^2+2D-3)y = 0$

$D^2(D+3)(D-1) \neq 0$

$D = 0, 0, -3, 1$

$y_c = (c_1 + c_2 x)e^0 + c_3 e^{-3x} + c_4 e^{x}$

$$y_p = \frac{2e^{3x}}{D^2(D^2+2D-3)} + \frac{4\sin x}{D^2(D^2+2D-3)}$$

$$= \frac{2e^{3x}}{9(9+6-3)} + \frac{4\sin x}{-1(-1+2D-3)}$$

$$= \frac{\cancel{2}e^{3x}}{9\times\cancel{12}6} + \frac{2\cancel{4}\sin x}{-\cancel{2}(D-2)}$$

$$= \frac{e^{3x}}{54} + \bar{2}\frac{(D+2)\sin x}{D^2-4}$$

$$= \frac{e^{3x}}{54} + \frac{2}{5}\left[2\sin x + \cos x\right]$$

Q7. $y'' + 3y' + 2y = 4\cos^2 x.$

$(D^2 + 3D + 2)y = 2(1 + \cos 2x)$

$= 2(e^{0x} + \cos 2x)$

$(D+2)(D+1) = 0$

$D = -2, -1$

$\therefore y_c = c_1 e^{-2x} + c_2 e^{-x}$

$$y_p = \frac{2(e^{0x} + \cos 2x)}{D^2+3D+2} = \frac{2e^{0x}}{D^2+3D+2} + \frac{2\cos 2x}{D^2+3D+2}$$

$$= \frac{2e^{0x}}{2} + \frac{2\cos 2x}{-4+3D+2}$$

$$= e^{0x} + \frac{2\cos 2x}{3D-2}$$

$$= e^{0x} + \frac{(D+2/3)\,2\cos 2x}{(D+2/3)(D-2/3)}$$

$$= e^{0x} + \frac{(D+2/3)\,2\cos 2x}{-4-\frac{4}{9}}$$

$$= e^{0x} + \frac{-9}{40}\left[\frac{4}{3}\cos 2x + 2\sin 2x\right]$$

$$y_p = 1 + \frac{-1}{40}(4\cos 2x - 12\sin 2x)$$

Q8. $y'' + y = \sin x \sin 2x$

$D^2 + 1 = 0$

$D = -1$

$D = \pm i$

$\sin x \sin 2x = \frac{1}{2}(2\sin x \sin 2x)$

$= \frac{1}{2}(\cos x - \cos 3x)$

$y_c = c_1\cos x + c_2\sin x$

$$y_p = \frac{1}{2}\left[\frac{\cos x}{D^2+1} - \frac{\cos 3x}{D^2+1}\right]$$

$$= \frac{1}{2}\left[\frac{-x\sin x}{2} + \frac{\cos 3x}{8}\right]$$

Q9. The solution of $k^2\frac{d^2y}{dx^2} = y - y_2$ satisfying the conditions

i) $y = y_1$ at $x = 0$

ii) $y = y_2$ at $x = \alpha$

k, y_1, y_2 are constants.

Au $k^2D^2y = y - y_2$

$(k^2D^2 - 1)y = -y_2$

$D = \pm\frac{1}{k}$

$y = c_1e^{x/k} + c_2e^{-x/k}$

$$y_p = \frac{-y_2 \times e^{0x}}{K^2D^2-1}$$

$$= \frac{\cancel{-}\, y_2 \times e^{0x}}{\cancel{-}1} = y_2$$

$$\therefore \; y = c_1 e^{x/K} + c_2 e^{-x/K} + y_2$$

$$y_1 = c_1 + c_2 + y_2$$

$$\cancel{y_2} = c_1 \alpha + c_2 0 + \cancel{y_2}$$

$$c_1 = 0$$

$$c_2 = \underline{\underline{y_1 - y_2}}$$

Case (iii)

Consider $f(D)y = x^m$ $\quad (m \in Z^+)$

then $y_p = \dfrac{x^m}{f(D)} = [f(D)]^{-1} x^m$

Expand $[f(D)]^{-1}$ as ascending powers of D.

By using Binomial expansions and then apply on x^m.

$(1-x)^{-1} = 1 + x + x^2 + x^3 + \dots$

$(1+x)^{-1} = 1 - x + x^2 - x^3 + \dots$

$(1 \mp x)^{-2} = 1 \pm 2x + 3x^2 \pm 4x^3 + \dots$

$(1 \pm x)^{-3} = 1 \pm 3x + 6x^2 \pm 10x^3 + \dots$

Q1. $y'' - 4y' + 4y = x^3$

$(D^2 - 4D + 4)y = x^3$

$(D-2)^2$

$D = 2, 2$

$y_c = (c_1 + c_2 x)e^{2x}$

$$y_p = \frac{x^3}{(D-2)^2} = \frac{x^3}{4\left(1-\frac{D}{2}\right)^2} = \frac{1}{4}\left(1+\frac{D}{2}\right)^{-2} x^3$$

$$= \frac{1}{4}\left[1+\cancel{2}\times\frac{D}{\cancel{2}} + 3\,\frac{D^2}{4} + \frac{\cancel{4}D^3}{\cancel{8}\,2}\right] x^3$$

$$= \frac{1}{4}\left[x^3 + 3x^2 + \frac{3}{4}\times 6x + \frac{1}{2}\times 6\right]$$

Q2. $\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} = x^2+2x+4$

$(D^2+D) = 0$

$D(D+1) = \cancel{\pm i}\ 0 \qquad D = 0, -1$

$y_c = \cancel{c_1\cos x + c_2\sin x} \qquad c_1e^{0x} + c_2e^{-x}$

$$y_p = \frac{x^2+2x+4}{D(D+1)}$$

$$= \frac{1}{D}(D+1)^{-1}(x^2+2x+4)$$

$$= \frac{1}{D}\left(1-D+D^2-D^3\right)\left(x^2+2x+4\right)$$

$$= \left(\frac{1}{D} - 1 + D - D^2\right)(x^2+2x+4)$$

$$= \frac{x^3}{3} + \frac{\cancel{2}x^2}{\cancel{2}} + 4x - (\cancel{x^2}+\cancel{2x}+4)$$

$$+ (\cancel{2x}+\cancel{2}) - \cancel{2}$$

$$y_p = \frac{x^3}{3} + 4x - 4$$

Q3. $\dfrac{d^2y}{dx^2} - \dfrac{3dy}{dx} + 2y = (x^2+x)$

$(D^2 - 3D + 2)y = x^3 + x$

$(D-1)(D-2) = 0$

$D = 1, 2$

$y_c = c_1e^x + c_2e^{2x}$

$$y_p = \frac{x^2+x}{(D^2-3D+2)} = \frac{x^2+x}{2\left(1-\frac{3}{2}D+\frac{D^2}{2}\right)}$$

$$= \frac{1}{2}\left[1-\left(\frac{3}{2}D+\frac{D^2}{2}\right)\right]^{-1}(x^2+x)$$

$$= \frac{1}{2}\left[1+\left(\frac{3}{2}D-\frac{D^2}{2}\right)+\left(\frac{3}{2}D+\frac{D^2}{2}\right)^2\right](x^2+x)$$

$$= \frac{1}{2}\left[1+\frac{3}{2}D-\frac{D^2}{2}+\frac{9}{4}D^2\right](x^2+x)$$

$$= \frac{1}{2}\left[x^2+x+\frac{3}{2}(2x+1)-\frac{1}{\cancel{2}}(\cancel{2})+\frac{9}{\cancel{4}\,2}\times\cancel{2}\right]$$

$$= \frac{1}{2}\left[x^2+x+3x+\frac{3}{2}-1+\frac{9}{2}\right]$$

$$= \frac{1}{2}\left[x^2+4x+5\right]$$

<u>05|07|12</u>

Q4. $\frac{d^2y}{dx^2} = 3x-2$ satisfying $y(0)=2$ and $y'(1)=-3$.

Ans

$$\frac{dy}{dx} = \frac{3x^2}{2} - 2x + C_1$$

$$y = \frac{\cancel{3}x^3}{2\times\cancel{3}} - \frac{2x^2}{2} + C_1x + C$$

$y'(1) = -3$

$$\Longrightarrow \quad -3 = \frac{3}{2} - 2 + C_1$$

$$C_1 = 5/2$$

$y(0) = 2$

$$\Longrightarrow \quad 2 = \frac{0}{2} + 0 + C$$

$$C = 2$$

$$\therefore y = \frac{x^3}{2} - x^2 + \frac{5}{2}x + 2$$

Case (iv) :

Consider the equation $f(D)y = e^{ax}.V$ where V is also a function of x. It may be $\sin(bx+c)$ is $\cos(bx+c)$ or x^m etc..

$$f(D)y = e^{ax}.V$$

$$y_p = \frac{e^{ax}.V}{f(D)} = e^{ax}\left[\frac{V}{f(D+a)}\right]$$

$$D[e^{ax}.V] = e^{ax}\,DV + aVe^{ax}$$

$$= e^{ax}[D+a]V$$

$$D^2[e^{ax}.V] = e^{ax}[D^2V + aDV] + (D+a)Ve^{ax}$$

$$= e^{ax}[D^2 + 2aD + a^2]V$$

$$= e^{ax}\,\underline{(D+a)^2}\,V$$

Q1. $(D^2 - 5D + 6)y = e^{2x}.x^3$

$(D-2)(D-3) = 0$

$D = 2, 3$

$\therefore y_c = C_1e^{2x} + C_2e^{3x}$

$$y_p = \frac{e^{2x}.x^3}{(D-2)(D-3)}$$

$$= e^{2x}\left[\frac{x^3}{(D+2-2)(D+2-3)}\right]$$

$$= -e^{2x}\left[\frac{1}{D}\;\frac{x^3}{(1-D)}\right]$$

$$= -e^{2x}\left[\frac{1}{D}(1-D)^{-1}\;x^3\right]$$

$$= -e^{2x}\left[\frac{1}{D}\left(1+D+D^2+D^3+D^4\right)(x^3)\right]$$

$$= -e^{2x}\Bigg[\Bigg(\frac{1}{D}\Big(+1+D+D^2+D^3\Big)\ x^3\Bigg]$$

$$= -e^{2x}\left[\frac{x^4}{4}+x^3+3x^2+6x+6\right]$$

Q2. $y''+4y = 2e^x\sin^2 x$

$$(D^2+4)y = e^x(1-\cos 2x)$$

$$= e^x - e^x\cos 2x$$

$$D = \pm 2i$$

$$y_c = c_1\cos 2x + c_2\sin 2x.$$

$$y_{p_1} = \frac{e^x}{D^2+4} = \frac{e^x}{5}$$

$$y_{p_2} = \frac{e^x\cos 2x}{D^2+4} = e^x\left[\frac{\cos 2x}{(D+1)^2+4}\right]$$

$$= e^x\left[\frac{\cos 2x}{D^2+2D+5}\right] = e^x\ \frac{\cos 2x}{\cancel{-4}+2D+1+\cancel{4}}$$

$$= e^x\ \frac{(2D-1)}{(2D-1)}\ \frac{\cos 2x}{(2D+1)}$$

$$= e^x\ \frac{(2D-1)\cos 2x}{4D^2-1}$$

$$= \frac{e^x}{-17}(2D-1)\cos 2x$$

$$= \frac{e^x}{-17}\left[2\times -\sin 2x\times 2-\cos 2x\right]$$

$$= \frac{e^x}{+17}\left[\cos 2x + 4\sin 2x\right]$$

$$\therefore y_p = y_{p_1}+y_{p_2}$$

$$y = y_c + y_p$$

Q3. $(D^2-2D+4)y = e^x \cos x$

Ans

Q4. $y'' - 7y' + 6y = e^{2x}(1+x)$

Q5. $\frac{d^2y}{dx^2} - \frac{2dy}{dx} + y = xe^x \sin x.$

$(D^2-2D+1)y = e^x (x\sin x)$

$(D-1)^2 y = e^x \, x\sin x$

$D = 1, 1$

$y_c = (C_1 + C_2 x)e^x$

$y_p = \frac{e^x (x\sin x)}{(D-1)^2} = e^x \left[\frac{x \sin x}{(D+\cancel{1}-\cancel{1})^2}\right]$

$= e^x \left[\frac{1}{D^2}(x\sin x)\right]$

$= e^x \left[\frac{1}{D}\left(x \times -\cos x - \int 1 \times -\cos x\right)\right]$

$= e^x \left[\frac{1}{D}(-x\cos x + \sin x)\right]$

$= e^x \left[-\left(x \times \sin x - \int 1 \times \sin x\right) + -\cos x\right]$

$= e^x [-x\sin x - \cos x - \cos x]$

$= -e^x [2\cos x + x\sin x]$

Q6. $(D^4-1)y = \sinh x \cos x.$

$(D^2-1)(D^2+1) =$ ~~sinhx~~ $\frac{1}{2}\left[e^x \cos x - e^{-x}\cos x\right]$

$D = \pm 1, \pm i$

$\therefore y_c = C_1 e^x + C_2 e^{-x} + C_3 \cos x + C_4 \sin x.$

$y_{p_1} = \frac{e^x \cos x}{(D^2-1)(D^2+1)} = e^x \left[\frac{\cos x}{((D+1)^2-1)((D+1)^2+1)}\right]$

$$= e^{x}\left[\frac{\cos x}{(\underset{-1}{D^2}+2D)(\underset{-1}{D^2}+2D+\underset{+1}{\cancel{2}})}\right]$$

$$= e^{x}\,\frac{\cos x}{(4D^2-1)} = e^{x}\,\underline{\underline{\frac{\cos x}{-5}}}$$

$$y_{p_2} = \frac{e^{-x}\cos x}{(D^2-1)}$$

Method of Variation of Parameters

Q1. $(D^2+a^2)y = \tan ax$.

Q2. $y'' + 4y = \sec 2x$

Q3. $\dfrac{d^2y}{dx^2} - 6\dfrac{dy}{dx} + 9y = \dfrac{e^{3x}}{x^2}$

Q4. $(D^2-1)y = e^{-2x}\sin(e^{-x})$

Q5. $\dfrac{d^2y}{dx^2} - y = \dfrac{2}{1+e^x}$

General description

Consider the equation $\dfrac{d^2y}{dx^2} + K_1\dfrac{dy}{dx} + K_2 y = X \longrightarrow ①$

where K_1, K_2 are constants and X is a function of x.

Now we can have the CF of ① in the form

$$y_c = c_1 y_1 + c_2 y_2 \longrightarrow ②$$

where c_1, c_2 are arbitrary constants or parameters and y_1, y_2 are functions of x.

By the method of variation of parameters it is possible to write the P.I of ① as in the form of ② given by $y_p = Ay_1 + By_2 \longrightarrow ③$ where

$$A = -\int \frac{X y_2}{W}\,dx \longrightarrow ④$$

$$B = \int \frac{X y_1}{W}\,dx \longrightarrow ⑤$$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \longrightarrow ⑥$$

[called wronskian of y_1, y_2]

Therefore the complete solution of ① is given by

$$y = y_c + y_p$$

$$= (C_1 + A)y_1 + (C_2 + B)y_2$$

Q1. $(D^2 + a^2)y = \tan ax$

$D = \pm ai$

$$y_c = C_1 \underbrace{\cos ax}_{y_1} + C_2 \underbrace{\sin ax}_{y_2}$$

$$W = \begin{vmatrix} \cos ax & \sin ax \\ -a\sin ax & a\cos ax \end{vmatrix} = a \cdot 1 = \underline{\underline{a}}$$

$$A = -\int \frac{X y_2}{W} dx = -\int \frac{\tan ax \times \sin ax}{a} dx$$

$$= \frac{-1}{a}\int \frac{\sin^2 ax}{\cos ax} dx = \frac{-1}{a}\int \frac{1 - \cos^2 ax}{\cos ax} dx$$

$$= \frac{-1}{a}\left[\left[\frac{\log(\sec ax + \tan ax)}{a}\right] - \frac{\sin ax}{a}\right]$$

$$= \frac{-1}{a^2}\left[\log(\sec ax + \tan ax) - \sin ax\right]$$

$$B = \frac{1}{a}\int \overset{\sin ax}{\cancel{\tan ax}} \, \cancel{\cos ax} \, dx$$

$$= \frac{1}{a}\int \sin ax \, dx$$

$$= \frac{1}{a} \times \frac{-\cos ax}{a} = \frac{-1}{a^2} \cos ax$$

$$y_p = Ay_1 + By_2$$

Q3. $(D^2 - 6D + 9)y = \dfrac{e^{3x}}{x^2}$

$(D-3)^2 y$

$D = 3, 3$

$$y_c = (C_1 + C_2 x)e^{3x} = C_1 \underbrace{e^{3x}}_{y_1} + C_2 \underbrace{xe^{3x}}_{y_2}$$

$$W = \begin{vmatrix} e^{3x} & xe^{3x} \\ 3e^{3x} & 3xe^{3x} + e^{3x} \end{vmatrix} = \underline{\underline{e^{6x}}}$$

$$A = -\int \frac{\cancel{e^{3x}}}{x^2} \cdot \frac{x\cancel{e^{3x}}}{\cancel{e^{6x}}}\, dx = -\log x$$

$$B = \int \frac{\cancel{e^{3x}}}{x^2} \times \frac{\cancel{e^{3x}}}{\cancel{e^{6x}}}\, dx = \underline{\underline{\frac{-1}{x}}}$$

$$y_p = \underline{\underline{Ay_1 + By_2}}$$

Q4 $(D^2-1)y = e^{-2x} \sin(e^{-x})$

$$D^2 - 1 = 0$$

$$D = \pm 1$$

$$y_c = c_1 \underbrace{e^{x}}_{y_1} + c_2 \underbrace{e^{-x}}_{y_2}$$

$$W = \begin{vmatrix} e^{x} & e^{-x} \\ e^{x} & -e^{-x} \end{vmatrix} = -1-1 = -2$$

$$A = +\int \frac{e^{-2x} \sin(e^{-x})\, e^{-x}}{+2}\, dx = \qquad e^{-x} = z, \quad dz = -e^{-x}dx$$

$$= \frac{-1}{2} \int z^2 \sin z\, dz$$

$$= \frac{-1}{2} \left[z^2 \times -\cos z + \int 2z \times \cos z\, dz \right]$$

$$= \frac{-1}{2} \left[z^2 \times -\cos z + 2 \times (z \sin z + \cos z) \right]$$

$$= \frac{+1}{2} \left[e^{-2x} \times \cos e^{-x} - \underline{\underline{2e^{-x} \sin e^{-x}}} + 2\cos e^{-x} \right]$$

$$B = \frac{-1}{2} \int e^{-2x} \sin e^{-x} \times e^{x} dx$$

$$= \frac{1}{2} \int \sin t\, dt = \underline{\frac{-1}{2} \cos(e^{-x})}$$

Equation Reducible To Linear Eqn with Constant Coeff. Form

Q1. $x^2 \frac{d^2y}{dx^2} + \frac{x\,dy}{dx} - 4y = 0$ satisfying $y(0) = 0$ on $y(1) = 1$

Q2. $x^2 \frac{d^2y}{dx^2} - 2y = x^2 + \frac{1}{x}$

Q3. $(x+1)^2 \frac{d^2y}{dx^2} + (x+1)\frac{dy}{dx} + y = 2x \sin(\log(x+1))$

Q4. $(2x+3)^2 \frac{d^2y}{dx^2} + 6(2x+3)\frac{dy}{dx} + 6y = \log(2x+3)$

Q5. $x^2 \frac{d^2y}{dx^2} - \frac{x\,dy}{dx} + 2y = x \log x$

Q6. $x^2 \frac{d^2y}{dx^2} + 4x\frac{dy}{dx} + 2y = e^x$.

Euler Cauchy's Form → 1, 2, 5, 6

Legender's Form → 3, 4

Solution

Q1- The given eqn is Euler Cauchy's form, To reduce this equation into constant coeff. form, we can substitute the following.

$$x = e^z \implies z = \log x \text{ \& } D = \frac{d}{dz}$$

then

$$x\frac{dy}{dx} = Dy$$

$$x^2\frac{d^2y}{dx^2} = D(D-1)y$$

$$x^3\frac{d^3y}{dx^3} = D(D-1)(D-2)y.$$

$$D(D-1)y + Dy - 4y = 0$$

$$(D^2 - D + D - 4)y = 0$$

$$(D+2)(D-2)y = 0$$

$$D = \underline{\underline{\pm 2}}$$

$$y = c_1 e^{-2z} + c_2 e^{+2z}$$

$$= \left(\frac{c_1}{x^2} + c_2 \times x^2 \right)$$

$$0 = c_1 x + c_2 \times 0 \implies c_1 = 0.$$

$$1 = c_2 \times 1 \qquad \therefore c_2 = 1$$

$$\therefore \; y = x^2$$

Q2. $$x^2 \frac{d^2y}{dx^2} - 2y = \left(x^2 + \frac{1}{x}\right)$$

$$D(D-1)y - 2y = x^2 + \frac{1}{x}$$

$$\cancel{D}(D^2 - D - 2)y = x^2 + \frac{1}{x} = e^{2z} + e^{-z}$$

$$(D-2)(D+1)y = 0$$

$$D = 2, -1$$

$$y_c = (c_1 e^{2z} + c_2 e^{-z})$$

$$y_p = \frac{e^{2z} + e^{-z}}{(D-2)(D+1)}$$

$$= \frac{z}{3} e^{2z} \mp \frac{z e^{-z}}{+3}$$

$$y = y_c + y_p = \left(a x^2 + \frac{c_2}{x} \right) + \frac{\log x}{3} \left(x^2 - \frac{1}{x} \right)$$

Q3. The given eqn is in Legendre's eqn form.

To reduce this eqn to constant coeff form, we can substitute the following.

Let $(ax+b) = e^z$

$z = \log(ax+b)$ and $D = \frac{d}{dz}$

$$(ax+b)\frac{dy}{dx} = aDy$$

$$(ax+b)^2\frac{d^2y}{dx^2} = a^2 D(D-1)y.$$

$$(ax+b)^3\frac{d^3y}{dx^3} = a^3 D(D-1)(D-2)y$$

$$(x+1)^2\frac{d^2y}{dx^2} + (x+1)\frac{dy}{dx} + y = 2\sin(\log(x+1))$$

$$D(D-1)y + Dy + y = 2\sin(z)$$

$$(D^2 - \cancel{D} + \cancel{D} + 1)y = -2\sin z$$

$$(D^2+1)y = 2\sin z.$$

$$D = \pm i$$

$$\therefore y_c = c_1\cos z + c_2\sin z$$

$$= c_1\cos(\log(x+1)) + c_2\sin(\log(x+1))$$

$$y_p = \frac{2\sin z}{D^2+1} \times z = -\frac{\cancel{2}\,\overset{z\cos z}{\cancel{\cos z}}}{\cancel{2}} = -\underline{\underline{\log(x+1)\cos(\log(x+1))}}$$

Q4 $2x+3 = e^z \implies z = \log(2x+3)$

$$4D(D-1)y + 6 \times 2Dy + 6y = z$$

$$\cancel{(2x+3)^2} \quad \cancel{2}(2D^2 \underset{4D}{-2D+6D} + 3)y = \frac{z}{2}$$

Q5. $D(D-1)y - Dy + 2y = e^{z} \cdot z$

$$(D^2 - D - D + 2)y = e^{z} \cdot z$$

$$(D^2 - 2D + 2)y = e^{z} \cdot z$$

Q6. $D(D-1)y + 4Dy + 2y = e^{e^z}$

$$(D^2 - 1D + 4D + 2)y = e^{e^z}$$

$$(D^2 + 3D + 2)y = e^{e^z}$$

PROBABILITY & STATISTICS

SYLLABUS

- Basics
- Probability
- Random Variable / Expectation
- Distribution → Discrete & Continuous
- Correlation and regression.

STATISTICS : According to PROF. R.A. FISHER statistics is defined as collection of data, analysis of data and interpretation of data.

Types Of datas :

- Grouped and ungrouped data.
- Open and closed.

Grouped data : If the data is in the form of class intervals and frequency together, then the data is known as grouped data or distributing the frequencies to their corresponding class intervals is known as freq. distribution.

Closed data : If the class intervals are in continuous form without any discontinuity, then the data is known as closed data. Otherwise open data.

Ungrouped data : If the data contains only observations without any class intervals, then the data is known as ungrouped data or RAW data

Mean [Average] :

$$\bar{X} = \frac{\sum_{i=1}^{n} x_i}{n}$$ For ungrouped data

$$\bar{X} = \frac{\sum f_i x_i}{N}$$ For grouped data

where $x_i = \frac{\text{Upper limit} + \text{lower limit}}{2}$

n = no. of observations

N = sum of frequencies.

Median

If n is odd → The middle observation itself is the median

If n is even → Average b/w the middle observations, provided

a) Data is rearranged either in ascending or in a descending order.

b) The no. of observations above the middle is equal to the no. of observations below the middle.

$$M_d = l + \left[\frac{N/2 - m}{f}\right] c$$

where l → lower limit for the ideal class.

f → frequency for ideal class.

m → cumulative frequency for above the ideal class.

c → Size of the class.

Q. Find the median for the following grouped data.

Class interval	Frequency	Cumulative Freq
0 - 10	3	3
10 - 20	5	8 → m
20 - 30	7 → f	15 → ideal class
30 - 40	2	17
40 - 50	1	18
	N = 18	

C = 30-20 = 10

N/2 = 18/2 = 9 lies in ideal class.

$$M_d = 20 + \frac{9-8}{7} \times 10 = 20 + 1.4 = \underline{21.4}$$

Note: If the first class itself is ideal, the cumulative frequency & frequency (f) are equal (m = f).

MODE : The most frequently repeated observation is known as the mode

eg : 1, 2, 3, 4, 5 , 2, 3, 6, 7, 2, 3, 11 , 14, 2, 21, 23, 3, 36

M = 2 ⟶ Unimodel

M = 2, 3 ⟶ Bimodel.

Relation b/w mode , median & mean ⟶ For grouped data only.

$$Mo = 3Md - 2\ mean$$

$$Mo = l + \left[\frac{\Delta_1}{\Delta_1 + \Delta_2}\right] C$$

$\Delta_1 = f - f_{-1}$

$\Delta_2 = f - f_{+1}$

C ⟶ Size of the class

l ⟶ lower limit of ideal class.

Q. Find the mode for the following freq & data.

C.I	freq
0 - 2	11
2 - 4	14
4 - 6	17
6 - 8	08
8 - 10	03

4 - 6 ⟶ Ideal class

$$Mo = 4 + \left[\frac{3}{3+9}\right] 2$$

17 - 14 = 3

17 - 8 = 9

$$= 4 + \frac{6}{12}$$

$$= \underline{\underline{4.5}}$$

Note :

- If the maximum frequencies are repeated, first, last and inbetween select in b/w as ideal.
- If the maximum frequencies are repeated in b/w select randomly.
- If all the frequencies are equal mode is undefined $(\frac{0}{0}$ form$)$
- If the maximum frequencies are repeated first and last, select randomly.

Measures of central Tendancies

Mean
Median
Mode
} Collectively called measures of central freq.

Mean is the best among the 3 measures of central freq.

The consistancy of measures of central freq. is less. So we consider measures of dispersion.

Measures Of Dispersion / Variability

To check the consistancy, uniformity etc of the data measures of dispersion is used. They are

- Range
- Quartile Deviation (QD)
- Mean Deviation (MD)
- Standard Deviation (SD)
- Coefficient Of Deviation (C.C.V)

Range : Maximum — Minimum

$$SD = \sqrt{Variance}$$

$$Variance = (SD)^2$$

Variance : The difference or deviations within the data is known as variance.

$$\sigma_x = \frac{\Sigma (x_i - \bar{x})^2}{n}$$

$$= \frac{1}{n}(\Sigma x_i^2) - (\bar{x})^2$$

- Variance of constant = 0
- Lesser variance is more consistant or uniform
- Variance can never be -ve.
- Sum of the deviations from the mean is always zero.

$$\Sigma (x_i - x) = 0$$

- Sum of the squares of the deviation from mean is minimum.
- If the variances are equal for the different groups greater mean is more consistant.

$$\sigma_x^2 = \frac{1}{N} \Sigma f_i x_i^2 - \left(\frac{1}{N} \Sigma f_i x_i\right)^2$$

$$= \frac{1}{N} \Sigma \; f_i \, (X_i - \bar{X})^2$$

- 6QD = 5MD = 4SD

6QD = 4SD

$QD = \frac{4}{6} SD = \frac{2}{3}\sigma$.

5MD = 4SD

$MD = \frac{4}{5} SD$

$= \frac{4}{5}\sigma$

- Coefficient of variation, $C.V = \frac{S.D}{mean} \times 100$

$$= \frac{\sigma}{\bar{X}} \times 100$$

Note : Lessen σ ⇒ lessen C.V ⇒ data is more consistant and uniform

S.D & C.V are the better measurements for identifying consistancy. Of these C.V is best.

Q. Find the mean and variance of 1st n natural no.s.

$$\bar{X} = \frac{1+2+3 \dots +n}{n} = \frac{1}{n}\left[\frac{n(n+1)}{2}\right] = \frac{n+1}{2}$$

$$\sigma_x^2 = \frac{1}{n} \Sigma x_i^2 - (\bar{x})^2$$

$$\frac{1}{n} \Sigma x_i^2 = \frac{1}{n}\left[1^2 + 2^2 + \dots + n^2\right] = \frac{1}{\cancel{n}} \frac{\cancel{n}(n+1)(2n+1)}{6} = \frac{(n+1)(2n+1)}{6}$$

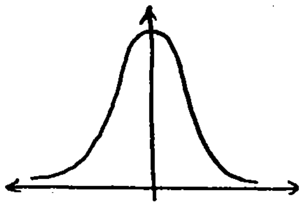
$$\sigma_x^2 = \frac{(n+1)(2n+1)}{6} - \left(\frac{n+1}{2}\right)^2$$

$$= \frac{n+1}{2}\left[\frac{2n+1}{3} - \frac{n+1}{2}\right] = \frac{n+1}{2}\left[\frac{4n+2-3n-3}{6}\right]$$

$$= \frac{n+1}{2} \times \left[\frac{n-1}{6}\right] = \frac{n^2-1}{12}$$

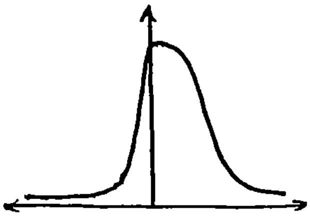
SKEWNESS

It is the measurement of lack of symmetry.



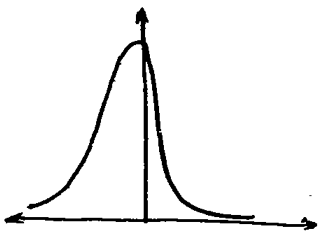
Symmetric

Mo = Md = mean



-ve skewed

Mo > Md > mean



+ve skewed

Mo < Md < mean

*** Pearson's Coeff. of Skewness

$$S_{kp} = \frac{M - M_o}{\sigma} = \frac{3(M - M_d)}{\sigma}$$

$$-3 \leq S_{kp} \leq +3$$

$-3 \leq S_{kp} \leq 0 \longrightarrow$ -ve skewed

$S_{kp} = 0 \longrightarrow$ Symmetric

$0 \leq S_{kp} \leq 3 \longrightarrow$ +ve skewed.

PROBABILITY

Random experiment : unpredictable outcomes of an exp. is known as a random experiment.

eg : Tossing an unbiased coin.

Rolling a die.

Sample space : The collection of all possible outcomes of an experiment is known as a sample space denoted by s.

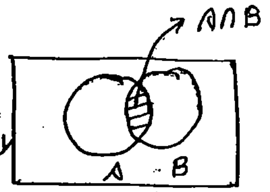
Event : The outcomes of an experiment is known as an event.

Event is a subset of sample space.

Probability : The probability of an event is defined as the ratio b/w the favourable cases of the event and the no. of outcomes of an experiment (the outcomes are mutually exclusive, exhaustive events). Therefore

$$P(E) = \frac{m}{n} \quad \text{where} \quad m \leq n$$

Axiomatic Approach / Rules of Probability / Prob. function



Dependent

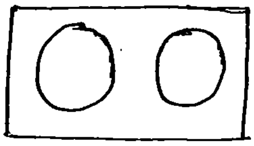
$P(S) = 1$

$0 \leq P(E) \leq 1$

$P(E) = 0 \longrightarrow$ Impossible event

$P(\phi) = 0$

$P(E) = 1 \longrightarrow$ Certain / Sure event.



Mutually exclusive

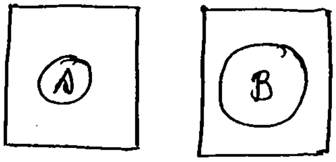
$$P\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} P(E_i)$$

E_i is disjoint / mutually exclusive.

eg : $n = 2$

$P(E_1 \cup E_2) = P(E_1) + P(E_2)$

$1 = 1/2 + 1/2$



Independent.

Note : Occurance of one event doesnot depend on the occurance of other events in the same sample space then those events are known as mutually exclusive events.

Let A & B are mutually exclusive events.

$A \cap B = \phi$ and $P(A \cap B) = 0$

- Occurance of one event doesnot depend on the occurance of same event in a different sample space, then those events are known as independent events.

- Mutually exclusive events never be equal to independent. Independent events also never be equal to mutually exclusive.

Results

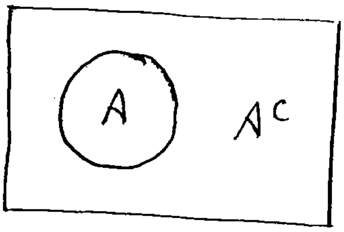
- $P(S) = 1$ $\quad S \longrightarrow$ Sample space.

$0 \leq P(E) \leq 1$

$P(A^c) = 1 - P(A)$

$P(A \cup A^c) = 1$

$P(A) + P(A^c) = 1$



If A & B are two events $P(A\cup B) = P(A) + P(B) - P(A\cap B)$

If A & B are mutually exclusive

$P(A\cup B) = P(A) + P(B)$

$P(A+B) = P(A) + P(B)$

$P(A+B+C) = P(A) + P(B) + P(C)$

If A and B are two events

$P(A\cap B) = P(A) . P(B/A) \longrightarrow$ Conditional prob.

$= P(B) . P(A/B)$

$\downarrow$ unknown event. $\hookrightarrow$ Known event

If A and B are two events.

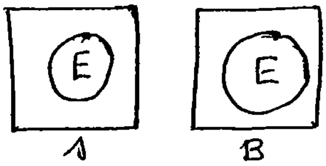
$P(A\cap B\wedge C) = P(A)\, P(B/A)\, P(C/A\cap B)$

Multiplication Theorem

If A & B are independent event iff $P(A\cap B) = P(A).P(B)$

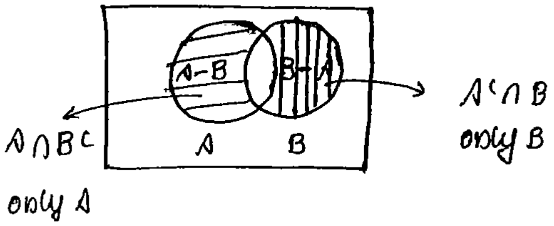
If $E_1, E_2 \ldots E_n$ are independent.

$P(E_1 \cap E_2 \ldots E_n) = P(E_1).P(E_2).P(E_3) \ldots P(E_n)$



If A & B are two events

$P(A\cap B^c) = P(A) - P(A\cap B)$



$P(A^c\cap B) = P(B) - P(A\cap B)$

$P(A^c \cap B^c) = P(\overline{A\cup B}) = 1 - P(A\cup B)$

$P(A\cap B)^c = P(A\cap B^c) + P(A^c\cap B)$

$P(A\cap B) = 0 \longrightarrow$ Mutually exclusive

$P(A\cap B) = P(A).P(B) \longrightarrow$ Independent

$P(A\cap B) = P(B)\, P(A/B) \longrightarrow$ dependent.

$$P(A^c/B) = \frac{P(A^c\cap B)}{P(B)} = \frac{P(B) - P(A\cap B)}{P(B)} = 1 - \frac{P(A\cap B)}{P(B)}$$

$$= \underline{\underline{1 - P(A/B)}}$$

$$P(A|B^c) = \frac{P(A\cap B^c)}{P(B^c)} = \frac{P(A)-P(A\cap B)}{1-P(B)} \quad \because P(B)\neq 1$$

$$P(A^c|B^c) = \frac{P(A^c\cap B^c)}{P(B^c)} = \frac{1-P(A\cup B)}{1-P(B)} \quad \because P(B)\neq 1$$

If A and B are independent events, the probability of $(A\cap B^c)$, $P(A^c\cap B)$ and $P(A^c\cap B^c)$ are also independent.

Baye's Theorem

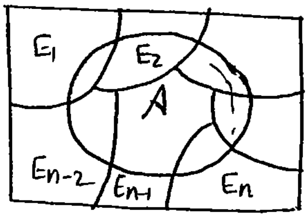
If $E_1, E_2 --- E_n$ are the mutually exclusive events $P(E_i)\neq 0$ such that A is an arbitrary event which is a subset of $\bigcup_{i=1}^{n} E_i$ then P(A) is

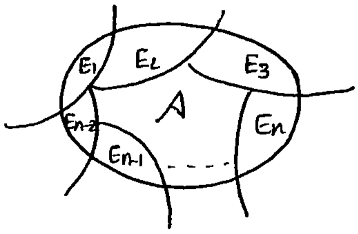
$$P(A) = P(A\cap E_1) + P(A\cap E_2) + ---- + P(A\cap E_n)$$

$$= P(E_1)/P(A/E_1) + P(E_2)/P(A/E_2) + \cdots + P(E_n)/P(A\cap E_n)$$

$$P(A) = \sum_{i=1}^{n} P(E_i)\,P(A|E_i) \longrightarrow$$ Total probability of unknown event.

Reverse probability $P(E_i|A) = \frac{P(E_i\cap A)}{P(A)} = \frac{P(E_i)\cdot P(A|E_i)}{\sum_{i=1}^{n} P(E_i)\,P(A|E_i)}$





Steps

1) Identify the known events in the data (mutually exclusive)
2) Select the unknown event (part of known events).
3) Write P(unknown) in terms of known.
4) Find the total Prob of unknown events.
5) Compute reverse probability for known event.

Results

1 coin ⟶ 2 outcome
2 " ⟶ 4 "
:
n " ⟶ 2^n "

Cards

		K	Q	J
13	H	1	1	1
13	D	1	1	1
13	C	1	1	1
13	S	1	1	1

Words	Addition	Multiplication
atleast → min → $\geq$	either/or	Simultaneously One after other
atmost → max → $\leq$	atleast once	As well as
and → product → $\cap$	or	Successively Alternatively.
or → sum → $\cup$		One by one and.

Q₁ If 3 coins are tossed at a time find P(getting atmost one head)

Ans $P(x \leq 1) = P(x<1) + P(x=1) = 1/8 + 3/8 = 4/8 = \underline{\underline{1/2}}$

Q₂ Find P(atleast 1 tail.

$P(x \geq 1) = 7/8 = 1 - P(x<1)$

Q₃. Find P(atleast one head and atleast 1 tail).

$P(x \geq 1)$ and $P(y \geq 1) = 6/8 = 3/4$ (HHT, HTH, THH, HTT, THT, TTH)

Q₄. Find P(atleast one head and atmost 1 tail)

Ans THH, HTH, HHT, HHT

P(atleast one head and atmost 1 tail) = $4/8 = \underline{\underline{1/2}}$

Q₅. 4 coins are tossed at a time, find P(getting atleast 2 heads & atmost 2 tails).

Ans { HHTT, TTHH, THHT, HTTH, THTH, HTHT } = 6 = $^4C_2$

$P(\overset{\text{atmost}}{\text{atleast}}\ 2\ \text{heads and}\ \overset{\text{atmost}}{\text{atleast}}\ 2\ \text{tails}) = \underline{\underline{6/16}}$

H		T
$^4C_0$ =	1	= $^4C_4$
$^4C_1$ =	4	= $^4C_3$
$^4C_2$ =	6	= $^4C_2$
$^4C_3$ =	4	= $^4C_1$
$^4C_4$ =	1	= $^4C_0$
	16	

(Atmost - atmost) no of heads = no of tails.

Q_6 A player tosses 6 coins. Find the P(no. of heads are more than the no. of tails).

Ans $n(S) = 2^6 = 64$

$^6C_0 \quad ^6C_1 \quad ^6C_2 \quad ^6C_3 \quad \underbrace{^6C_4 \quad ^6C_5 \quad ^6C_6}_{\text{Favourable cases.}}$

x x x x ✓ ✓ ✓

$P(H>T) = \frac{^6C_4}{64} + \frac{^6C_5}{64} + \frac{^6C_6}{64} = \frac{15+6+1}{64} = 22/64$

Q_7. A coin is tossed 7 times. Find P(head appears in the odd turns).

Ans $n(S) = 2^7 = 128$.

$^7C_1 \quad ^7C_3 \quad ^7C_5 \quad ^7C_7$

7 35 21 1

P(Head in odd turns) $= \frac{64}{128} = 1/2$

OR

Note : $n(S) = 2^n$

$^nC_1 + {^nC_3} + {^nC_5} + \dots + {^nC_{n-1}} = 2^{n-1}$

$^nC_0 + {^nC_2} + {^nC_6} + \dots + {^nC_n} = 2^{n-1}$

∴ Req. probability $= \frac{2^{n-1}}{2^n} = 1/2$

Q_8 Two dice are thrown 2 times. Find P(getting a sum of 7).

i) atleast once

ii) only once.

iii) T~~hr~~wice.

Ans A : Sum 7 first time $P(A) = 6/36 = 1/6$; $P(A^c) = 5/6$

B : Sum 7 2nd time $P(B) = 6/36 = 1/6$; $P(B^c) = 5/6$

i) P(Atleast once) $= P(A \cup B) = 1 - P(A^c \cap B^c)$

$= 1 - [P(A^c) \cdot P(B^c)]$

$= 1 - \frac{5}{6} \times \frac{5}{6} = 11/36$

ii) P(only once) $= P(A \cap B^c) + P(A^c \cap B)$

$= P(A)\,P(B^c) + P(A^c)\,P(B) = 1/6 \times 5/6 + 5/6 \times 1/6$

$= 10/36$

$P(\text{twice}) = P(A \cap B)$

$= P(A)P(B) = 1/6 \times 1/6 = \underline{\underline{1/36}}$

Q9 Two dice are rolled. Find P(1st die should contain a prime no. or a total of 8).

Ans A : prime no on 1st $P(A) = 18/36$.

B : Sum of 8 $P(B) = 5/36$

$P(A \cap B) = 3/36$.

$P(A \cup B) = 18/36 + 5/36 - 3/36 = \underline{\underline{20/36}}$

Q10 Two dice are rolled once. Find P(neither sum 9 nor sum 11).

Ans 9 : (4,5) (5,4) (6,3) (3,6)

11 : (5,6) (6,5)

$P(9) = 4/36$

$P(11) = 2/36$

$P(9^c \cap 11^c) = 1 - P(9 \cup 11)$

$= 1 - \{P(9) + P(11)\}$ $P(A \cap B) = 0$

$= 1 - \{4/36 + 2/36\}$

$= \underline{\underline{30/36}}$

Q11 A determinant is chosen from a set of all determinants of order 2 with the elements 0 (Eo) or 1. Find the P(the chosen determinant is nonzero)

Ans $n(S) = 16$.

$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\Delta = ad - bc \neq 0$

Case(i) : +1 [a = d = 1 and atleast one of b = c = 0]

$\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}, \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix}, \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 3$

Case (ii) $\Delta = -1$ [$b = c = 1$ atleast one of $a = d = 0$]

$$\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}, \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix}, \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = 3$$

P(non zero Δ) = $\frac{3}{16} + \frac{3}{16} = 6/16$ P(zero Δ) = $1 - 6/16 = \underline{\underline{10/16}}$

P(no. neg. Δ) = $1 - 3/16 = \underline{\underline{13/16}}$

Q12 Four cards are drawn at random from the pack of 52 cards.

i) Find P(all 4 cards are drawn from same suit).

ii) P(no two cards are drawn from same suit)

Ans P(4 cards same suit) = $\dfrac{\overset{H}{13C_4} + \overset{D}{13C_4} + \overset{C}{13C_4} + \overset{D}{13C_4}}{52C_4}$ [at a time]

P(no two cards from same suit) = $\dfrac{13C_1 \times 13C_1 \times 13C_1 + 13C_1}{52C_4}$ [one after other]

Q13. A card is drawn from the pack of 52 cards. Find P(neither a diamond nor a face card).

Ans $P(D) = 13/52$

$P(fc) = 4 \times 3/52 = 12/52.$

$P(D \cap fc) = 3/52$

$P(D^c \cap fc^c) = 1 - (D \cup fc) = 1 - [13/52 + 12/52 - 3/52]$

$= 1 - 22/52 = \underline{\underline{30/52}}$

Q14. A and B are the two players rolling a die on the condition that one who get the 2 first ~~get~~ wins the game. If A starts the game what are the winning chances of player A, B.

Ans $P(2) = 1/6$ $P(2^c) = 5/6$
(p) (q)

$P(B\text{ win}) = qp + q^2qp + q^2q^2qp + q^2q^2q^2qp + \ldots$

$= qp\,[1 + q^2 + (q^2)^2 + (q^2)^3 + \ldots]$

$= qp\left[\frac{1}{1-q^2}\right] = \frac{1}{6} \times \frac{5}{6}\left[\frac{1}{1 - 25/36}\right] = \underline{\underline{5/11}}$

Q15. A, B & C are tossing a coin on the condition that one who gets the head first wins the game. If A start the game, what are the winning chances of C in the 3rd trial.

Ans 3rd trial.

$$\underset{\substack{\downarrow \\ p}}{P(H)} = 1/2 \qquad \underset{\substack{\downarrow \\ q}}{P(T)} = 1/2$$

$$P(\text{win } C) = q^3 q^3 q\, q\, p$$

$$= (1/2)^3 \times (1/2)^3 \times 1/2 \times 1/2 \times 1/2 = \underline{\underline{1/512}}$$

Problems On Conditional Prob. & Bayes Theorem

Q16 A die is rolled. If the no is odd no. find P{odd no}.

Ans.
$$S = \left\{ \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \\ O & & O & & O & \\ & & P & & P & \end{array} \right\}$$

$$P(O) = 3/6$$

$$P(P|O) = \frac{P(P \cap O)}{P(O)} = \frac{2/6}{3/6} = \underline{\underline{2/3}}$$

Q17 A no. is chosen from the 100 nos those are 0, 1, 2, 3, ... 99. Let x denote the sum of digits on a number and y denotes the product of the digits on the no. Find $P(x=9$ given $y=0)$.

Ans
$$P(x=9|y=0) = \frac{P(x=9 \cap y=0)}{P(y=0)} = \frac{2/100}{19/100} = \underline{\underline{2/19}}$$

00	10
01	20
02	30
03	40
04	50
05	60
06	70
07	80
08	90
09	

Q18. 60% of the employees of the company are college graduates. Of these 10% are in the sales dept. Of the employees who didnot graduate from the college 80% are in the sales dept. A person is ~~called~~ selected at random. Find probability that.

i) The person is in the sales dept.

ii) Neither in the sales dept nor a college graduate.

$P(E|T) = 2/8.$

$P(E|C) = 1/7$

$P(E) = P(T\cap E) + P(C\cap E) = P(T) \cdot P(E|T) + P(C) \cdot P(E|C)$

$= 1/2 \times 2/8 + 1/2 \times 1/7 = \underline{\underline{11/56}}$

$$P(C|E) = \frac{P(C\cap E)}{P(E)} = \frac{1/2 \times 1/7}{11/56} = \underline{\underline{4/11}}$$

Q23. There are 3 bags which contain blue, red and green color balls in the form of.

	B	R	G
A	1	2	3
B	2	3	1
C	3	1	2

A bag is drawn at random and two balls are drawn from it. They are found to be 1 blue and 1 red. Find the probability that the selected balls are from bag C.

Ans $P(A) = P(B) = P(C) = 1/3$

E : Getting one blue and one red.

$$P(E|A) = \frac{1C_1 \times 2C_1}{6C_2} = 2/15$$

$$P(E|B) = \frac{2C_1 \times 3C_1}{6C_2} = 6/15$$

$$P(E|C) = \frac{8C_1 \times 1C_1}{6C_2} = 3/15$$

$P(E) = P(A\cap E) + P(B\cap E) + P(C\cap E)$

$= P(A)P(E|A) + P(B)P(E|B) + P(C)P(E|C)$

$= 1/3\left[2/15 + 6/15 + 3/15\right] = \underline{\underline{11/45}}$

$$P(C|E) = \frac{P(C\cap E)}{P(E)} = \frac{1/3 \times 3/15}{11/45} = \underline{\underline{3/11}}$$

Some Standard Qns

- P(53 sundays in leap year) = 2/7.

  366 ⟶ 52 + 2 day

  S-M, M-T, T-W, ~~W-Th~~, W-T, T-F, F-S, S-S ⟹ 2/7

- P(53 sundays in non leap yr) = 1/7.

  365 ⟶ 52 weeks + 1 day

  S, M, T, W, T, F, S ⟶ 1/7

- Two dice

  P(difference = 0) = 6/36 (1,1) (2,2) . . . (6,6).

- Three dice

  P(triplet) = $6/6^3$ = 1/36 (1,1,1) (2,2,2). . . . . (6,6,6)

- {1, 2, ... 200}

  P(div. by 6 or 8)

  P(div 6) = 33/200 200/6 = 33

  P(div 8) = 25/200 200/8 = 25

  P(div 6 ∩ div 8) = L.C.M (6,8) = 24

  P (div 6 or 8) = $\frac{33}{200} + \frac{25}{200} - \frac{8}{200} = \frac{50}{200}$ = 1/4

Random Variable or Expectation

Random variable : Connecting the outcomes of an exp. with the real values is known as Random variable (R.V) (1-D R.V). The corresp. data is known as univariate data.

2-D R.V. : Connecting the two outcomes to a real value provided those two outcomes are drawn from same sample space. The corresp. data is known as Bivariate data.

Random Variable

- → Discrete R.V → Prob. mass fn → Discrete distn
  - Binomial
  - Poisson
- → Continuous R.V → Prob. density fn → Continuous distn
  - Normal
  - uniform

Total = 100

Graduates = 60

Grad. & Sales = 6.

Non Grad. = 40

N. Grad in sales = 32.

i) P(sales) = 38/100

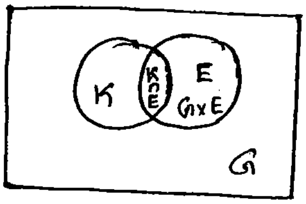
ii) P(N-S & N-G) = 8/100

Q19. In answering a question on multiple choice test, the students either knowing the answer or guessing the answer. Let P be the prob. that student knowing the answer and 1-P that of guessing the answer. Assume that the student gues the answer to a qn. will be correct with a probability 1/5. what is the conditional probability that the student knew the answer to a qn given that he answered correctly.

Ans By Baye's Theorem.

$P(K) = P$; $P(G) = 1-P$ ⟶ mutually exclusive.

E : Answering correctly.



$P(E/K) = 1$

$P(E|G) = 1/5.$

$$P(E) = P(K \cap E) + P(G \cap E) = P(K)\,P(E|K) + P(G)\,P(E|G)$$

$$= P \times 1 + (1-P)\,1/5 = \underline{\underline{\frac{4P+1}{5}}}$$

$$P(K/E) = \frac{P(K \cap E)}{P(E)} = \frac{P(K)\,P(E|K)}{P(E)}$$

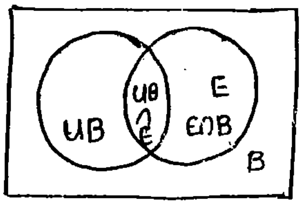
$$= \frac{P \times 1}{\frac{4P+1}{5}} = \underline{\underline{\frac{5P}{4P+1}}}$$

Q20 There are 3 coins out of which 2 are unbiased and one is a biased coin with two heads. A coin is drawn at random and tossed 2 times. It appears head on both the times. Find the probability that the selected coin is a biased coin?

Ans Unbiased = 2

Biased = 1

P(UB) = 2/3 , P(B) = 1/3



E : Getting a head two times.

$P(E/UB) = 1/2 \times 1/2 = 1/4$

$P(E|B) = 1 \times 1 = 1$

$P(E) = P(UB \cap E) + P(B \cap E) = P(UB)P(E/UB) + P(B)P(E|B)$

$= 2/3 \times 1/4 + 1/3 \times 1 = 2/12 + 1/3 = 6/12 = \underline{\underline{1/2}}$

$P(B|E) = \frac{P(B\cap E)}{P(E)} = \frac{1/3 \times 1}{1/2} = \underline{\underline{2/3}}$

Q21. Player A speaking truth 4/7 times. A card is drawn from the pack of 52 cards, he reports that there is a diamond. What is the probability that actually there was a diamond.

Ans $P(T) = 4/7$; $P(L) = 3/7$

E : Reporting a diamond.

$P(E|T) = 13/52 = 1/4$

$P(E|L) = 39/52 = 3/4$

$P(E) = P(T \cap E) + P(L \cap E) = P(T)P(E|T) + P(L)P(E|L)$

$= 4/7 \times 1/4 + 3/7 \times 3/4 = \underline{\underline{13/28}}$

$P(T|E) = \frac{P(T\cap E)}{P(E)} = \frac{4/7 \times 1/4}{13/28} = \underline{\underline{4/13}}$

Q22. A letter is known to have come from TATA NAGAR (or) CALCUTTA. On the envelope, the just two consecutive letters T & A are visible. Find the probability that the letter has come from CALCUTTA.

Ans $P(T) = 1/2$

$P(C) = 1/2$

E : Getting a 'TA'.

T : $\overset{7}{T}\overset{6}{A}TA\ \overset{5}{N}\overset{4}{A}\overset{3}{G}\overset{2}{A}\overset{1}{R}$

C : $\overset{1}{C}\overset{2}{A}\overset{3}{L}\ \overset{4}{C}\overset{5}{U}\overset{6}{T}\overset{7}{T}A$

Dis^n fn | Cumulative dis^n fn

$$\frac{d}{dx} F(x) = f(x) \quad \text{prob. density fn.}$$

$$F(x) = \int_{-\infty}^{x} f(x)\,dx.$$

Expectation

$$E(x) = \sum_{x=0}^{n} x \cdot p(x) \qquad x : D.R.V$$

$$= \int_{-\infty}^{\infty} x f(x)\,dx \qquad x : C.R.V$$

$$\sum p(x) = 1 \quad , \quad \int_{-\infty}^{\infty} f(x)\,dx = \underline{1}$$

Variance :

$$V(x) = E(X^2) - [E(X)]^2$$

$$= E[X - E(X)]^2$$

$$V(x) = \sum x^2 p(x) - (\sum x p(x))^2 \qquad x : DRV$$

$$= \int x^2 f(x)\,dx - \left[\int x \cdot f(x)\,dx\right]^2 \qquad x : CRV$$

$$\sigma_x^2 = \frac{1}{N} \sum f x^2 - \left(\frac{1}{N} \sum f x\right)^2 = \frac{1}{N} \sum f(x - \bar{x})^2$$

$$\frac{1}{N} \sum f x^1 = E(X_1)$$

$$\frac{1}{N} \sum f x^2 = E(X^2)$$

$$\vdots$$

$$\frac{1}{N} \sum f x^r = \underline{E(X^r)}$$

Properties Of Expectation

- If 'x' a r.v and 'a' constant $E(ax) = a\,E(x)$
- If X & Y are r.v's.

$$E(X+Y) = E(X) + E(Y)$$

$$E(X-Y) = E(X) - E(Y)$$

- If X & Y are r.v's

$$E(X \cdot Y) = E(X)\,E(Y/X)$$

$$= E(Y)\,E(X/Y) \longrightarrow \text{Conditional expectation.}$$

- X & Y are r.vs then $E(X.Y) = E(X)E(Y)$
- If $Y = ax+b$, $a, b \rightarrow$ constants.

  $$E(Y) = E(aX+b) = E(aX)+E(b)$$
  $$= aE(x)+b \qquad E(\text{const.}) = \text{const.}$$
- $E(E(E(x))) = E(X)$

Properties of Variance

- If X is a r.v and 'a' a constant

  $$V(aX) = a^2 V(x).$$
  $$V(-Y) = (-1)^2 V(Y) = V(Y)$$
- If X & Y are independent R.Vs

  $$V(X+Y) = V(X)+V(Y).$$
  $$V(X-Y) = V(X)+V(-Y) = \underline{\underline{V(X)+V(Y)}}$$
  $$V(X \pm Y) = V(X)+V(Y)$$
- If a & b are constants, X & Y Independent R.Vs.

  $$V(ax-by) = a^2 V(x) + b^2 V(y)$$
  $$V(x/a - y/b) = \frac{1}{a^2} V(x) + \frac{1}{b^2} V(y)$$
- If $Y = ax+b$ where a, b are constants.

  $$V(Y) = V(ax+b) = V(ax)+V(b) = a^2 V(x) \qquad V(\text{const}) = 0$$
- If X & Y are r.vs

  $$V(x+y) = V(x)+V(y)+2\,\mathrm{Cov}(x,y)$$
  $$V(x-y) = V(x)+V(y)-2\,\mathrm{cov}(x,y)$$
  $$\boxed{\mathrm{Cov}(x,y) = E(xy) - E(x)E(y)}$$
  $$\mathrm{Cov}(a,b) = E(ab) - E(a)E(b) = ab - ab = \underline{\underline{0}} \qquad a \text{ \& } b \rightarrow \text{const.}$$
- If X & Y are independent R.V, covariance of X & Y $\mathrm{Cov}(x,y) = 0$

  But the converse of statement is not true.
- Variance and covariance are independent of change of origin and dependent of change of scale.

- Expectation (mean) dependent of change of origin as well as dependent of change of scale.

SKEWNESS

Skewness is defined as the lack of symmetry.

$\beta_1 = \frac{\mu_3^2}{\mu_2^3}$ μ_3 : 3rd central moment.

μ_2 : variance.

- If μ_3 value is -ve, then the curve is known as -vely skewed.
- If μ_3 value is +ve, then the curve is known as +vely skewed.
- If $\mu_3 = 0$, then the curve is known as symmetry ($\beta_1 = 0$)

Problems

Q1 Find the expectation of the no. on a die when it is thrown.

x R.V	1	2	3	4	5	6
P(x)	1/6	1/6	1/6	1/6	1/6	1/6

$E(X) = \text{mean} = \sum_{x=1}^{6} x\,P(x) = 1 \times P(1) + 2P(2) + 3P(3) + 4P(4) + 5P(5) + 6P(6)$

$= \frac{1}{6}[1+2+3+4+5+6] = 21/6 = \underline{\underline{7/2}}$

Q1. Find the variance for single dice.

$V(x) = E(X^2) - [E(X)]^2$

$E(x^2) = \sum_{x=1}^{6} x^2\,P(x) = \frac{1}{6}[1^2+2^2+3^2+4^2+5^2+6^2] = \underline{\underline{91/6}}$

$V(x) = 91/6 - (7/2)^2 = \frac{91}{6} - \frac{49}{4} = \frac{182-147}{12} = \underline{\frac{35}{12}}$

- The mean and variance for the sum of the no's on the dice is

$$\boxed{\begin{aligned} E(X) &= \frac{7n}{2} \\ V(X) &= \frac{35}{12}n \end{aligned}}$$

$n \longrightarrow$ no. of dice.

Q2. 3 unbiased dice are thrown. Find the mean and variance of the sum of no.s on them.

$E(X) = \frac{7}{2} \times 3 = \underline{\underline{21/2}}$

$V(X) = \frac{35}{12} \times 3 = \underline{\underline{35/4}}$

Q3. when two dice are rolled, find the expectation for sum 7.

Ans $E(X) = x \cdot P(x)$

$E(7) = 7 \cdot P(7) = 7 \cdot \frac{6}{36} = \underline{7/6}$

Q4. A player tosses 3 coins. He wins Rs 500 if 3 heads occur. Rs 800 if 2 heads occur, Rs 100 if only 1 head occur. On the other hand, if 3 tails occur he looses Rs 1500. Find expected value of the game.

Ans

x no of head	+500 3	+800 2	+100 1	-1500 0
p(x)	1/8	3/8	3/8	1/8
	$3C_3$	$3C_2$	$3C_1$	$3C_0$

Value of the game = Gain - loss

$= 500 \times \frac{1}{8} + 800 \times \frac{3}{8} + 100 \times \frac{3}{8} - 1500 \times \frac{1}{8}$

$= \frac{1700}{8} - \frac{1500}{8} = \underline{\text{Rs } 25/-}$

Note ; If the game is said to be balanced or fade, then the value of game = 0 (No loss & No gain).

Q5 A man was given n keys of which 1 fits the lock. He tries them successively without replacement to open the lock. what is the prob. that the lock will be open after the nth trial. Also determine mean and variance.

Ans With ~~out~~ replacement ⟹ Independent trials.

without replacement ⟹ dependent trials.

P (opening lock), 1st trial = $1/n$

" , 2nd trial = $1/n-1$

" , 3rd trial = $1/n-2$

⋮

P(opening lock, 1st success, 2nd trial) = $(1 - 1/n)\frac{1}{n-1} = \frac{n-1}{n} \times \frac{1}{n-1} = \underline{1/n}$

P(" " , " , 3rd trial) = $(1 - 1/n)(1 - 1/n-1)(1 - 1/n-2)$

$= \underline{1/n}$

III'ly P(opening lock 1st success, rth trial) = 1/n.

$E(X) = \frac{n+1}{2}$

$V(X) = \frac{n^2-1}{12}$

Q6 A man was given 100 - 999 keys. Find the prob. that the lock will be open at 450th trial by with and without replacement.

Ans 450th trial lock open WOR = 1/n = 1/900

Lock 450th trial W.R

$$P(450\text{th trial } WR) = \underbrace{(1 - 1/900)^{449}}_{P(\text{previous failures})} \times 1/900$$

Note : The probability for the 1st success at Nth trial with replacement is $P(x=n)_{WR} = q^{n-1} \cdot p$

a ⟶ failure probability.

p ⟶ success probability.

Q7. If x is a continuous R.V and $f(x) = K e^{-x^2/2}$ $-\infty < x < +\infty$ find i) K ii) E(X), V(x).

Ans

Note : $\sqrt{n} = \int_0^\infty e^{-x} x^{n-1} dx \quad n > 0$

$\sqrt{n} = n-1(\sqrt{n-1}) = (n-1)!$

$\sqrt{1} = 1$

$\sqrt{1/2} = \sqrt{\pi}$

$\sqrt{0}$: doesnot exist.

Since $\int_{-\infty}^{\infty} f(x)dx = 1$

$\int_{-\infty}^{\infty} K e^{-x^2/2} dx = 1$

$K \int_{-\infty}^{\infty} e^{-x^2/2} dx = 1$

$2K \int_0^{\infty} e^{-x^2/2} dx = 1$

Put $x^2/2 = u$

$x^2 = 2u$

$2x\,dx = 2du$

$dx = \frac{\cancel{2}du}{\cancel{2}x} = \frac{du}{\sqrt{2u}}$

$$2k\int_0^\infty e^{-u}\, u^{-1/2}\, \frac{du}{\sqrt{2}} = 1$$

$$\sqrt{2}k \int_0^\infty e^{-u}\, u^{-1/2}\, du = 1$$

$$\sqrt{2}k \int_0^\infty e^{-u}\, u^{1/2-1}\, du = 1$$

$$\sqrt{2}k\, \Gamma 1/2 = 1$$

$$\sqrt{2}k\sqrt{\pi} = 1$$

$$k = \frac{1}{\sqrt{2\pi}}$$

$$E(X) = \int_{-\infty}^{\infty} x f(x)dx = \frac{-1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} x \cdot x^{-x^2/2}\, dx = 0$$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} x^2 e^{-x^2/2}dx$$

$$= \frac{2}{\sqrt{2\pi}}\int_0^\infty \cancel{x^2 e}\; x^2 e^{-x^2/2}dx = \frac{\cancel{\sqrt{2}}}{\sqrt{\pi}}\int_0^\infty 2u e^{-u}\, u^{-1/2}\frac{du}{\cancel{\sqrt{2}}}$$

$$= \frac{2}{\sqrt{\pi}}\int_0^\infty u^{1/2}\, e^{-u}du = \frac{2}{\sqrt{\pi}}\,\Gamma 3/2 = \frac{2}{\sqrt{\pi}} \times \frac{1}{2} \times \Gamma 1/2 = \underline{1}$$

$$V(X) = E(X^2) - [E(X)]^2 = 1 - 0 = \underline{\underline{1}}$$

Q8 If X is a continuous R.V and $f(x) = |x|$, $-1 < x < 1$ find $V(x)$.

Ans $E(X) = \int_{-1}^{1} x f(x)\, dx = \int_{-1}^{1} x|x|dx = 0$

$$E(X^2) = \int_{-1}^{1} x^2 f(x)dx = \int_{-1}^{1} x^2 |x| dx = 2\int_0^1 x^2 \cdot x\, dx$$

$$= 2\left[\frac{x^4}{4}\right]_0^1 = 2/4 = \underline{\underline{1/2}}$$

$$V(X) = 1/2 - 0 = \underline{\underline{1/2}}.$$

Q9. If X & Y are the R.V, mean(X) = 10, variance(X) = 25, find the values of a, b such that $y = ax - b$ has $E(Y) = 0$, $V(Y) = 1$.

Ans $V(Y) = a^2 V(x)$

$1 = 25a^2$

$a^2 = 1/25$

$a = 1/5$

$E(Y) = a\,E(x) - b$

$0 = 10a - b$

$b = 10a = 10 \times 1/5 = \underline{\underline{2}}$

Bivariate Data :

Case I : Continuous R.V

If X & Y are 2-D continuous R.Vs and its probability fn is known as joint prob. density fn and is denoted by $f(x,y)$.

The marginal density fns are.

- $f(x) = \int_y f(x,y)\,dy$
- $f(y) = \int_x f(x,y)\,dx$

If X & Y are 2-D, continuous, independent R.V iff $f(x,y) = f(x) \cdot f(y)$.

ie JPDF = mdf(x) mdf(y)

- Relation b/w JDF and JPDF is

$$\frac{d^2}{dxdy} F(x,y) = f(x,y)$$

$$\text{or } F(x,y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f(x,y)\,dx\,dy.$$

Conditional PDF is $f(x/y) = \dfrac{f(x,y)}{f(y)}$ $(f(y) \neq 0)$

Conditional expectation $E(x/y) = \dfrac{E(xy)}{E(y)}$ $(E(y) \neq 0)$

Case II : Discrete R.V

If X & Y are 2-D discrete ~~and~~ R.V and its probability fn is known as joint prob. mass fn denoted by $P(x,y)$.

The marginal mass fn are

$$P(x) = \sum_{y} P(x,y)$$

$$P(y) = \sum_{x} P(x,y)$$

Q1 If x & y are 2-D continuous R.V and its corresp. joint probability fn is $f(x,y) = x \cdot y$. $x \rightarrow 0$ to 1, $y \rightarrow 0$ to 1.

i) Find mean and variance of y.

ii) $E(X,Y)$, $Cov(X,Y)$.

iii) $f(x/y)$, $E(X/Y)$

iv). Check X & Y are independent variable or not.

Ans $f(x,y) = \int_{y} f(x,y)dy = \int_{0}^{1} xy\,dy = x\int_{0}^{1} y\,dy = x\left[\frac{y^2}{2}\right]_0^1 = \underline{\underline{x/2}}$

$f(y) = y/2$

$E(X) = \int_{0}^{1} x\, f(x)\,dx = \int_{0}^{1} x \cdot \frac{x}{2}\,dx = \underline{\underline{1/6}}$

$E(Y) = 1/6$

$V(Y) = E(Y^2) - [E(Y)]^2$

$E(Y^2) = \int_{0} y^2 f(y)\,dy = \int_{0}^{1} y^2 \cdot \frac{y}{2}\,dy = \underline{\underline{1/8}}$

$V(Y) = \frac{1}{8} - \left(\frac{1}{6}\right)^2 = \frac{1}{8} - \frac{1}{36} = \frac{9-2}{72} = \underline{\underline{7/72}}$

$E(XY) = \int_{0}^{1}\int_{0}^{1} xy\, f(x,y)\,dxdy = \int_{0}^{1}\int_{0}^{1} x^2y^2\,dxdy = \underline{\underline{1/9}}$

$Cov(X,Y) = E(X \cdot Y) - E(X)E(Y) = \frac{1}{9} - \frac{1}{6} \times \frac{1}{6} = \frac{1}{9} - \frac{1}{36} = \frac{4-1}{36} = \underline{\underline{1/12}}$

$f(x|y) = \frac{f(x,y)}{f(y)} = \frac{xy}{y/2} = 2x$

$E(X|Y) = \frac{E(xy)}{E(y)} = \frac{1/9}{1/6} = \underline{\underline{2/3}}$

$f(x,y) = f(x)\,f(y)$

$xy \neq \frac{x}{2} \cdot \frac{y}{2}$

∴ X & Y are <u>dependent</u> R.V

Q2. If X & Y are 2-D discrete R.V , the corresp. prob. fn is.

X \ Y	-1	0	+1
-1	1/4		
0		1/2	
1			1/4

Find $P(x+y=2 / x-y=0)$

Ans $$P(x+y=2 | x-y=0) = \frac{P(x+y=2 \cap x-y=0)}{P(X-Y=0)}$$

$$= \frac{P(X=1, y=1)}{P(x=-1, y=-1)+P(x=0, y=0)+P(x=1, y=1)} = \frac{1/4}{1/4+1/2+1/4} = \underline{\underline{1/4}}$$

BINOMIAL DISTRIBUTION

If X is said to be Binomial R.V, it allows the values from 0 → n with the parameters n, p and its probability mass fn is.

$$B(x ; n,p) = p(x) = \begin{cases} (nC_x)\, p^x q^{n-x} & 0 \le x \le n \\ 0 & \text{otherwise.} \end{cases}$$

$p+q=1$

$q=1-p$

Conditions

- Observations are independent (n is small)
- p (success) is constant (p is large)
- Mean is greater than variance.

Properties

- $E(X) =$ mean $= np$
- $V(x) = \mu_2 = npq$
- $\mu_3 = npq\,(q-p) = npq\,(1-2p)$
- $$\beta_1 = \frac{\mu_3^2}{\mu_2^2} = \frac{n^2p^2q^2\,(1-2p)^2}{n^3\,p^3q^3} = \underline{\underline{\frac{(1-2p)^2}{npq}}}$$

Moment Generating fn

- $M_x(t) = E(e^{tx}) = (q+pe^t)^n$

Characteristic fn

- $\phi_x(t) = E(e^{itx}) = (q+pe^{it})^n$

Note

- For $p = 1/2$, then the binomial distribution is symmetric ($\mu_3 = 0$)
- For $p < 1/2$, $\Longrightarrow \mu_3$ is +ve, then the curve is +vely skewed.
- For $p > 1/2$ $\Longrightarrow \mu_3$ is -ve, then the curve is -vely skewed.
- Sum of the independent binomial R.Vs is also a Binomial R.V.
- The moment generating fn is used to find addition and difference b/w R.V with their prob. fn.
- The characteristic fn is used for finding the convolution b/w the R.V and ratio b/w the R.V.

Problems

Q1 Find the probability of getting a 9 exactly twice in 3 times with a pair of dice.

Ans $n = 3$

$x = 2$

$P(\text{sum} = 9) = 4/36 = 1/9$

$q = (1 - 1/9) = 8/9$

$P(X = 2) = 3C_2 (1/9)^2 (8/9)^1 = \underline{\underline{8/243}}$

Q2. The prob. of a man hitting the target is 1/3.

i) If he fires 5 times, what is the probability of his hitting the target atleast twice.

ii) How many times must he fire so that the probability of his hitting the target atleast once is more than 90%.

Ans $p = 1/3$

$q = 2/3$

i) $n = 5$

$P(x \geq 2) = 1 - P(x < 2)$

$= 1 - [P(x=0) + P(x=1)] = 1 - [(2/3)^5 + 5C_1(1/3)^1(2/3)^4]$

$= \underline{\underline{131/243}}$

ii) $P(x \geq 1) \geq 90\%$

$1 - P(x=0) > 0.9$

$P(x=0) < 0.1$

$q^n < 0.1$

$(2/3)^n < 0.1$

$n = 5.72 \simeq \underline{\underline{6}}$

Q3 Two dice are rolled 120 times. Find the average no. of times in which the no. on the first dice exceeds the no. on 2nd dice.

Ans $n = 120.$

(2,1) (3,1) (3,2) (4,1) (4,2) (4,3) (5,1) (5,2) (5,3) (5,4) (6,1) (6,2) (6,3) (6,4) (6,5)

$P(15/36)$

$E(X) = np = 120 \times \frac{15}{36} = \underline{\underline{50}}$

Q4. If x & y are the binomial R.V $x \sim B(2,p)$, $y \sim B(4,p)$ if $P(x \geq 1) = 5/9$, find $P(y \geq 1)$?

Ans $n_x = 2$

$n_y = 4$

$P(x \geq 1) = 5/9$

$1 - P(x=0) = 5/9$

$P(x=0) = 4/9$

$q^n = 4/9$

$q^2 = 4/9$

$q = 2/3 \implies p = 1/3$

$P(Y \geq 1) = 1 - P(Y=0)$

$= 1 - q^n = 1 - (1/3)^4 = 1 - \frac{16}{81} = \underline{\underline{\frac{65}{81}}}$

Q5. If x is a binomial R.V, then find the value of $\sum_{x=0}^{n} \frac{x}{n} (nCx) p^x q^{n-x}$

Ans $= \frac{1}{n} \sum_{x=0}^{n} x (nCx) p^x q^{n-x} = \frac{1}{n} \left[\sum_{x=0}^{n} x \cdot P(x \right]$

$= \frac{1}{n} E(X) = \frac{1}{n} np = \underline{p}$

Q6. If x is a binomial R.V & $E(X)=4$, $V(X)=4/3$ find i) $x \leq 2$

2) Comment on β_1.

Ans $E(X) = np = 4$

$V(X) = npq = 4q = 4/3$

$\therefore q = 1/3$

$p = 2/3$

$np = 4$

$\therefore n = \frac{3}{2} \times 4 = \underline{\underline{6}}$

$$P(x \leq 2) = P(X=0) + P(X=1) + P(X=2) = (1/3)^6 + 6C_1(1/3)^5(2/3) + 6C_2(1/3)^4(2/3)^2$$

$$= \frac{1}{729}[1+12+60] = \underline{\underline{\frac{73}{729}}}$$

ii) Given data is -vely skewed ($p = 2/3 > 1/2$)

POISSON DIS^n :

→ arrival rate

→ Rare occurance

→ Defect probability.

→ Evalutionary process

[non stationary / Time dep]

If X is a Poisson R.V defined in the interval $0 \rightarrow \infty$ with a parameter $\lambda (>0)$ and its probability mass fn is

$$P(x : \lambda > 0) = P(x) = \begin{cases} \frac{e^{-\lambda}\lambda^x}{x!} & \lambda > 0 \\ & 0 \leq x \leq \infty \\ 0 & \text{otherwise} \end{cases}$$

Conditions

- Observation are infinitely large ($n \rightarrow \infty$)
- probability of success is very small ($p \rightarrow 0$)
- $np = \lambda \implies p = \frac{\lambda}{n}$
- $P(x : np) = \frac{e^{-np}(np)^x}{x!}$

Approximation of Binomial.

Poisson Process : $P(x : \lambda, t > 0) = \frac{e^{-\lambda t}(\lambda t)^x}{x!}$

Properties

- $E(X) = \text{mean} = \lambda$
- $V(X) = \mu_2 = \lambda$
- $\mu_3 = \lambda$
- $\beta_1 = \frac{\mu_3^2}{\mu_2^3} = \frac{\lambda^2}{\lambda^3} = 1/\lambda \longrightarrow$ B_1 always > 0

∴ Positively skewed.

MGF :

- $M_x(t) = \cancel{e^{\lambda} f e^t} \quad e^{\lambda(e^t - 1)}$

Char. fn

- $\phi_x(t) = e^{\lambda(e^{it} - 1)}$

Note :

- In Poisson disn mean = variance = parameter = λ
- It is always positively skewed.
- Sum of the independent Poisson R.V is also Poisson R.V.
- The difference b/w the independent Poisson R.V is not a Poisson R.V

Q1. A telephone switchboard receives 20 calls on an average during an hour. Find the probability that for a period of 5 mins.

i) no. call is received.

ii) Exactly 3 calls are received.

iii) Atleast 2 calls are received.

60 min λ 20

1 min $= \frac{20}{60} = .33$

5 " $= 5 \times .33$

$= 1.65$

Ans i) $P(X=0) = \frac{e^{-1.65}(1.65)^0}{0!} = \underline{\underline{e^{-1.65}}}$

ii) $P(X=3) = \frac{e^{-1.65}(1.65)^3}{3!}$

iii) $P(X \geq 2) = 1 - P(X<2)$

$= 1 - [P(X=0) + P(X=1)]$

$= 1 - [e^{-1.65} + (1.65)e^{-1.65}] = \underline{\underline{.491}}$

Q2. If X & Y are two independent Poisson R.V such that $P(X=1) = P(X=2)$ & $P(Y=2) = P(Y=3)$, find $V(3X-4Y)$?

Ans

$$P(X=1) = P(X=2) \qquad\qquad P(X=2) = P(Y=3)$$

$$\frac{e^{-\lambda}\lambda^1}{1!} = \frac{e^{-\lambda}\lambda^2}{2!} \qquad\qquad \frac{e^{-\theta}\theta^2}{2!} = \frac{e^{-\theta}\theta^3}{3!}$$

$$1 = \lambda/2 \qquad\qquad 1 = \theta/3$$

$$\lambda = 2 \qquad\qquad \theta = 3$$

$$E(X) = 2 \qquad\qquad E(Y) = 3$$

$$V(X) = 2 \qquad\qquad V(Y) = 3$$

$$V(3X-4Y) = 3^2V(X) + 4^2V(Y) = 9\times 2 + 16\times 3 = \underline{\underline{66}}$$

Q3. If X_1 & X_2 are two independent R.V with variances 1, 2. find $P(X_1+X_2=4)$.

Ans

$$P(X_1+X_2=K) = \frac{e^{-(\lambda_1+\lambda_2)}(\lambda_1+\lambda_2)^K}{K!}$$

Since sum of Poisson R.V is a Poisson R.V

$\lambda_1 = 1,\ \lambda_2 = 2.$

$$P(X_1+X_2=4) = \frac{e^{-(1+2)}(1+2)^4}{4!} = \underline{\underline{\frac{e^{-3}3^4}{4!}}}$$

Q4. If X is a Poisson R.V then find the value of $\sum_{x=0}^{\infty}\frac{x}{\lambda}\frac{e^{-\lambda}\lambda^x}{x!}$

Ans

$$= \frac{1}{\lambda}\sum_{x=0}^{\infty}\frac{xe^{-\lambda}\lambda^x}{x!} = \frac{1}{\lambda}\sum_{x=0}^{\infty}x\,P(x)$$

$$= \frac{1}{\lambda}E(X) = 1/\lambda \times \lambda = \underline{\underline{1}}$$

Q5. If X is a Poisson R.V and $E(X^2) = 6$ find $V(X)$

Ans

$$V(X) = E(X^2) - [E(X)]^2$$

$$\lambda = 6 - \lambda^2 \qquad\qquad \because V(X) = E(X) = 6$$

$$\lambda^2 + \lambda - 6 = 0$$

$$(\lambda+3)(\lambda-2) = 0$$

$$\lambda = \underset{\times}{-3}, 2 \qquad \lambda = 2 \Longrightarrow V(X) = \underline{\underline{2}}$$

NORMAL/ GAUSSIAN DIS^N

It is called Parent dis^n because it covers whole natural numbers.

- If x is said to be a normal R.V defined in $(-\infty, \infty)$ with mean $= \mu$ & variance $= \sigma^2$, then the R.V is known as normal R.V and its density fn is

$$N(x : \mu, \sigma^2) = f(x) = \begin{cases} \frac{1}{\sigma\sqrt{2\pi}} e^{-1/2\left(\frac{x-\mu}{\sigma}\right)^2} & -\infty < x < +\infty \\ & -\infty < \mu < +\infty \\ & 0 < \sigma < \infty \\ 0 & \text{otherwise} \end{cases}$$

Standard Normal R.V

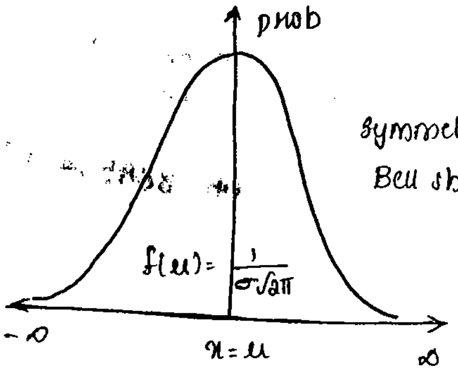
If x is a normal R.V with mean $= 0$ and variance $= 1$, then the R.V is known as standard normal R.V and its density fn is.

$$N(0,1) = f(x) = \frac{1}{\sqrt{2\pi}} e^{-1/2\, x^2}$$

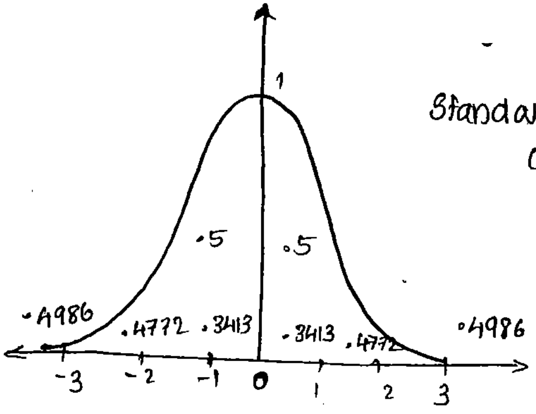
Mathematically the standard normal variate is denoted by 'Z' and is given by.

$$Z = \frac{X - E(X)}{\sqrt{V(x)}}$$

$$-3 \leq Z \leq +3$$



Symmetry
Bell shaped



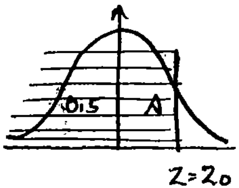
Standard
curve.

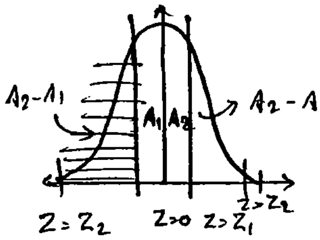
x & $B(x, p)$

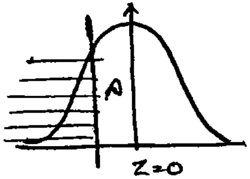
$$z = \frac{x - np}{\sqrt{npq}}$$

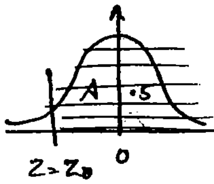
Areas under Normal curve :

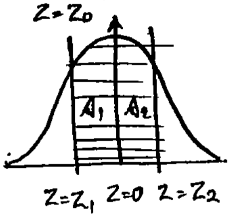
- $P(Z \leq Z_0) = 0.5 + A(z_0 \text{ +ve})$
- $P(Z \leq Z_0) = 0.5 - A(z_0 \text{ -ve})$
- $P(Z_1 \leq Z \leq Z_2) = A_1 + A_2 \begin{cases} Z_1 \text{ -ve} \\ Z_2 \text{ +ve} \end{cases}$
- $P(Z_1 \leq Z \leq Z_2) = A_1 - A_2 \{ Z_1 \,\&\, Z_2$ (+ve or -ve)
- $P(Z \geq Z_0) = 0.5 + A \quad (Z_0 \text{ -ve})$
- $P(Z \geq Z_0) = 0.5 - A \quad (Z_0 \text{ +ve})$

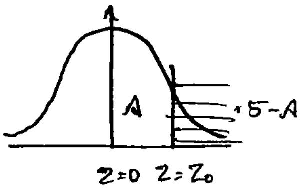












Q1. If 6 is distributed with S.D = 3.33, $\sigma = 20$, find the prob b/w 21.11 & 26.66. The area under the curve Z=0 to Z = 0.33 is .1293.

Ans $P(\overset{x_1}{21.11} \leq x \leq \overset{x_2}{26.66})$

Z=0 to Z = 0.33 is 0.1293.

$$Z_1 = \frac{x_1 - \mu}{\sigma} = \frac{21.11 - 20}{3.33} = \frac{1.11}{3.33} = \underline{.33}$$

$$Z_2 = \frac{x_2 - \mu}{\sigma} = \frac{26.66 - 20}{3.33} = \frac{6.66}{3.33} = 2$$

$$P(21.11 \leq x \leq 26.66) = P(0.33 \leq Z \leq 2)$$

$$= 0.4772 - 0.1293 = \underline{.3479}$$

Q2. A die is rolled 180 times. Using the normal distn find the probability that the face 4 will turn up atleast 35 times. The area under the normal curve Z=0 to Z=1 is .3413.

Ans $n = 180$

$p = 1/6 \quad q = 5/6$

$E(X) = np = 180 \times 1/6 = 30$

$V(X) = npq = 180 \times 1/6 \times 5/6 = 25$

$$Z = \frac{x - m}{\sigma} = \frac{35 - 30}{\sqrt{25}} = 5/5 = 1$$

$$P(x \geq 35) = P(Z \geq 1) = 0.5 - .3413 = \underline{0.1587}$$

Q3. If X is normally distributed with mean = 30 and standard deviation = 5 find $P(|x - 30| > 5)$

Ans $P(|x - 30| > 5) = 1 - P(|X - 30| \leq 5)$

$P(|x - 30| < 5) = P(-5 \leq x - 30 \leq 5) = P(25 \leq x \leq 35)$

$$Z_1 = \frac{x_1 - \mu}{\sigma} = \frac{25 - 30}{5} = -1, \quad Z_2 = \frac{35 - 30}{5} = 1$$

$P(25 \le x \le 35) = P(-1 \le z \le +1) = .3413 + .3413 = \underline{\underline{.6826}}$

$P(|x-30| > 5) = 1 - P(|x-30| \le 5)$

$= 1 - .6826 = \underline{\underline{.3174}}$

Properties

- $E(x) = \text{Mean} = \mu$
- $V(x) = \mu_2 = \sigma^2$
- $\mu_3 = 0$
- $\beta_1 = 0$ [symmetry]

Moment Generating fn : $M_x(t) = e^{t\mu + \frac{t^2\sigma^2}{2}}$

Characteristic fn : $\phi_x(t) = e^{it\mu - \frac{t^2\sigma^2}{2}}$

$X \sim N(0,1)$

$M_x(t) = e^{t^2/2}$

$\phi_x(t) = e^{-t^2/2}$

- Sum of the independent R.V is also a normal R.V.
- The difference b/w the independent normal R.V is also a normal R.V. (linear combination)

Uniform Distribution

[Rectangular Distn]

If X is a uniform R.V in (a,b) where $a<b$ and its probability density fn is

$$u(a,b) = f(x) = \begin{cases} \frac{1}{b-a} & a<x<b \\ & a<b \\ 0 & \text{otherwise} \end{cases}$$

Mean and Variance are.

- Mean $E(x) = \frac{a+b}{2}$
- Variance $V(x) = \frac{(b-a)^2}{12}$

Note: If X is a uniform R.V in $(-a,a)$ its density fn is

$f(x) = 1/2a$

$E(X) = 0$ $V(X) = \frac{a^2}{3}$

Correlation & Regression

Correlation : The relation b/w the 2-D R.V in ~~which~~ bivariable data is known as correlation. That means the changes in one variable is affecting the changes in ~~an~~other variable parallely, then those variables are known as correlated variable.

Type Of Correlation

1) +ve correlation
2) −ve correlation.

1) If the changes in both variables are in same direction (↑ or ↓) then those variables are known as +vely correlated variables.

2) If the changes in the one variable is affecting the changes of the other variable parallely in the reverse dirn ~~also~~ then those variables are known as −vely correlated variables.

Karl Pearson's Correlations ;

$$r(x,y) = \frac{Cov(x,y)}{\sigma_x \sigma_y} \qquad \text{where } cov(x,y) = E(XY) - E(X)E(Y)$$

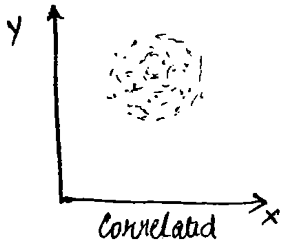
$$= \frac{1}{n}\Sigma xy - \bar{x}\bar{y}$$

$$-1 \leq \overset{r}{\cancel{n}} \leq 1$$

Scatter Diagram

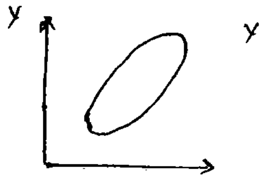
It is a graphical representation of correlation. If the points are very closer and very thick on the x-y plane, then their points are known as correlated pts.

If the points are widely spreaded, ~~they~~ they are said to be uncorrelated.

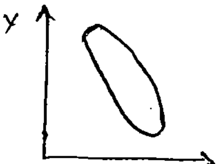


Correlated

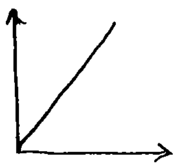
Uncorrelated



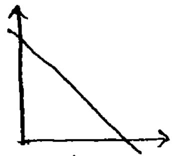
+ve



−ve



perfect +ve



perfect −ve

If the R.V are independent $\Longrightarrow \text{Cov}(x,y) = 0$.

$\Longrightarrow \gamma(x,y) = 0$

$\longrightarrow$ They are highly uncorrelated

Regression

The linear relationship b/w the 2-D R.V in bivariate data is known as regression

Lines Of Regression

- Y on X

$$(Y - \bar{Y}) = \gamma \times \underbrace{\frac{\sigma_y}{\sigma_x}}_{b_{YX}} (X - \bar{X})$$

- X on Y

$$(X - \bar{X}) = \gamma \underbrace{\frac{\sigma_x}{\sigma_y}}_{b_{XY}} (Y - \bar{Y})$$

Properties

Correlation coeff is geometric mean b/w.

$$b_{YX} \cdot b_{XY} = \gamma \frac{\sigma_y}{\sigma_x} \times \gamma \frac{\sigma_x}{\sigma_y} = \gamma^2 \qquad \therefore \gamma = \pm\sqrt{b_{XY} \times b_{YX}}$$

Note :
- Both the regression coeff must have same sign.

ie if both are +ve $\Longrightarrow \gamma$ is +ve

both are -ve $\Longrightarrow \gamma$ is -ve

- If $b_{YX} > 1 \Longrightarrow b_{XY} < 1$ & vice versa.
- If the regression coeff are equal $\Longrightarrow$ Variance also equal.

$$b_{xy} = b_{yx} \Longrightarrow \gamma \frac{\sigma_y}{\sigma_x} = \gamma \frac{\sigma_x}{\sigma_y}$$

$$\Longrightarrow \sigma_y^2 = \sigma_x^2$$

- Regression eqn satisfies the points $\bar{X}, \bar{Y}$.
- Angle b/w the regression lines $\theta = \tan^{-1}\left[\frac{1-\gamma^2}{|\gamma|} \quad \frac{\sigma_x \sigma_y}{\sigma_x^2 + \sigma_y^2}\right]$

$\gamma = 0 \Longrightarrow \theta = \pi/2$ $\quad \perp r$

$\gamma = 1 \Longrightarrow \theta = \pi$ $\quad \parallel$al

Q. The regression eqns are $x+2y=0$ & $2x+y=1$. Find i) r ii) $\bar{x},\bar{y}$.

Ans i) Y on X

$x+2y=0$

$2y=-x$

$y=\frac{-x}{2}$

$b_{yx} = -1/2$

X on Y

$2x+y=1$

$2x=1-y$

$x = 1/2 - y/2$

$b_{xy} = -1/2$

$r = -\sqrt{1/2 \times 1/2} = -1/2$

ii)

$2\bar{x} + 4\bar{y} = 0$

$2\bar{x} + \bar{y} = 1$

$3\bar{y} = -1$

$\bar{y} = -1/3$

$\bar{x} = 2/3$

$\therefore (\bar{x},\bar{y}) = (2/3, -1/3)$

Q. $x-3y=4$ (y on x)

$2x-y=1$ (x on y)

Find i) r ii) $\bar{x},\bar{y}$

Ans i) Y on X

$x-3y=4$

$-3y = 4-x$

$y = -\frac{4}{3} + \frac{x}{3}$

$b_{yx} = 1/3$

X on Y

$2x-y=1$

$2x = 1+y$

$x = 1/2 + y/2$

$b_{xy} = 1/2$

$r = \sqrt{1/3 \times 1/2} = \sqrt{1/6}$

ii)

$2\bar{x} - 6\bar{y} = 8$

$2\bar{x} - \bar{y} = 1$

$-5\bar{y} = 7$

$\bar{y} = \frac{-7}{5}$

$\bar{x} = -1/5$

$(\bar{x},\bar{y}) = (-1/5, -7/5)$

LINEAR ALGEBRA

Determinants — MATRICES - A. R. Vasistha

Adjoint and determinant of a matrix

Rank of a matrix

Solution of Homogeneous and non homogeneous linear eqns.

Eigen values and Eigen vectors.

Cayley Hamilton Theorem.

Properties of Determinants

The value of the determinant will not be changed if rows and columns are interchanged.

ie $|A| = |A|^T$

$$|A| = \begin{vmatrix} 5 & 4 \\ 1 & 2 \end{vmatrix} = 10-4 = 6$$

$$|A^T| = \begin{vmatrix} 5 & 1 \\ 4 & 2 \end{vmatrix} = 10-4 = 6$$

The value of the determinant will be multiplied by '-1' if two rows or two columns are interchanged

$$|A| = \begin{vmatrix} 5 & 4 \\ 1 & 2 \end{vmatrix} = 10-4 = 6$$

$$|A| = \begin{vmatrix} 1 & 2 \\ 5 & 4 \end{vmatrix} = 4-10 = -6$$

The value of determinant can be zero in the following cases.

i) The elements in two rows or two columns are identical.

$$|A| = \begin{vmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{vmatrix} = 0$$

ii) The elements in two rows and two columns are proportional to each other.

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 3 & 6 & 9 \end{vmatrix} = 0 \qquad \because R_3 = 3R_1$$

All elements in any row or any column are zeros.

$$|A| = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 0 & 0 & 0 \end{vmatrix} = 0$$

If the elements in the determinant are of consecutive order. (valid for 3rd and higher order)

$$|A| = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 0$$

The first row of each element starts from the 2nd element of previous row such that the elements in that determinant are of consecutive order.

$$\begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix} = 0$$

The determinant of upper Δ^{lr}, lower Δ^{lr}, diagonal, scalar or identity matrix is the sum of its diagonal elements.

Upper Δ^{lr} : $|A| = \begin{vmatrix} 1 & 2 & 3 \\ 0 & 3 & 5 \\ 0 & 0 & 7 \end{vmatrix} = 1 \times 3 \times 7 = 21$

Lower Δ^{lr} : $|A| = \begin{vmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 5 & 6 \end{vmatrix} = 1 \times 3 \times 6 = 18$

Diagonal matrix : $\begin{vmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{vmatrix} = 1 \times 2 \times 3 = 6$

Scalar matrix : A diagonal matrix with same diagonal elements.

$$|A| = \begin{vmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{vmatrix} = 2 \times 2 \times 2 = \underline{\underline{8}}$$

Only skew symmetric matrices (Skew Hermitian Matrices) have their diagonal elements zeros.

Skew Symmetric

$\begin{bmatrix} 0 & -3 \\ 3 & 0 \end{bmatrix} \longrightarrow$ Sum of all elements of skew symmetric matrices will be zero.

$\begin{bmatrix} 0 & -3 & -8 \\ 3 & 0 & -2 \\ 8 & 2 & 0 \end{bmatrix}_{3\times 3} \longrightarrow$ Sum $= 0$.

$\therefore$ Sum of all elements of any order skew symmetric matrix is zero.

The number of non zero elements in 2×2 skew symmetric matrix is $2\times2-2 = 2$

No of non zero elements in 3×3 skew symmetric matrix is $3\times3-3 = 6$.

The no. of non zero elements in $n\times n$ skew symmetric matrix is $n\times n - n = \underline{\underline{n(n-1)}}$

Q4. The A has m rows and $(m+5)$ columns and B has n rows and $(11-n)$ columns, the orders of A & B if AB and BA are defined is ?

Ans AB defined $\Longrightarrow$ $m+5 = n$

BA defined $\Longrightarrow$ $11-n = m$

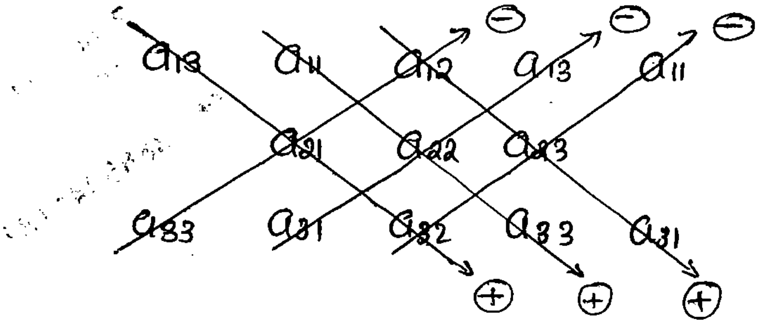
$$\begin{aligned} m-n &= -5 \\ m+n &= 11 \\ \hline 2m &= 6 \\ m &= \underline{\underline{3}} \\ n &= \underline{\underline{8}} \end{aligned}$$

Note :

Consider a 3x3 matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

$$|A| = a_{11}(a_{22} \times a_{33} - a_{32} \times a_{23}) - a_{12}(a_{21} \times a_{33} - a_{23} \times a_{31})$$
$$+ a_{13}(a_{21}a_{32} - a_{31}a_{22})$$

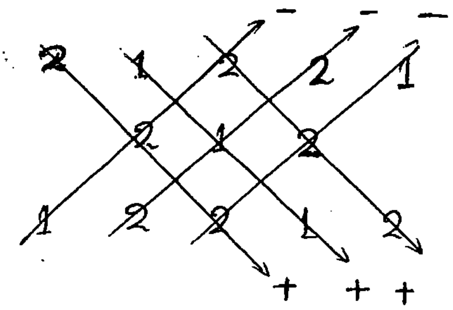
Simple way for calculating $|A|$



$$|A| = a_{13} \times a_{21} \times a_{32} + a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} - a_{12}a_{21}a_{33}$$
$$- a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32}$$

Q. Find determinant of $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$

Ans



$$|A| = 8 + 1 + 8 - 4 - 4 - 4 = \underline{\underline{5}}$$

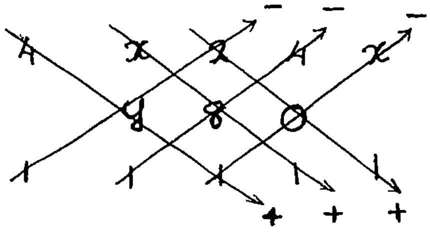
Q. The following represents equation of a straight line. The line passes through

$$\begin{vmatrix} x & 2 & 4 \\ y & 8 & 0 \\ 1 & 1 & 1 \end{vmatrix} = 0$$

a) 0,0 b) 3,4 c) 4,3 d) 4,4.

Ans

$|A| =$



If A is an $n \times n$ matrix then

$$|KA| = K^n |A|$$

Consider the matrix $|A| = \begin{vmatrix} 4 & 2 \\ 2 & 4 \end{vmatrix}_{2\times2}$

$$\therefore |3A| = 3^2 \times \begin{vmatrix} 4 & 2 \\ 2 & 4 \end{vmatrix} = 9 \times 12 = \underline{108}$$

If each element of a determinant contains sum of two elements, then that determinant should be expressed as sum of two determinants of the same order.

Eg : $|A| = \begin{vmatrix} a & a^2 & a^3+3 \\ b & b^2 & b^3+3 \\ c & c^2 & c^3+3 \end{vmatrix} = \begin{vmatrix} a & a^2 & a^3 \\ b & b^2 & b^3 \\ c & c^2 & c^3 \end{vmatrix} + \begin{vmatrix} a & a^2 & a3 \\ b & b^2 & 3 \\ c & c^2 & 3 \end{vmatrix}$

Q1. If $A = (a_{ij})_{3\times3}$, $B = (b_{ij})_{3\times3}$ such that $b_{ij} = 2^{i+j} \cdot a_{ij} \; \forall \; i,j$ and $|A| = 2$ then $|B| =$ ____ ?

a) 2^{10} b) 2^{11} c) 2^{12} d) 2^{13}

Ans Given $b_{ij} = 2^{i+j} \cdot a_{ij} \; \forall \; i,j$

$$\therefore |B| = \begin{vmatrix} 2^2 a_{11} & 2^3 a_{12} & 2^4 a_{13} \\ 2^3 a_{21} & 2^4 a_{22} & 2^5 a_{23} \\ 2^4 a_{31} & 2^5 a_{32} & 2^6 a_{33} \end{vmatrix}$$

$$= 2^2 \times 2^3 \times 2^4 \begin{vmatrix} a_{11} & 2a_{12} & 2^2 a_{13} \\ a_{21} & 2a_{22} & 2^2 a_{23} \\ a_{31} & 2a_{32} & 2^2 a_{33} \end{vmatrix}$$

$$= 2^2 \times 2^3 \times 2^4 \times 2 \times 2^2 \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$|A| = 2 \Longrightarrow |B| = 2^{12} \times 2 = \underline{\underline{2^{13}}}$$

Q2. If $A = a_{ij\,m\times n}$ such that $a_{ij} = i+j \ \forall\ i,j$, then sum of all elements of A is ____ ?

Ans $$|A| = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{vmatrix}$$

$$A = \begin{bmatrix} 1+1 & 1+2 & \cdots & 1+n \\ 2+1 & 2+2 & \cdots & 2+n \\ \vdots & & & \\ m+1 & m+2 & \cdots & m+n \end{bmatrix}_{m\times n}$$

$$= \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 2 & 2 & \cdots & \cdots & 2 \\ \vdots & & & & \\ m & m & \cdots & \cdots & m \end{bmatrix}_{m\times n} + \left[\begin{array}{ccccc} 1 & 2 & 3 & \cdots & n \\ 1 & 2 & 3 & \cdots & n \\ \vdots & & & & \\ 1 & 2 & 3 & \cdots & n \end{array}\right] \begin{array}{c} \frac{n(n+1)}{2} \\ + \\ \frac{n(n+1)}{2} \\ \vdots \\ \frac{n(n+1)}{2} \end{array}$$

$$\frac{m(m+1)}{2} + \frac{m(m+1)}{2} + \cdots + \frac{m(m+1)}{2} \Longrightarrow \frac{nm(m+1)}{2} \qquad \Downarrow \ \frac{mn(n+1)}{2}$$

Sum of m numbers $= \frac{m(m+1)}{2}$

$\therefore$ Sum of $[A] = \frac{nm(m+1)}{2} + \frac{mn(n+1)}{2}$

$$= \frac{mn}{2}\left[m+n+2\right]$$

Q3. Given $A = (a_{ij})_{n\times n}$ such that

i) $a_{ij} = i^2 - j^2 \ \forall\ i,j$

ii) $a_{ij} = i - j \ \forall\ i,j$

Find sum of the elements of the matrix in each case.

Ans $$A = \begin{bmatrix} 0 & -3 & -8 & \cdots & 1-n^2 \\ 3 & 0 & -5 & \cdots & 2^2-n^2 \\ +8 & 5 & 0 & \cdots & 3^2-n^2 \\ n^2-1^2 & n^2-2^2 & \cdots & \cdots & 0 \end{bmatrix} \Longrightarrow \text{Skew Symmetric Matrix}$$

$\therefore$ Sum of elements $= 0$

$$4y + 8x - 2y - 32 - x = 0$$

$$2y + 8x = 32$$

$$8x + 2y = 32$$

Substituting the options we get $\underline{\underline{(3,4)}}$

Q. Find the determinant of $A = \begin{bmatrix} 5 & 0 & 2 \\ 0 & 3 & 0 \\ 2 & 0 & 1 \end{bmatrix}$

Ans $|A| = 5 \times 3 + 2 \times -6 = 15 - 12 = \underline{\underline{3}}$

Note :

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

$$|A| = a_{11} \begin{vmatrix} a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{22} & a_{24} \\ a_{31} & a_{33} & a_{34} \\ a_{41} & a_{43} & a_{44} \end{vmatrix}$$

$$+ a_{13} \begin{vmatrix} a_{21} & a_{22} & a_{24} \\ a_{31} & a_{32} & a_{34} \\ a_{41} & a_{42} & a_{44} \end{vmatrix} - a_{14} \begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{vmatrix}$$

By using either row or column or both operation make all elements above and below or to the left and right of the selected element are zeros. And then expand the determinant along the row or column.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & b_{22} & b_{23} & b_{24} \\ 0 & b_{32} & b_{33} & b_{34} \\ 0 & b_{42} & b_{43} & b_{44} \end{bmatrix}$$

$$|A| = a_{11} \underbrace{\begin{vmatrix} b_{22} & b_{23} & b_{24} \\ b_{32} & b_{33} & b_{34} \\ b_{42} & b_{42} & b_{44} \end{vmatrix}}_{\text{new elements after row \& column opern.}}$$

Eg :
$$\begin{bmatrix} 1 & 2 & -1 & 5 \\ 0 & 4 & 5 & 7 \\ 1 & 1 & 1 & 3 \\ 2 & 3 & 4 & 9 \end{bmatrix}$$

$$R_3 \longrightarrow R_3 - R_1$$
$$R_4 \longrightarrow R_4 - 2R_1$$

$$A = \begin{bmatrix} 1 & 2 & -1 & 5 \\ 0 & 4 & 5 & 7 \\ 0 & -1 & 2 & -2 \\ 0 & -1 & 6 & -1 \end{bmatrix}$$

$$|A| = 1 \times \begin{vmatrix} 4 & 5 & 7 \\ -1 & 2 & -2 \\ -1 & 6 & -1 \end{vmatrix} = \underline{\underline{17}}$$

Q1. The value of the determinant $A = \begin{vmatrix} 1+b & b & 1 \\ b & 1+b & 1 \\ 1 & 2b & 1 \end{vmatrix}$

Ans $C_1 \longrightarrow C_1 + C_2$

$$\begin{vmatrix} 1+2b & b & 1 \\ 1+2b & 1+b & 1 \\ 1+2b & 2b & 1 \end{vmatrix}$$

proportional $\qquad C_1 = (1+2b)C_3$

$$\therefore \underline{|A| = 0}$$

Q2. Find the value of $\begin{vmatrix} 1/a & a & bc \\ 1/b & b & ca \\ 1/c & c & ab \end{vmatrix}$

Ans $C_1 \longrightarrow abc \times C_1$

$$\begin{vmatrix} bc & a & bc \\ ac & b & ca \\ ab & c & ab \end{vmatrix} = 0 \quad \text{or} \quad \begin{vmatrix} \frac{bc}{abc} & a & bc \\ \frac{ac}{abc} & b & ca \\ \frac{ab}{abc} & c & ab \end{vmatrix} = \frac{1}{abc}\Bigg| \qquad \Bigg| = \underline{\underline{0}}$$

Q3. $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}$ $\qquad$ $C_2 \longrightarrow C_2 - C_1$, $C_3 \longrightarrow C_3 - C_1$ $\begin{vmatrix} 1 & 0 & 0 \\ a & b-a & c-a \\ a^2 & b^2-a^2 & c^2-a^2 \end{vmatrix}$

$$= 1\left((b-a)(c^2-a^2)-(c-a)(b^2-a^2)\right)$$
$$= (b-a)\left[(c^2-a^2)-(c-a)(b+a)\right]$$

$$= (b-a)(c-a)\left[c+a-b-a\right] = (b-a)(c-a)(c-b)$$

$$= \underline{\underline{(a-b)(b-c)(c-a)}}$$

<u>Note</u> :
$$\begin{vmatrix} a^0 & b^0 & c^0 & d^0 \\ a^1 & b^1 & c^1 & d^1 \\ a^2 & b^2 & c^2 & d^2 \\ a^3 & b^3 & c^3 & d^3 \end{vmatrix} = (a-b)(b-c)(c-d)(d-a)$$

when (5×5) $|A| = (a-b)(b-c)(c-d)(d-e)(e-a)$

Q4.
$$\begin{vmatrix} x & a & a & a \\ a & x & a & a \\ a & a & x & a \\ a & a & a & x \end{vmatrix}$$

$$C_1 \longrightarrow C_1+C_2+C_3+C_4$$

$$|A| = \begin{vmatrix} x+3a & a & a & a \\ x+3a & x & a & a \\ x+3a & a & x & a \\ x+3a & a & a & x \end{vmatrix} = (x+3a)\begin{vmatrix} 1 & a & a & a \\ 1 & x & a & a \\ 1 & a & x & a \\ 1 & a & a & x \end{vmatrix}$$

$$R_2 \longrightarrow R_2 - R_1 \quad , R_3 \longrightarrow R_3 - R_1 \; , \quad R_4 \longrightarrow R_4 - R_1$$

$$|A| = (x+3a)\begin{vmatrix} 1 & a & a & a \\ 0 & (x-a) & 0 & 0 \\ 0 & 0 & (x-a) & 0 \\ 0 & 0 & 0 & (x-a) \end{vmatrix}$$

$$\therefore \; |A| = (x+3a)(x-a)(x-a)(x-a) = \underline{\underline{(x+3a)(x-a)^3}}$$

Short <u>cut method</u>

$$\begin{vmatrix} x & a & a & a \\ a & x & a & a \\ a & a & x & a \\ a & a & a & x \end{vmatrix}$$

- when diagonal elements are same and non diagonal elements are same (other category) then
  1) Select the first row.
  2) Add all element of selected row $(x+a+a+a)$
  3) Take product of (1st - 2nd)(1st - 3rd)(1st - 4th)

$$|A| = (x+a+a+a)\underline{(x-a)}(x-a)(x-a)$$

Q5. Find the determinant of

1)
$$\begin{vmatrix} 1+x & 2 & 3 & 4 \\ 1 & 2+x & 3 & 4 \\ 1 & 2 & 3+x & 4 \\ 1 & 2 & 3 & 4+x \end{vmatrix}$$

$$C_1 \longrightarrow C_1+C_2+C_3+C_4$$

$$\begin{vmatrix} 10+x & 2 & 3 & 4 \\ 10+x & 2+x & 3 & 4 \\ 10+x & 2 & 3+x & 4 \\ 10+x & 2 & 3 & 4+x \end{vmatrix} = (x+10)\begin{vmatrix} 1 & 2 & 3 & 4 \\ 1 & 2+x & 3 & 4 \\ 1 & 2 & 3+x & 4 \\ 1 & 2 & 3 & 4+x \end{vmatrix}$$

$R_2 \longrightarrow R_2 - R_1$
$R_3 \longrightarrow R_3 - R_1$
$R_4 \longrightarrow R_4 - R_1$

$$= (x+10)\begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & x & 0 & 0 \\ 0 & 0 & x & 0 \\ 0 & 0 & 0 & x \end{vmatrix} = (x+10)(1\times x\times x\times x)$$

$$= \underline{x^3(x+10)}$$

Adjoint & Inverse Of a Matrix

Minor

$$A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 2 & -1 \\ 1 & 0 & 5 \end{bmatrix}_{3\times 3}$$

Minor : Minor of an element a_{ij} is denoted by M_{ij} is $(n-1)^{th}$ order determinant.

Cofactor : Cofactor of an element a_{ij} is denoted by $A_{ij} = (-1)^{i+j} M_{ij}$

eg: Consider A given above.

Minor of $(a_{13}) = 3$ is $M_{13} = \begin{vmatrix} -1 & 2 \\ 1 & 0 \end{vmatrix}_{2\times 2} = 0-2 = \underline{-2}$

Cofactor of a_{13}, $A_{13} = (-1)^{1+3} M_{13} = 1\times -2 = -2$

~~Cofactor~~

Adjoint Of a matrix

Let B be the cofactor element of the matrix A, then adj of $A = B^T$

Singular Matrix :

A matrix is said to be singular matrix if $|A| = 0$.
non singular if $|A| \neq 0$.

Invertible Matrix :

A matrix A is said to be invertible if we can find some other matrix B such that $AB = BA = I$, then B is called Inverse of the matrix A.

Note : • $A^{-1} = \dfrac{\text{adj } A}{|A|}$

$|A| \cdot A^{-1} = \text{adj } A$

Premultiply both sides by A

$A\,|A| \cdot A^{-1} = A \cdot \text{adj } A$

$|A|\,(A \cdot A^{-1}) = A \cdot \text{adj } A$

ie $\underline{A \cdot \text{adj } A = |A| \cdot I}$

- $A^{-1} = \frac{\text{adj}\, A}{|A|}$ 174

Premultiply or post multiply both sides by $(\text{adj}\, A)^{-1}$

$$(\text{adj}\, A)^{-1} \cdot A^{-1} = \frac{\text{Adj}\, A}{|A|} \cdot (\text{adj}\, A)^{-1} = \frac{I}{|A|}$$

Post multiply both sides by A.

$$(\text{adj}\, A)^{-1} \cdot A^{-1} \cdot A = \frac{I \cdot A}{|A|}$$

$$(\text{adj}\, A)^{-1} = \frac{A}{|A|}$$

- $A^{-1} = \frac{\text{adj}\, A}{|A|}$ $A \longrightarrow n \times n$ matrix

$$|A|\, A^{-1} = \text{adj}\, A$$

$$\Longrightarrow \Big| \underset{K}{\underset{\downarrow}{|A|}}\, A^{-1} \Big| = |\text{adj}\, A|$$

$$|A|^n |A^{-1}| = |\text{adj}\, A|$$

$$|A|^n\, |A|^{-1} = |\text{adj}\, A|$$

$$|A|^{n-1} = |\text{adj}\, A|$$

$|AB| = |A|\,|B|$

$|KA| = K^n |A|$

$A \to n \times n$ $|A^{-1}| = |A|^{-1}$

- $A^{-1} = \frac{\text{adj}\, A}{|A|}$

We know that $|\text{adj}\, A| = |A|^{n-1}$

Replace A by $\text{adj}\, A$ $\Longrightarrow |\text{adj}\,(\text{adj}\, A)| = |\text{adj}\, A|^{n-1}$

$$= (|A|^{n-1})^{n-1}$$

ie $|\text{adj}\,(\text{adj} A)| = |A|^{(n-1)^2}$

$|||^{ly}$ $|\text{adj}\,(\text{adj}\,(\text{adj}\, A))| = |A|^{(n-1)^3}$

Note : we know that

$$A\, \text{adj}\, A = |A|\, I$$

Replacing A by $\text{adj}\, A$

$$\text{adj}\,(\text{adj}\,(\text{adj} A)) = |\text{adj} A|\, I$$

Let $A \longrightarrow n \times n$ matrix

$$\Longrightarrow \text{adj} A\ (\text{adj}\,(\text{adj} A)) = |A|^{n-1}\, I$$

$$A \text{ adj } A \,(\text{adj}(\text{adj } A)) = A\,|A|^{n-1}.I$$

$$|A|\; I(\text{adj}(\text{adj } A)) = |A|^{n-1}.A$$

$$\text{adj}.\text{adj } A = |A|^{n-2}.A$$

- If A is an orthogonal matrix, then $A \times A^T = A^T.A = I$

$$A.A^T = I$$

Premultiplying with A^{-1}

$$A^{-1}.A.A^T = A^{-1}.I$$

$$I.A^T = A^{-1}.I$$

$$\text{ie } A^T = A^{-1}$$

- If A is an orthogonal matrix, then A^{-1} & A^T are also orthogonal matrix

Q₁. For the matrix $M = \begin{bmatrix} 3/5 & 4/5 \\ x & 3/5 \end{bmatrix}$ and $M^T = M^{-1}$. Find x.

Ans If M is an orthogonal matrix, its rows and columns must be pairwise orthogonal and periodic orthonormal.

$$\frac{3}{5}x + \frac{4}{5} \times \frac{3}{5} = 0$$

$$\cancel{\frac{3}{5}}x = -\frac{4}{\cancel{5}} \times \frac{\cancel{3}}{5}$$

$$x = -4/5$$

Q₂. If $A = (a_{ij})_{5\times 5}$ such that.

i) $a_{ij} = i - j$

ii) $a_{ij} = i^2 - j^2 \quad \forall\, i, j$

Find A^{-1} in each case.

Symmetric Matrix

A matrix A is said to be symmetric when.

1) ⓐ $A = A^T$
ⓑ $a_{ij} = a_{ji}$ } $A \longrightarrow$ Symmetric

2) ⓐ $A^T = -A$
ⓑ $a_{ij} = -a_{ji}$ } $A \longrightarrow$ Skew symmetric.

Solution

i) $a_{ij} = i-j = -(j-i)$

$a_{ij} = -a_{ji} \Rightarrow A \longrightarrow$ skew symmetric

$A^T = -A$

$|A^T| = |-A|$

$|A| = (-1)^5 |A| = -|A|$

$|A| + |A| = 0$

$2|A| = 0$

$|A| = 0 \Rightarrow$ singular matrix

$\therefore A^{-1}$ doesnot exist

Note : • Every odd order skew symmetric matrix is a singular matrix $|A|_{n\times n} = 0$ $n \longrightarrow$ odd and its inverse doesnot exist.

- Every even order skew symmetric matrix is a non singular matrix $|A|_{n\times n} \neq 0$ $n \longrightarrow$ even. and hence its inverse exists.
- The determinant of even order skew symmetric matrix is always perfect square.

Eg : $\begin{vmatrix} 0 & -3 \\ 3 & 0 \end{vmatrix} = 9 = 3^2 \neq 0$

Q1. If x & y are two non zero matrices of the same order, such that XY=0, then.

a) $|X| \neq 0$, $|Y| = 0$

b) $|X| = 0$, $|Y| \neq 0$

c) $|X| \neq 0$, $|Y| \neq 0$

d) $|X| = 0$, $|Y| = 0$

Ans For $XY = 0$ either $X = 0$, $Y = 0$ or $X = Y = 0$.

So (c) $\longrightarrow$ wrong.

Let $|X| = 0$ and $|Y| \neq 0 \Rightarrow Y^{-1}$ exists.

$XY = 0$

$XYY^{-1} = 0Y^{-1}$

$\Rightarrow X = 0$ is false since it is given X & Y are non zero matrices.

$\therefore |Y| \neq 0$ is wrong $\Longrightarrow |Y| = 0$

So option is (d) ie $|x| = 0, |Y| = 0$

Note : Product of two non zero matrices is a null matrix if both of them are singular.

Q2. If A, B, C, D, E, F, G are non singular matrices of same order such that $CEDBGAF = I$ then $B^{-1} = \_\_\_?$

Ans $CEDBGAF = I$

Premultiplying with C^{-1}

$$C^{-1}CEDBGAF = C^{-1}I \Longrightarrow EDBGAF = C^{-1}$$

$$E^{-1}EDBGAF = E^{-1}C^{-1}$$

$$D^{-1}DBGAF = D^{-1}E^{-1}C^{-1}$$

$$BGAF = D^{-1}E^{-1}C^{-1}$$

Now post multiplying both sides with F^{-1}

$$BGAFF^{-1} = D^{-1}E^{-1}C^{-1}F^{-1}$$

$$BGA = D^{-1}E^{-1}C^{-1}F^{-1}$$

|||ly $B = D^{-1}E^{-1}C^{-1}F^{-1}A^{-1}G^{-1}$

$$B^{-1} = (D^{-1}E^{-1}C^{-1}F^{-1}A^{-1}G^{-1})^{-1}$$

Now $(AB)^{-1} = B^{-1}A^{-1}$

$$\therefore B^{-1} = (G^{-1})^{-1}(A^{-1})^{-1}\ldots(D^{-1})^{-1} = \underline{GAFCED}$$

Shortcut

$B^{-1} = \underline{CED}\,\underline{BGAF} = \underline{GAFCED}$

Interchange

$G^{-1} = AFCEDB$

$A^{-1} = FCEDBG$

Q1. Let K be a positive real number and let

$$A = \begin{bmatrix} (2K-1) & 2\sqrt{K} & 2\sqrt{K} \\ 2\sqrt{K} & 1 & -2K \\ -2\sqrt{K} & 2K & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 2K-1 & \sqrt{K} \\ 1-2K & 0 & 2\sqrt{K} \\ -\sqrt{K} & -2\sqrt{K} & 0 \end{bmatrix}$$

Find i) $|\text{adj } B|$ ii) $|\text{adj } A|$ iii) If $|\text{adj } A| = 10^6$, the value of K is ___ ?

Ans i) Here B is an odd order skew symmetric matrix.

$$\therefore |B| = 0 \implies |\text{adj } B| = |B|^{n-1} = \underline{\underline{0}}$$

ii) $|\text{adj } A| = |A|^{n-1} = |A|^2 \longrightarrow ①$

$$|A| = \begin{vmatrix} 2K-1 & 2\sqrt{K} & 2\sqrt{K} \\ 2\sqrt{K} & 1 & -2K \\ -2\sqrt{K} & 2K & -1 \end{vmatrix}$$

$$R_2 \longrightarrow R_2 + R_3 \implies \begin{vmatrix} 2K-1 & 2\sqrt{K} & 2\sqrt{K} \\ 0 & 1+2K & -(1+2K) \\ -2\sqrt{K} & 2K & -1 \end{vmatrix}$$

$$C_2 \longrightarrow C_2 + C_3 \implies \begin{vmatrix} 2K-1 & 4\sqrt{K} & 2\sqrt{K} \\ 0 & 0 & -(1+2K) \\ -2\sqrt{K} & 2K-1 & -1 \end{vmatrix}$$

$$= (1+2K)\left[(2K-1)^2 + 8K\right]$$

$$= (1+2K)\left[4K^2 \cancel{-4K} + 1 + \overset{4K}{\cancel{8K}}\right]$$

$$= (1+2K)(1+2K)^2 = \underline{(2K+1)^3}$$

$$\therefore (1) \implies |\text{adj } A| = |A|^2 = \left[(2K+1)^3\right]^2 = \underline{\underline{(2K+1)^6}}$$

iii) $(2K+1)^6 = 10^6$

ie $2K+1 = 10 \implies 2K = 9 \implies K = \underline{9/2}$

Q3. Find the inverse of following matrices.

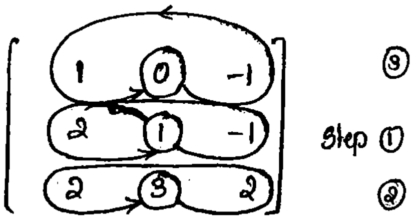
1) $\begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & -1 \\ 2 & 3 & 2 \end{bmatrix}$ 2) $\begin{bmatrix} 5 & 0 & 7 \\ 0 & 3 & 0 \\ 2 & 0 & 1 \end{bmatrix}$ 3) $\begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

Ans 1) $|A| = 1(2+3) - 1(6-2) = 5-4 = \underline{\underline{1}}$

Method to find adjA (only for 3rd order)

- Select the middle row and middle number.
- Write the numbers in anti c-wise dirn completing a cycle. The corresponding elements will be written column wise.

$$\begin{bmatrix} 1 & 3 & 0 & 1 \\ -1 & 2 & -1 & -1 \\ 2 & 2 & 1 & 2 \\ 1 & 3 & 0 & 1 \end{bmatrix}$$



- Now select the 3rd row and repeat the same.
- Now select 1st row " " " " "
- Repeat the 1st column
- Now select each 2x2 matrix and find determinant

$$\text{adj } A = \begin{bmatrix} 5 & -3 & 1 \\ -6 & 4 & -1 \\ 4 & -3 & 1 \end{bmatrix}$$

$$A^{-1} = \frac{\text{adj} A}{|A|} = \underline{\begin{bmatrix} 5 & -3 & 1 \\ -6 & 4 & -1 \\ 4 & -3 & 1 \end{bmatrix}}$$

$$A^{-1} = \frac{\text{adj}}{|A|}$$

Q4. Given an orthogonal matrix $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}$. The $(A \times A^T)^{-1}$ is ___?

Ans $(A \times A^T) = I \quad \therefore (A \times A^T)^{-1} = I^{-1} = \underline{I}$

Q5. Find the inverse of matrix $A = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Ans A is an orthogonal matrix.

Since its rows and columns are pairwise orthogonal

$A^T = A^{-1}$

$$\therefore A^{-1} = \underline{\begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

RANK OF A MATRIX

- Submatrix

A matrix is obtained after deleting some rows or columns is called a submatrix.

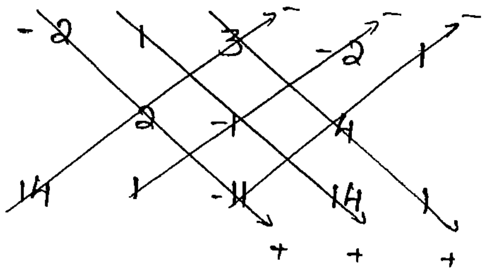
Eg: $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 5 & 6 & 7 \\ 1 & -1 & 0 & 2 \end{bmatrix}$

$$\left.\begin{array}{l} B_1 = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}_{2\times 3} \\[2ex] B_3 = \begin{bmatrix} 3 & 4 \\ 6 & 7 \end{bmatrix}_{2\times 2} \\[2ex] B_3 = \begin{bmatrix} 5 & 6 & 7 \\ -1 & 0 & 2 \end{bmatrix}_{2\times 3} \end{array}\right\} \text{ Submatrices Of } A$$

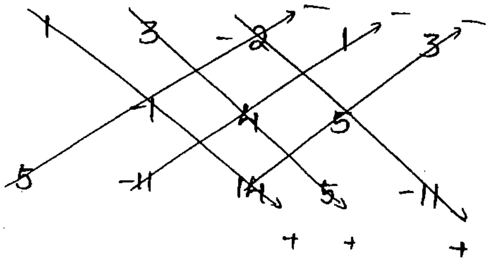
- <u>MINOR</u> : The determinant of a square submatrix.
- <u>RANK</u> : If the determinant of atleast one highest possible square sub matrix is non zero then order of the determinant is called Rank of a matrix.

Eg ① : $A = \begin{bmatrix} 1 & 3 & -2 & 1 \\ 2 & -1 & 4 & 5 \\ 1 & -11 & 14 & 5 \end{bmatrix}_{3\times 4}$ Find rank.

Ans Highest possible sq. matrix is 3×3.



$|B| = 44 - 14 + 12 - 84 - 2 + 44 = \underline{0}$



$|B| = 14 + 60 + 110 - 10 + 44 - 210 \neq \underline{0}$

$\therefore \underline{\rho(A) = 3}$

Properties Of Rank Of a matrix

- $\rho(O_{n\times n}) = 0$
- $\rho(I_{n\times n}) = n$
- $\rho(adj(I_{n\times n})) = n$
- $\rho(A+B) \leq \rho(A) + \rho(B)$
- $\rho(A-B) \geq \rho(A) - \rho(B)$
- If A is an $m\times n$ matrix, then $\rho(A) \leq \min(m,n)$
- $\rho(AB) \leq \min\{\rho(A), \rho(B)\}$

- $\rho(A\,B) \geq \rho(A) + \rho(B) - n$ If A & B are $n \times n$ matrices.
- $\rho(A) = \rho(A^T)$
- If $\rho(A_{n\times n}) = n$ then $\rho(\operatorname{adj} A) = n$
- If $\rho(A_{n\times n}) = n-1$ then $\rho(\operatorname{adj} A) = 1$

Eg ① : Consider $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & -1 \\ 0 & 0 & 0 \end{bmatrix}_{3\times 3} \Longrightarrow \rho(A) = 3-1 = \underline{\underline{2}}$

$$\therefore \quad \rho(\operatorname{adj} A) = \underline{\underline{1}}$$

② $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & -1 \\ 0 & 0 & 0 \end{bmatrix}$

$$\operatorname{adj}(A) = \begin{bmatrix} 5 & 0 & 2 & 5 \\ -1 & 0 & 3 & -1 \\ 0 & 0 & 1 & 0 \\ 5 & 0 & 2 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -17 \\ 0 & 0 & 1 \\ 0 & 0 & 5 \end{bmatrix}$$

$R_1 \longrightarrow R_1 + 17R_2$, $R_3 \longrightarrow R_3 - 5R_2$

$$\operatorname{adj} A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore \ \rho(\operatorname{adj} A) =$ no. of non zero rows $= \underline{\underline{1}}$

If rank $(A_{n\times n}) \leq n-2$, then $\rho(\operatorname{adj} A) = 0$

ECHELON FORM

Row Echelon Form : A matrix is said to be in Echelon Form if it satisfies the following conditions.

- All zero rows should occupy last rows, if any.
- The number of zeros before a non zero entry of each row is less than the no. of such zeros before a non zero entry of the next row. ie zeros increases row by row.

Rank of a matrix in Echelon Form = no. of non zero rows.

*** To reduce to Row Echelon Form use only row operations.

The no. of non zero rows in row Echelon Form are also called linearly Independent vectors. or LI rows.

Eg :
$$\begin{array}{r} \\ 1\text{ zero}\rightarrow \\ 2\text{"}\rightarrow \\ 3\text{"}\rightarrow \end{array}\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 3 & 4 \\ 0 & 0 & 6 & 7 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Bigg\} \text{ 3 LI vectors.}$$

Every upper triangular matrix will be a row Echelon matrix but converse may or may not be true.

$$\begin{array}{r} \\ 1\leftarrow \\ 2\leftarrow \end{array}\begin{bmatrix} 3 & -2 & 2 \\ 0 & -2 & 1 \\ 0 & 0 & 5 \end{bmatrix} \longrightarrow \text{Upper } \Delta^{lr} \text{ matrix}$$

Echelon form as it satisfies the two conditions.

$$A = \begin{bmatrix} 0 & -2 & 5 & 6 \\ 0 & 0 & 5 & 2 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \longrightarrow \text{Echelon Form}$$

But not upper Δ^{lr} matrix as the diagonal elements are all zero.

Q. Find the rank of the following matrices.

1) $A = \begin{bmatrix} 1 & 2 & 2 & -1 \\ 2 & 1 & 2 & 0 \\ 2 & 2 & 1 & 3 \end{bmatrix}$

$R_2 \longrightarrow R_2 - 2R_1$

$R_3 \longrightarrow R_3 - 2R_1$

$$\therefore A = \begin{bmatrix} 1 & 2 & 2 & -1 \\ 0 & -3 & -2 & 2 \\ 0 & -2 & -3 & 5 \end{bmatrix}$$

Now $R_3 \longrightarrow 3R_3 - 2R_2$

$$A = \begin{bmatrix} 1 & 2 & 2 & -1 \\ 0 & -3 & -2 & 2 \\ 0 & 0 & -5 & 11 \end{bmatrix} \Bigg\} \text{ 3 LI vectors} \qquad \underline{\underline{\rho(A) = 3}}$$

2) $A = \begin{bmatrix} 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 5 \\ 1 & 1 & 2 & 3 \end{bmatrix}$

$R_3 \longrightarrow 2R_3 - R_1$ $\quad A = \begin{bmatrix} 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 5 \\ 0 & 1 & 3 & 6 \end{bmatrix}$

$R_3 \longrightarrow R_3 - R_2$ $\quad A = \begin{bmatrix} 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 5 \\ 0 & 0 & 2 & 1 \end{bmatrix} \Bigg\} \text{ 3 L.I vectors.}$

$\underline{\rho(A) = 3}$

Note : If $\rho(A) = n$ then it has n linearly independent rows and n linearly independent columns.

Q1. If $A = (a_{ij})_{m\times n}$ such that $a_{ij} = i\times j$ $\forall$ i,j then $\rho(A) =$ ____?

Ans

$$A = \begin{bmatrix} 1 & 2 & 3 & \cdots & n \\ 2 & 4 & 6 & \cdots & 2n \\ \vdots & & & & \\ m & 2m & 3m & \cdots & nm \end{bmatrix}$$

$$= 1\times 2\times 3\times \cdots \times m \begin{bmatrix} 1 & 2 & 3 & \cdots & n \\ 1 & 2 & 3 & \cdots & n \\ \vdots & & & & \\ 1 & 2 & 3 & \cdots & n \end{bmatrix}$$

$R_2 \longrightarrow R_2 - R_1$

$R_3 \longrightarrow R_3 - R_1$

$\vdots$

$R_n \longrightarrow R_n - R_1$

$$= m! \begin{bmatrix} 1 & 2 & 3 & \cdots & n \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

No. of non zero rows = 1

$\therefore$ $\rho(A) = 1 \implies 1$ L.I Vectors

Q2. $X = (x_1, x_2 ---- x_n)^T$ is an n-tuple non zero vector. Then find.

i) $P(XX^T)$

ii) $\rho(X^T X)$

Ans

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1} \qquad X^T = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}_{1\times n}$$

$$\implies P[\underset{\underset{n\times 1}{\downarrow}}{X}\underset{\underset{1\times n}{\downarrow}}{X^T}] \leq \min\{\rho(X), \rho(X^T)\}$$

$$\implies \rho[XX^T] \leq \min\{1, 1\}$$

Since X is a non zero vector $\rho(X) \neq 0$

$\therefore$ $\underline{\underline{\rho(X) = 1}}$

ii) |||ly $\underline{\underline{\rho(x) = 1}}$

Q3. The rank of 5×6 matrix is 4. Then which of the following statement is true Q.

a) Q has 4 linearly independent rows and 4 linearly independent columns.

b) Q has 4 LI rows and 5 LI columns.

c) $Q \cdot Q^T$ is invertible.

d) $Q^T \cdot Q$ is invertible

Note : If rank of a matrix is 4 then it should have 4 LI rows and 4 LI columns. These 4 LI rows can be obtained by reducing the matrix into Echelon form and 4 LI columns can be obtained by reducing the matrix into Echelon form.

LINEARLY DEPENDENT & INDEPENDENT VECTORS

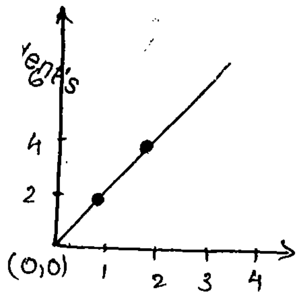
- **Vector** : If the elements are written in a horizontal line or in vertical line is called an vector.

Eg : $(1 \ 2 \ 3) \longrightarrow$ 3 vector.

Two vectors X_1 & X_2 are said to be linearly dependent if it is possible to express one of the vectors as multiple of other vector.

$$X_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad X_2 = \begin{bmatrix} 2 \\ 4 \end{bmatrix} = 2X_1$$

$$\text{or } X_1 = \frac{1}{2} X_1$$



Two vectors in R^2 (two dimensional) are said to be linearly dependent iff they are colinear.

|||ly 3 vectors in R^3 (3-D) are said to be L.D iff they are coplanar.

If it is not possible to express 1 vector as multiple of other vector, the two vectors are said to be linearly independent.

Eg : $X_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad X_2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

$$X_1 \neq kX_2$$

- **L.D vectors**

A set of r n-vectors, $X_1, X_2 \ldots X_r$ are said to be L.D if there exists r scalars $K_1, K_2 \ldots K_r$ such that $K_1X_1 + K_2X_2 + \ldots + K_rX_r = 0$ where $K_1, K_2 \ldots K_r$, not all zeros.

(Atleast 1 K should be a non zero number).

- **L I Vectors :** 'r' vectors $X_1, X_2 \dots X_r$ are L I vectors if there exists 'r' scalars $K_1, K_2 \dots K_r$ such that $K_1X_1 + K_2X_2 + \dots + K_rX_r = 0$ where $K_1, K_2 \dots K_r$ are all zeros.

Note : • If $\rho(A) <$ no of given vectors or $|A| = 0$ } Vectors are said to be L·D

• $\rho(A) =$ no of given vectors or $|A| \neq 0$ } Vectors are said to be L·I.

Note :

- A set of 'r' vectors with $r < n$ components should be L·I.

Eg : Consider vectors
$$(1 \quad 2 \quad 3 \quad 4)$$
$$(2 \quad 1 \quad -1 \quad 7)$$
$$(2 \quad 3 \quad 4 \quad 5)$$

No of vectors $r = 3$

No of component $n = 4$

$r < n$
$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & -1 & 7 \\ 2 & 3 & 4 & 5 \end{bmatrix}_{3\times4}$$

Then the given vectors are L·I.

$R_2 \longrightarrow R_2 - 2R_1$

$R_3 \longrightarrow R_3 - 2R_1$
$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -7 & -1 \\ 0 & -1 & -2 & 1 \end{bmatrix}$$

$R_3 \longrightarrow R_3 - 3R_2$
$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -7 & -1 \\ 0 & 0 & 19 & 4 \end{bmatrix} \Bigg\} \text{ 3 LI vectors.}$$

$\rho(A) = 3 =$ no. of given vectors.

There fore the given vectors are L·I

- If a set of 'r' vectors with $r \gtrless n$ components, then the vectors are said to be linearly dependent.

Consider the vector $X_1 = (1 \quad 2 \quad 3)$

$X_2 = (1 \quad 0 \quad 3)$

$X_3 = (1 \quad 1 \quad -1)$

$X_4 = (0 \quad 1 \quad 2)$

No of vectors $r = 4$

No of components $n = 3$

$r > n$ So the given vectors are L.D

- Suppose the vectors are L.D, then any 1 of the vectors can be expressed as linear combination of other vectors.
- If the given vectors are L.I, then it is impossible to express any 1 of the vector as a linear combination of other vectors.
- Every non zero vector must be a L.I vectors.

Eg : $X = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

$KX = 0$ ie $K \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = 0$

$\therefore K = 0$ $\therefore X$ is a L.I vector.

Q. $X_1 = (1 \; 0 \; 0)$

$X_2 = (0 \; 1 \; 0)$

$X_3 = (0 \; 0 \; 1)$

$K_1 X_1 + K_2 X_2 + K_3 X_3 = 0$

$K_1(100) + K_2(0 \; 1 \; 0) + K_3(0 \; 0 \; 1) = (0 \quad 0 \quad 0)$

ie $(K_1 \quad K_2 \quad K_3) = (0 \quad 0 \quad 0)$

$\Longrightarrow K_1 = K_2 = K_3 = 0$

- $\underline{\text{II}^{nd}\text{ Method}}$

$$|A| = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 1 \neq 0$$

$\rho(A)$ = no. of non zero rows = $\underline{\underline{3}}$

The set of unit vectors must be L.I

- The set of vectors having atleast one vector is a null vector, the vectors are said to be linearly dependent.

Eg: consider the vectors $X_1 = (2 \quad 3 \quad 4)$

$X_2 = (1 \quad -1 \quad 2)$

$X_3 = (0 \quad 0 \quad 0)$

$$|A| = \begin{vmatrix} 2 & 3 & 4 \\ 1 & -1 & 2 \\ 0 & 0 & 0 \end{vmatrix} = 0$$

$\rho(A) = 2$

No of given vectors = 3.

$\rho(A) < n.$

$\therefore$ Given vectors are L.D

- Two vectors x_1 & x_2 are orthogonal if $X_1^T X_2 = X_1 X_2^T = 0$

The set of orthogonal vectors in R^n (R vectors with n elements) are L.I.

DIMENSION & BASIS

- DIMENSION : The no. of LI vectors is called its dimension.

Dimension = no of L.I vectors = no of non zero rows. after reducing the matrix into Echelon form.

- Basis : The set of LI vectors is called its basis.

Basis = Set of LI vectors = set of non zero rows after reducing the matrix into Echelon form.

Note: If the number of vectors = no. of components, then the given vectors are LI or LD.

Q1. Test whether the following vectors are LD or L.I. Also find dimension & basis

① (1 1 -1 0) (4 4 -3 1) (-6 2 2 2) (9 9 -6 3)

② (1 2 3) (1 1 3) (2 4 9)

③ (1 -1 0) (4 5 7) (0 0 1)(1 2 5)

④ (1 -1 0 5) (2 3 4 9) (1 2 3 1)

Ans

① $\begin{bmatrix} 1 & 1 & -1 & 0 \\ 4 & 4 & -3 & 1 \\ -6 & 2 & 2 & 2 \\ 9 & 9 & -6 & 3 \end{bmatrix}$

$R_2 \longrightarrow R_2 - 4R_1$

$R_3 \longrightarrow R_3 + 6R_1$

$R_4 \longrightarrow R_4 - 9R_1$

$$A = \begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 8 & -4 & 2 \\ 0 & 0 & 3 & 3 \end{bmatrix}$$

$$R_2 \Longleftrightarrow R_3 \implies A = \begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 8 & -4 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 3 & 3 \end{bmatrix}$$

$$R_4 \longrightarrow R_4 - 3R_3 \implies A = \begin{bmatrix} 1 & -1 & 0 & 1 \\ 0 & 8 & -4 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \left.\vphantom{\begin{matrix}1\\1\\1\end{matrix}}\right\} \text{3 L.I vectors.}$$

$\therefore \rho(A) = 3 \longrightarrow$ Dimension = 3

Basis = {(1 -1 0 1),(0 8 -4 2)(0 0 1 1)}

Nullity of a Matrix

$$\boxed{N(A) = n(A) - \rho(A)}$$

- Nullity of a matrix is denoted by N(A) and is defined as the difference b/w order and rank of the matrix.

  ie $N(A) = n(A) - \rho(A)$

- Nullity of an non singular matrix of any order is always zero.

  Let us consider an nxn matrix (non singular) A.

  The $\rho(A) = n$.

  $\therefore N(A) = n(A) - \rho(A) = n - n = \underline{\underline{0}}$

Q1. The nullity of $A = \begin{bmatrix} k & 1 & 2 \\ 1 & -1 & -2 \\ 2 & 1 & 1 \end{bmatrix}$ is 1. The value of k is ?

Ans $N(A) = 1$. $\therefore n(A) = 3$

$\Longrightarrow \quad N(A) = n(A) - \rho(A)$

$1 = 3 - \rho(A)$

$\therefore \quad \rho(A) = 3 - 1 = 2$

$\Longrightarrow \quad |A| = 0$

$$|A| = 0 \Longrightarrow \begin{vmatrix} K & 1 & 2 \\ 1 & -1 & -2 \\ 2 & 1 & 1 \end{vmatrix} = 0 \qquad R_2 \longrightarrow R_2 + R_3$$

$$\Longrightarrow \begin{vmatrix} K & 1 & 2 \\ 3 & 0 & -1 \\ 2 & 1 & 1 \end{vmatrix} \Longrightarrow 3(1-2) - 1(K-2) = 0$$

$\therefore \quad -3 - K + 2 = 0$

$\underline{K = -1}$

Q2. The rank of a matrix is 5 and nullity of the matrix is 3. Then what is the order of the matrix.

Ans $N(A) = n(A) - \rho(A)$

$3 = n(A) - 5$

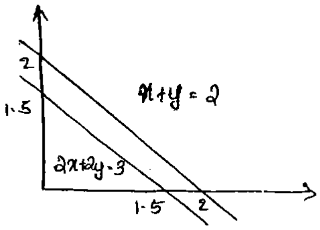
$\underline{\underline{n(A) = 8}}$

SYSTEM OF NON HOMOGENEOUS & HOMOGENEOUS LINEAR EQNS

Consider a non homogeneous s/m of eqns, containing two eqns and two variables.

$ax + by = e$

$cx + dy = f$.



The above s/m of eqns have

- no solution if $\frac{a}{c} = \frac{b}{d} \neq \frac{e}{f}$ $\Longrightarrow$ Lines ||al to each other.
- Unique solution if $\frac{a}{c} \neq \frac{b}{d}$ $\Longrightarrow$ Lines intersect at 1 particular point only.
- Infinite no. of solutions if $\frac{a}{c} = \frac{b}{d} = \frac{e}{f}$

$\Longrightarrow$ The given s/m of eqns can be brought to the form of no. of eqns < no. of variables.

Q1. The s/m of equations $4x+2y=7$, $2x+y=6$ have

Ans $$\frac{a}{c} = \frac{b}{d} \neq \frac{e}{f}$$

∴ No solution

Q2. The s/m of equations $x+3y=5$, $2x+5y=-3$ have

Ans $$\frac{1}{2} \neq \frac{3}{5}$$

∴ Unique solution.

Note : • Consider a s/m of m eqns and n variables.

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{21}x_3 + \dots + a_{2n}x_n = b_2$$
$$\vdots \qquad\qquad\qquad\qquad \vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \dots + a_{mn}x_n = b_m$$

The above s/m of eqns can be put in matrix form.

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$$AX = B$$

where $X \longrightarrow$ solution matrix

If we write the elements of matrix B in the last column of matrix A, the resulting matrix is called Augmented matrix and is denoted by (A/B).

$$[A/B] = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & & & & \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{bmatrix}$$

PROCEDURE

i) Reduce (A/B) to Row Echelon Form.

ii) Find $\rho(A/B)$ & $\rho(A)$

iii) If $\rho(A) < \rho(A/B)$ or $\rho(A/B) \neq \rho(A)$ the given s/m of eqns have no soln.
ie Inconsistant

iv) If $\rho(AB) = \rho(A) =$ no. of unknown, the given slm of eqns have unique solution

ie **Consistant**

v) If $\rho(A|B) = \rho(A) <$ no of unknown, the given slm of eqns have infinite no. of solutions.

vi) If the given slm of equations are consistant (unique solution or infinite solution), then the solution variables $x_1, x_2 \ldots x_n$ can be found using backward substitution.

Note : If the no. of eqns < no. of variables $(r<n)$, the given slm of eqns will have infinite no. of non zero solutions. This non zero solutions can be found by assigning $(n-r)$ variables or arbitrary constants. These $n-r$ solution are said to be L.I soln.

- **Inconsistant** : The given slm of eqns are said to be inconsistant if they have no. solution.
- **Consistant** : The given slm of eqns are said to be consistant if they have a solution (unique or infinite).

Note : If the no. of eqns < no. of variables (or) no. of variables exceeds no. of eqns, the slm of eqns have ∞ no. of solutions.

Homogeneous Slm Of Linear Eqn

Consider the following homogeneous slm of linear eqn of n variables

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = 0$$

$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = 0$$

$$\vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = 0$$

In matrix form

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \text{ie} \quad AX = 0$$

$R_3 \longrightarrow R_3 + R_2$

$$[A|B] = \begin{bmatrix} 1 & 2 & -2 & 4 \\ 0 & -3 & 5 & -10 \\ 0 & 0 & 2 & -4 \end{bmatrix}$$

* * *

(Backward Substitution)

$$0 \times x_1 + 0x_2 + 2x_3 = -4$$

$$x_3 = \underline{\underline{-2}}$$

$$-3x_2 + 5x_3 = -10$$

$$-3x_2 = 0$$

$$x_2 = \underline{\underline{0}}$$

$$x_1 + 2x_2 - 2x_3 = 4$$

$$x_1 + 0 + 4 = 4$$

$$x_1 = \underline{\underline{0}}$$

Q5. How many no. of solution does the following sYm of eqns have.

$-x + 5y = -1, \quad x - y = 2, \quad x + 3y = 3.$

a) ∞ b) exactly 2 [✓ c) unique soln] d) no. solution.

Q6. For the set of eqn.

$$x_1 + 2x_2 + x_3 + 4x_4 = 2$$

$$3x_1 + 6x_2 + 3x_3 + 12x_4 = 6$$

Ans No of eqns < no of variables

$\therefore$ Infinite no. of non zero soln

Q7. Find the value of λ & μ for the sYm of eqns.

$x + y + z = 6, \quad x + y + 6z = 20, \quad x + 4y + \lambda z = \mu$ to have

a) no solution b) unique solution c) Rhoto no. of non zero sol^n.

Ans

$$[A/B] = \begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & 1 & 6 & 20 \\ 1 & 4 & \lambda & \mu \end{bmatrix}$$

$R_2 \longrightarrow R_2 - R_1$

$R_3 \longrightarrow R_3 - R_1$

$$[A|B] = \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 3 & 5 & 14 \\ 0 & 3 & \lambda-6 & \mu-6 \end{bmatrix}$$

$R_3 \longrightarrow R_3 - R_2 \qquad [A|B] = \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 3 & 5 & 14 \\ 0 & 0 & \lambda-6 & \mu-20 \end{bmatrix}$

a) no. solution

$\rho(A) < \rho(A|B)$

ie $\lambda - 6 = 0$, $\mu - 20 \neq 0$

$\lambda = 6$ & $\mu \neq 20$

b) Unique soln

$\rho(A|B) = \rho(A)$ = no. of unknowns.

$\rho(A|B) = \rho(A) = 3$

$\lambda - 6 \neq 0$, $\mu - 20 \neq 0$ or $\mu - 20 = 0$

$\therefore \lambda \neq 6$, μ = any value

c) ∞ no of solution

$\rho(A|B) = \rho(A) <$ no. of ~~solution~~ unknowns.

$\lambda - 6 = 0$ & $\mu - 20 = 0$

$\therefore \lambda = 6$ and $\mu = 20$

Q7. For what value of α & β, the following sim of eqns $x+y+z=5$, $x+3y+3z=9$; $x+2y+\alpha z = \beta$. have infinite no. of solutions.

Ans $\rho(A|B) = \rho(A) <$ no. of unknowns

$[A|B] = \begin{bmatrix} 1 & 1 & 1 & 5 \\ 1 & 3 & 3 & 9 \\ 1 & 2 & \alpha & \beta \end{bmatrix}$

$R_2 \longrightarrow R_2 - R_1$

$R_3 \longrightarrow R_3 - R_1$

$[A|B] = \begin{bmatrix} 1 & 1 & 1 & 5 \\ 0 & 2 & 2 & 4 \\ 0 & 1 & \alpha-1 & \beta-\overset{5}{\cancel{10}} \end{bmatrix}$

$R_3 \longrightarrow R_3 - R_1 \qquad [A|B] = \begin{bmatrix} 1 & 1 & 1 & 5 \\ 0 & 2 & 2 & 4 \\ 0 & 0 & \alpha-2 & \beta-10 \end{bmatrix}$

$\alpha = 2$, $\beta = 10$

Q8 For what values of λ, μ the following s/m of eqns $x+2y+3y=6$;

$x+3y+5z=9$, $2x+5y+\lambda z=\mu$ have.

i) no solution ii) unique solution iii) ∞ no of solution.

Q9. Find the condition on a, b, c for which the following s/m of equations.

$3x+4y+5z=a$

$4x+5y+6z=b$

$5x+6y+7z=c$ donot have a solution.

Ans $[A|B]$ = $\begin{bmatrix} 3 & 4 & 5 & a \\ 4 & 5 & 6 & b \\ 5 & 6 & 7 & c \end{bmatrix}$ $\underset{\substack{R_2 \to 3R_2 - 4R_1 \\ R_3 \to 3R_3 - 5R_1}}{=}$ $\begin{bmatrix} 3 & 4 & 5 & a \\ 0 & -1 & -2 & 3b-4a \\ 0 & -2 & -4 & 3c-5a \end{bmatrix}$

$R_3 \longrightarrow R_3 - 2R_2$ $[A|B] = \begin{bmatrix} 3 & 4 & 5 & a \\ 0 & -1 & -2 & 3b-4a \\ 0 & 0 & 0 & 3(ac-2b) \end{bmatrix}$

$3(a+c-2b)$

$3c-5a \neq 0$

Q10. For what values of λ does the s/m of equations have two linearly independent solution.

$x+y+z=0$, $(\lambda+1)y+(\lambda+1)z=0$, $(\lambda^2-1)z=0$.

Ans $n-r=2$ $A\ C_0$

$3-r=2$

$r=3-2=1$

If rank is 1, determinant of the matrix should be zero.

$$\Longrightarrow |A| = \begin{vmatrix} 1 & 1 & 1 \\ 0 & \lambda+1 & \lambda+1 \\ 0 & 0 & \lambda^2-1 \end{vmatrix}_{3\times 3} = 0$$

$$\Longrightarrow 1\left[(\lambda+1)(\lambda^2-1) - 0\right] = 0$$

$$(\lambda+1)(\lambda+1)(\lambda-1) = 0$$

$$\lambda = +1, -1$$

For $\lambda = -1$, rank $= 1$

If $\lambda = 1$, rank $= 2$

$\therefore \lambda = -1$ is correct

Q11. The rank of $A_{3\times3}$ is 1. The s/m of equations $AX = 0$ has.

a) only trivial solution b) 1 L.I solution

c) 2 L.I solution d) 3 LI solution.

Ans $A_{3\times3}$

Rank $\longrightarrow$ $r = 1$

Order $\longrightarrow$ $n = 3$

If $r < n$, $n - r = 3-1 = 2$ LI solution

Q12. The s/m of equations.

$2x_1 + x_2 + x_3 = 0$

$x_2 - x_3 = 0$

$x_1 + x_2 = 0$ has.

a) no. non real solution b) unique non real solution.

c) 2 non real solution d) Infinite no of non trivial solution.

Ans $[A] = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & -1 \\ 1 & 1 & 0 \end{bmatrix}$

$|A| = 2(0+1) + 1(-1-1) = 2-2 = 0$

Therefore the given s/m of equations have infinite no. of non real solution.

Q13. The s/m of equations $x + 3y - 2z = 0$, $2x - y + 4z = 0$, $x - 11y + 14z = 0$ have

a) Trivial solution b) two non trivial solution.

c) 5 non trivial solution d) ∞ no of non trivial solution.

Ans $$|A| = \begin{array}{ccccc} -2 & 1 & 3 & -2 & 1 \\ & 2 & -1 & 4 & \\ -14 & 1 & -11 & 14 & 1 \end{array} = 0$$

Then it have ∞ no. of non trivial solution

Q14. Find the value of K for which the s/m of equations.

$$(3K-8)x + 3y + 3z = 0$$

$$3x + (3K-8)y + 3z = 0$$

$$3x + 3y + (3K-8)z = 0$$ have non trivial solution.

Ans

$$\begin{vmatrix} 3K-8 & 3 & 3 \\ 3 & 3K-8 & 3 \\ 3 & 3 & 3K-8 \end{vmatrix} = 0$$

ie $(3K-8+6)(3K-8-3)(3K-8-3) = 0$

$$(3K-2)(3K-11)^2 = 0$$

$$K = 2/3, \quad \underline{\underline{K = 11/3, 11/3}}$$

Q15. Find the value of λ for which the s/m of equations $8x+2y+3z = \lambda x$, $3x+y+2z = \lambda y$, $2x+3y+z = \lambda z$ have non trivial solution.

EIGEN VALUES & EIGEN VECTORS

Let A be an $n \times n$ matrix and λ is a scalar. The matrix $(A-\lambda I)$ is called a characteristic ~~equation~~ matrix.

$|A-\lambda I|$ is called characteristic determinant or characteristic polynomial. The eqn $|A-\lambda I| = 0$, is called as characteristic eqn.

The roots of this charac. eqn are called characteristic roots or Eigen values or proper values or latent roots.

The set of Eigen values of a matrix 'A' is called as Spectrum of A.

Eg: $A = \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$ $\lambda \longrightarrow$ Scalar.

$$A - \lambda I = \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 4-\lambda & 2 \\ 2 & 4-\lambda \end{bmatrix} \longrightarrow \text{charac. matrix.}$$

$$|A-\lambda I| = \begin{vmatrix} 4-\lambda & 2 \\ 2 & 4-\lambda \end{vmatrix} \longrightarrow \text{charac. polynomial.}$$

$$|A-\lambda I| = 0 \longrightarrow \text{charac. eqns.}$$

$$\begin{vmatrix} 4-\lambda & 2 \\ 2 & 4-\lambda \end{vmatrix} = 0 \implies (4-\lambda)^2 - 4 = 0$$

$$16 + \lambda^2 - 8\lambda - 4 = 0$$

$$\lambda^2 - 8\lambda + 12 = 0$$

$$\underline{\underline{\lambda = 6, 2}} \longrightarrow \text{Eigen Value}$$

<u>Eigen Vectors</u> : If λ is an Eigen value of the matrix A, then there exists a non zero vector X such that $AX = \lambda X$. The X is called an Eigen vector corresponding to the Eigen value λ.

Eg : Consider the matrix $A = \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4+2 \\ 4+2 \end{bmatrix} = \begin{bmatrix} 6 \\ 6 \end{bmatrix} = \underline{6\begin{bmatrix} 1 \\ 1 \end{bmatrix}}$

$\therefore \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an Eigen Vector corresponding to the Eigen value $\lambda = 6$

III^{ly} $\begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 4+4 \\ 8+2 \end{bmatrix} = \begin{bmatrix} 8 \\ 10 \end{bmatrix} = \underline{\underline{\begin{bmatrix} 4 \\ 5 \end{bmatrix}}}$

$\therefore \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is not an Eigen Vector

If any vector exists along the <u>line OA</u>, it is an Eigen vector

Q1. Find the Eigen values and Eigen vectors of matrix $A = \begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix}$.

<u>Ans</u> $A = \begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix}$

$$|A - \lambda I| = 0 \Longrightarrow \begin{vmatrix} 3-\lambda & 4 \\ 4 & -3-\lambda \end{vmatrix} = 0$$

$$(3-\lambda)(-3-\lambda) - 16 = 0$$

$$-9 - 3\lambda + 3\lambda + \lambda^2 - 16 = 0$$

$$\lambda^2 - 25 = 0$$

$$\lambda^2 = 25$$

$$\underline{\underline{\lambda = \pm 5}}$$

<u>Case(i)</u>: when $\lambda = 5$

$$AX = \lambda X$$

$$AX = \lambda X I$$

$$AX - \lambda X I = 0$$

$$\Longrightarrow (A - \lambda I)X = 0$$

$$A - \lambda I = \begin{bmatrix} 3-\lambda & 4 \\ 4 & -3-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\lambda = 5 \implies \begin{bmatrix} -2 & 4 \\ 4 & -8 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-2x_1 + 4x_2 = 0 \longrightarrow ①$$

$$4x_1 - 8x_2 = 0 \longrightarrow ②$$

$$\frac{②}{-2} \implies -2x_1 + 4x_2 = 0$$

$$\therefore 2x_1 = 4x_2$$

$$\frac{x_1}{x_2} = \frac{2}{1}$$

Therefore the Eigen vector corresponding to Eigen value $\lambda = 5$

$$X_1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \underline{\underline{\begin{bmatrix} 2 \\ 1 \end{bmatrix}}}$$

Case (ii) : when $\lambda = -5$

$$A - \lambda X = 0$$

$$\begin{bmatrix} 3-\lambda & 4 \\ 4 & -3-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 8 & 4 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$8x_1 + 4x_2 = 0$$

$$4x_1 + 2x_2 = 0$$

$$4x_1 + 2x_2 = 0$$

ie $4x_1 = -2x_2$

$$x_1 = -\frac{x_2}{2}$$

$$\therefore \frac{x_1}{x_2} = \frac{-1}{2}$$

$$\therefore X_2 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \underline{\begin{bmatrix} -1 \\ 2 \end{bmatrix}}$$

Therefore the Eigen vectors are $X_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $X_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

$$A = \begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix} \implies A^T = \begin{bmatrix} 3 & 4 \\ 3 & -3 \end{bmatrix} \implies A^T = A \longrightarrow \text{Symmetric}$$

- Two vectors are said to be orthogonal if $X_1 X_2^T = 0$ & $X_1^T X_2 = 0$

$$X_1^T X_2 = \begin{bmatrix} 2 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \end{bmatrix} = -2 + 2 = \underline{0}$$

Remark :

The Eigen vectors corresponding to distinct Eigen values of a real symmetric matrix are always orthogonal.

$$|X_1\ X_2| = \begin{vmatrix} 2 & -1 \\ 1 & 2 \end{vmatrix} = 4-(-1) = 5 \neq 0 \implies \underline{L.I}$$

- The Eigen vectors corresponding to distinct Eigen values of any square matrix are always L.I.

The Eigen values corresponding to repeated Eigen values may be LI or LD vectors.

Q. Find the Eigen values and corresponding Eigen vectors of the following.

i) $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

$\lambda = 1, 1, 1$ ($A \longrightarrow$ Upper Δ^{r} matrix)

$(A - \lambda I)X = 0$

$\begin{bmatrix} 1-\lambda & 1 & 0 \\ 0 & 1-\lambda & 1 \\ 0 & 0 & 1-\lambda \end{bmatrix} \implies$ Augmented matrix $(A-\lambda I)$

$$\begin{bmatrix} 1-\lambda & 1 & 0 \\ 0 & 1-\lambda & 1 \\ 0 & 0 & 1-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\lambda = 1 \implies$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_2 = x_3 = 0$, $x_1 \longrightarrow$ Can't say. Can be any value.

No of eqns < no. of variables $\implies$ Infinite no. of non zero solutions.

$r = 2$, $n = 3$

If $r < n$, put $n - r = 3 - 2 = 1 \longrightarrow$ Arbitrary constant.

- Zero vectors cannot be Eigen vectors.

$\therefore$ Eigen vectors $= \begin{bmatrix} c \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ Can be 1, 2 . . . .

Corresponding to 3 repeated Eigen values, there exists only 1 L·I Eigen vector.

$$X_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}$$

ii) $\begin{bmatrix}1 & 1 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$

Ans $\lambda = 1, 1, 1$ (upper Δ^{lr} matrix)

$(A - \lambda I)X = 0$

$$\Longrightarrow \begin{bmatrix}1-\lambda & 1 & 0\\0 & 1-\lambda & 0\\0 & 0 & 1-\lambda\end{bmatrix}[X] = 0$$

$\lambda = 1$ $\begin{bmatrix}0 & 1 & 0\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = 0$

$r = 1,\ n = 3$ $x_2 = 0$

$r < n$, put $n - r = 3 - 1 = 2 \longrightarrow$ Two variables can be assumed as Arbitrary const.

Put $x_1 = c_1,\ x_2 = 0,\ x_3 = c_2$

$$X = \begin{bmatrix}c_1\\0\\c_2\end{bmatrix} = c_1\underset{X_1}{\underset{\downarrow}{\begin{bmatrix}1\\0\\0\end{bmatrix}}} + c_2\underset{X_2}{\underset{\downarrow}{\begin{bmatrix}0\\0\\1\end{bmatrix}}}$$

No vector is expressed as multiple of other vector.
ie $X_1 \neq kX_2$ or $X_2 \neq kX_1$

Therefore corresponding to 3 equal Eigen values, there exists two L·I Eigen vectors $X_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}$ $X_2 = \begin{bmatrix}0\\0\\1\end{bmatrix}$

iii) $\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$

$\lambda = 1, 1, 1$

$(A - \lambda I)X = 0$ $\lambda = 1$

ie $\begin{bmatrix}0 & 0 & 0\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = 0$

$r = 0,\ n = 3$

$r < n$, put $n - r = 3 - 0 = 3 \longrightarrow$ 3 arbitrary constants can be assumed.

$x_1 = c_1,\ x_2 = c_2,\ x_3 = c_3$.

$$X = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = c_1 \underset{\substack{\downarrow \\ X_1}}{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}} + c_2 \underset{\substack{\downarrow \\ X_2}}{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}} + c_3 \underset{\substack{\downarrow \\ X_3}}{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}$$

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1 \neq 0$$

∴ The vectors are linearly independent.

∴ There exists 3 LI Eigen vectors.

$$X_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},\ X_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},\ X_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Corresponding to 3 equal Eigen values

Note :
- If an Eigen value λ is repeated m times, then the corresponding Eigen vectors are always L.I and is given by

  $p = n - r$ where $1 \leq p \leq m$

  ↓

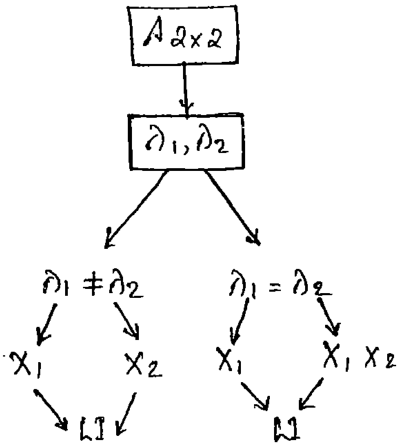
  no. of L.I Eigen vector.

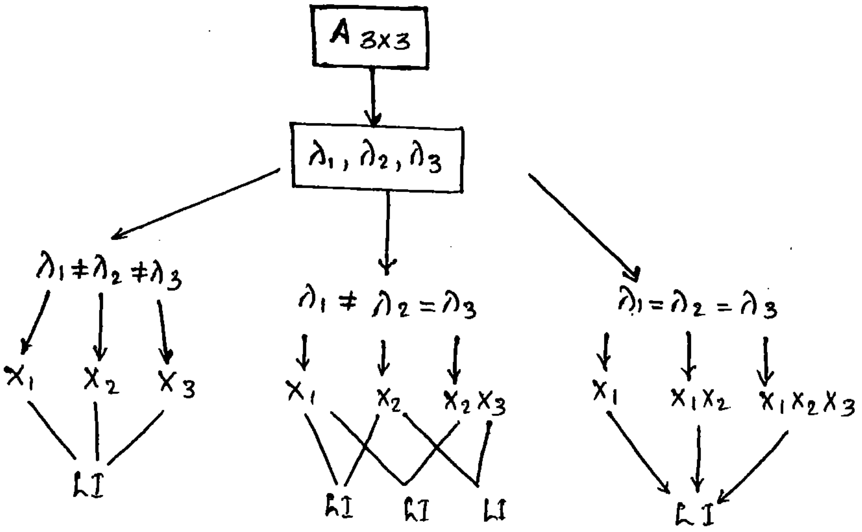
  $n \longrightarrow$ order of matrix or no. of variables.

  $r \longrightarrow \rho(A - \lambda I)$

  $m \longrightarrow$ no. of times an Eigen value is repeated.

- If some Eigen values are repeated and some are non repeated then the corresponding Eigen vectors may be L.I or L.D.





Q1 The no. of LI Eigen vectors of $A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$

Ans $\lambda = 2, 2.$

$(A - \lambda I)X = 0$

$\begin{bmatrix} 2-\lambda & 1 \\ 0 & 2-\lambda \end{bmatrix} = A - \lambda I$

$\lambda = 2 \implies \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

$p = n - r$, where $1 \le p \le 2$

$n = 2, \ r = 1$

$\therefore \ \underline{\underline{p = 2 - 1 = 1}}$ L.I Eigen vector

Q2. The no. of LI Eigen vector of $A = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$

Ans $\lambda = 3, 3.$

$(A - \lambda I)X = 0$

$\lambda = 3 \implies \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$p = n - r = \underline{\underline{2 - 0 = 2}}$ LI Eigen Vectors

Properties Of Eigen Values and Eigen vectors

Let A be an n×n matrix. The Eigen values of A are $\lambda_1, \lambda_2 \dots \lambda_n$.

- $\text{Trace}(A) = \lambda_1 + \lambda_2 + \dots + \lambda_n$.

  ie Sum of the diagonal elements of the matrix = Sum of the Eigen values.
- $\det(A) = \lambda_1 \cdot \lambda_2 \cdot \lambda_3 \dots \lambda_n$

  ie product of the Eigen values = det. of a matrix.

- The Eigen values of a upper Δ^{lr} or lower Δ^{lr} or diagonal or scalar or identity matrix is its diagonal elements.
- The Eigen values of A & A^T are same.
- The Eigen values of A and $P^{-1}AP$ (P is a non singular matrix)are same).
- The Eigen values of orthogonal matrix are of unit modulus

$$|\lambda| = 1$$

- If λ is an Eigen value of an orthogonal matrix then $1/\lambda$ is also one of its Eigen value.
- The Eigen values of real symmetric matrix are real.
- The Eigen values of skew symmetric matrix are purely imaginary or zeros.
- For a real matrix if $a+ib$ is an Eigen value then $a-ib$ is also another Eigen value of the same matrix.
- The Eigen vectors of A & A^{-1} are same.
- The Eigen vectors of A & A^m are same.
- If λ is an Eigen value of a matrix A then
    - $k\lambda$ is an Eigen value of kA.
    - $1/\lambda$ is an Eigen value of A^{-1}
    - λ^2 is an Eigen value of A^2
    - λ^m is an " " " A^m
    - $\frac{|A|}{\lambda}$ " " " " " $Adj\ A$
    - $\lambda \pm k$ " " " " " $A \pm kI$
    - $(\lambda \pm k)^2$ " " " " " $(A \pm kI)^2$
    - $\frac{1}{\lambda \pm k}$ " " " " " $(A \pm kI)^{-1}$

Note : • A matrix A is said to be singular

ie $\det A = 0$ $[|A| = 0]$ if one of its eigen value is zero.

Its converse is also true.

- If one of the eigen value of a matrix A is 0, then the homogeneous sim of equations have infinite number of non zero solutions.

Q1. The Eigen values of $A = \begin{bmatrix} -1 & 3 & 5 \\ -3 & -1 & 6 \\ 0 & 0 & 3 \end{bmatrix}$ are.

a) $3, 8+5j, -6-j$ b) $-6+5j, 3-j, 3+j$

c) $3-j, 5+j, 3+j$ d) $3, -1+3j, -1-3j$

Trace [A].

$Tr[A] = \lambda_1 + \lambda_2 + \lambda_3 = (-1)+(-1)+3 = \underline{1}$

ie (d) $\longrightarrow$ $3, \underline{-1+3j}, -1-3j$

Q2. The Eigen values and eigen vectors of a 2x2 matrix are given by.

Eigen value

$\left.\begin{matrix} \lambda_1 = 8 \\ \lambda_2 = 4 \end{matrix}\right\} 8+4 = 12$

Eigen vector

$\gamma_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\gamma_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

The matrix is a) $\begin{bmatrix} 6 & 2 \\ 2 & 6 \end{bmatrix}$ b) $\begin{bmatrix} 4 & 6 \\ 6 & 4 \end{bmatrix}$ c) $\begin{bmatrix} 2 & 4 \\ 4 & 2 \end{bmatrix}$ d) $\begin{bmatrix} 4 & 8 \\ 8 & 4 \end{bmatrix}$

$6+6=12$ $4+4=8$ $2+2=4$ $4+4=8$

- If all sum are made same go for determinant.

Q3. The vector $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ is an Eigen vector of $A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$. The Eigen value corresponding to the Eigen vector is.

$(A - \lambda I)X = 0$

$$\begin{bmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & 0-\lambda \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-1 + -2\times 2 + -\lambda \times -1 = 0$

$-1-4+\lambda = 0$

$\underline{\underline{\lambda = 5}}$

Q3. For the matrix $\begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$, the Eigen value corresponding to the Eigen vector $\begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$ is ___ ?

Ans $[A - \lambda I]X = 0$

$$\begin{bmatrix} 4-\lambda & 2 \\ 2 & 4-\lambda \end{bmatrix} \begin{bmatrix} 101 \\ 101 \end{bmatrix} = 0$$

$$2 \times 101 + (4-\lambda)101 = 0$$

$$101(2 + 4 - \lambda) = 0$$

$$6 - \lambda = 0$$

$$\underline{\underline{\lambda = 6}}$$

Q4. The minimum and maximum Eigen values of $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$ are -2 and 6 respectively. The 3rd Eigen value is ___ ?

Ans $Tr(A) = \lambda_1 + \lambda_2 + \lambda_3$

$\downarrow$

Sum of diagonal :

$$1 + 5 + 1 = -2 + 6 + \lambda_3$$

$$7 = 4 + \lambda_3$$

$$\lambda_3 = \underline{3}$$

Q5. The matrix $\begin{bmatrix} 1 & 2 & 4 \\ 3 & 0 & 6 \\ 1 & 1 & p \end{bmatrix}$ has an Eigen value $= 3$. The sum of other two Eigen value is ___ ?

Ans $1 + 0 + p = 3 + \lambda_2 + \lambda_3$

$$p + 1 - 3 = \lambda_2 + \lambda_3$$

$$\lambda_2 + \lambda_3 = \underline{\underline{p - 2}}$$

Q6. The Eigen values ~~and Eigen vectors~~ of a 2×2 matrix $A = \begin{bmatrix} 2 & 3 \\ x & y \end{bmatrix}$ are given by.

Eigen Value

$$\left.\begin{matrix} \lambda_1 = 8 \\ \lambda_2 = 4 \end{matrix}\right\} 8 + 4 = 12$$

· The values of x & y are.

Ans $2 + y = 12 \qquad y = 12 - 2 = \underline{\underline{10}}$

Using property $|A| = \lambda_1 \lambda_2$

$$2y - 3x = 4 \times 8$$

$$2 \times 10 - 3x = 32$$

$$3x = 20 - 32 = -12$$

$$\underline{\underline{x = -4}}$$

Q7. Given $A = \begin{bmatrix} 1 & 2 & -3 \\ 0 & 3 & 2 \\ 0 & 0 & -2 \end{bmatrix}$. The Eigen values of $3A^3 + 5A^2 - 6A + 2I$ is ?

Ans $\lambda = 1, 3, -2$ ($A \longrightarrow$ upper Δ^{lr} matrix)

$$3A^2 + 5A^2 - 6A + 2I \longrightarrow 3\lambda^3 + 5\lambda^2 - 6\lambda + 2$$

$$\lambda = 1 \implies 3(1)^3 + 5(1)^2 - 6(1) + 2 = 4$$

$$\lambda = 3 \implies 3(3)^3 + 5(3)^2 - 6 \times 3 + 2 = 110$$

$$\lambda = -2 \implies 3(-2)^3 + 5(-2)^2 - 6 \times -2 + 2 = \underline{\underline{10}}.$$

Q8 The Eigen value of a 3×3 matrix are given by 1, 2, 3.

Find i) $Tr\,(A^2 + A^{-1} + adj\,A)$

ii) $det\,(A^2 + A^{-1} + adj\,A)$

Ans $\lambda = 1, 2, 3$

$$|A| = 1 \times 2 \times 3 = 6$$

$$A^2 + A^{-1} + adj\,A = \lambda^2 + \frac{1}{\lambda} + \frac{|A|}{\lambda} = \lambda^2 + 1/\lambda + 6/\lambda = \underline{\underline{\lambda^2 + 7/\lambda}}$$

$$\lambda = 1 \implies 1^2 + 7/\lambda = 8$$

$$\lambda = 2 \implies 4 + 7/2 = 15/2$$

$$\lambda = 3 \implies 9 + 7/3 = 34/3$$

i) $Tr(A^2 + A^{-1} + adj\,A)$ = Sum of Eigen values of matrix

$$= 8 + \frac{15}{2} + \frac{34}{3}$$

ii) $det(A^2 + A^{-1} + adj\,A) = \underline{\underline{8 \times \frac{15}{2} \times \frac{34}{3}}}$

Q9. The Eigen values of a 2×2 matrix are given by -2 & -3 resp. The Eigen values of $(X + I)^{-1}(X + 5I)$ are ?

Ans $(X + I)^{-1}(X + 5I) = (X + I)^{-1} \cdot (X + I + 4I) = (X + I)^{-1}(X + I) + 4I(X + I)^{-1}$

$$= I + 4(\underset{\overset{\downarrow}{A}}{X} \underset{\overset{\downarrow}{K}}{+} \underset{\overset{\downarrow}{I}}{I})^{-1}$$

$$\frac{1}{\lambda+K} = (A+KI)^{-1} \Longrightarrow 1+\frac{4}{\lambda+1}$$

$$\lambda = -2 \Longrightarrow 1+\frac{4}{-2+1} = 1-4 = \underline{\underline{-3}}$$

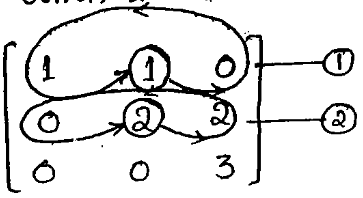
$$\lambda = -3 \Longrightarrow 1+\frac{4}{-3+1} = \underline{\underline{-1}}$$

Q_{10} Find the Eigen values & Eigen vectors of the following.

① $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$ ② $\begin{bmatrix} 3 & -2 & 2 \\ 0 & -2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ ③ $\begin{bmatrix} 1 & 2 & 3 \\ 0 & -4 & 2 \\ 0 & 0 & 7 \end{bmatrix}$

Ans ① $\lambda = 1, 2, 3$

when $\lambda = 1$



$$= \begin{array}{cccc} x_1 & x_2 & x_3 & \\ \left[\begin{matrix} 1 & 0 & \underset{(1-\lambda)}{0} & 1 \\ \\ \underset{(2-\lambda)}{1} & 2 & 0 & \underset{(2-\lambda)}{1} \end{matrix}\right] \end{array}$$

Start from row middle element Rotate clock wise or (counter c.w). Once you encounter a diagonal element subtract λ value from it and put in b/w x_1, x_2, x_3. find determinant.

$$\frac{x_1}{2-0} = \frac{x_2}{0-0} = \frac{x_3}{0-0}$$

$$X_1 = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} \longrightarrow \underline{\underline{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}}$$

$\underline{\lambda = 2}$

$$\begin{array}{cccc} x_1 & x_2 & x_3 & \\ 1 & 0 & -1 & 1 \\ 0 & 2 & 0 & 0 \end{array}$$

$$\frac{x_1}{2} = \frac{x_2}{2} = \frac{x_3}{0}$$

$$X_2 = \begin{bmatrix} 2 \\ -2 \\ 0 \end{bmatrix} \longrightarrow \underline{\underline{\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}}}$$

$\underline{\lambda = 3}$

$$\begin{array}{cccc} x_1 & x_2 & x_3 & \\ 1 & 0 & -2 & 1 \\ -1 & 2 & 0 & -1 \end{array}$$

$$\frac{x_1}{2} = \frac{x_2}{4} = \frac{x_3}{2} \qquad X_3 = \begin{bmatrix} 2 \\ +4 \\ 2 \end{bmatrix} = \underline{\underline{\begin{bmatrix} 1 \\ +2 \\ 1 \end{bmatrix}}}$$

109

Q11. The Eigen vectors of a 2x2 matrix $\begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$ are given by $\begin{bmatrix} 1 \\ a \end{bmatrix}, \begin{bmatrix} 1 \\ b \end{bmatrix}$. What is a+b ?

Ans $\lambda = 1, 2.$

$$(A - \lambda I) X = 0$$

$$\begin{bmatrix} 1-\lambda & 2 \\ 0 & 2-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

For $\lambda = 1$

$$\begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$2x_2 = 0$ ——①

$x_2 = 0$ ——②

$r = 1, \quad n = 2$

If $r < n$, $n - r = 2 - 1 = 1 \longrightarrow$ Arbitrary constant.

$$X_1 = \begin{bmatrix} c \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ a \end{bmatrix} \text{ or } \begin{bmatrix} 1 \\ b \end{bmatrix} \quad a = 0 \text{ or } b = 0$$

$\left(\frac{1}{a}\right)$ or $\left(\frac{1}{b}\right)$ can be assumed. a & b may be changed but a+b doesnot change.

For $\lambda = 2$.

$$\begin{bmatrix} -1 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$

$$-x_1 + 2X_2 = 0$$

$$\frac{x_1}{x_2} = \frac{2}{1}$$

$$X_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1/2 \end{bmatrix}$$

ie $b = 1/2$

$$\therefore a + b = 0 + 1/2 = \underline{\underline{1/2}}$$

Method ②

Eigen vector, $AX = \lambda X$

$\lambda = 1, 2$

$$\lambda = 1 \Longrightarrow \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ a \end{bmatrix} = 1 \begin{bmatrix} 1 \\ a \end{bmatrix}$$

$$\begin{bmatrix} 1+2a \\ 2a \end{bmatrix} = \begin{bmatrix} 1 \\ a \end{bmatrix}$$

$$1+2a = 1$$

$$2a = 0$$

$$a = 0$$

$\lambda = 2$ $$\begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ b \end{bmatrix} = 2 \begin{bmatrix} 1 \\ b \end{bmatrix}$$

$$\begin{bmatrix} 1+2b \\ 2b \end{bmatrix} = \begin{bmatrix} 2 \\ 2b \end{bmatrix}$$

$$1+2b = 2$$

$$2b = 1$$

$$\underline{\underline{b = 1/2}}$$

$$\therefore \underline{\underline{a+b = 1/2}}$$

CAYLEY HAMILTON THEOREM

Every square matrix satisfies its own characteristic eqn. Using Cayley Hamilton Theorem, we can find.

- Inverse of a matrix
- Higher powers of a matrix.

eg: Consider matrix $A = \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$. Its characteristic eqn is

$$\lambda^2 - 8\lambda + 12 = 0$$

By cayley Hamilton Theorem we have,

$$A^2 - 8A + 12I = 0$$

<u>Note</u>: The constant term of any characteristic eqn of any order square matrix is equal to the determinant of a matrix.

Positive Powers of A

Consider $A^2 - 8A + 12I = 0$

$\Longrightarrow A^2 = 8A - 12I$

$A^3 = 8A^2 - 12A$

$A^4 = 8A^3 - 12A^2$

- - - - - -

- - - - - -

- - - - - -

Negative Powers of A

$A^2 - 8A + 12I = 0$

$12I = 8A - A^2$

$I = \frac{1}{12}\left[-A^2 + 8A\right]$

$A^{-1} = \frac{1}{12}\left[-A + 8I\right]$

$A^{-2} = \frac{1}{12}\left[-I + 8A^{-1}\right]$

$A^{-3} = \frac{1}{12}\left[-A^{-1} + 8A^{-2}\right]$

- - - - - -

- - - - - -

- Consider $A_{2\times 2} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Charac. eqn is $|A - \lambda I| = 0$

ie $\begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = 0$

$(a-\lambda)\times(d-\lambda) - bc = 0$

$ad - a\lambda - \lambda d + \lambda^2 - bc = 0$

$\lambda^2 - \lambda(a+d) + (ad-bc) = 0$

$\Longrightarrow \lambda^2 - \lambda\left[\text{Tr}(A)\right] + |A| = 0 \longrightarrow$ char. eqn.

According to Cayley Hamilton Theorem we have.

$\lambda \longrightarrow A$

$\Longrightarrow A^2 - A(\text{Tr}(A)) + |A|I = 0$

- Consider a 3×3 matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

char. eqn : $\lambda^3 - \lambda^2\left[\text{Tr}(A)\right] + \lambda\left\{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} + \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}\right\}$

$+ |A| = 0$

Q1 Find A^8 if $A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$ using Cayley Hamilton Theorem.

Ans $|A| = -1-4 = -5$

Char eqn : $\lambda^2 - \lambda(1+(-1)) + -5 = 0$

$\lambda^2 - 5 = 0$

By C.H Theorem

$A^2 - 5I = 0$

$A^2 = 5I$

$(A^2)^4 = (5I)^4$

$= 625\,I = 625 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 625 & 0 \\ 0 & 625 \end{bmatrix}$

Q2. The char. eqn of a 3x3 matrix P is given by

$\alpha(\lambda) = |\lambda I - P| = \lambda^3 + \lambda^2 + 2\lambda + 1 = 0$ where $I \longrightarrow$ Identity matrix

The inverse of matrix P will be ___? a) P^2+P+2I b) P^2+P+I c) $-(P^2+P+I)$ d) $-(P^2+P+2I)$

Ans By C.H. Theorem

$\lambda \longrightarrow P$ $P^3 + P^2 + 2P + I = 0$

$I = -(P^3 + P^2 + 2P)$

$P^{-1} I = -(P^2 + P + 2I)$

$P^{-1} = -(P^2 + P + 2I)$

Q3. Given $A = \begin{bmatrix} -5 & -3 \\ 2 & 0 \end{bmatrix}$ & $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Express A^3 as a linear polynomial.

Ans Char eqn is $\lambda^2 + 5\lambda + 6I = 0$

$A^2 + 5A + 6I = 0$

$A^2 = -(5A + 6I)$ ——①

$A^3 = -(5A^2 + 6A)$

$= -5(-5A - 6I) - 6A$ From ①

$= 19A + 30I$

Q4. ~~Bey~~ C.H Theorem states that every square matrix satisfies its own charac. eqn. Consider the matrix $A = \begin{bmatrix} -3 & 2 \\ -1 & 0 \end{bmatrix}$

i) A satisfies the relation.

a) $A + 3I + 2A^{-1} = 0$ b) $A^2 + 2A + 2I = 0$

c) $(A+I)(A+2I) = 0$ d) $\exp(A)$

2) $A^9 =$

a) $511A + 510I$ b) $309A + 104I$

c) $154A + 155I$ d) $\exp(A)$

Ans 1) $\lambda^2 + 3\lambda + 2 = 0$

$A^2 + 3A + 2I = 0$ $\lambda = -1, -2$

$\underline{(A+I)(A+2I) = 0}$

2) $A^2 + 3A + 2I = 0$

Method I

$A^2 = -3A - 2I$

$A^2 \times A^2 = (-3A - 2I)(-3A - 2I)$

$A^4 = 9A^2 + 12A + 4I$

$= 9(-3A - 2I) + 12A + 4I = -27A - 18I + 12A + 4I = -15A - 14I$

$A^4 \times A^4 = -(15A + 14I) \times -(15A + 14I) = 225A^2 + 420A + 196I$

$= 225(-3A - 2I) + 420A + 196I$

$= -675A - 450I + 420A + 196I$

$A^8 = -255A - 254I$

$A^9 = -(255A + 254I)(A) = -255A^2 - 254A = -255(-3A - 2I) - 254A$

$= 765A + 510I - 254A = \underline{\underline{511A + 510I}}$

Method II

$A^2 = -3A - 2I \longrightarrow ①$

$A^3 = -3A^2 - 2A = -3(-3A - 2I) - 2A = 7A + 6I \longrightarrow ②$

$\Longrightarrow A^4 = -15A - 14I \longrightarrow ③$

$A^5 = -15A^2 - 14A = -15(-3A-2I) - 14A$

$= \underline{31A + 30I}$

Method III

Consider $\lambda^n = a\lambda + b$

where a & b are constants.

$\lambda^n = a\lambda^2 + b\lambda + c$ for 3x3

$\lambda^n = a\lambda^3 + b\lambda^2 + c\lambda + d$ for 4x4

λ^9, $\lambda^2 + 3\lambda + 2$

$\lambda^n = a\lambda + b$

$(-1)^n = a \times -1 + b$ for $(\lambda = -1) \longrightarrow ①$

$(-2)^n = -2a + b$ for $(\lambda = -2) \longrightarrow ②$

$(-1)^n - (-2)^n = a$ $\quad b = 2(-1)^n - (-2)^n$

Substituting in ①

$\lambda^n = [(-1)^n - (-2)^n]\lambda + [2(-1)^n - (-2)^n]$

$A^n = [(-1)^n - (-2)^n]A + [2(-1)^n - (-2)^n]I$

$n = 9 \implies A^9 = [(-1)^9 - (-2)^9]A + [2(-1)^9 - (-2)^9]I$

$= \underline{511A + 510I}$

If Eigen values are repeated.

$\lambda^n = a\lambda + b$ —— ①

$n\lambda^{n-1} = a$ —— ②

Solve ① & ②

Q. Find A^{15} if $A = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$

TRANSFORM THEORY

LAPLACE TRANSFORMS

If f(t) is defined ∀ $t \in R$ then

$$I\{f(t)\} = \int_{-\infty}^{\infty} f(t)\, K(s,t)\, dt = \bar{f}(s)$$

$K(s,t) \longrightarrow$ Kernel of Integral Transform

i) If $K(s,t) = e^{ist}$ or e^{-ist} $\longrightarrow$ Fourier Transform. s may be real or complex.

ii) If $K(s,t) = e^{-st}$ $\longrightarrow$ Laplace Transform

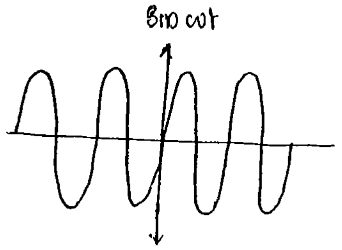
Fourier Transform cannot be applied to periodic fns.

If $K(s,t) = e^{-st}$ then $L\{f(t)] = \int_{-\infty}^{\infty} f(t) e^{-st} dt = \bar{f}(s)$ $\longrightarrow$ Bilateral L.T.

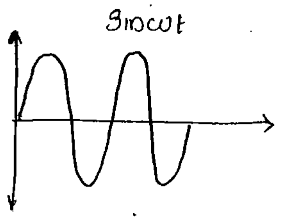
Bilateral L.T fails for exp. and sin sles.

If f(t) is defined ∀ $t \geq 0$ then

$$L\{f(t)\} = \int_{0}^{\infty} e^{-st} f(t)\, dt = \bar{f}(s) \longrightarrow \text{Unilateral L.T.}$$



Here unilateral L.T fails as $-\infty$ to ∞



Semi periodic Sine.

Necessary Condition for Existance of L.T

- Laplace of f(t) exists if $\int_{0}^{\infty} | f(t) e^{-st} dt |$ is convergent integral ie finite

$\int_{0}^{\infty} \sin t\, dt = [-\cos t]_0^{\infty} = \pm 1 + \cos 0 \longrightarrow$ Oscillating (b/w two finite values)

∴ It is not convergent integral.

Sufficient Condition For Existancy

- f(t) is a continuous fn.
- f(t) is a fn of exponential order.

ie $\lim_{t\to\infty} e^{-st} f(t) = 0$ or finite

Note: The L.T exists for $1/\sqrt{t}$ even if it is not piecewise continuous.

L.T of Some Elementary Fns

- $L(1) = 1/s \; ; \; s > 0$

$L\{e^{at}\} = \frac{1}{s-a} \; ; \; s > a$

$L\{e^{-at}\} = \frac{1}{s+a} \; ; \; s > -a$

$L\{\sin at\} = \frac{a}{s^2+a^2} \; ; \; s > 0$

$L\{\cos at\} = \frac{s}{s^2+a^2} \; ; \; s > 0$

$L\{\sinh at\} = \frac{a}{s^2-a^2} \; ; \; s > |a|$

$L\{\cosh at\} = \frac{s}{s^2-a^2} \; ; \; s > |a|$

$$L\{t^n\} = \begin{cases} \frac{n!}{s^{n+1}} & ; n \in Z^+ \\ \frac{\Gamma{n+1}}{s^{n+1}} & ; n \notin Z^+ \end{cases} \qquad s > 0$$

Gamma Function : Use only when $n > -1$

$\Gamma{n+1} = n\Gamma{n} = n(n-1)(n-2)\ldots$ for +ve values of n.

$\Gamma{n+1} = n! \; ; \; n \in Z^+$

$\Gamma{1} = 1$ & $\Gamma{1/2} = \sqrt{\pi}$

$\sqrt{n} = \frac{\sqrt{n+1}}{n}$ for -ve values of n.

$\sqrt{0}, \sqrt{-1}, \sqrt{-2}, \sqrt{-3} \ldots$ are not defined.

First Shifting Theorem of L.T

If $L\{f(t)\} = \bar{f}(s)$, then

$$L\{e^{at} f(t)\} = \bar{f}(s-a)$$

$$L\{e^{-at} f(t)\} = \bar{f}(s+a)$$

Second Shifting Theorem of L.T

If $L\{f(t)\} = \bar{f}(s)$ and $G(t) = \begin{cases} f(t-a) & ; t \geq a \\ 0 & ; t < a \end{cases}$

then $L\{G(t)\} = e^{-as} \bar{f}(s)$

Scaling

$$L\{f(at)\} = \frac{1}{a} \bar{f}(s/a)$$

$$L\{f(t/a)\} = a \bar{f}(as)$$

Multiplication by t^n $(n \in Z^+)$

$$L\{t f(t)\} = \frac{-d}{ds} \bar{f}(s)$$

$$L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} \bar{f}(s)$$

Division by t^n $(n \in Z^+)$

$$L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty \bar{f}(s) ds$$

$$L\left\{\frac{f(t)}{t^n}\right\} = \int_s^\infty \int_s^\infty \ldots \bar{f}(s)(ds)^n$$

L.T of Derivatives

$$L\{f'(t)\} = s\bar{f}(s) - f(0)$$

$$L\{f^n(t)\} = s^n \bar{f}(s) - s^{n-1} f(0) - s^{n-2} f'(0) \ldots f^{n-1}(0)$$

L.T of Integrals

$$L\left\{\int_0^t f(t)dt\right\} = \frac{1}{s}\bar{f}(s)$$

$$L\left\{\int_0^t f(t)dt^n\right\} = \frac{1}{s^n}\bar{f}(s)$$

L.T of Periodic fn

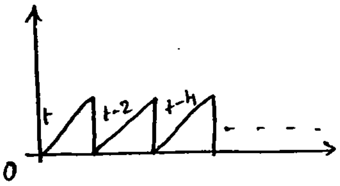
If $f(t) = f(t+T)\ \forall t$

then $L\{f(t)\} = \frac{\int_0^T e^{-st}f(t)dt}{1-e^{-sT}}$

eg: $f(t) = t \quad ; \quad 0<t<2$

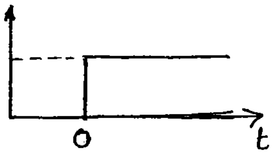
$f(t-2) = f(t)\ \forall t$

$f(t) = t-6 \quad ; \quad 6<t<8$



Unit Step fn / Heavisides fn

$$u(t-a) \text{ or } u_a(t) \text{ or } H(t-a) = \begin{cases} 1 & ; t \geq a \\ 0 & ; t<a \end{cases}$$



$$L\{u(t-a)\} = \frac{e^{-as}}{s}$$

$$L\{u(t)\} = 1/s$$

S.S Theorem

$$L\{f(t-a)u(t-a)\} = e^{-as}L\{f(t)\}$$

$$L\{f(t)\,u(t-a)\} = e^{-as}L\{f(t+a)\}$$

$$f(t) = \begin{cases} f_1(t) & ; 0 \leq t<a \\ f_2(t) & ; t \geq a \end{cases}$$

$$\Longrightarrow f(t) = f_1(t) + \{f_2(t)-f_1(t)\}u(t-a)$$

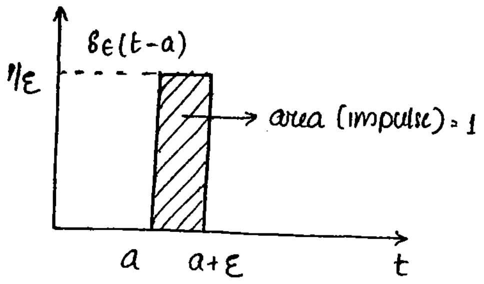
$$f(t) = \begin{cases} f_1(t) & ; 0 \leq t < a_1 \\ f_2(t) & ; a_1 \leq t < a_2 \\ \vdots & \\ f_n(t) & ; t \geq a_{n-1} \end{cases}$$

$$= f_1(t) + [f_2(t)-f_1(t)]u(t-a_1) + [f_3(t)-f_2(t)]u(t-a_2) - - - - -$$

$$- - - + [f_n(t)-f_{n-1}(t)]u(t-a_{n-1})$$

Unit Impulse / Dirac's Delta fn

If we apply large forces in a short time, then the product of force and time is called unit Impulse and the limiting form of force is known as impulse fn.



$$\delta(t-a) = \lim_{\varepsilon\to 0}\left\{\delta_\varepsilon(t-a)\right\}$$

$$= \lim_{\varepsilon\to 0}\begin{cases} 1/\varepsilon & , a \le t \le a+\varepsilon \\ 0 & , \text{otherwise} \end{cases}$$

$$\delta(t-a) = \begin{cases} \infty & ; t=a \\ 0 & ; t\neq a \end{cases}$$

$$L\{\delta(t-a)\} = e^{-as}$$

$$L\{\delta(t)\} = 1$$

$$L\{f(t)\,\delta(t-a)\} = e^{-as} f(a)$$

$$L\{u'(t-a)\} = L\{\delta(t-a)\} = e^{-as}$$

$$\int_0^\infty f(t)\,\delta(t-a)\,dt = \int_{-\infty}^{\infty} f(t)\,\delta(t-a)\,dt = f(a)$$

Q. $L\{a^t\}$; $a>0$

$$= L\{e^{\log a^t}\} = L\{e^{t\log a}\} = \underline{\underline{\frac{1}{s-\log a}}}$$

Q. $L\{t^{7/2} + t^{-1/2} + t^3\} = \frac{\Gamma(7/2+1)}{s^{7/2+1}} + \frac{\Gamma(-1/2+1)}{s^{-1/2+1}} + \frac{3!}{s^4}$

$$= \frac{7/2\cdot 5/2\cdot 3/2\cdot 1/2\cdot \Gamma 1/2}{s^{9/2}} + \frac{\Gamma 1/2}{s^{1/2}} + \frac{6}{s^4}$$

Q $L\left\{\left(\sqrt{t} + \frac{1}{\sqrt{t}}\right)^2\right\} = L\left\{t + \underline{\frac{1}{t}} + 2\right\} = \underline{\underline{\text{Doesnot exist}}}$

Q. $L\left\{\left(\sqrt{t} + \frac{1}{\sqrt{t}}\right)^3\right\} = L\left\{t\sqrt{t} + \frac{1}{t\sqrt{t}} + 3t\times\frac{1}{\sqrt{t}} + 3\sqrt{t}\times\frac{1}{t}\right\}$

$$= L\{t^{3/2} + \underline{t^{-3/2}} + 3t^{1/2} + 3t^{-1/2}\} = \underline{\underline{\text{Doesnot exist}}}$$

Q. $L\{\sin\sqrt{t}\} = L\left\{ t^{1/2} - \frac{t^{3/2}}{3!} + \frac{t^{5/2}}{5!} - \frac{t^{7/2}}{7!} + --- \right\}$

$= \frac{\sqrt{1/2+1}}{s^{3/2}} - \frac{\sqrt{3/2+1}}{3!\,s^{5/2}} + \frac{\sqrt{5/2+1}}{5!\,s^{7/2}} - - - - -$

$= \frac{1/2\sqrt{\pi}}{s^{3/2}} - \frac{3/2\cdot 1/2\sqrt{\pi}}{3!\ s^{5/2}} + \frac{5/2\cdot 3/2\cdot 1/2\cdot\sqrt{\pi}}{5!\ s^{7/2}} - - - - -$

$= \frac{\sqrt{\pi}}{2s^{3/2}}\left[1 - \frac{1}{4s} + \frac{1}{2!(4s)^2} - \frac{1}{3!(4s)^3} + - - - -\right]$

$= \frac{\sqrt{\pi}}{2s^{3/2}}\, e^{-1/4s} \quad (s>0)$

Q. $L\{e^{t^2}\}$

↓

does not exist as $\int$al is not absolutely converging

$= L\left\{1+t^2+\frac{t^4}{2!}+\frac{t^6}{3!}+---\right\}$

$= \frac{1}{s}+\frac{2!}{s^3}+\frac{4!}{2!\,s^5}+\frac{6!}{3!\,s^7}+----$

$= \sum_{n=0}^{\infty}\frac{(2n)!}{n!\,s^{2n+1}} \longrightarrow$ Divergent series.

Since all are increasing terms.

These transformations does not exist

Q. $L\{\sin 3t\cos 4t\} = \frac{1}{2}L\{\sin 7t + \sin(-t)\}$

$= \frac{1}{2}\left[\frac{7}{s^2+49} - \frac{1}{s^2+1}\right] = \frac{1}{2}\left\{\frac{6s^2-42}{(s^2+49)(s^2+1)}\right\}$

$= \frac{3s^2-21}{(s^2+49)(s^2+1)}$

Q. $L\{\cos t\cos 2t\cos 3t\} = \frac{1}{2}L\{(\cos 3t+\cos(-t))\cos 3t\}$

$= \frac{1}{2}L\{\cos^2 3t + \cos t\cos 3t\} = \frac{1}{4}L\{1+\cos 6t + \cos 4t + \cos(-2t)\}$

$= \frac{1}{4}\left\{\frac{1}{s}+\frac{s}{s^2+36}+\frac{s}{s^2+16}+\frac{s}{s^2+4}\right\}$

$\cos^3 2t = \frac{(\cos 6t + 3\cos 2t)}{4}$

Q. $L\{\cos^3 2t\} = L\left\{\frac{\cos 6t + 3\cos 2t}{4}\right\} = \frac{1}{4}\left\{\frac{s}{s^2+36}+\frac{3s}{s^2+4}\right\}$

Q. $L\{\cos^3 2t \sin 3t\} = L\left\{\frac{(\cos 6t + 3\cos 2t)}{4} \sin 3t\right\}$

$= \frac{1}{4} L\left\{\frac{1}{2}(\sin 9t + \sin(-3t)) + \frac{3}{2}(\sin 5t + \sin(-t))\right\}$

$= \frac{1}{8}\left\{\frac{9}{s^2+81} - \frac{3}{s^2+9} + \frac{3\times 5}{s^2+25} - \frac{3}{s^2+1}\right\}$

Q $L\{\sin^4 t\} = L\{(\sin^2 t)^2\} = L\left\{\left(\frac{1-\cos 2t}{2}\right)^2\right\}$

$= \frac{1}{4} L\{1+\cos^2 2t - 2\cos 2t\} = \frac{1}{4} L\left\{1 + \frac{1+\cos 4t}{2} - 2\cos 2t\right\}$

$= \frac{1}{8} L\{3 + \cos 4t - 4\cos 2t\} = \frac{1}{8}\left\{\frac{3}{s} + \frac{s}{s^2+16} - 4\times\frac{s}{s^2+4}\right\}$

Q. $L\{\sin(\omega t+\alpha)\} = L\{\sin\omega t \cos\alpha + \cos\omega t \sin\alpha\}$

$= \cos\alpha \times \frac{\omega'}{s^2+\omega^2} + \sin\alpha \times \frac{s}{s^2+\omega^2}$

Q $L\{\sinh at \sin at\} = L\left\{\frac{e^{at}-e^{-at}}{2} \times \sin at\right\} = \frac{1}{2}\left\{\frac{a}{(s-a)^2+a^2} - \frac{a}{(s+a)^2+a^2}\right\}$

$= \frac{a}{2}\left\{\frac{4as}{(s^2+2a^2-2as)(s^2+2a^2+2as)}\right\} = \frac{2a^2 s}{(s^2+2a^2)^2-(2as)^2} = \frac{2a^2 s}{s^4+4a^4}$

Q. If $L\{f(t)\} = \bar{f}(s)$, $L\{f(t)\cos at\} = ?$

Ans $L\{f(t)\cos at\} = L\left\{f(t)\frac{e^{iat}+e^{-iat}}{2}\right\} = \frac{1}{2} L\{e^{iat}f(t) + e^{-iat}f(t)\}$

$= \frac{1}{2}\{\bar{f}(s-ia) + \bar{f}(s+ia)\}$

Q. $L\{e^{-3t}(3\cos 4t + 2\sin 4t)\}$

$= 3L\{e^{-3t}\cos 4t\} + 2L\{e^{-3t}\sin 4t\} = \frac{3(s+3)}{(s+3)^2+16} + \frac{2(4)}{(s+3)^2+16} = \frac{3s+17}{(s+3)^2+16}$

Q. If $L\{f(t)\} = \frac{e^{-1/s}}{s}$ then $L\left\{e^{-3t}\int_0^t f(3t)dt\right\} =$ ____ ?

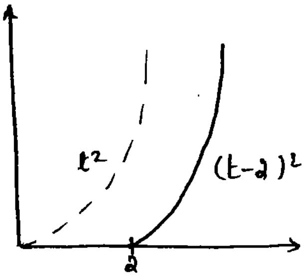
Ans $L\{f(3t)\} = \frac{1}{3}\bar{f}(s/3) = \frac{\cancel{3}e^{-3/s}}{\cancel{3}s} = \frac{e^{-3/s}}{s}$

$L\left\{\int_0^t f(3t)dt\right\} = \frac{1}{s}\frac{e^{-3/s}}{s} = \frac{e^{-3/s}}{s^2}$

$L\left\{e^{-3t}\int_0^t f(3t)dt\right\} = \frac{e^{-3/s+3}}{(s+3)^2}$

Q. $f(t) = \begin{cases} (t-2)^2 & ; t \geq 2 \\ 0 & ; t<2 \end{cases}$. Find $L\{f(t)\}$.

$\Downarrow$

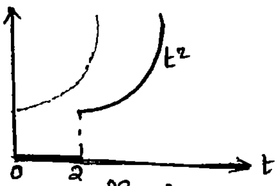


Ans $f(t) = (t-2)^2 u(t-2)$

$L\{t^2\} = \frac{2}{s^3}$

$L\{(t-2)^2\} = e^{-2s} \times \frac{2}{s^3}$

Q. $f(t) = \begin{cases} t^2 & ; t \geq 2 \\ 0 & ; t<2 \end{cases}$ Find $L\{f(t)\}$

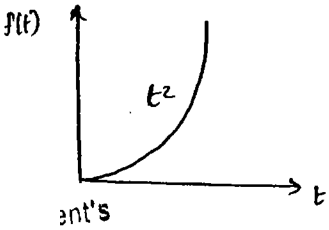


Ans. $f(t) = t^2 u(t-2)$

$L\{f(t)\} = e^{-2s} L\{(t+2)^2\} = e^{-2s} L\{(t+2)^2\} = e^{-2s} L\{t^2+4t+4\}$

$= e^{-2s}\{2/s^3 + 4/s^2 + 4/s\}$

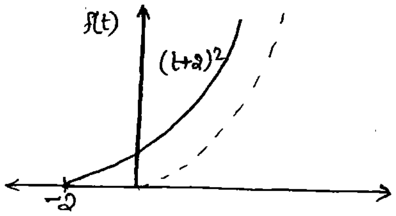
Q. $f(t) = \begin{cases} t^2 & t \geq 0 \\ 0 & t<0 \end{cases}$



Ans $L\{f(t)\} = L\{t^2 u(t)\} = \frac{2!}{s^3}$

ent's

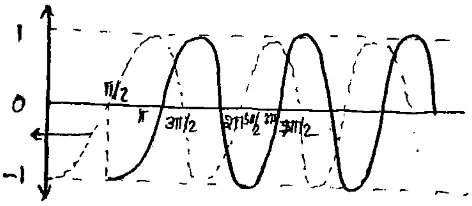
Q. $f(t) = \begin{cases} (t+2)^2 & ; t \geq -2 \\ 0 & ; t<-2 \end{cases}$ ACc



Ans $f(t) = (t+2)^2 u(t+2)$

$L\{f(t)\} = e^{2s} L\{(t+2-2)^2\} = e^{2s} L\{t^2\} = e^{2s} \times \frac{2!}{s^3}$

Q. Find L.T of given wave.



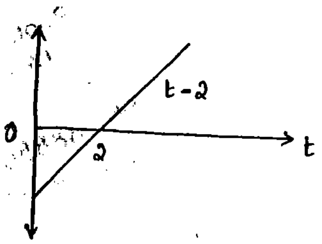
Ans – Cos wave shifted π/2

$\therefore\ f(t) = -\sin t\, u(t-\pi/2)$

$L\{f(t)\} = -e^{-\pi/2\, s} L\{\sin(t+\pi/2)\}$

$= -e^{-\pi/2\, s} L\{\cos t\} = -e^{-\pi/2\, s}\left(\frac{s}{s^2+1}\right)$

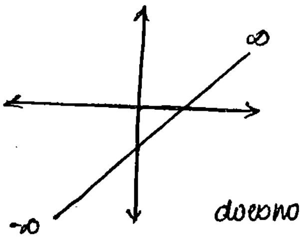
Q.



Ans $f(t) = (t-2)u(t)$

$\mathcal{L}\{f(t)\} = \frac{1}{s^2} - \frac{2}{s}$

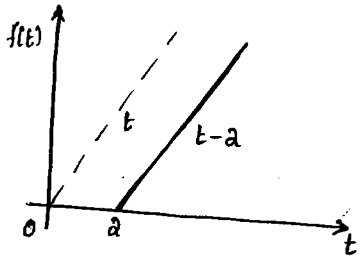
***



doesnot exist $(-\infty)$

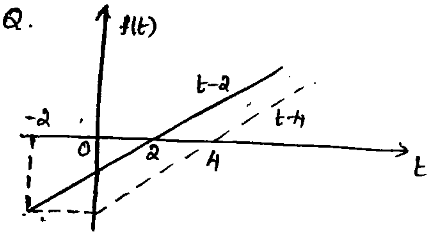
Laplace defined only from 0 to ∞

Q.



$\mathcal{L}\{(t-2)u(t-2)\} = e^{-2s}\mathcal{L}(t) = e^{-2s}\cdot\frac{1}{s^2}$

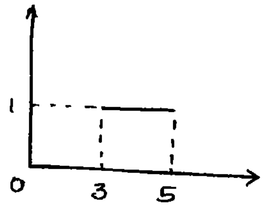
Q.



$e^{2s}\mathcal{L}\{f(t-2)\} = e^{2s}\mathcal{L}\{t-4\}$

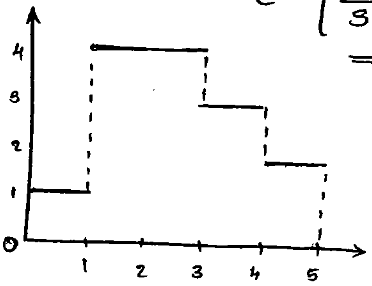
$= e^{2s}\left[\frac{1}{s^2} - \frac{4}{s}\right]$

Q.



i) $f(t) = u(t-3) - u(t-5)$

$\bar{f}(s) = \frac{e^{-3s} - e^{-5s}}{s}$



ii) $f(t) = u(t) + 3(u(t-1)) - u(t-3) - u(t-4) - 2u(t-5)$

$\bar{f}(s) = \frac{1}{s}\left[1 + 3e^{-s} - e^{-3s} - e^{-4s} - 2e^{-5s}\right]$

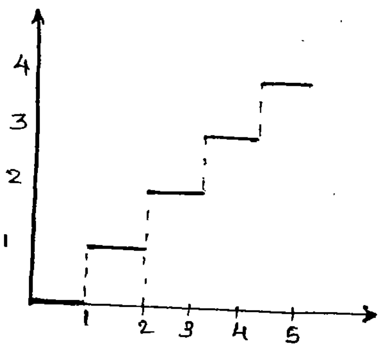
Q.

Staircase / Step / Integral Fn :-

$[x] = n \quad ; \; n \le x < n+1$

$n \in Z$

$$[t] = \begin{cases} 0 & 0 \le t < 1 \\ 1 & 1 \le t < 2 \\ 2 & 2 \le t < 3 \\ \vdots & \end{cases}$$



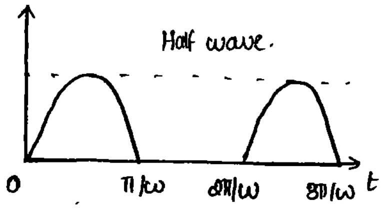
$$f(t) = u(t-1) + u(t-2) + u(t-3) + \cdots$$

$$L\{f(t)\} = \frac{e^{-s}}{s} + \frac{e^{-2s}}{s} + \frac{e^{-3s}}{s} + \cdots$$

$$= \frac{e^{-s}}{s}\left[1 + e^{-s} + e^{-2s} + e^{-3s} + \cdots\right]$$

$$= \frac{e^{-s}}{s}\,(1-e^{-s})^{-1} = \underline{\underline{\frac{e^{-s}}{s(1-e^{-s})}}}$$

Q. $f(t) = \begin{cases} \sin\omega t & ; 0 \le t < \pi/\omega \\ 0 & ; \pi/\omega \le t < 2\pi/\omega \end{cases}$



$$f(t+2\pi/\omega) = f(t) \;\forall\, t.$$

$T = 2\pi/\omega.$

$$L\{f(t)\} = \frac{1}{(1-e^{-s2\pi/\omega})}\left[\int_0^{\pi/\omega} e^{-st}\sin\omega t\,dt + \int_{\pi/\omega}^{2\pi/\omega} \cancel{e^{-st}\times}\overset{0}{0}\,dt\right]$$

$$= \frac{1}{(1-e^{-s2\pi/\omega})}\left[\frac{e^{-st}}{s^2+\omega^2}(-s\sin\omega t - \omega\cos\omega t)\right]_0^{\pi/\omega}$$

$$= \frac{1}{(1^2 - e^{-2s\pi/\omega})}\left[\frac{e^{-s\pi/\omega}}{s^2+\omega^2}\left(-s(0) - \omega(-1)\right) - \frac{1}{s^2+\omega^2}\left(-s(0) - \omega(1)\right)\right]$$

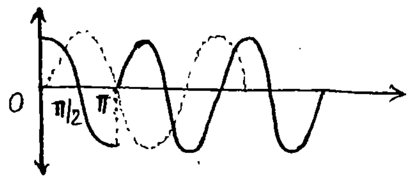
$$= \frac{\omega}{s^2+\omega^2}\;\frac{\cancel{1+e^{-s\pi/\omega}}}{(1-e^{-s\pi/\omega})\cancel{(1+e^{-s\pi/\omega})}} = \underline{\frac{\omega}{s^2+\omega^2} \times \frac{1}{1-e^{-s\pi/\omega}}}$$

For Fullwave

$$L\{f(t)\} = \frac{\omega}{s^2+\omega^2}\;\frac{\cancel{e^{-s\pi/2\omega}}}{\cancel{e^{-s\pi/2\omega}}}\;\frac{(e^{s\pi/2\omega} + e^{-s\pi/2\omega})}{(e^{s\pi/2\omega} - e^{-s\pi/2\omega})}$$

$$= \underline{\underline{\frac{\omega}{s^2+\omega^2}\,\coth(s\pi/2\omega)}}$$

Q. $f(t) = \begin{cases} \sin t & ; t > \pi \\ \cos t & ; 0 < t < \pi \end{cases}$



$$f(t) = \cos t + [\sin t - \cos t]\,u(t-\pi)$$

$$L\{f(t)\} = \frac{s}{s^2+1} + e^{-\pi s} L\{\underset{\underset{-\sin t}{\downarrow}}{\sin(t+\pi)} - \underset{\underset{-\cos t}{\downarrow}}{\cos(t+\pi)}\} = \frac{s}{s^2+1} + e^{-\pi s}\underline{\underline{\left[\frac{-1}{s^2+1} + \frac{s}{s^2+1}\right]}}$$

Q. $f(t) = |t+1| + |t-1| \quad ; t \geq 0$

Ans $f(t) = \begin{cases} (t+1)-(t-1) & ; 0 \leq t < 1 \\ (t+1)+(t-1) & ; t \geq 1 \end{cases}$

$|x| = x \quad ; x > 0$

$\quad = -x \quad ; x < 0$

$|x-1| = x-1 \quad ; x \geq 1$

$\quad = -(x-1) \; ; x \leq 1$

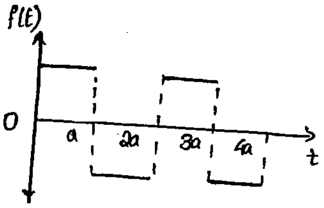
$$= \begin{cases} 2 & ; \quad 0 \leq t < 1 \\ 2t & ; \quad t \geq 1 \end{cases}$$

$f(t) = 2 + [2t-2]\,u(t-1) \qquad = 2 + e^{-s}\underbrace{[2(t+1)-2]}_{2t}$

$L\{f(t)\} = \frac{2}{s} + e^{-s} \times \frac{2}{s^2}$

Q. $f(t) = \begin{cases} 1 & ; 0 \leq t < a \\ -1 & ; a \leq t < 2a \end{cases}$

$f(t+2a) = f(t) \; \forall \; t$



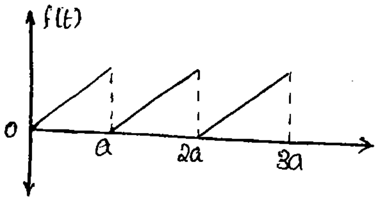
1) $\frac{1}{s}\tanh(as)$ 2) $\frac{1}{s}\tanh\left(\frac{as}{2}\right)$ 3) $\frac{1}{s}\tanh\left(\frac{as}{4}\right)$ 4) None.

Ans $L\{f(t)\} = \dfrac{L\,[u(t) - 2u(t-a) + u(t-2a)]}{(1-e^{-2as})}$

$$= \frac{1/s - \frac{2e^{-as}}{s} + \frac{e^{-2as}}{s}}{1-e^{-2as}} = \frac{\frac{1}{s}(1-e^{-as})^{\cancel{2}}}{\cancel{(1-e^{-as})}(1+e^{-as})}$$

$$= \frac{1}{s} \; \frac{e^{-as/2}}{e^{-as/2}} \left[\frac{e^{as/2} - e^{-as/2}}{e^{as/2} + e^{-as/2}} \right]$$

Q.



$f(t) = \frac{K}{a} t \qquad ; 0 \leq t < a$

$T = a$

$$\bar{f}(s) = \frac{\frac{K}{a}\int_0^a t e^{-st}\,dt}{1-e^{-sa}}$$

Q. $L\{t \cos at\} = \frac{-d}{ds}\left[\frac{s}{s^2+a^2}\right]$

$$= -\left[\frac{(s^2+a^2)\times 1 - s.2s}{(s^2+a^2)^2}\right]$$

$$= \frac{s^2-a^2}{(s^2+a^2)^2}$$

Q. $\mathcal{L}\{t^2 \sin at\} = (-1)^2 \dfrac{d^2}{ds^2} \mathcal{L}\{\sin at\}$

$$= \frac{d^2}{ds^2}\left[\frac{a}{s^2+a^2}\right] = \frac{d}{ds}\left[\frac{-2as}{(s^2+a^2)^2}\right]$$

$$= -2a\left[\frac{(s^2+a^2)^2 \times 1 - s \times 2(s^2+a^2)2s}{(s^2+a^2)^4}\right] = (-2a)\left[\frac{\cancel{(s^2+a^2)}[s^2+a^2-4s^2]}{(s^2+a^2)^{\cancel{4}3}}\right]$$

$$= (-2a)\ \frac{a^2-3s^2}{(s^2+a^2)}$$

Q. $\mathcal{L}\{t^6 e^{3t}\}$.

Ans: $\mathcal{L}(t^6) = \dfrac{6!}{s^7}$

$\mathcal{L}\{e^{3t} t^6\} = \dfrac{6!}{(s-3)^7}$

Q. $\mathcal{L}\{t^3 \cosh 2t\} = \mathcal{L}\left\{t^3\left[\dfrac{e^{2t}+e^{-2t}}{2}\right]\right\} = \dfrac{1}{2}\left[\dfrac{3!}{(s-2)^4} + \dfrac{3!}{(s+2)^4}\right]$

Q. $\mathcal{L}\{t \cos 3t\} = \dfrac{-d}{ds}\left[\dfrac{s}{s^2+9}\right] = -\dfrac{(s^2+9)1 - s \times 2s}{(s^2+9)^2} = \dfrac{s^2-9}{(s^2+9)^2}$

Q. $\mathcal{L}\{f(t)\} = \dfrac{1}{s^2+s+1}$. $\mathcal{L}\{tf(t)\} = ?$

Ans a) $\dfrac{2s+1}{(s^2+s+1)^2}$ b) $\dfrac{-(2s+1)}{(s^2+s+1)^2}$ c) $\dfrac{-1}{(s^2+s+1)^2}$ d) None.

Ans $\dfrac{-d}{dt}\left[\dfrac{1}{s^2+s+1}\right] = \dfrac{2s+1}{(s^2+s+1)^2}$

Q. $\mathcal{L}\left\{\dfrac{t^{n-1}}{1-e^{-t}}\right\} = \mathcal{L}\{(1-e^{-t})^{-1} t^{n-1}\}$

$$= \mathcal{L}\{t^{n-1}(1 + e^{-t} + e^{-2t} + e^{-3t} + \ldots)\}$$

$$= \mathcal{L}\left\{t^{n-1} \sum_{k=0}^{\infty} e^{-kt}\right\} = \sum_{k=0}^{\infty} \mathcal{L}\{t^{n-1} e^{-kt}\}$$

$$\mathcal{L}\{t^{n-1}\} = \frac{(n-1)!}{s^n}$$

$$\therefore \sum_{k=0}^{\infty} L\{t^{n-1}e^{-kt}\} = \sum_{k=0}^{\infty} \frac{\sqrt{n}}{(s+k)^n}$$

Q. $L\left\{\frac{\sin at}{t}\right\}$

1) $\tan^{-1}(s/a)$ 2) $\tan^{-1}(a/s)$ 3) $\cot^{-1}(a/s)$ 4) None.

Ans $L\left\{\frac{\sin at}{t}\right\} = \int_s^{\infty}\left[\frac{a}{s^2+a^2}\right]ds$

$= \left[\tan^{-1}(s/a)\right]_s^{\infty} = \pi/2 - \tan^{-1}(s/a)$ $\cot^{-1}x = \tan^{-1}(1/x)$

Q. $L\left\{\frac{\cos at}{t}\right\} = \int_s^{\infty}\left\{\frac{s}{s^2+a^2}\right\}ds$ $= \cot^{-1}(s/a) = \tan^{-1}(a/s)$

$= \frac{1}{2}\left[\log(s^2+a^2)\right]_s^{\infty}$ = Doesnot exist ∵ $\log_e \infty$ is not defined

IIIly $L\left[\frac{e^{-at}}{t}\right] \longrightarrow$ doesnot exist

$L\{1/t\} \longrightarrow$ " "

Q. $L\left\{\frac{e^{-at}-e^{-bt}}{t}\right\} = \int_s^{\infty}\left(\frac{1}{s+a} - \frac{1}{s+b}\right) = \left.\log(s+a) - \log(s+b)\right]_s^{\infty}$

$\infty - \infty \rightarrow$ Indeterminant

$= \log\left(\frac{s+a}{s+b}\right)_s^{\infty} \longrightarrow \frac{\infty}{\infty} \rightarrow$ indeterminant

$= \log\left[\frac{s(1+a/s)}{\cancel{s}(1+b/s)}\right]_s^{\infty} = \log(1) - \log\left[\frac{1+a/s}{1+b/s}\right]$

$= -\log\left(\frac{s+a}{s+b}\right)$ or $\log\left(\frac{s+b}{s+a}\right)$

IIIly $L\left\{\frac{\cos at - \cos bt}{t}\right\} = \frac{1}{2}\left(\frac{s^2+b^2}{s^2+a^2}\right)$

Q. $L\left\{\frac{\sin t \sin 4t}{t}\right\}$

$= \frac{1}{2} L\left\{\frac{\cos 3t - \cos 5t}{t}\right\} = \frac{1}{2} \times \frac{1}{2} \log\left[\frac{s^2+25}{s^2+9}\right]$

Q If $L\{\sin\sqrt{t}\} = \frac{\sqrt{\pi}\, e^{-1/4s}}{2s^{3/2}}$, find $L\left\{\frac{\cos\sqrt{t}}{\sqrt{t}}\right\}$.

Ans $f(t) = \sin\sqrt{t} \implies f(0) = 0$

$f'(t) = \frac{\cos\sqrt{t}}{2\sqrt{t}}$ $L\{f'(t)\} = sL\{f(t)\} - f(0)$.

$$\mathcal{L}\left[\frac{\cos\sqrt{t}}{\cancel{2}\sqrt{t}}\right] = \frac{s\sqrt{\pi}\, e^{-1/4s}}{\cancel{2}s^{3/2}}$$

$$\therefore \quad = \underline{\underline{\sqrt{\pi/s}\; e^{-1/4s}}}$$

Q. $\mathcal{L}\left\{e^{-3t}\int_0^t \frac{1-e^{-t}}{t}\,dt\right\} = \left\{\mathcal{L}\left[\int_0^\infty \frac{1-e^{-t}}{t}\,dt\right]\right\}_{s\to s+3}$

$$= \left[\frac{1}{s}\,\mathcal{L}\left[\frac{1-e^{-t}}{t}\,\cancel{dt}\right]\right]_{s\to s+3} = \left[\frac{1}{s}\log\left(\frac{s+1}{s}\right)\right]$$

$$= \underline{\underline{\frac{1}{s+3}\log\frac{s+4}{s+3}}}$$

Q. $\mathcal{L}\left\{\int_0^t e^{-2t}\frac{\sin t}{t}\,dt\right\} = \frac{1}{s}\left\{\mathcal{L}\left[e^{-2t}\frac{\sin t}{t}\right]\right\}$

$$= \frac{1}{s}\left\{\mathcal{L}\left\{\frac{\sin t}{t}\right\}_{s\to s+2}\right\} = \frac{1}{s}\left[\tan^{-1}(1/s)\right]_{s\to s+2}$$

$$= \underline{\underline{\frac{1}{s}\tan^{-1}\left(\frac{1}{s+2}\right)}}$$

Q. $\mathcal{L}\{t^4\,\delta(t-2)\} = \underline{\underline{2^4\,e^{-2s}}}$

$$= \underline{\underline{16\,e^{-2s}}}$$

$\mathcal{L}\{f(t)\,\delta(t-a)\} = e^{-as}f(a)$

Q. $\mathcal{L}\{(t^2+3)\,\delta(t-1)\} = e^{-s}(1+3) = \underline{\underline{4e^{-s}}}$

Q. $\mathcal{L}\{t\,\delta(t)\} = 0$

Q. $\mathcal{L}\{\cos t\,\delta(t)\} = 1$

Q $\int_0^\infty \frac{\sin t}{t}\,dt = \cot^{-1}(s)$

$$\mathcal{L}\left\{\frac{\sin t}{t}\right\} = \cot^{-1}s$$

$$\int_0^\infty e^{-st}\frac{\sin t}{t}\,dt = \underline{\underline{\cot^{-1}(s)}}$$

For $s=0$ $\int_0^\infty \frac{\sin t}{t}\,dt = \cot^{-1}(0) = \underline{\underline{\pi/2}}$

Q. $\int_0^\infty t^4 e^{-3t}\,dt = \mathcal{L}\{t^4\}$ for $s=3$

$$= \left.\frac{4!}{s^5}\right|_{s=3} = \underline{\underline{\frac{4!}{3^5}}}$$

Q. $\int_0^\infty t e^{-4t}\cos 3t\,dt = L\{t\cos 3t\}_{\text{for } s=4} = \frac{s^2-9}{(s^2+9)^2} = \underline{\underline{\frac{7}{625}}}$

Q. $\int_0^\infty \frac{e^{-4t}-e^{-3t}}{t}\,dt = L\left[\frac{e^{-4t}-e^{-3t}}{t}\right]_{\text{for } s=0} = \log\left(\frac{s+3}{s+4}\right)\Big]_{s=0}$

$= \underline{\underline{\log(3/4)}}$

Q. $\int_0^\infty \cos t\ \delta(t-\pi/4)\,dt = \cos(\pi/4) = \underline{\underline{1/\sqrt{2}}}$

Q. $\int_{-\infty}^{\infty} t\ \delta(t-4)\,dt = \underline{\underline{4}}$

- $\int_0^\infty \cos t\,dt$ & $\int_0^\infty \sin t\,dt$. Laplace fails as fns are divergent. Laplace Transformation is applicable to determine the improper integrals. incase of integrant is convergent as $t \to \infty$.

INVERSE LAPLACE TRANSFORMATIONS

If $L\{f(t)\} = \bar{f}(s) \Longrightarrow f(t) = L^{-1}[\bar{f}(s)]$

- $L^{-1}\{a\bar{f}(s) \pm b\bar{g}(s)\} = a\,L^{-1}\{\bar{f}(s)\} \pm b\,\bar{L}\{\bar{g}(s)\}$
- $L^{-1}[1/s] = 1$
- $L^{-1}\left[\frac{1}{s-a}\right] = e^{at}$
- $L^{-1}\left[\frac{1}{s+a}\right] = e^{-at}$
- $L^{-1}\left[\frac{1}{s^2+a^2}\right] = \frac{1}{a}\sin at$
- $L^{-1}\left[\frac{s}{s^2+a^2}\right] = \cos at$
- $L^{-1}\left[\frac{1}{s^2-a^2}\right] = \frac{1}{a}\sinh at$
- $L^{-1}\left[\frac{s}{s^2-a^2}\right] = \cosh at$
- $L^{-1}\left[\frac{1}{s^n}\right] = \begin{cases} \frac{t^{n-1}}{(n-1)!} & : n \in Z^+ \\ \frac{t^{n-1}}{\Gamma n} & : n \notin Z^+ \end{cases}$ $\quad n>0$

Freq. Shifting Theorem of I.L.T

$$L^{-1}\{\bar{f}(s-a)\} = e^{at} L^{-1}[\bar{f}(s)]$$

$$L^{-1}\{\bar{f}(s+a)\} = e^{-at} L^{-1}[\bar{f}(s)]$$

Second Shifting Theorem:

If $L^{-1}\{\bar{f}(s)\} = f(t) \Longrightarrow L^{-1}\{e^{-as}\bar{f}(s)\} = f(t-a)u(t-a)$

Scaling

$$L^{-1}\{\bar{f}(as)\} = \frac{1}{a} f(t/a)$$

$$L^{-1}\{\bar{f}(s/a)\} = a\, f(at)$$

Multiplication by s^n $(n \in Z^+)$

$$L^{-1}\{s\bar{f}(s)\} = f'(t) \quad ; f(0) = 0$$

$$L^{-1}\{s^n\bar{f}(s)\} = f^n(t) \quad ; f(0) = f'(0) = \_\_\_\_ = f^{n-1}(0) = 0$$

Division by s^n $(n \in Z^+)$

$$L^{-1}\left\{\frac{1}{s}\bar{f}(s)\right\} = \int_0^t f(t)dt$$

$$L^{-1}\left\{\frac{1}{s^n}\bar{f}(s)\right\} = \int_0^t f(t)(dt)^n$$

I.L.T of Derivatives

$$L^{-1}\left[\frac{d}{ds}\bar{f}(s)\right] = (-t)\, L^{-1}[\bar{f}(s)]$$

$$L^{-1}\left[\frac{d^n}{ds^n}\bar{f}(s)\right] = (-t)^n L^{-1}[\bar{f}(s)]$$

I.L.T of Integrals

$$L^{-1}\left\{\int_0^\infty \bar{f}(s)ds\right\} = \frac{1}{t} f(t)$$

$$L^{-1}\left\{\int_s^\infty \bar{f}(s)ds\right\} = \frac{1}{t^n} f(t)$$

Convolution Theorem

If $L^{-1}\{\bar{f}(s)\} = f(t)$

$L^{-1}\{\bar{g}(s)\} = g(t)$

$$\Longrightarrow L^{-1}\{\bar{f}(s)\bar{g}(s)\} = f(t)*g(t) = \int_0^1 f(x)g(t-x)dx$$

Initial Value Theorem / Final Value Theorem

If $L\{f(t)\} = \bar{f}(s)$

then (i) $\lim_{t\to 0} f(t) = \lim_{s\to\infty} s\bar{f}(s)$

(ii) $\lim_{t\to\infty} f(t) = \lim_{s\to 0} s\bar{f}(s)$

Initial value Theorem is applicable if $\bar{f}(s)$ is proper. Final value theorem is applicable if poles of $\bar{f}(s)$ are on the left hand side of s plane; ie in the case of right hand side pole or marginal poles, it is not applicable.

Q. $$L^{-1}\left\{\frac{1}{\sqrt{4s-3}}\right\} = \frac{1}{\sqrt{4}} L^{-1}\left\{\frac{1}{(s-3/4)^{1/2}}\right\}$$

$$= \frac{1}{\sqrt{4}} e^{3/4t} L^{-1}\left\{\frac{1}{s^{1/2}}\right\} = \frac{1}{2} e^{3/4t} \frac{t^{1/2-1}}{\sqrt{1/2}} = \frac{e^{3/4t}}{2\sqrt{\pi t}}$$

$$L^{-1}\left\{\frac{1}{(as+b)^n}\right\} = \frac{1}{a^n} e^{-b/at} \frac{t^{n-1}}{\sqrt{n}}$$

Q. $$L^{-1}\left\{\frac{3s-2}{4s^2-16} - \frac{2s-1}{9s^2+25}\right\} = \frac{1}{4} L^{-1}\left\{\frac{3s}{s^2-4} - \frac{2}{s^2-4}\right\} - \frac{1}{9} L^{-1}\left\{\frac{2s}{s^2+\frac{25}{9}} - \frac{1}{s^2+\frac{25}{9}}\right\}$$

$$= \frac{1}{4}\left\{3\cosh 2t - \sinh 2t\right\} - \frac{1}{9}\left\{2\cos\frac{5}{3}t - \frac{1}{5/3}\sin\frac{5}{3}t\right\}$$

Q. $$L^{-1}\left\{\frac{s}{(2s+3)^2}\right\} = \frac{1}{2^2} L^{-1}\left\{\frac{(s+3/2)-3/2}{(s+3/2)^2}\right\} = \frac{1}{4} e^{-3/2t} L^{-1}\left\{\frac{s-3/2}{s^2}\right\}$$

$$= \frac{1}{4} e^{-3/2t} L^{-1}\left\{\frac{1}{s} - \frac{3/2}{s^2}\right\} = \frac{1}{4} e^{-3/2t}\left(1 - \frac{3}{2}t\right)$$

Q. $\mathcal{L}^{-1}\left[\frac{s^2}{(s-2)^2}\right] = \mathcal{L}^{-1}\left[\frac{((s-2)+2)^2}{(s-2)^2}\right] = e^{+2s}\,\mathcal{L}^{-1}\left[\frac{(s+2)^2}{s^2}\right]$

$= e^{2s}\,\mathcal{L}^{-1}\left[\frac{s^2+4s+4}{s^2}\right] = e^{2s}\,\mathcal{L}^{-1}\left[1+\frac{4}{s}+\frac{4}{s^2}\right].$

$= e^{2s}\left\{\delta(t) + 4 + 4t\right\}$

Q. $\mathcal{L}^{-1}\left\{\frac{2s+3}{s^2-4s+3}\right\}$

Note: If Dr is quadratic check for answer options. If options are exponential, go for partial fraction.

$\frac{2s+3}{s^2-4s+3} = \frac{A}{s-3} + \frac{B}{s-1}$

$2s+3 = A(s-1) + B(s-3)$

$s=1 \Longrightarrow \quad B = 5/2$

$s=3 \Longrightarrow \quad A = 9/2$

$\therefore\ \mathcal{L}^{-1}\left\{\frac{9/2}{s-3} + \frac{5/2}{s-1}\right\} = \frac{9}{2}e^{+8t} + \frac{5}{2}e^{+t}$

Q. $\mathcal{L}^{-1}\left\{\frac{s+3}{s^3-2s^2-8s}\right\} = \mathcal{L}^{-1}\left[\frac{s+3}{s(s-4)(s+2)}\right] = \mathcal{L}^{-1}\left\{\frac{-3/8}{s} + \frac{7/24}{s-4} + \frac{1/12}{s+2}\right\}$

$= -3/8 + \frac{7}{24}e^{4t} + \frac{1}{12}e^{-2t}$

Q $\mathcal{L}^{-1}\left\{\frac{3s+1}{s^2+5s+2}\right\} = \mathcal{L}^{-1}\left\{\frac{3(s+\frac{5}{2}) - \frac{15}{2}+1}{(s+\frac{5}{2}) - \frac{25}{4}+2}\right\}$

$= e^{-5/2t}\,\mathcal{L}^{-1}\left[\frac{3s-13/2}{s^2-17/4}\right] = e^{-5/2t}\left[3\cosh\frac{\sqrt{17}}{2}t - \frac{13}{2}\times\frac{1}{\frac{\sqrt{17}}{2}}\sinh\frac{\sqrt{17}t}{2}\right]$

$= e^{-5/2t}\left[3\cosh\frac{\sqrt{17}}{2}t - \frac{13}{\sqrt{17}}\sinh\frac{\sqrt{17}}{2}t\right]$

Q. $\mathcal{L}^{-1}\left\{\frac{s-1}{s^2+s+1}\right\} = \mathcal{L}^{-1}\left\{\frac{(s+1/2) - 1/2 - 1}{(s+1/2)^2 - \frac{1}{4} + 1}\right\} = e^{-t/2}\,\mathcal{L}^{-1}\left[\frac{s-3/2}{s^2+3/4}\right]$

$= e^{-t/2}\left[\cos\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{\cancel{2}}\,\frac{1}{\cancel{\sqrt{3}/2}}\ \sin\frac{\sqrt{3}}{2}t\right]$

Q. $$L^{-1}\left\{\frac{s}{(s+3)(s^2+4)}\right\} = L^{-1}\left\{\frac{-3/13}{s+3} + \frac{Bs+C}{s^2+4}\right\}$$

Coeff. of $s^2 \Longrightarrow B - 3/13 = 0 \Longrightarrow B = 3/13$

const. $\Longrightarrow \frac{-12}{13} + 3C = 0 \Longrightarrow C = 4/13$

$$= \frac{-3}{13}e^{-3t} + \frac{3}{13}\cos 2t + \frac{4}{13}\,\frac{\sin 2t}{2}$$

Q. $$L^{-1}\left\{\frac{1}{s^3(s^2+1)}\right\}$$

By convolution property, $\frac{t^2}{2!} \sin t = \int_0^t \frac{x^2}{2!} \sin(t-x)\,dx$

$$= \int_0^t\int_0^t\int_0^t \sin t\,(dt)^3 = \int_0^t\int_0^t (-\cos t)_0^t\,dt^2 = \int_0^t\int_0^t (1-\cos t)\,dt$$

$$= \int_0^t (t - \sin t)_0^t\,dt = \left.\frac{t^2}{2} + \cos t\right]_0^t = \frac{t^2}{2} + \cos t - 1$$

Q. $$L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right]$$

$$\bar{f}(s) = \frac{1}{s^2+a^2}$$

$$\frac{d}{ds}\bar{f}(s) = \frac{-2s}{(s^2+a^2)^2}$$

$$L^{-1}\left[\frac{\cancel{-}2s}{(s^2+a^2)^2}\right] = \cancel{-}t\,\frac{1}{a}\sin at$$

$$L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right] = \frac{t}{2a}\sin at$$

$$L^{-1}\left[\frac{d}{ds}\bar{f}(s)\right] = -t\,L^{-1}\left[\bar{f}(s)\right]$$

<u>OR</u> By conversion

$$= \frac{1}{a}\sin at * \cos at$$

$$= \int_0^t \frac{1}{a}\sin ax \cos a(t-x)\,dx$$

$$= \frac{1}{2a}\int_0^t \left[\sin at + \sin(2ax - at)\right]dx$$

$$= \frac{1}{2a}\left[x\sin at - \frac{\cos(2ax-at)}{2a}\right]_0^t$$

$$= \frac{1}{2a}(t\sin at)$$

Q. $$L^{-1}\left[\frac{s^2}{(s^2+a^2)^2}\right] = L^{-1}\left\{s\cdot\frac{s}{(s^2+a^2)^2}\right\} = \frac{d}{dt}\left\{\frac{t\sin at}{2a}\right\} = \frac{1}{2a}\left\{ta\cos at + \sin at\right\}$$

Q. $$L^{-1}\left[\frac{1}{(s^2+a^2)^2}\right] = L^{-1}\left[\frac{1}{s}\cdot\frac{s}{s^2+a^2}\right] = \int_0^t \frac{t\sin at}{2a}\,dt = \frac{1}{2a}\left[\left.\frac{t(-\cos at)}{a}\right]_0^t - \int\frac{-\cos at\,dt}{a}\right.$$

$$= \frac{1}{2a}\left[\frac{-t\cos at}{a} + \frac{\sin at}{a^2}\right]$$

$$L^{-1}\left\{\frac{s}{(s^2+a^2)(s^2+b^2)}\right\} = \frac{1}{b^2-a^2} L^{-1}\left\{\frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}\right\} = \frac{1}{b^2-a^2}\left[\cos at - \cos bt\right]$$

Q. $$L^{-1}\left\{\frac{1}{(s^2-4)(s^2+16)}\right\} = \frac{1}{20} L^{-1}\left\{\frac{1}{s^2-4} - \frac{1}{s^2+16}\right\} = \frac{1}{20}\left\{\frac{1}{2}\sinh 2t - \frac{1}{4}\sin 4t\right\}$$

Q. $$L^{-1}\left\{\frac{s^2}{(4-s^2)(s^2+25)}\right\} = \frac{-1}{29} L^{-1}\left\{\frac{s^2}{s^2-4} - \frac{s^2}{s^2+25}\right\}$$

$$= \frac{-1}{29} L^{-1}\left\{\left(1+\frac{4}{s^2-4}\right) - \left(1-\frac{25}{s^2+25}\right)\right\}$$

$$= \frac{-1}{29}\left\{2\sinh 2t + 5\sin 5t\right\}$$

Q. $$L^{-1}\left\{\frac{s}{s^4+s^2+1}\right\} = L^{-1}\left\{\frac{s}{(s^2+1)^2-s^2}\right\} = L^{-1}\left\{\frac{s}{(s^2+1-s)(s^2+1+s)}\right\}$$

$$= \frac{1}{2} L^{-1}\left\{\frac{1}{s^2+1-s} - \frac{1}{s^2+1+s}\right\} = \frac{1}{2} L^{-1}\left\{\frac{1}{(s-1/2)^2+3/4} - \frac{1}{(s+1/2)^2+3/4}\right\}$$

F.S Theorem

Q. $$L^{-1}\left\{\frac{s}{(s^4+4a^4)}\right\} = L^{-1}\left\{\frac{s}{(s^2+2a^2)^2-(2as)^2}\right\} = L^{-1}\left\{\frac{s}{(s^2+2a^2-2as)(s^2+2a^2+2as)}\right\}$$

$$= \frac{1}{4a} L^{-1}\left\{\frac{1}{(s^2+2a^2-2as)} - \frac{1}{(s^2+2a^2+2as)}\right\}$$

$$= \frac{1}{2} L^{-1}\left\{\frac{1}{(s-1/2)^2+3/4} - \frac{1}{(s+1/2)^2+3/4}\right\}$$

$$= \frac{1}{2}\left\{e^{t/2}\,\frac{\sin \sqrt{3}/2\,t}{\sqrt{3}/2} - e^{-t/2}\,\frac{\sin \sqrt{3}/2\,t}{\sqrt{3}/2}\right\}$$

$$= \frac{1}{\cancel{2}}\;\frac{1}{\sqrt{3}/\cancel{2}}\left\{\left(\sin\frac{\sqrt{3}}{2}t\right)\left(2\sinh\frac{t}{2}\right)\right\}$$

Q. $$L^{-1}\left\{\frac{2+5s}{s^4 e^{4s}}\right\} = L^{-1}\left\{e^{-4s}\left(\frac{2}{s^4}+\frac{5}{s^3}\right)\right\}$$

S.S Theorem $\Longrightarrow \left(\frac{2t^3}{3!} + \frac{5t^2}{2!}\right)_{t=4}$

$$= \left(\frac{2(t-4)^3}{3!} + \frac{5(t-4)^2}{2!}\right) u(t-4)$$

Q. $L^{-1}\left\{\frac{1}{s(1-e^{-s})}\right\} = L^{-1}\left\{\frac{1}{s}(1-e^{-s})^{-1}\right\}$

$$= L^{-1}\left\{\frac{1}{s}(1+e^{-s}+e^{-2s}+e^{-3s}+----)\right\}$$

$$= u(t)+u(t-1)+u(t-2)+-----$$

$$= \sum_{k=0}^{\infty} u(t-k)$$

Q. If $L^{-1}\{\bar{f}(s)\} = \frac{t\sin 3t}{3}$ find $L^{-1}\left\{e^{-3s}\int_s^{\infty}\bar{f}(3s+5)ds\right\} = ?$

Ans $L^{-1}\{\bar{f}(3s+5)\} = L^{-1}\{\bar{f}(3(s+5/3))\}$

F.S Theorem $\Longrightarrow$ $= e^{-5/3t} \, L^{-1}\{\bar{f}(3s)\}$

$\downarrow$

Scaling

$$= \frac{e^{-5/3t} \times \frac{1}{3} \times \frac{t\sin 3t/3}{3}}{3} = \frac{e^{-5/3t}\, t\sin t}{27}$$

$$L^{-1}\left\{\int_s^{\infty}\bar{F}(3s+5)ds\right\} = \frac{1}{t}\left\{e^{-5/3t}\frac{t\sin t}{27}\right\}$$

Q. $L^{-1}\{1/s \cot^{-1}(s)\}$

Ans Let $\bar{f}(s) = \cot^{-1}(s)$

$$\frac{d}{ds}\bar{f}(s) = \frac{-1}{s^2+1}$$

$$L^{-1}\left\{\frac{d}{ds}\bar{f}(s)\right\} = -t\, L^{-1}\{\bar{f}(s)\}$$

$$L^{-1}\left\{\frac{\cancel{-}1}{s^2+1}\right\} = \cancel{-}t\, L^{-1}\{\cot^{-1}(s)\}$$

$$u \quad \frac{\sin t}{t} = L^{-1}\{\cot^{-1}(s)\}$$

$$\therefore L^{-1}\left[\frac{1}{s}\cot^{-1}s\right] = \int_0^t \frac{\sin t}{t}dt$$

Q. $L^{-1}\left\{\frac{1}{s}\sin\frac{1}{s}\right\} = L^{-1}\left\{\frac{1}{s}\left(\frac{1}{s}-\frac{1}{3!s^3}+\frac{1}{5!s^5}----\right)\right\}$

$$= L^{-1}\left\{\frac{1}{s^2}-\frac{1}{3!s^4}+\frac{1}{5!s^6}-----\right\}$$

$$= t - \frac{t^3}{(3!)^2} + \frac{t^5}{(5!)^2} - \frac{t^7}{(7!)^2} + ----$$

Q. $L^{-1}\left\{\frac{1}{s+3}\log\left(\frac{s+5}{s+3}\right)\right\}$

F.S Theorem $\Longrightarrow$ $e^{-3t} L^{-1}\left\{\frac{1}{s}\log\left(\frac{s+2}{s}\right)\right\}$ A

$\bar{f}(s) = \log\left(\frac{s+2}{s}\right) = \log(s+2) - \log s$

$\frac{d}{ds}\bar{f}(s) = \left(\frac{1}{s+2} - \frac{1}{s}\right)$

$L^{-1}\{\bar{f}(s)\} = \frac{-1}{t} L^{-1}\left\{\frac{d}{ds}\bar{f}(s)\right\} = \frac{-1}{t}\left[e^{-2t}-1\right] = \frac{1-e^{-2t}}{t}$

$\therefore L^{-1}\{\quad\} = e^{-3t}\int_0^t \left(\frac{1-e^{-2t}}{t}\right)dt.$

Q. $L\{f(t)\} = \frac{2s+3}{s^2-2s+2}$ then $\lim_{t\to\infty} f(t) = ?$

① 0 ② 2 ③ 3/2 ④ None.

Ans Final value theorem not applicable as poles are on right side

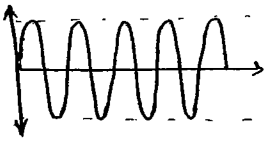
So the s/m is unstable $\Longrightarrow$ undefined.

$L^{-1}\left\{\frac{2s+3}{s^2-2s+3}\right\} = L^{-1}\left\{\frac{2(s-1)+5}{(s-1)^2+1}\right\} = \underset{\text{Infinite}}{\underset{\downarrow}{e^{-t}}}\underbrace{(2\cos t + \sin t)}_{\text{Finite}}$

Infinite × Finite = Infinite.

Q. $\lim_{t\to\infty} f(t) = \lim_{s\to 0} \frac{s(2s+3)}{s^2+2s+2} = 0 \times 3/2 = 0$

Q. $L\{f(t)\} = \frac{\omega_0}{s^2+\omega_0}$ then $\lim_{t\to\infty} f(t) = ?$



① 0 ② 1 ③ $-1 \le f(t) \le 1$ ④ None.

Ans The given s/m is marginally stable i.e it oscillates b/w two finite values.

Therefore the answer can be ③ (or) not defined.

Applications Of L.T to LDE

Q. $\frac{d^2y}{dt^2} + y = u(t)$; $y(0)=0$; $y'(0)=1$

Ans $L\left\{\frac{d^2y}{dt^2} + y = u(t)\right\}$

$s^2\bar{y}(s) - s\overset{0}{y(0)} - y'(0) + \bar{y}(s) = 1/s$

$\bar{y}(s)\left[s^2+1\right] = \frac{1}{s} + y'(0) = 1/s + 1 = \frac{s+1}{s}$

$\bar{y}(s) = \frac{s+1}{s(s^2+1)}$

$$y(t) = L^{-1}\left\{\frac{s+1}{s(s^2+1)}\right\} = L^{-1}\left\{\frac{1}{s^2+1} + \frac{1}{s(s^2+1)}\right\}$$

$$\frac{1}{s(s^2+1)} = \frac{1}{s} - \frac{s}{s^2+1}$$

$$= \underline{\underline{\sin t + 1 - \cos t}}$$

Q. $\frac{d^2y}{dt^2} - \frac{3dy}{dt} + 2y = \delta(t)$; $y\Big|_{t=0^-} = 0$, $\frac{dy}{dt}\Big|_{t=0^-} = 1$

Then $\frac{dy}{dt}\Big|_{t=0^+} = ?$

Ans $s^2\bar{y}(s) - sy(0) - y'(0) - 3[s\bar{y}(s) - y(0)] + 2\bar{y}(s) = 1$

$(s^2 - 3s+2)\bar{y}(s) = 1 + y'(0) = 1+1 = 2$

$\bar{y}(s) = \frac{2}{s^2-3s+2}$

$$y(t) = L^{-1}\left\{\frac{2}{s^2-3s+2}\right\} = L^{-1}\left\{\frac{-2}{s-1} + \frac{2}{s-2}\right\}$$

$$= \underline{\underline{(-2e^t + 2e^{2t})}}u(t)$$

$\frac{dy}{dt} = -2e^t + 4e^{2t}$

$\frac{dy}{dt}\Big|_{t=0^+} = -2+4 = \underline{\underline{2}}$

Z TRANSFORMATIONS

Z transformations also has similar properties of L.T but the main difference is that it operates only on discrete fns but not on continuous fns.

Defn : If $f(n)$ is defined $\forall\, n \geq 0$ $(n \in Z)$ and $f(n) = 0$ $\forall\, n < 0$.

$$Z\{f(n)\} = \sum_{n=0}^{\infty} f(n)z^{-n} = \bar{f}(z) \longrightarrow \text{Right Sided Transform.}$$

Here z is complex.

$z^{-n} \longrightarrow$ Kernel of Z.T

If $f(n)$ is defined $\forall\, n \leq 0$ $(n \in Z)$ and $f(n) = 0$ $\forall\, n \geq 0$.

$$\Longrightarrow Z\{f(n)\} = \sum_{n=-\infty}^{-1} f(n)z^{-n} = \bar{f}(z).$$

$\longrightarrow$ Left sided Transform.

Note : Bilateral transformation is also possible.

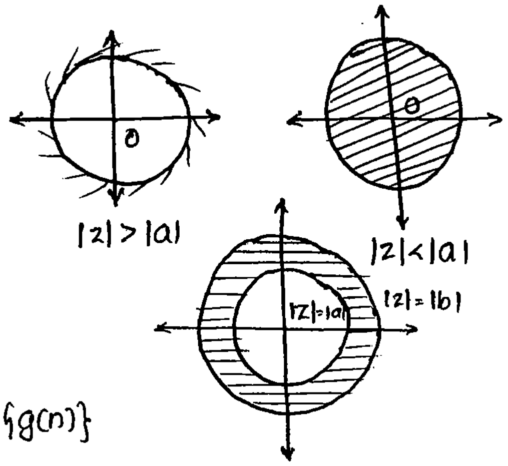
The transformation exists if the series is convergent.

ROC :

① For right sided $|z| > |a|$

② For left sided $|z| < |a|$

③ For double sided $|a| < |z| < |b|$



Properties

Linearity : $Z\{af(n) \pm bg(n)\} = a\,Z\{f(n)\} + b\,Z\{g(n)\}$

Some Standard Fns

ROC

$Z\{1\} = \frac{z}{z-1}$; ~~RO~~ $|z| > 1$

$Z\{a^n\} = \frac{z}{z-a}$; $|z| > 1$

$Z\{a^{-n}\} = \frac{z}{z-1/a}$; $|z| > \frac{1}{|a|}$

$= \frac{az}{az-1}$

$Z\{(-a)^n\} = \frac{z}{z+a}$; $|z| > |a|$

$Z\{k\} = k \times \frac{z}{z-1}$; $|z| > 1$

$Z\{u(n)\} = \frac{z}{z-1}$; $|z| > 1$

$Z\{\delta(n)\} = 1$

$Z\{\delta(n-k)\} = z^{-k}$; $|z| > 0$

Q. $Z\left\{\frac{1}{n!}\right\} = \sum_{n=0}^{\infty} \frac{1}{n!} z^{-n} = 1 + \frac{1}{z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \cdots$

$= \underline{\underline{e^{1/z}}}$; $|z| > 0$ or $|z| \neq 0$

Q. $Z\left\{\frac{1}{(n+1)!}\right\} = \sum_{n=0}^{\infty} \frac{1}{(n+1)!} z^{-n} = 1 + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \cdots$

$= z\left\{\frac{1}{z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \cdots\right\}$

$$= z\left\{1+\frac{1}{z}+\frac{1}{2!z^2}+\frac{1}{3!z^3}+\cdots-1\right\} = z\{e^{1/z}-1\} \text{ for } |z|>0$$

|||ly $$z\left\{\frac{1}{(n-1)!}\right\} = \sum_{n=1}^{\infty}\frac{1}{(n-1)!}z^{-n} = \frac{1}{z}e^{1/z}$$

$$\underbrace{z(n!)}_{\Downarrow} = \sum_{n=0}^{\infty} n!\,z^{-n} \longrightarrow \text{Divergent Series.}$$

doesnot exist.

Q. $$z\{1/n\} = \sum_{n=1}^{\infty}\frac{1}{n}z^{-n} \Longrightarrow \text{At } n=0 \text{ it is not defined.}$$

$$= \frac{1}{z}+\frac{1}{2z^2}+\frac{1}{3z^3}+\cdots$$

Note : $$\log(1-x) = -\left(x+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{4}+\cdots\right)$$

$$= -\log(1-1/z) \quad \text{for } |1/z|<1$$

$$= \log\left(\frac{z}{z-1}\right) \quad \text{for } |z|>1$$

Q. $$z\left\{\frac{1}{n+1}\right\} = \sum_{n=2}^{\infty}\frac{1}{(n+1)}z^{-n}$$ 4

$$= \frac{1}{3z^2}+\frac{1}{4z^3}+\frac{1}{5z^4}+\cdots$$

$$= \frac{1}{z}\left\{\frac{1}{3z}+\frac{1}{4z^2}+\frac{1}{5z^3}+\cdots\right\} = z^{-1}\log\left(\frac{z}{z-1}\right)$$

Q. $$z\{n\} = \sum_{n=0}^{\infty} n z^{-n}$$

$$= \frac{1}{z}+\frac{2}{z^2}+\frac{3}{z^3}+\cdots$$

$$= \frac{1}{z}\left[1+\frac{2}{z}+\frac{3}{z^2}+\cdots\right] = \frac{1}{z}\left\{\left(1-\frac{1}{z}\right)^{-2}\right\} = \frac{z}{(z-1)^2} : |z|>1.$$

Q. Recurrence Relation

$$z\{n^p\} = -z\frac{d}{dz}\left[z(n^{p-1})\right] \quad (p\in z+)$$

Eg : $$z\{n^2\} = -z\frac{d}{dz}\left[z(n)\right] = -z\frac{d}{dz}\left[\frac{z}{(z-1)^2}\right] = \frac{z(z+1)}{(z-1)^3} \quad |z|>1$$

Q $$z\{\cos n\theta + i\sin n\theta\}$$

$$z\{\cos n\theta\} = ?$$

$$z\{\sin n\theta\} = ?$$

$$Z\{\cos n\theta + i\sin n\theta\} = Z\{e^{in\theta}\}$$

$$= Z\{(e^{i\theta})^n\} = \frac{z(z-e^{-i\theta})}{(z-e^{i\theta})(z-e^{-i\theta})} \quad \text{for } |z| > |e^{i\theta}| = 1$$

$$= \frac{z^2 - z(\cos\theta - i\sin\theta)}{z^2 - z(e^{i\theta}+e^{-i\theta}) + \cancel{e^{i\theta}e^{-i\theta}} \to 1} = \frac{z^2 - z\cos\theta + iz\sin\theta}{z^2 - 2z\cos\theta + 1}$$

Comparing Real & Imaginary parts.

$$Z\{\cos n\theta\} = \frac{z^2 - z\cos\theta}{z^2 - 2z\cos\theta + 1}$$

$$Z\{\sin n\theta\} = \frac{z\sin\theta}{z^2 - 2z\cos\theta + 1}$$

III'ly $Z\{\sinh n\theta\} = \dfrac{z\sinh\theta}{z^2 - 2z\cosh\theta + 1}$

$$Z\{\cosh n\theta\} = \frac{z^2 - z\cosh\theta}{z^2 - 2z\cosh\theta + 1}$$

Properties

Damping Rule :

If $z(u_n) = \bar{u}(z)$

$\Longrightarrow$ ① $z(a^n u_n) = \bar{u}(z/a)$

② $z(a^{-n} u_n) = \bar{u}(az)$

First Shifting Property / Right Shifting

If $z\{u_n\} = \bar{u}(z)$ then $z\{u_{n-k}\} = z^{-k}\bar{u}(z)$; $(n \geq k)$

Second Shifting / Left Shifting

$$z\{u_{n+k}\} = z^k\left[\bar{u}(z) - u_0 - \frac{u_1}{z} - \frac{u_2}{z^2} - \cdots \frac{u_{k-1}}{z^{n-1}}\right]$$

Multiplication by n^p $(p \in Z^+)$:-

If $z\{u_n\} = \bar{u}(z)$

then ① $z\{nu_n\} = -z\dfrac{d}{dz}\{\bar{u}(z)\}$

② $z\{n^p u_n\} = \left(-z\dfrac{d}{dz}\right)^p\{\bar{u}(z)\}$

Division by n

$$Z\left\{\frac{u_n}{n}\right\} = -\int_0^z z^{-1}\,\bar{u}(z)\,dz$$

Convolution Property

If $f(n)$ and $g(n)$ are two discrete fns then $f(n)*g(n) = \sum_{m=0}^{n} f(m)g(n-m)$

$$Z\{f(n)*g(n)\} = \bar{f}(z)\cdot\bar{g}(z)$$

Initial Value Theorem

If $Z(u_n) = \bar{u}(z)$

$$\bar{u}(z) = u_0 + \frac{u_1}{z} + \frac{u_2}{z^2} + \cdots$$

then ① $u_0 = \lim_{z\to\infty} \bar{u}(z)$

② $u_k = \lim_{z\to\infty} z(u_{n+k})$

Eg: $u_1 = \lim_{z\to\infty} z(u_{n+1}) = \lim_{z\to\infty} z\{u(z) - u(0)\}$

$u_2 = \lim_{z\to\infty} z(u_{n+2}) = \lim_{z\to\infty} z^2\left\{\bar{u}(z) - u_0 - \frac{u_1}{2}\right\}$

Final Value Theorem

$$\lim_{z\to\infty} u_n = \lim_{z\to 1} (z-1)\bar{u}(z)$$

Q. $Z\{2n^2 + \overset{(-1)^n}{\cos n\pi} + a^3\} = 2\,\frac{z^2+z}{(z-1)^3} + \frac{z}{z+1} + a^3 \times \frac{z}{z-1}$ ROC : $|z|>1$

Q. $Z\{\sin n\pi/2\} = \frac{z}{z^2+1}$

$Z\{\cos n\pi/2\} = \frac{z^2}{z^2+1}$

Q. $Z\{\sin n\theta \cos n\theta\} = \frac{1}{2} Z(\sin n(2\theta)) = \frac{1}{2}\left\{\frac{z\sin 2\theta}{z^2 - 2z\cos 2\theta + 1}\right\}$

Q $Z\{\sin(n+1)\theta\}$

$u_n = \sin n\theta$

$z(u_{n+1}) = ?$

$$Z(u_{n+1}) = z\{\bar{u}(z) - u_0\} = z\left\{\frac{z\sin\theta}{z^2 - 2z\cos\theta + 1} - 0\right\} = \frac{z^2\sin\theta}{z^2 - 2z\cos\theta + 1}$$

Q. $Z\{\sin(n-1)\theta\} = Z\{\sin n\theta\cos\theta - \cos n\theta\sin\theta\} = \dfrac{Z\sin\theta\cos\theta - (Z^2 - Z\cos\theta)\sin\theta}{Z^2 - 2Z\cos\theta + 1}$

$$= \frac{-Z^2\sin\theta + 2Z\sin\theta\cos\theta}{Z^2 - 2Z\cos\theta + 1}$$

(or) $Z(u_{n-1}) = Z(\bar{u}(z)) \qquad n \geq 1$

$$= Z\left[\frac{Z\sin\theta}{Z^2 - 2Z\sin\theta + 1}\right] = \underline{\underline{\frac{Z^2\sin\theta}{Z^2 - 2Z\sin\theta + 1}}} \qquad n \geq 0$$

0 not included

Q. $Z\left\{a^n + \frac{1}{n}\right\} \qquad |z| > \max(|a|, 1)$

$$= \sum_{n=1}^{\infty}\left(a^n + \frac{1}{n}\right)z^{-n} = \sum_{n=1}^{\infty} a^n z^{-n} + \sum_{n=1}^{\infty}\frac{1}{n}z^{-n}$$

Q. $Z\left\{\frac{1}{n(n+1)}\right\} = Z\left\{\frac{1}{n} - \frac{1}{n+1}\right\} \qquad n \geq 1$

$$= \log\left(\frac{z}{z-1}\right) - \left\{z\log\left(\frac{z}{z-1}\right) - 1\right\} = \underline{\underline{(1-z)\log\left(\frac{z}{z-1}\right) + 1}}$$

Q. $\sum_{n=1}^{\infty}\frac{1}{n+1}z^{-n} = Z\left\{\frac{1}{z} + \frac{1}{2z^2} + \frac{1}{3z^3} + \cdots\right\}$

$$= z\log\left(\frac{z}{z-1}\right) - 1$$

Q. $Z\left\{\frac{1}{n(n-2)}\right\} = Z\left\{\frac{-1/2}{n} + \frac{1/2}{n-2}\right\} \qquad n \geq 3$

$$= \frac{-1}{2}\sum_{n=3}^{\infty}\frac{1}{n}z^{-n} + \frac{1}{2}\sum_{n=3}^{\infty}\frac{1}{n-2}z^{-n}$$

$$= \frac{-1}{2}\left[\log\left(\frac{z}{z-1}\right) - \frac{1}{z} - \frac{1}{2z^2}\right] + \frac{1}{2}\left[z^{-2}\log\left(\frac{z}{z-1}\right)\right]$$

***

	Domain	Transform	ROC
$Z[a^n u(n)]$	$n \geq 0$	$\frac{z}{z-a}$	$\lvert z\rvert > \lvert a\rvert$
$Z[a^n u(n-1)]$	$n > 0$	$\frac{a}{z-a}$	$\lvert z\rvert > \lvert a\rvert$
$Z[a^n u(-n)]$	$n \leq 0$	$\frac{a}{a-z}$	$\lvert z\rvert < \lvert a\rvert$
$Z[a^n u(-n-1)]$	$n < 0$	$\frac{z}{a-z}$	$\lvert z\rvert < \lvert a\rvert$

Q. $z\{a^{|n|}\} = z\{a^n u(n) + a^{-n} u(-n-1)\}$

$$= \frac{z}{z-a} + \frac{z}{\frac{1}{a} - z} \quad ; \text{ ROC } |z|>|a| \ \& \ |z| < \frac{1}{|a|}$$

$$\Longrightarrow \underline{|a|<|z|<\frac{1}{|a|}}$$

$$\Longrightarrow |a| < \frac{1}{|a|}$$

i.e. $|a|^2<1 \Longrightarrow \underline{|a|<1}$

Note : $z\{2^{|n|}\} \longrightarrow \underline{\text{doesnot exist}}$

Q. ROC of $z\{(\frac{1}{3})^{|n|} + (\frac{1}{2})^n u(n)\}$

ⓐ $|z| > 1/2$ ⓑ $|z| > 1/3$ ⓒ $\frac{1}{3} < |z| < \frac{1}{2}$ ⓓ $\frac{1}{2} < |z| < 3$

Ans

$$= Z\begin{cases} (\frac{1}{3})^n + (\frac{1}{2})^n & n \geq 0 \\ (1/3)^{-n} & n<0 \end{cases}$$

$$= \frac{z}{z-1/3} + \underline{\frac{z}{z-1/2}} + \frac{z}{3-z}$$

$|z| > 1/3$ & $|z| > 1/2$ & $|z| < 3$

$$\Longrightarrow |z| > 1/2 \ \& \ |z| < 3$$

$$\Longrightarrow \underline{\frac{1}{2} < |z| < 3}$$

Q. $z\{4^n + 5^n\} = \frac{z}{z-4} + \frac{z}{z-5}$; ROC $|z|>5$

Q. $z\{(4^n+5^n)\, u(-n)\}$

$$= \underset{|z|<4}{\underset{\downarrow}{\frac{4}{4-z}}} + \underset{|z|<5}{\underset{\downarrow}{\frac{5}{5-z}}} \qquad \underline{\text{ROC}} \quad |z|<4$$

Q. If $u(n) = \begin{cases} 4^n & ; n<0 \\ 5^n & ; n \geq 0 \end{cases}$

Doesnot exist ($\because |z|<4$ & $|z|>5 \Longrightarrow$ Null set)

Since common ROC doesnot exist transformation doesnot exist

Q. $u(n) = \begin{cases} 3^n & ; n<0 \\ 2^n & ; n \geq 0 \end{cases}$

$$\Longrightarrow \frac{z}{3-z} + \frac{z}{z-2}$$

$$|z|<3 \qquad |z|>2 \quad \Longrightarrow \quad \underline{\underline{2 \leq |z| \leq 3}}$$

Q. $z\{u_6\}$ where $\{u_6\} = \{1, 3, 5, 7, 0, 9\}$

$$z\{u_6\} = \sum_{n=0}^{5} u_n z^{-n} = 1 + \frac{3}{z} + \frac{5}{z^2} + \frac{7}{z^3} + \frac{9}{z^5}$$

$$\text{ROC} : \underline{\underline{|z|>0}}$$

Q. $z\{u_6\}$ where $\{u_6\} = \{1, 3, \overset{\downarrow}{5}, 7, 0, 9\}$

$-u_2\ -u_1\ u_0\ u_1\ u_2\ u_3$

$$z\{u_6\} = \sum_{n=-1}^{3} u_n z^{-n} = z^2 + 3z + 5 + \frac{7}{z} + \frac{9}{z^3}$$

$$\text{ROC} : \underline{\underline{0 < |z| < \infty}}$$

Q. If $u_K = \frac{1}{4^K}$; $-2 \leq K \leq 2$ then $z(u_K) = ?$

Ans $\sum_{K=-2}^{2} \frac{1}{4^K} z^{-K} = 16z^2 + 4z + 1 + \underline{\underline{\frac{1}{4z}}} + \frac{1}{16z^2}$

Q $z\{nC_K\}$; n is a scalar.

Ans $z\{nC_K\} = \sum_{K=0}^{n} nC_K z^{-K}$

$nC_r : 0 \leq r \leq n$

$$= nC_0 + \frac{nC_1}{z} + \frac{nC_2}{z^2} + \frac{nC_3}{z^3} + \cdots + \frac{nC_n}{z^n} \; ; \text{ for } |z|>0$$

Q. $z\{nC_K\}$ where K is a scalar.

$$= \sum_{n=K}^{\infty} nC_K z^{-n} \qquad \because nC_r = nC_{n-r}$$

$$= \sum_{n=K}^{\infty} nC_{n-K} z^{-n} = KC_0 z^{-K} + (K+1)C_1 z^{-(K+1)} + (K+2)C_2 z^{-(K+2)} + \cdots$$

$$= z^{-K}\{KC_0 + (K+1)C_1 z^{-1} + (K+2)C_2 z^{-2} + \cdots\}$$

$$= z^{-K}\left\{\left(1 - \frac{1}{z}\right)^{-(K+1)}\right\} \quad \text{for } \underline{\underline{|z|>1}}$$

Note : $(1-x)^n = 1+nx + \frac{n(n+1)}{2!}x^2 + \frac{n(n+1)(n+2)}{3!}x^3 + \dots$

$(1-x)^{-(k+1)} = 1+(k+1)x + \frac{(k+1)(k+2)}{2!}x^2 + \frac{(k+1)(k+2)(k+3)}{3!}x^3 + \dots$

Q1. Find $z\{(n+k)C_k\}$ n is a scalar.

$$= \sum_{k=0}^{\infty}(n+k)C_k z^{-k} = \underline{\underline{\left(1-\frac{1}{z}\right)^{-(n+1)}}}$$

$(n+k)C_k = (n+k)C_n$

$$\therefore z\{(n+k)C_k\} = \underline{\underline{z\{(n+k)C_n\}}}$$

Q2. $$\frac{1}{n!} * \frac{1}{n!} = \sum_{m=0}^{n} \frac{1}{m!}\frac{1}{(n-m)!}$$

$$= \frac{1}{n!} + \frac{1}{(n-1)!} + \frac{1}{2!(n-2)!} + \dots + \frac{1}{n!}$$

$$= \frac{1}{n!}\left\{1+n+\frac{n(n-1)}{2!} + \dots + 1\right\}$$

$$= \frac{1}{n!}\left\{nC_0 + nC_1 + nC_2 + \dots + nC_n\right\}$$

$$= \frac{1}{n!}\{2^n\} = \underline{\underline{\frac{2^n}{n!}}}$$

** $Z\left\{\frac{1}{n!} * \frac{1}{n!}\right\} = e^{1/z} \times e^{1/z} = \underline{\underline{e^{2/z}}}$

|||ly $z\left\{\frac{1}{n!} * \frac{1}{n!} * \frac{1}{n!}\right\} = \underline{\underline{\frac{3^n}{n!}}}$

Q. $1*1 = \sum_{m=0}^{n}(1)(1) = \underline{\underline{n+1}}$

$u(n) * u(n) = \underline{n+1}$

$u(t) * u(t) = \underline{\underline{t}}$ $\quad \int_0^t 1\times 1\, dx = (x)_0^t = \underline{\underline{t}}$

Q. $z\{a^n n u(n)\}$

$z\{nu(n)\} = \frac{z}{(z-1)^2}$

Applying Damping Rule

$$z\{a^n \times n u(n)\} = \frac{z/a}{(z/a-1)^2} = \underline{\underline{\frac{az}{(z-a)^2}}}$$

Q. $z\{a^n u(n)\} = \frac{z}{z-a}$

$\downarrow$

$$z\{n\,a^n u(n)\} = -z\frac{d}{dz}\left\{\frac{z}{z-a}\right\} = \underline{\underline{\frac{az}{(z-a)^2}}}$$

Q. $z\left\{\frac{a^n e^{-a}}{n!}u(n)\right\}$

$$= e^{-a} z\left\{\frac{a^n}{n!}u(n)\right\}$$

$z\{\frac{1}{n!}\} = e^{1/z}$

Damping Rule.

$$= e^{-a} e^{1/z/a} = \underline{\underline{e^{-a+a/z}}}$$

Q. $z\{\sinh an \sin n\theta\} = z\left\{\left(\frac{e^{an}-e^{-an}}{2}\right)\sin n\theta\right\}$

$$= \frac{1}{2}\left\{z\{e^{an}\sin\theta\} - z\{e^{-an}\sin n\theta\}\right\}$$

$$= \underline{\underline{\frac{1}{2}\left[\left\{\frac{z/e^a \sin\theta}{(z/e^a)^2 - 2(z/e^a)\cos\theta+1}\right\} - \left\{\frac{(ze^a)\sin\theta}{(ze^a)^2 - 2(ze^a)\cos\theta+1}\right\}\right\}}}$$

Q. $z\{n\sin n\theta\} = -z\frac{d}{dz}\left\{\frac{z\sin\theta}{(z^2-2z\cos\theta+1)}\right\}$

$$= -z\left\{\frac{(z^2-2z\cos\theta+1)\sin\theta - z\sin\theta(2z+2\cos\theta)}{z^2-2z\cos\theta+1}\right\}$$

$$= \underline{\underline{\frac{z^3\sin\theta - z\sin\theta}{z^2-2z\cos\theta+1}}}$$

Q. $z\{(n-1)^2\}$

① $\frac{z^2+z}{(z-1)^3}$ ② $\frac{z+1}{(z-1)^3}$ ③ $\frac{z^3+z}{(z-1)^3}$ ④ None.

Ans $z\{n^2-2n+1\} = \frac{z^2+z}{(z-1)^3} - \frac{2z}{(z-1)^2} + \frac{z}{z-1}$

$$= \frac{z^2+z-2z(z-1)+z(z-1)^2}{(z-1)^3}$$

$$= \underline{\underline{\frac{z^3-3z^2+4z}{(z-1)^3}}}$$

Q. If $u_n = n^2$, $z(u_{n-1}) = ?$

Ans F.S Theorem $\longrightarrow z\{u_{n-1}\} = z^{-1}\, z(n^2)$; $n \geq 1$

$$z\{(n-1)^2 u(n-1)\} = z^{-1}\,\frac{z^2+z}{(z-1)^3} = \underline{\underline{\frac{z+1}{(z-1)^3}}}\ ^{A}C_0$$

* For right shifting donot apply F.S Theorem.

$$z\{u_{n+1}\} = z[\bar{u}(z) - u_0] = z\left[\frac{z^2+z}{(z-1)^3} - 0\right] = \underline{\underline{\frac{z^3+z^2}{(z-1)^3}}}$$

Q. If $z\{u_n\} = \frac{z}{z-1} + \frac{z}{z^2+1}$ then $z(u_{n-2}) = ?$, $z(u_{n+2}) = ?$

Ans $z\{u_{n-2}\} = z^{-2}\,\bar{u}(z) = z^{-2}\left\{\frac{z}{z-1} + \frac{z}{z^2+1}\right\}$

$$z\{u_{n+2}\} = z^2\left\{\bar{u}(z) - u_0 - \frac{u_1}{z}\right\} \longrightarrow ①$$

$$u_0 = \lim_{z\to\infty} \bar{u}(z) = \lim_{z\to\infty} \frac{z}{z-1} + \frac{z}{z^2+1} = 1+0 = \underline{\underline{1}}$$

$$u_1 = \lim_{z\to\infty} z\{\bar{u}(z) - u_0\}$$

$$= \lim_{z\to\infty} z\left\{\frac{z}{z-1} + \frac{z}{z^2+1} - 1\right\}$$

$$= \lim_{z\to\infty} z\left\{\frac{z-(z-1)}{z-1} + \frac{z}{z^2+1}\right\}$$

$$= \lim_{z\to\infty} \left\{\frac{z}{z-1} + \frac{z^2}{z^2+1}\right\} = 1+1 = \underline{\underline{2}}$$

$$f(z) = \frac{z}{z-1} + \frac{z}{z^2+1}$$

$$u_n = 1 + \sin\frac{n\pi}{2} \Longrightarrow u_{n+2} = 1 + \sin(n+2)\pi/2 = 1 - \sin n\pi/2$$

$$z(u_{n+2}) = \frac{z}{z-1} + \frac{z^2}{z^2+1} = 1+1 = \underline{\underline{2}}$$

Q. If $z(u_n) = \frac{z^2-3z+4}{(z-3)^3}$; for $|z| < 3$ then $u_3 = ?$

Ans $|z| < 3$

So u_n is defined for all $-ve$ values and $u_n = 0\ \forall n \geq 0$

$\therefore\ \underline{\underline{u_3 = 0}}$

Q. If $z(u_n) = \dfrac{z^2-3z+4}{(z-3)^3}$; for $|z|>3$ then $u_3 = ?$

Ans $\dfrac{z^2-3z+4}{(z-3)^3}$ $\quad \left|\frac{3}{z}\right| < 1$

$$= \frac{z^2-3z+4}{z^3(1-3/z)^3} = \frac{z^2-3z+4}{z^3}\left[1-\frac{3}{z}\right]^{-3}$$

$$= \frac{z^2-3z+4}{z^3}\left\{1+3\times\frac{3}{z}+6\times\left(\frac{3}{z}\right)^2+10\left(\frac{3}{z}\right)^3+\cdots\right\}$$

$$\bar{u}(z) = u_0+\frac{u_1}{z}+\frac{u_2}{z^2}+\frac{u_3}{z^3}+\cdots$$

$$u_3 = \text{Coeff of } \frac{1}{z^3} = 54-27+4 = 31$$

$$(z^2-3z+4)\left[\frac{1}{z^3}+\frac{9}{z^4}+\frac{54}{z^5}+\cdots\right]$$

Inverse Z- Transformations

If $z\{f(n)\} = \bar{f}(z) \Longrightarrow f(n) = z^{-1}(\bar{f}(z))$

Linearity : $z^{-1}(a\bar{f}(z) \pm b\bar{g}(z)) = a\,z^{-1}(\bar{f}(z)) \pm b\,z^{-1}(\bar{g}(z))$

Method : ① Standard Formula [Based on Formula]

② Power Series | Division method.

In this method expand $\bar{f}(z)$ as infinite series in powers of z as per ROC by using McLauren's, Lorentz or ~~Fourier~~ Taylor series. Thus it can be expressed as $\sum_n f(n)z^{-n}$. Hence f(n) is the required fn.

③ Partial Fraction.

Here split the partial fraction for $\frac{\bar{f}(z)}{z}$, not for $\bar{f}(z)$ directly.

④ Convolution

$$z^{-1}(\bar{f}(z)\,\bar{g}(z)) = f(n)*g(n) = \sum_{m=0}^{n} f(m)g(n-m)$$

⑤ ~~Co~~

⑥ Contour Integration ·

$$Z^{-1}(\bar{f}(z)) = \frac{1}{2\pi i}\oint_{c} z^{n-1}\,\bar{f}(Z)\,dz$$

By Residue Theorem

$= \frac{1}{2\pi i} \times 2\pi i$ [sum of residues of $Z^{n-1}f(z)$ at each pole which are inside of c_1]

$$= \underline{\underline{\sum_{i} r_i}}$$

1) $Z = a$ is a simple pole (order 1)

$$\text{Res}\{f(z) : z = a\} = \lim_{z \to a} (z-a)\,f(z)$$

2) $Z = a$ is a pole of order m.

$$\text{Res} = \frac{1}{(m-1)!}\lim_{z \to a}\frac{d^{m-1}}{dz^{m-1}}\{(z-a)^m \cdot f(z)\}$$

Note: If C is not specified consider all poles are inside.

$Z^{-1}(1) = \delta(n)$

$Z^{-1}\left(\frac{z}{z-1}\right) = u(n)$

$Z^{-1}\left(\frac{z}{z-a}\right) = a^n u(n)$

$Z^{-1}\left(\frac{z}{z+a}\right) = (-a)^n u(n)$

$Z^{-1}(z^k) = \delta(n+k) = u(n+k) - u(n+(k+1))$

$Z^{-1}(z^{-k}) = \delta(n-k) = u(n-k) - u(n-(k+1))$

Q. $Z^{-1}(e^{3/z}) = Z^{-1}(e^{1/(z/3)}) = 3^n Z^{-1}(e^{1/z}) = \underline{\underline{3^n \frac{1}{n!}}}$

(or) $Z^{-1}(e^{3/z}) = Z^{-1}\left\{1 + \frac{3}{z} + \frac{1}{2!}\left(\frac{3}{z}\right)^2 + \frac{1}{3!}\left(\frac{3}{z}\right)^3\right\}$

$$= Z^{-1}\left\{\sum_{n=0}^{\infty}\frac{3^n}{n!}z^{-n}\right\} = \underline{\underline{3^n \frac{1}{n!}}}$$

Q2. $Z^{-1}\left\{\log\left(\frac{z}{z+1}\right)\right\} = -Z^{-1}\left\{\log\left(\frac{z+1}{z}\right)\right\}$

$$= -Z^{-1}\left\{\log\left(1+\frac{1}{z}\right)\right\}$$

$$= -Z^{-1}\left\{\frac{1}{z} - \frac{1}{2z^2} + \frac{1}{3z^3} - \frac{1}{4z^4} + \cdots\right\}$$

$$= -Z^{-1}\left\{\sum_{n=1} \frac{(-1)^{n-1}}{n} z^{-n}\right\}$$

$$= -\frac{(-1)^{n-1}}{n} u(n-1) = \frac{(-1)^n}{n} u(n-1)$$

Q3. $Z^{-1}\left[\frac{z}{(z-5)(z+4)}\right]$

$$\frac{f(z)}{z} = \frac{1}{(z-5)(z+4)} = \frac{1/9}{z-5} - \frac{1/9}{z+4} = \frac{1}{9}(5)^n u(n) - \frac{(-4)^n}{9} u(n)$$

By Residue Theorem.

or $Z^{-1}\left[\frac{z \times z^{n-1}}{(z-5)(z+4)}\right] = z^n$

$$= r_1|_{z=5} + r_2|_{z=-4} = \left[\frac{5^n}{9} + \frac{(-4)^n}{9}\right] u(n)$$

Q4. $Z^{-1}\left[\frac{1 \times z^{n-1}}{(z-2)(z+3)}\right]$

$$= r_1/z=2 + r_2/z=-3$$

$$= \frac{2^{n-1}}{5} + \frac{(-3)^{n-1}}{-5}$$

Q. $Z^{-1}\left[\frac{8z^2}{(4z+1)(2z-1)}\right]$

$$\frac{f(z)}{\cancel{z}} = \frac{8z^2}{2\times4(z+1/4)(z-1/2)}$$

By Residue Theorem

$$= Z^{-1}\left\{\frac{z^2 z^{n-1}}{(z+1/4)(z-1/2)}\right\} = Z^{-1}\left\{\frac{z^{n+1}}{(z+1/4)(z-1/2)}\right\}$$

$$= r_1 / z = -1/4 \quad + r_2 / z = 1/2$$

$$= \left[\frac{(-1/4)^{n+1}}{-3/4} + \frac{(+1/2)^{n+1}}{3/4} \right] u(n)$$

Q. $z^{-1}\left\{\frac{z^2}{(z+4)(z+5)(3z+2)}\right\}$

$$= z^{-1}\left\{\frac{z^2 \times z^{n-1}}{3(z+4)(z+5)(z+2/3)}\right\} = \frac{1}{3} z^{-1}\left\{\frac{z^{n+1}}{(z+4)(z+5)(z+2/3)}\right\}$$

$$= \frac{1}{3}\left[r_1/z=-4 \quad + r_2/z=-5 \quad + r_3/z=-2/3\right]$$

$$= \frac{1}{3}\left[\frac{(-4)^{n+1}}{1(-10/3)} + \frac{(-5)^{n+1}}{(-1)(-13/3)} + \frac{(-2/3)^{n+1}}{(10/3)(13/3)}\right]$$

$$= \left[\frac{(-4)^{n+1}}{-10} + \frac{(-5)^{n+1}}{13} + \frac{3(-2/3)^{n+1}}{130}\right] u(n)$$

Q. $z^{-1}\left[\frac{z}{z^2+2z+2}\right] = z^{-1}\left\{\frac{z \times z^{n-1}}{z^2+2z+2}\right\}$

$z = -1 \pm i$

$z_1 = -1+i$,

$$= z^{-1}\left\{\frac{z^n}{(z-z_1)(z-z_2)}\right\}$$

$$= r_1 | z=z_1 + r_2 | z=z_2 = \frac{(-1+i)^n - (-1-i)^n}{2i}$$

$$= (\sqrt{2})^n\left\{\frac{(-1/\sqrt{2} + i/\sqrt{2})^n - (-1/\sqrt{2} - i/\sqrt{2})^n}{2i}\right\}$$

$$= (\sqrt{2})^n \sin(3n\pi/4)\, u(n)$$

Q. $z^{-1}\left\{\frac{z}{(z-2)^2(z+2)}\right\} = z^{-1}\left\{\frac{z \times z^{n-1}}{(z-2)^2(z+2)}\right\}$

$$= \lim_{z\to 2} \frac{d}{dz}\left(\frac{z^n}{z+2}\right) + \frac{(-2)^n}{16} = \lim_{z\to 2} \frac{(z+2)z^{n-1} - z(1)}{(z+2)^2} + \frac{(-2)^n}{16}$$

$$= \frac{4 \times 2^{n-1} - 2^n}{16} + \frac{(-2)^n}{16}$$

Q. $Z^{-1}\left[\frac{z^3}{(z-5)^4}\right] = Z^{-1}\left[\frac{z^3 \times z^{n-1}}{(z-5)^4}\right] = \frac{1}{(4-1)!}\underset{z\to 5}{lt}\frac{d^3}{dz^3}(z^{n+1})$

$$= \frac{1}{6}\left[(n+1)n(n-1)5^{n-2}\right]u(n) \,/\, u(n-2)$$

$$= \frac{1}{6}\left[(n+3)(n+2)(n+1)5^{n}\right]u(n-2) \,/\, u_n$$

Q. $Z^{-1}\left(\frac{1}{z-2}\right)$ for $|z|<2$

Ans $\left|\frac{z}{2}\right| < 1$

$$= Z^{-1}\left[\frac{1}{-2\left(1-\frac{z}{2}\right)}\right] = \frac{-1}{2}Z^{-1}\left[\left(1-\frac{z}{2}\right)^{-1}\right]$$

$$= \frac{-1}{2}Z^{-1}\left[1+\frac{z}{2}+\left(\frac{z}{2}\right)^{2}+\cdots\right] = \frac{-1}{2}\left[\sum_{n=-\infty}^{0} 2^{n}z^{-n}\right]$$

$$= \frac{-1}{2}2^{n}u(-n) = -2^{n-1}u(-n)$$

Q. $Z^{-1}\left[\frac{z}{(z-3)(z-4)}\right]$ for $|z|<3$

Ans $Z^{-1}\left\{\frac{z \times z^{n-1}}{(z-3)(z-4)}\right\} = z^{n}$

$$= r_1\big|_{z=3} + r_2\big|_{z=4} = \left(\frac{3^n}{-1}+\frac{4^n}{1}\right)u(n)$$

$$= (-3^n+4^n)\,u(n) \quad \text{for } |z|>4 \ \& \ |z|>3$$

$$u_n = -(-3^n+4^n)\,u(-n-1) \quad \text{for } |z|<3$$

***

$$-3^n u(n) + 4^n u(n)$$

$$\frac{-z}{z-3} + \frac{z}{z-4}$$

$$= \frac{z}{(z-3)(z-4)}$$

$$3^n u(-n-1) - 4^n u(-n-1)$$

$$\frac{z}{3-z} - \frac{z}{4-z}$$

$$= \frac{z}{(3-z)(4-z)}$$

Q. $Z^{-1}\left\{\dfrac{z}{(z-3)(z-4)}\right\}$ for $3<|z|<4$

$-3^n u(n) + 4^n(u(n))$ for $|z|>4$ & $|z|>3$

$\Longrightarrow$ $\underline{\underline{-3^n u(n) - 4^n u(n)}}$ for $3<|z|<4$

Q. $Z^{-1}\left\{\dfrac{1}{(z+4)(z-6)}\right\}$ for $4<|z|<6$

Ans $Z^{-1}\left\{\dfrac{z^{n-1}}{(z+4)(z-6)}\right\} = \dfrac{(-4)^{n-1}}{-10}u(n-1) - \dfrac{(6)^{n-1}}{10}u(-n)$

$$= \begin{cases} -\dfrac{(-4)^{n-1}}{10} & ; n>0 \\ \dfrac{-6^{n-1}}{10} & ; n\leq 0 \end{cases}$$

Q. $a^n * a^n = \_\_\_?$

Ⓐ $a^{2n}u(n)$ Ⓑ $2a^n u(n)$ Ⓒ $na^n u(n)$ Ⓓ None

Ans $a^n * a^n = \sum_{m=0}^{n} a^m a^{n-m} = a^n \sum_{m=0}^{n}(1) = \underline{\underline{a^n(n+1)\,u(n)}}$

Q. (OR) $Z^{-1}\left(\dfrac{z}{z-a}\right)\left(\dfrac{z}{z-a}\right) = Z^{-1}\left\{\dfrac{z^2 z^{n-1}}{(z-a)^2}\right\} = z^{n+1}$

$= \lim_{z\to a} \dfrac{d}{dz} z^{n+1} = \underline{\underline{(n+1)a^n u(n)}}$

Q. $a^n * b^n = Z^{-1}\left(\dfrac{z}{z-a}\cdot\dfrac{z}{z-b}\right)$

$= \dfrac{a^{n+1}}{a-b} + \dfrac{b^{n+1}}{b-a} = \underline{\underline{\dfrac{a^{n+1}-b^{n+1}}{a-b}}}u(n)$

Q. If $Z\{f(n)\} = \dfrac{z}{(z-1/2)(z+3)(z-2)}$ in $|z|=1$ then $f(2) = \_\_?$

Ans $f(n) = Z^{-1}\left\{\dfrac{z\times z^{n-1}}{(z-1/2)(z+3)(z-2)}\right\} = r_1/_{z=1/2}$ Since only $z=1/2$ is inside $|z|=1$

$= \dfrac{(1/2)^n}{(3+1/2)(1/2-2)} = \underline{\underline{\dfrac{-4}{21}(1/2)^n}}$

$f(2) = \dfrac{-4}{21}\left(\dfrac{1}{2}\right)^2 = \underline{\underline{-1/21}}$

Q. $Z\{u_n\} = \dfrac{z^2-3z+4}{(z-3)^3}$ for $|z|<3$ then $u_3 = ?$

Ans $$u_n = z^{-1}\left\{\frac{(z^2-3z+4)\,z^{n-1}}{(z-3)^3}\right\} = z^{n+1} - 3z^n + 4z^{n-1}$$

$$r_1\Big|_{z=3} = \frac{1}{(3-1)!}\left[(n+1)n\,3^{n-1} - 3n(n-1)\,3^{n-2} + 4(n-1)(n-2)\,3^{n-3}\right]u(n-1)$$

$$u_3 = \frac{1}{2}\left[4\times3\times9 - 3\times3\times2\times3 + 4(2)\times1\times1\right]$$

$$= \frac{1}{2}\left[108 - 54 + 8\right] = \underline{\underline{31}}$$

Application Of Z Transform to Differential Eqns

Q. Solve $u_{n+1} + u_n = 0$; $u_0 = 1$

Ans $$Z\{u_{n+1} + u_n\} = 0$$

$$z\{\bar{u}(z) - u_0\} + \bar{u}(z) = 0 \implies (z+1)\bar{u}(z) = u_0 z$$

$$\bar{u}(z) = \frac{u_0 z}{z+1}$$

$$u_n = z^{-1}\left(\frac{z}{z+1}\right) = \underline{\underline{(-1)^n}}$$

Q. $Z\left[u_{n+2} - 3u_{n+1} + 2u_n = u_n\right]$; $u_0 = 0,\ u_1 = 1$

$$z^2\left\{\bar{u}(z) - \underset{0}{u_0} - \frac{u_1}{z}\right\} - 3z\left\{\bar{u}(z) - \underset{0}{u_0}\right\} + 2\bar{u}(z) = \frac{z}{z-1}$$

$$\bar{u}(z)\left\{z^2 - 3z + 2\right\} = \frac{z}{z-1} + u_1 z = \frac{z}{z-1} + z = \underline{\underline{\frac{z^2}{z-1}}}$$

FOURIER TRANSFORMATIONS

If a fn f(x) is a periodic fn and is satisfying the conditions of Dirichlets then it can be expressed as a Fourier series. By extending this concept for non periodic fns it can be expressed as integral form.

Definition : If f(x) is defined on (-c, c) and satisfying the conditions of Dirichlet's as $c \longrightarrow \infty$

$$f(x) = \frac{1}{\pi}\int_{-\infty}^{\infty}\int_{0}^{\infty} f(t) \cos \lambda (t-x)\, dt\, d\lambda \longrightarrow \text{Fourier Integral.}$$

Here λ is a parameter.

i) If f(x) is an odd fn

$$f(x) = \frac{2}{\pi}\int_{0}^{\infty} \sin \lambda x \int_{0}^{\infty} f(t) \sin \lambda t\, dt\, d\lambda \longrightarrow \text{Sine Integral.}$$

ii) If f(x) is an even fn.

$$f(x) = \frac{2}{\pi}\int_{0}^{0} \cos \lambda x \int_{0}^{\infty} f(t) \cos \lambda t\, dt\, d\lambda \longrightarrow \text{Cosine Integral.}$$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\lambda x}\int_{-\infty}^{\infty} f(t)\, e^{i\lambda t}\, dt\, d\lambda \longrightarrow ①$$

$\longrightarrow$ Complex form of Integral.

$$\bar{F}(\lambda) = \int_{-\infty}^{\infty} f(t)\, e^{i\lambda t}\, dt$$

$$\bar{f}(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{j\omega t}\, dt \Longrightarrow \text{in } ①$$

[Fourier Transform]

$$\bar{F}(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt \Longrightarrow \text{in } ②$$

[Inverse Fourier Transform]

$$\bar{f}(s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\, e^{-ist}\, dt$$

$$f(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ist}\, \bar{f}(s)\, ds.$$

FOURIER TRANSFORMATIONS

$$F\{f(t)\} = \bar{f}(s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\, e^{ist}\, dt \longrightarrow F.T$$

$$f(t) = F^{-1}\{\bar{f}(s)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \bar{f}(s) e^{-ist} ds \longrightarrow \text{IFT}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{f}(s) e^{ist} ds \longrightarrow \text{IFT}$$

Fourier Sine & Cosine Transformations.

$$F_s\{f(t)\} = \bar{f}_s(s) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(t) \sin st\,dt \longrightarrow \text{FST}$$

$$f(t) = F_s^{-1}(\bar{f}_s(s) = \sqrt{\frac{2}{\pi}}\int_0 \bar{f}_s(s) \sin st\,ds \longrightarrow \text{IFST}$$

$$F_c\{f(t)\} = \bar{f}_c(s) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(t) \cos st\,dt \longrightarrow \text{FCT}$$

$$f(t) = F_c^{-1}(\bar{f}_c(s)) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} \bar{f}_c(s) \cos st\,ds \longrightarrow \text{IFCT}.$$

Finite Fourier Sine/ Cosine Transformations

If $f(x)$ is defined in $(0, l)$ then

$$F(f(x)) = \int_0^1 f(x) \sin \frac{s\pi x}{l}\,dx \longrightarrow \text{FFST}$$

$$= \int_0^l f(x) \cos \frac{s\pi x}{l}\,dx \longrightarrow \text{FFCT}$$

Here s is being integer.

Q. F.T of $f(t) = \begin{cases} 1 & ; |t| \leq a \\ 0 & ; |t| > a \end{cases}$

$$F\{f(t)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t) e^{ist} dt = \frac{1}{\sqrt{2\pi}}\int_{-a}^{a} (1)\, e^{ist} dt = \frac{1}{\sqrt{2\pi}}\left(\frac{e^{ist}}{is}\right)_{-a}^{a}$$

$$= \frac{1}{\sqrt{2\pi}} \frac{e^{isa} - e^{-isa}}{is} = \frac{1}{\sqrt{2\pi}} \frac{2\sin as}{s}$$

(or) $1 \cdot \frac{2\sin as}{s}$ or $\frac{1}{2\pi}\frac{2\sin as}{s}$

$$f(t) = \int_{-\infty}^{\infty} \frac{1}{2\pi} \frac{2\sin as}{s} e^{-ist} ds$$

$$t = 0 \longrightarrow 1 = \int_{-\infty}^{\infty} \frac{1}{\pi} \frac{\sin as}{s} ds.$$

$$1 = \frac{2}{\pi}\int_0^\infty \frac{\sin as}{s}\,ds \Longrightarrow \int_0^\infty \frac{\sin as}{s} = \underline{\pi/2}$$

Q. F.T of $f(t) = \begin{cases} 1-t^2 & ; |t|\leq 1 \\ 0 & ; |t|>1 \end{cases}$

$$F\{f(t)\} = \frac{1}{\sqrt{2\pi}}\int_{-1}^{1}(1-t^2)e^{ist}dt = \frac{1}{\sqrt{2\pi}}\left[\left[\frac{e^{ist}}{is}(1-t^2) - (-2t)\left(\frac{e^{ist}}{(is)^2}\right) + (-2)\frac{e^{ist}}{(is)^3}\right]_{-1}^{1}\right.$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{2e^{is}}{-s^2} - \frac{2e^{is}}{-is^3} - 2(-1)\frac{e^{-is}}{-s^2} + \frac{2e^{-is}}{-is^3}\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{-2}{s^2} - \frac{2e^{is}}{-is^3} - 2(-1)\frac{e^{-is}}{-s^2} + \frac{2e^{-is}}{-is^3}\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{-2}{s^2}(e^{is}+e^{-is}) + \frac{2}{is^3}(e^{is}-e^{-is})\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{-2}{s^2}(2\cos s) + \frac{2}{is^3}(2i\sin s)\right] = \frac{1}{\sqrt{2\pi}}\,\frac{4}{s^3}\,\underline{[\sin s - s\cos s]}$$

Q. $e^{-a|t|}$; $(a>0)$

$$f(s) = \int_{-\infty}^{0} e^{at}e^{ist}dt + \int_0^\infty e^{-at}e^{ist}dt. = \left.\frac{e^{(a+is)t}}{(a+is)}\right]_{-\infty}^{0} + \left[\frac{e^{-(a-is)t}}{-(a-is)}\right]_0^\infty$$

$$= \frac{1}{a+is} + \frac{1}{a-is} = \underline{\frac{2a}{a^2+s^2}}$$

F (sin x) doesnot exist as sin x is divergent ($-\infty$ to ∞). But F(sin x) do exist in finite intervals.

Q. Find FST of e^{-ax}.

Ans $f_s(s) = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}\sin sx\,dx = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}\frac{e^{sx}-e^{-sx}}{2}dx = \sqrt{\frac{2}{\pi}}\ \underline{\frac{s}{s^2+a^2}}$

Q. $F_s\left(\frac{e^{-ax}}{x}\right) = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{e^{-ax}}{x}\sin sx\,dx = \cot^{-1}(a/s)$

$$** \int_0^\infty e^{-sx}\frac{\sin ax}{x}ds = \cot^{-1}(s/a)$$ s & a are interchanged

Q. $\frac{d}{ds}\bar{f}_s(s) = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{e^{-ax}}{\cancel{x}}\times\cancel{x}\cos sx\,dx = \sqrt{\frac{2}{\pi}}\ \frac{a}{s^2+a^2}$

Integrate w.r.t s $\bar{f}_s(s) = \sqrt{\frac{2}{\pi}}\tan^{-1}(s/a)$ (or) $\underline{\sqrt{2/\pi}\cot^{-1}(a/s)}$

Q. F.S.T of xe^{-ax}

Ans $\bar{f}_s(s) = F_s\left(\frac{e^{-ax}}{x}\right) = \sqrt{\frac{2}{\pi}}\int_0^\infty x e^{-ax}\sin sx\,dx$

Integrate w.r.t s.

$$\int \bar{f}_s(s)\,ds = \sqrt{\frac{2}{\pi}}\int_0^\infty x e^{-ax}\left(\frac{-\cos sx}{\cancel{x}}\right)dx$$

$$\int \bar{f}_s(s)\,ds = -\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}\cos sx\,dx = -\sqrt{\frac{2}{\pi}}\,\frac{a}{s^2+a^2}$$

Diff. w.r.t.

$$f_s(s) = \sqrt{2/\pi}\left[\frac{2as}{(s^2+a^2)^2}\right]$$

Q. F.S.T of $1/x$.

Ans $\bar{f}_s(s) = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{1}{x}\sin sx\,dx = \sqrt{\frac{2}{\pi}}\times\frac{\pi}{2} = \sqrt{\pi/2}$

By L.T

Q. F.S.T of $1/\sqrt{x}$

Ans $f_s(s) = \sqrt{2/\pi}\int_0^\infty \frac{1}{\sqrt{x}}\sin sx\,dx$

$$\sqrt{n} = \int_0^\infty e^{-x}x^{n-1}\,dx \qquad n>0$$

$$f_s(s) = -\text{ Imag. part of } \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{1}{\sqrt{x}}e^{-isx}\,dx$$

Put $isx = t$

$x = t/is \Longrightarrow dx = \frac{dt}{is}$

$$= -\text{ Imag. part of } \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{1}{\sqrt{t/is}}\cdot e^{-t}\cdot\frac{dt}{is}$$

$$= -\text{ Imag. part of } \frac{\sqrt{2}}{\sqrt{\pi}}\,\frac{1}{\sqrt{is}}\int_0^\infty t^{1/2}e^{-t}\,dt = \int_0^\infty t^{1/2-1}e^{-t}\,dt.$$

$$= \sqrt{1/2} = \sqrt{\pi}$$

$$= -\text{ Imag. part of } \frac{\sqrt{2}}{\cancel{\sqrt{\pi}}}\times\frac{1}{\sqrt{s}}\times\frac{\cancel{\sqrt{\pi}}}{\sqrt{i}}$$

$$= \frac{\cancel{\sqrt{2}}}{\sqrt{s}}\times\frac{1}{\cancel{\sqrt{2}}} = 1/\sqrt{s}$$

$$\frac{1}{\sqrt{i}} = \frac{1}{(e^{i\pi/2})^{1/2}}$$

$$= e^{-i\pi/4}$$

$$= \cos\pi/4 - i\sin\pi/4$$

$$= \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}$$

Self Reciprocal Functions

$F_s(1/\sqrt{x}) = 1/\sqrt{s}$
$|||^{ly}$ $F_c(1/\sqrt{x}) = 1/\sqrt{s}$ } If the transformation of a fn is the fn itself then it is called self reciprocal.

Gauss function is also self reciprocal.

$$F(e^{-x^2/2}) = e^{-s^2/2}$$

Q₁. Finite F.S.T of $f(x)=1$ in $(0,\pi)$

$$F_s(f(x)) = \int_0^{l} f(x)\sin\frac{s\pi x}{l}\,dx = \int_0^{\pi}(1)\sin\frac{s\pi x}{\pi}\,dt = \left(-\frac{\cos sx}{s}\right)_0^{\pi} = \left(-\frac{\cos s\pi+1}{s}\right)$$

$$= -\frac{(-1)^s+1}{s}$$

Q₂. $$\int_0^{\infty} f(x)\sin tx\,dx = \begin{cases} 1 & ; 0\le t\le 1 \\ 2 & ; 1\le t<2 \\ 0 & ; t\geq 2 \end{cases} \Longrightarrow f(x) = ?$$

Ans $$f(x) = F_s^{-1}(\bar{f}_s(t)) = \frac{2}{\pi}\int_0^{\infty}\bar{f}_s(t)\sin tx\,dt$$

$$= \frac{2}{\pi}\left[\int_0^1 1\sin tx\,dt + \int_1^2 2\sin tx\,dx + \int_2^{\infty} 0\,dt\right]$$

$$= \frac{2}{\pi}\left[\left(-\frac{\cos tx}{x}\right)_{t=0}^{1} + \left(-\frac{2\cos tx}{x}\right)_1^2\right]$$

$$= \frac{2}{\pi}\left[-\frac{\cos x}{x} + \frac{1}{x} - \frac{2\cos 2x}{x} + \frac{2\cos x}{x}\right]$$

Properties

$F(\delta(t-a)) = e^{isa}$ (or) e^{-isa}

$F_s(\delta(t-a)) = \sin as$

$F_c(\delta(t-a)) = \cos as$

$F(\delta(t)) = 1$

$F_s(\delta(t)) = 0$

$F_c(\delta(t)) = 1$

Linearity : $F\{a\pm b\} = F\{a\} \pm F\{b\}$

Scaling : If $F(f(t)) = \bar{f}(s)$

① $F\{f(at)\} = \frac{1}{|a|}\bar{f}(s/a)$ ② $F\{f(t/a)\} = |a|\bar{f}(as)$

Shifting Property.

If $F\{f(t)\} = \bar{f}(s)$

		e^{ist}	e^{-ist}
① $F\{f(t)e^{iat}\}$	$=$	$\bar{f}(s+a)$	$\bar{f}(s-a)$
② $F\{f(t)e^{-iat}\}$	$=$	$\bar{f}(s-a)$	$\bar{f}(s+a)$
③ $F\{f(t-a)\}$	$=$	$e^{isa}\bar{f}(s)$	$e^{-isa}\bar{f}(s)$

Modulation Properties

If $F(f(t)) = \bar{f}(s)$

$$\Longrightarrow F(f(t)\cos at) = \frac{1}{2}\left[\bar{f}(s+a) + \bar{f}(s-a)\right]$$

i) $F_s(f(t)\sin at) = \frac{1}{2}\left[\bar{f}_c(s-a) - \bar{f}_c(s+a)\right]$

ii) $F_s(f(t)\cos at) = \frac{1}{2}\left[\bar{f}_s(s+a) + \bar{f}_s(s-a)\right]$

iii) $F_c(f(t)\sin at) = \frac{1}{2}\left[\bar{f}_s(s+a) - \bar{f}_s(s-a)\right]$

iv) $F_c(f(t)\cos at) = \frac{1}{2}\left[\bar{f}_c(s+a) + \bar{f}_c(s-a)\right]$

Relation B/w L.T & F.T

$$\text{If } f(t) = \begin{cases} e^{-xt}g(t) & ; t \geq 0 \\ 0 & ; t < 0 \end{cases}$$

then $F(f(t)) = \underline{L(g(t))}$

$$F(f(t)) = \int_0^\infty e^{-xt} g(t) e^{-j\omega t}\,dt = \int_0^\infty g(t) e^{-(x+j\omega)}\,dt$$

Convolution Theorem

If $F^{-1}(\bar{f}(s)) = f(t)$ & $F^{-1}(\bar{g}(s)) = g(t)$

$$\Longrightarrow F^{-1}(\bar{f}(s)\cdot\bar{g}(s)) = f(t) * g(t)$$

$$= \int_{-\infty}^{\infty} f(x)\,g(t-x)\,dx.$$

Parsevals Identities

$$F(s) \xrightarrow{F.T} f(t)$$

$$G(s) \xrightarrow{"} G(t)$$

$$\bar{G}(s) \xrightarrow{\text{Conjugate}} G(s)$$

$$\bar{g}(t) \xrightarrow{"} g(t)$$

① $$\frac{1}{2\pi}\int_{-\infty}^{\infty} F(s)\,\bar{G}(s)\,ds = \int_{-\infty}^{\infty} f(t)\,\bar{g}(t)\,dt$$

② $$\frac{1}{2\pi}\int_{-\infty}^{\infty} |F(s)|^2\,ds = \int_{-\infty}^{\infty} |f(t)|^2\,dt$$

} Fourier Parsevals Identities

③ $$\frac{2}{\pi}\int_{0}^{\infty} F_s(f(x))\,F_s(g(x))\,ds = \int_{0}^{\infty} f(x)\,g(x)\,dx$$ Sine identity

④ $$\frac{2}{\pi}\int_{0}^{\infty} F_c(f(x))\,F_c(g(x))\,ds = \int_{0}^{\infty} f(x)\,g(x)\,dx$$ Cosine identity.

$$F\{x^n f(x)\} = i^n \frac{d^n}{ds^n}(\bar{f}(s))$$

$$\bar{f}(s) = \int_{-\infty}^{\infty} f(x)\,e^{-isx}\,dx$$

$$\frac{d}{ds}\bar{f}(s) = \int_{-\infty}^{\infty} -ix\,f(x)\,e^{-isx}\,dx.$$

$$i\frac{d}{ds}\bar{f}(s) = \int_{-\infty}^{\infty} x\,f(x)\,e^{-isx}\,dx.$$

$$i^n\frac{d^n}{ds^n}\bar{f}(s) = \int_{-\infty}^{\infty} x^n f(x)\,e^{-isx}\,dx$$

$$F_s(xf(x)) = \frac{-d}{ds}\left[\bar{f}_c(s)\right]$$

$$F_c(xf(x)) = \frac{d}{ds}\left[\bar{f}_s(s)\right]$$

Q. $$\int_{0}^{\infty} \frac{x^2}{(x^2+a^2)^2}\,dx = \int_{0}^{\infty} \frac{s}{s^2+a^2}\cdot\frac{s}{s^2+a^2}\,ds$$

$$= \frac{\pi}{2}\int_{0}^{\infty} e^{-ax} \times e^{-ax}\,dx = \frac{\pi}{2}\left[\frac{e^{-2ax}}{-2a}\right]_0^{\infty} = \underline{\pi/4a}$$

(or) $x = a\tan\theta$ $\quad dx = a\sec^2\theta$

$$\int_{0}^{\pi/2} \frac{a^2\tan^2\theta}{a^4\sec^4\theta}\,a\sec^2\theta\,d\theta = \frac{1}{a}\int_{0}^{\pi/2}\sin^2\theta\,d\theta = \frac{1}{a}\int_{0}^{\pi/2}\left(\frac{1-\cos 2\theta}{2}\right)d\theta$$

$$= \frac{1}{2a}\left(\theta - \frac{\sin 2\theta}{2}\right)_0^{\pi/2} = \underline{\frac{\pi}{4a}}$$

DIFFERENTIAL EQUATIONS

Solution Of A First Order First Degree Ordinary Differential Eqn

The general form of a 1st order 1st degree differential equation is given by

$$M + N\frac{dy}{dx} = 0 \quad \text{——} ①$$

$$M\,dx + N\,dy = 0 \quad \text{——} ②$$

where M, N are functions of variable x & y.

Suppose it is possible to express ① as

$$\frac{dy}{dx} = \frac{f(x)}{g(y)}$$

then $f(x)dx = g(y)dy$

$\int f(x)dx = \int g(y)dy$ gives the solution of eq ①.

This method is known as Variable Seperable Method.

Q. A Spherical Naphthalene ball exposed to air, losses its volume proportional to its instantaneous surface area due to evapouration. If the initial diameter of the ball is 2cm, reduces to 1cm after 3 months, the ball completely evapourates in ____?

Ans Volume, $V_1 = \frac{4}{3}\pi r^3$

Area, $A = 4\pi r^2$

$t_1 \longrightarrow V_1$

$t_2 \longrightarrow V_2$

$\frac{dV}{dt} \propto A$ ie $\frac{dV}{dt} = -kA$ ('−' → area decreasing)

ie $\frac{d}{dt}\left(\frac{4}{3}\pi r^3\right) = -k(4\pi r^2)$

$$\frac{\cancel{4\pi}}{\cancel{3}} \times \cancel{3r^2}\frac{dr}{dt} = -K\,\cancel{4\pi r^2}$$

$$\frac{dr}{dt} = -K \quad \text{——} ①$$

i) At $t=0 \longrightarrow r=1$

ii) At $t = 3$ months $\longrightarrow r = \frac{1}{2}$ cm

iii) At $t = n \longrightarrow r=0$

From ① $\frac{dr}{dt} = -K \longrightarrow ①$

$$dr = -K\,dt$$

$$\int ing \Longrightarrow r = -Kt + C \longrightarrow ②$$

$$(i) \text{ in } ② \Longrightarrow 1 = C \longrightarrow ③$$

$$Kt = 1 - r \longrightarrow ④ \quad [③ \text{ in } ②]$$

$$3K = 1 - 1/2 = 1/2 \longrightarrow ⑤ \quad [(ii) \text{ in } ④]$$

$$⑤ \text{ in } ④ \Longrightarrow \frac{t}{6} = (1-r) \longrightarrow ⑥$$

Put $r=0$

$$\therefore \; t = 6(1-0) = \underline{6 \text{ months}}$$

Q2. A body originally 60°c cools down to 40°c in 15 minutes while kept in air at a temperature of 25°c Then what will be the temperature of body after 30 minutes?

a) 20°c b) 25°c c) 28.7°c d) 31.4°c.

Ans To Solve the above problem we can apply.

Newton's Law Of Cooling.

According to this law, the temperature of a body changes proportional to the difference b/w the body temperature and surrounding medium temp- Assume that θ be the temperature of body and θ_0 the surrounding medium temperature.

then $\frac{d\theta}{dt} \propto \theta - \theta_0$

$$\frac{d\theta}{dt} = K(\theta - \theta_0) \longrightarrow ①$$

where $K \longrightarrow$ proportionality constant and '-ve' sign indicates body temp decreases as time ↑.

i) at $t = 0$, $\theta = 60°c$ ——(i)

$t = 15$, $\theta = 40°c$ (ii)

$t = 30$, $\theta = ?$ (iii)

From (1) $$\frac{d\theta}{dt} = -k(\theta - \theta_0)$$

ie $$\frac{d\theta}{\theta - \theta_0} = -k\,dt$$

$$\int \frac{d\theta}{\theta - \theta_0} = -k\int dt$$

$\theta_0 = 25$ $$\log(\theta - 25) = -kt + c \text{ ——} ②$$

① in ② $$\log 35 = c \text{ ——} ③$$

③ in ② $$\log(\theta - 25) = -kt + \log 35$$

$$kt = \log 35 - \log(\theta - 25) \text{ ——} ④$$

(ii) in ④ $$15k = \log 35 - \log 15 \text{ ——} ⑤$$

$\frac{④}{⑤}$ $$\therefore \frac{t}{15} = \frac{\log\left(\frac{35}{\theta - 25}\right)}{\log\left(\frac{35}{15}\right)}$$

put $t = 30$

$$\frac{30}{15} \log \frac{35}{15} = \log \frac{35}{\theta - 25}$$

$$\left(\frac{35}{15}\right)^2 = \frac{35}{\theta - 25}$$

ie $$\theta - 25 = \frac{45}{7} = 6.43.$$

$$\therefore \theta = \underline{\underline{31.43°c}}$$

Q. If a thermometer is taken outdoor where the temperature is 0°c. from a room in which temp. is 21°c and the reading drops to 10°c in 1 min. Then how long after its removal will the temp. reading be 5°c.

Ans $\theta_0 = 0°c$.

i) at $t = 0$, $\theta = 21°c$

ii) at $t = 1m$, $\theta = 10°c$

iii) at $t = ?$, $\theta = 5°c$.

$$\int \frac{d\theta}{\theta - \theta_0} = -k\int dt$$

$\theta_0 = 0 \Longrightarrow \quad \log \theta = -Kt + C$ ——①

(i) $\longrightarrow \log 21 = C$ ——②

$\log \theta = -Kt + \log 21$ ——③

$Kt = \log 21 - \log \theta$ ——③

$K \times 1 = \log 21 - \log 10$ ——④

$$\frac{③}{④} \Longrightarrow t = \frac{\log 21/\theta}{\log 21/10}$$

$$\theta = 5°c \longrightarrow t = \frac{\log 21/5}{\log 21/10} = 1 \text{ min } 56 \text{ sec} = 1.98 \text{ min}$$

Q4. Radium decomposes at a rate proportional to the amount of radium present at that instant. If 5% of the original amount disappears after 50 years then howmuch will remain after 100 years?

Ans i) at $t = 0 \longrightarrow x$ original amount.

ii) at $t = 50 \longrightarrow .95x = y$.

iii) at $t = 100 \longrightarrow$? 95% of $y = .95 \times .95x = 90.25\%$ of x

Q5. The rate at which ice melts is proportional to the amount of ice present at that moment. Find the amount of ice leftover after 2 hrs if half the quantity melts in 30 min.

Ans i) at $t = 0 \longrightarrow x$ units

ii) at $t = 30 \longrightarrow \frac{x}{2}$ 50% of x

iii) at $t = 2 \longrightarrow$?

at $t = 1$ 50% of 50% of $x = x/4 = 25\%$ of x

at $t = 2$ 25% of 25% of $x = x/16$

Q6. The amount of bacteria in a culture grew at a rate proportional to N itself. If the original number doubles in 2 hours, then in how many hours will it triple?

Ans i) at $t = 0 \longrightarrow x = N$

ii) at $t = 2$ hrs $\longrightarrow x = 2N$

iii) at $t = ? \longrightarrow x = 3N$

To solve the above problem we can appy Law of Natural Growth (or decay). This law states that the rate of growth or decay of any material at any time is proportional to the amount of material present at that time.

Assume that x be the quantity of material available at any time t, then

$$\frac{dx}{dt} \propto x$$

$$\frac{dx}{dt} = kx \longrightarrow \text{growth}$$

$$= -kx \longrightarrow \text{decay.}$$

From growth formula $\frac{dx}{dt} = kx$

$$\frac{dx}{x} = k\,dt$$

Ing $\quad \log x = kt + c$ ——①

(i) in ① $\Longrightarrow$ $\log N = c$

$$kt = \log x - \log N = \log(x/N) \text{ ——③}$$

(ii) in ③ $\Longrightarrow$ $\log\left(\frac{2N}{N}\right) = 2k$ ——④

$\frac{③}{④} \Longrightarrow$ $\frac{2}{t} = \frac{\log(x/N)}{\log 2}$ ——⑤

(iii) in ⑤ $\Longrightarrow$ $\frac{2}{t} = \frac{\log 3}{\log 2}$

$$t = \frac{2 \times \log 2}{\log 3} = 3.16 \text{ hours}$$

Remark

We know that from Newton's Law of Cooling

$$\frac{d\theta}{dt} = -k(\theta - \theta_0) \longrightarrow ①$$

Suppose $\theta_0 = 0$

Then ① $\Longrightarrow$ $\frac{d\theta}{dt} = -k\theta \longrightarrow ②$

ie Newton's Law of cooling is reduced to Law of Natural Decay

Law Of Natural Decay

Q7. The solution of $\frac{dy}{dx} + 2y = 0$ satisfying $y = 5$ at $x = 1$.

Ans

$$\frac{dy}{dx} = -2y$$

$$\frac{dy}{y} = -2dx$$

$$\log y = -2x + C$$

$y = 5$ at $x = 1$

$$\log 5 = -2 + C$$

$$C = 2 + \log 5$$

$$\underline{\log y = -2x + 2 + \log 5}$$

Q8. $$x^4 \frac{dy}{dx} + x^3 y + \text{Cosec}\, xy = 0$$

$$x^3\left(x\frac{dy}{dx} + y\right) + \text{Cosec}\, xy = 0.$$

let $xy = z$

$$\frac{dz}{dx} = x\frac{dy}{dx} + y.$$

$$x^3\frac{dz}{dx} + \text{Cosec}\, z = 0$$

$$x^3\frac{dz}{dx} = -\text{Cosec}\, z.$$

$$\frac{dx}{x^3} = \frac{dz}{-\text{Cosec}\, z} = -\sin z\, dz$$

Sing $$\int \sin z\, dz = -\int \frac{dx}{x^3}$$

$$-\cos z = -\left(\frac{-1}{2}x^{-2}\right) + C$$

$$\underline{\cos z = \frac{-1}{2}x^{-2} + C}$$

Q9. $$\frac{dz}{dx} + \cancel{x} - x\tan z = \cancel{1}$$

$$\frac{dz}{dx} = x\tan z$$

$$\frac{dz}{\tan z} = x\,dx$$

140

$$\int \cot z\,dx = \int x\,dx.$$

$$\log(\sin z) = \frac{x^2}{2} + C$$

Q10. $\frac{dy}{dx} = (4x+y+1)^2$ with $y(0) = 1$.

An Let $4x+y+1 = z$

$$\frac{dz}{dx} = 4 + \frac{dy}{dx} = 4 + z^2$$

$$\frac{dz}{4+z^2} = dx$$

∫ing $\frac{1}{2}\tan^{-1}(z/2) = x + c$

$$\frac{1}{2}\tan^{-1}\frac{(4x+y+1)}{2} = x + c$$

Substituting $\Rightarrow \frac{1}{2}\tan^{-1}(1) = c$

$$c = \frac{1}{2} \times \frac{\pi}{4} = \pi/8.$$

$$\therefore \frac{1}{2}\tan^{-1}\left(\frac{4x+y+1}{2}\right) = x + \pi/8$$

Q11. $y - \frac{x\,dy}{dx} = a\left(y^2 + \frac{dy}{dx}\right)$

$$y - ay^2 = (a+x)\frac{dy}{dx}$$

$$\int \frac{dy}{y - ay^2} = \int \frac{dx}{a+x}$$

$$\log(a+x) = \int \frac{ay + (1-ay)}{y(1-ay)}\,dy = \frac{a\log(1-ay)}{-a} + \log(yc)$$

$$\log(a+x) + \log(1-ay) = \log(yc) \qquad = -\log(1-ay) + \log(yc)$$

Antilog $(a+x)\times(1-ay) = yc$

Q12. $(x+y)^2\frac{dy}{dx} = K^2$

$x + y = z \longrightarrow \frac{dz}{dx} = 1 + \frac{dy}{dx} \Longrightarrow \frac{dy}{dx} = \frac{dz}{dx} - 1$

$$Z^2\left(\frac{dz}{dx}-1\right) = K^2$$

$$Z^2\frac{dz}{dx} = K^2+Z^2$$

$$\frac{K^2+Z^2}{Z^2} = \frac{dz}{dx}$$

$$\int dx = \int\frac{Z^2dZ}{K^2+Z^2} = \int\left(1-\frac{K^2}{K^2+Z^2}\right)dZ \qquad \frac{Z^2}{K^2+Z^2} = \frac{(K^2+Z^2)-K^2}{K^2+Z^2}$$

$$x+C = Z - K\tan^{-1}\left(\frac{Z}{K}\right)$$

$$x+C = x+y - K\tan^{-1}\left(\frac{x+y}{K}\right)$$

Model II :- Linear Differential Eqns.

The general form of 1st order linear diff. eqn in variable y is given by

$$\frac{dy}{dx} + P.y = Q. \longrightarrow ①$$

where P & Q are fns of x.

multiply ① with R. $\quad R \longrightarrow$ someother fn of x

$$R\frac{dy}{dx} + RPy = RQ.$$

Let $RP = \frac{dR}{dx}$

$$\frac{d}{dx}[R.y] = RQ$$

$$Ry = \int RQ\,dx \longrightarrow ②$$

$$y \times e^{\int Pdx} = \int Q e^{\int Pdx} dx + C \longrightarrow ④$$

$$\because RP = \frac{dR}{dx}$$

$$\frac{dR}{R} = \int Pdx$$

$$\therefore R = e^{\int Pdx}$$

$$\frac{dR}{R} = \int Pdx$$

Solve the following.

Q1. The solution of $t\frac{dx}{dt} + x = t$ satisfying the condition $x(1) = 0.5$.

Ans $\quad t\frac{dx}{dt} + x = t$

$\div$ by t, $\quad \frac{dx}{dt} + \frac{x}{t} = 1$

$$IF = e^{\int 1/t\,dt} = e^{\log t} = t$$

Solution is $x \times t = \int 1 \times t \, dt = \frac{t^2}{2} + C$

$\cdot 5 = \frac{1}{2} + C \implies C = 0$

$xt = \frac{t^2}{2}$

$x = \underline{t/2}$

Q2. $\frac{dy}{dx} + 2y \tan x = \sin x$ given $y = 0$ when $x = \pi/3$.

Ans IF = ~~ef~~ $e^{\int 2\tan x d} = e^{2\log\sec x}$

$y \times \sec^2 x = \int \sin x \times \sec^2 x \, dx + C$

$= \int \tan x \sec x \, dx + c = \underline{\underline{\sec x + c}}$

$0 = 2 + C$

$\therefore C = -2$

$\therefore y \times \underline{\sec^2 x = \sec x} - 2$

Q3. $x(x-1)\frac{dy}{dx} - y = x^2(x-1)^2$

$\implies \frac{dy}{dx} - \frac{y}{x(x-1)} = x(x-1)$

$IF = e^{\int \frac{-1}{x(x-1)} dx} = e^{\int \frac{(x-1)-x}{x(x-1)} dx} = e^{\log x - \log(x-1)} = \underline{\underline{\frac{x}{x-1}}}$

$y \times \frac{x}{x-1} = \int x \cancel{(x-1)} \times \frac{x}{\cancel{x-1}} \, dx + C$

$\underline{\underline{y \times \frac{x}{x-1} = \frac{x^3}{3}}} + C$

Q4. $x \log x \frac{dy}{dx} + y = 2 \log x$

Ans

$\frac{dy}{dx} + \frac{y}{x\log x} = \frac{2}{x}$

$IF = e^{\int \frac{1}{x\log x} dx} = e^{\int \frac{1}{t} dt} = e^{\log t} \underline{\underline{= t}}$

$\log x = t$

$\frac{1}{x} dx = dt$

$y \times \log x = \int \frac{2}{x} \log x \, dx + C$

$y \log x = 2\int t \, dt + c = 2\left[\frac{t^2}{2}\right] + c$

$y \log x = \underline{\underline{(\log x)^2}} + c$

Q5. $x\sin x \frac{dy}{dx} + y(x\cos x + \sin x) = \sin x$

$$\frac{dy}{dx} + y\left(\cot x + \frac{1}{x}\right) = \frac{1}{x}$$

IF = $e^{\int(\cot x + \frac{1}{x})dx} = e^{\log \sin x + \log x} = \underline{\underline{x\sin x}}$

$$\therefore y \times \sin x = \int \frac{1}{x} x\sin x\, dx = -\cos x + c$$

$$\underline{\underline{y\sin x = -\cos x + c}}$$

Q6. $(1+y^2)dx = (\tan^{-1}y - x)dy$

$$\frac{dx}{dy} = \frac{\tan^{-1}y - x}{1+y^2}$$

$$\frac{dx}{dy} + \frac{x}{1+y^2} = \frac{\tan^{-1}y}{1+y^2}$$

IF = $e^{\int \frac{1}{1+y^2}dy} = e^{\tan^{-1}y}$

$$x \times e^{\tan^{-1}y} = \int \frac{\tan^{-1}y}{1+y_2} \times e^{\tan^{-1}y}\, dy$$

$t = \tan^{-1}y$.

$dt = \frac{1}{1+y^2}dy$

$$= \int t \times e^t dt = t \times e^t - \int e^t$$

$$= e^t(t-1)$$

$$\underline{\underline{= e^{\tan^{-1}y}(e^{\tan^{-1}y} - 1)}}$$

Q7. $2\frac{dy}{dx} - y\sec x = y^3\tan x$

÷ by y^3

$$\frac{2}{y^3}\frac{dy}{dx} - \frac{\sec x}{y^2} = \tan x$$

× by -1

$$\frac{-2}{y^3}\frac{dy}{dx} + \frac{\sec x}{y^2} = -\tan x$$

$$\boxed{f(y)\frac{dy}{dx} + \frac{\sec x}{y^2} = -\tan x}$$

General form of an eqn that can be converted to linear eqn.

$$\frac{1}{y^2} = z$$

$$\frac{-2}{y^3} = \frac{dz}{dy}$$

$$\therefore \frac{dz}{dy} + \sec x \times z = -\tan x$$

$$IF = e^{\log(\sec x + \tan x)} = \sec x + \tan x$$

$$\therefore \frac{1}{y^2} \times (\sec x + \tan x) = \int (\sec x + \tan x) \times -\tan^2 x \, dx$$

$$= -\int \sec x \tan x + \tan^2 x \, dx \qquad \tan^2 x = \sec^2 x - 1$$

$$= -(\sec x + \tan x - x) + c$$

Q8. $2xy' = (10x^3y^5 + y)$

$\div$ by $2x$

$$y' = 5x^2y^5 + \frac{y}{2x}$$

$$\frac{dy}{dx} - \frac{1}{2x} \times y = 5x^2y^5$$

$\div$ by y^5

$$\frac{1}{y^5}\frac{dy}{dx} - \frac{1}{2xy^4} = 5x^2$$

$\times$ by -4

$$\underbrace{\frac{-4}{y^5}\frac{dy}{dx}}_{-\frac{dz}{dx}} + \frac{2}{x} \times \underbrace{\frac{1}{y^4}}_{z} = -20x^2$$

$$e^{\int 2x \frac{1}{x} dx} = e^{\log x^2} = x^2$$

$$zx^2 = -20\int x^2 \times x^2 dx = -20 \times \frac{x^5}{5} + c = -4x^5 + c$$

Q9. $\frac{dy}{dx} + x \sin 2y = x^3 \cos^2 y$

Ans $\div$ by $\cos^2 y$

$$\sec^2 y \frac{dy}{dx} + x \frac{2\sin y \cos y}{\cos^2 y} = x^3$$

$$\sec^2 y \frac{dy}{dx} + 2x \tan y = x^3$$

$$\tan y = z \Rightarrow \frac{dz}{dx} = \sec^2 y \frac{dy}{dx}$$

$$\frac{dz}{dx} + 2xz = x^3$$

$$IF = e^{\int 2x dx} = e^{x^2}$$

$$\therefore y \times e^{x^2} = \int x^3 e^{x^2} dx = \frac{1}{2} e^{x^2}(x^2 - 1) + c$$

Q10. $$\frac{dZ}{dx} + \frac{Z}{x}\log Z = \frac{Z}{x^2}(\log Z)^2 .$$

Ans ÷ by $Z(\log z)^2$

$$\frac{1}{Z(\log z)^2}\frac{dz}{dx} + \frac{1}{x} \times \frac{1}{\log Z} = \frac{+1}{x^2}$$

× by −1

$$\underbrace{\frac{-1}{Z(\log z)^2}\frac{dz}{dx}}_{\frac{dy}{dx}} + \underbrace{\frac{-1}{x}}_{P} \times \underbrace{\frac{1}{\log z}}_{y} = \underbrace{\frac{-1}{x^2}}_{Q}$$

$$IF = e^{\int P dx} = e^{\int -\frac{1}{x}dx} = e^{-\log x \cancel{dx}} = \underline{\underline{1/x}}$$

$$\therefore \quad \frac{1}{\log Z} \times \frac{1}{x} = -\underline{\underline{\int \frac{1}{x^2} \times \frac{1}{x}\,dx}} = \frac{1}{2x^2} + C$$

Q11. $$\tan y \frac{dy}{dx} + \tan x = \cos y \cos^3 x$$

÷ by $\cos y$.

$$\underbrace{\tan y \sec y \frac{dy}{dx}}_{\frac{dz}{dx}} + \underbrace{\tan x}_{P} \underbrace{\sec y}_{z} = \underbrace{\cos^3 x}_{Q}$$

$$z(\sec x) = \int \cos^{\overset{2}{\cancel{3}}} x \times \cancel{\sec x}\, dx = \int \frac{1+\cos 2x}{2}\,dx = \frac{1}{2}\left(x + \underline{\underline{\frac{\sin 2x}{2}}}\right) + C$$

Q12. $$\frac{dy}{dx} + \underbrace{\left(\frac{2}{x}\right)}_{P} y = \underbrace{\frac{2\log x}{x^3}}_{Q}$$ where $y(1) = 0$. Then $y(e) = ?$

$$IF = e^{\int 2/x\, dx} = x^2 .$$

$$y \times x^2 = 2\int \frac{\log x}{x^{\cancel{3}}} \times \cancel{x^2}\,dx = 2\,\frac{(\log x)^2}{2} + C = \underline{\underline{(\log x)^2 + C}}$$

$$y \times e^2 = (\log x)^2 + C$$

$$y \times e^2 = \underline{\underline{(\log e)^2 = 1}}$$

Q18. $r\sin\theta\, d\theta + (r^3 - 2r^2\cos\theta + \cos\theta)\, dr = 0$

$(r^3 - 2r^2\cos\theta + \cos\theta)dr = -r\sin\theta\, d\theta$

$Z = \cos\theta$

$dz = -\sin\theta\, d\theta$

$$\frac{(r^3 - 2r^2 z + z)\,dr}{r} = dz$$

$$\frac{dz}{dr} = r^2 - 2rz + \frac{z}{r}$$

$$\underbrace{r^2}_{Q} = \frac{dz}{dr} + \underbrace{\left(2r - \frac{1}{r}\right)}_{P} z$$

$$IF = e^{\int(2r - \frac{1}{r})dr} = e^{r^2 - \log r} = \frac{e^{r^2}}{r}$$

$$Z \times \frac{e^{r^2}}{r} = \int r^2 \times \frac{e^{r^2}}{r}\, dr = \frac{1}{2} e^{r^2} + C$$

$r^2 = t$

$dt = 2r\,dr.$

Inspection Method

Consider the eqn.

$M\,dx + N\,dy = 0 \longrightarrow$ ①

Suppose it is possible to rearrange the terms on LHS of ① by observing the following formulae we can solve ① through inspection

Some useful Formulae

- $d[\log(xy)] = \frac{x\,dy + y\,dx}{xy}$
- $d\left[\log\left(\frac{x}{y}\right)\right] = \frac{1}{(x/y)}\left[\frac{y\,dx - x\,dy}{y^2}\right] = \frac{y\,dx - x\,dy}{xy}$
- $d\left[\log\left(\frac{y}{x}\right)\right] = \frac{x\,dy - y\,dx}{xy}$
- $d\left[\tan^{-1}\left(\frac{x}{y}\right)\right] = \left[\frac{y\,dx - x\,dy}{x^2 + y^2}\right]$
- $\frac{d}{d'}\ d[\tan^{-1}(y/x)] = \frac{x\,dy - y\,dx}{x^2 + y^2}$

- $d[\log(x^2+y^2)] = \left[\frac{2(xdx+ydy)}{x^2+y^2}\right]$
- $d\left(\frac{e^x}{y}\right) = \left(\frac{ye^x dx - e^x dy}{y^2}\right)$
- $d\left(\frac{e^y}{x}\right) = \left(\frac{xe^y dy - e^y dx}{x^2}\right)$
- $d\left(\frac{x^2}{y}\right) = \left[\frac{2xy\,dx - x^2 dy}{y^2}\right]$
- $d\left(\frac{y^2}{x}\right) = \left[\frac{2xy\,dy - y^2 dx}{x^2}\right]$
- $d\left(\frac{x^2}{y^2}\right) = \left[\frac{2xy^2 dx - 2x^2 y\,dy}{y^4}\right]$
- $d\left(\frac{y^2}{x^2}\right) = \left[\frac{2x^2 y\,dy - 2xy^2 dx}{x^4}\right]$

Q1. $\left[y + \cos y + \frac{1}{2\sqrt{x}}\right]dx + (x - x\sin y - 1)dy = 0$

$$\Longrightarrow (ydx + xdy) + (\cos y\,dx - x\sin y\,dy) + \frac{1}{2\sqrt{x}}dx - dy = 0$$

$$d(xy) + d(x\cos y) + \frac{1}{2\sqrt{x}}dx - dy = 0$$

$$\underline{\underline{xy + x\cos y + \sqrt{x} - y = 0}}$$

Q2. $(y + 1 + x^2)dx + (x^2\sin y - x)dy = 0$

$$(ydx - xdy) + dx + x^2dx + x^2\sin y\,dy = 0$$

÷ by x^2

$$\frac{ydx - xdy}{x^2} + \frac{dx}{x^2} + dx + \sin y\,dy \neq 0$$

$$-d(y/x) + \frac{1}{x^2}dx + dx + \sin y\,dy = 0$$

$$\underline{\underline{-(y/x) - \frac{1}{x} + x - \cos y = c}}$$

Q3. $y(2x^2y + e^x)dx - (e^x + y^3)dy = 0$

$$2x^2y^2 dx + e^x y\,dx - e^x dy - y^3 dy = 0$$

÷ by y^2

$$2x^2 dx + \frac{ye^x dx - e^x dy}{y^2} - y\,dy = 0$$

$$\frac{2x^3}{3} + \left(\frac{e^x}{y}\right) - \frac{y^2}{2} = c$$

Q4. $(x^2+y^2+2x)dx - 2(x^2+y^2-y)dy = 0$

$$x^2dx + y^2dx + 2xdx - 2x^2dy \mp 2y^2dy + 2ydy = 0$$

$$(x^2+y^2)(dx - 2dy) + 2(xdx+ydy) = 0$$

÷ by x^2+y^2.

$$dx - 2dy + 2\left(\frac{xdx+ydy}{x^2+y^2}\right) = 0$$

$$x - 2y + \log(x^2+y^2) = c$$

Q5. $(x^4e^x - 2mxy^2)dx + 2mx^2ydy = 0$

$$x^4e^xdx - m(2xy^2dx - 2x^2ydy) = 0$$

÷ by x^4

$$e^xdx + m\left(\frac{-2xy^2dx + 2x^2ydy}{x^4}\right) = 0$$

$$e^xdx + md\left(\frac{y^2}{x^4}\right) = 0$$

∫ing $$e^x + m\left(\frac{y^2}{x^2}\right) = c$$

Q6. $(1+xy)ydx + (1-xy)xdy = 0$

$$ydx + xy^2dx + xdy - x^2ydy = 0$$

$$xdy + ydx + xy^2dx - x^2ydx = 0$$

÷ by $(xy)^2$

$$\frac{xdy+ydx}{(xy)^2} + \frac{ydx - xdy}{xy} = 0$$

$$\frac{1}{(xy)^2}d(xy) + d\left[\log\left(\frac{x}{y}\right)\right] = 0$$

$$\frac{-1}{xy} + \log\left(\frac{x}{y}\right) \mp c$$

Q7. $$\frac{xdx+ydy}{xdy-ydx} = \sqrt{\frac{a^2-x^2-y^2}{x^2+y^2}}$$

$$\frac{xdx+ydy}{xdy-ydx} = \sqrt{\frac{a^2}{x^2+y^2} - 1}$$

$$\frac{d\left(\frac{x^2+y^2}{2}\right)}{(xdy-ydx)} = \sqrt{\frac{a^2-(x^2+y^2)}{x^2+y^2}}$$

$$\frac{1}{2}\frac{\sqrt{(x^2+y^2)}}{\sqrt{a^2-(x^2+y^2)}}\, d(x^2+y^2) = xdy - ydx$$

× by $\frac{1}{x^2+y^2}$ $$\frac{1}{2}\frac{1}{\sqrt{x^2+y^2}\sqrt{a^2-(x^2+y^2)}}\, d(x^2+y^2) = \frac{xdy-ydx}{x^2+y^2}$$

Let $\sqrt{x^2+y^2} = z \Longrightarrow dz = \frac{1}{2\sqrt{x^2+y^2}} \times d(x^2+y^2)$

$$\therefore \frac{dz}{\sqrt{a^2-(x^2+y^2)}} = d\tan^{-1}(y/x)$$

$$\frac{dz}{\sqrt{a^2-z^2}} = d\tan^{-1}(y/x)$$

$$\frac{1}{a}\sin^{-1}(z/a) = d(\tan^{-1}(y/x))$$

$$\frac{1}{a}\sin^{-1}(z/a) = \underline{\tan^{-1}(y/x)} + c$$

MODEL IV - Exact Differential Eqn

Consider the eqn $Mdx + Ndy = 0 \longrightarrow ①$

If it is possible to express ① as

$$d\, f(x,y) = 0$$

then by Integration $f(x,y) = c \longrightarrow ②$

is the solution of ①.

Then we say that ① has an exact differential eqn

Ex : $ydx + xdy = 0$

ie $d(xy) = 0$

$$\Longrightarrow \underline{xy = c}$$

② $ydx - xdy = 0$

$$\frac{ydx - xdy}{y^2} = 0$$

$$d(x/y) = 0$$

$$x/y = c$$

In the above Eqn ② if the given equation is made an exact eqn by multiplying by $1/y^2$, therefore it is called an Integrating Factor (IF).

we can also observe that $-1/x^2$, $1/xy$, $-1/xy$, $1/(x^2+y^2)$, $-1/x^2+y^2$ etc are suitable IF's of ②. Therefore the IF of a differential eqn is not unique

③ $\frac{2x}{y^3}dx + \left(\frac{y^2-3x^2}{y^4}\right)dy = 0$

Remarks

The necessary and sufficient condition for the eqn $Mdx + Ndy = 0$ is said to be exact is given by .

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

③ $\frac{\partial M}{\partial y} = 2x \times \frac{-3}{y^4} = \frac{6x}{y^4}$

$$\frac{\partial N}{\partial x} = 0 - \frac{3}{y^4} \times 2x = \frac{6x}{y^4}$$

To get the required solution we can apply any one of the formulae.

- $\int\limits_{(y=\text{const})} M dx + \int$ terms of N not containing 'x' dy $= c$
- $\int N dy + \int$ terms of M not containing 'y' dx $= c$ (treat x as const.)

If we apply formula 1 on eq ③ we get the solution as

$$\frac{2}{y^3} \times \frac{x^2}{2} + \int y^{-2} dy = c$$

$$\frac{\cancel{2}}{y^3} \times \frac{x^2}{\cancel{2}} + -1 \times y^{-1} = c \longrightarrow ①$$

Formula ② on ③ we get

$$\frac{-1}{y} + 3x^2 \times \left(\frac{+1}{3y^3}\right) = c \longrightarrow ②$$

② & ③ are same.

Q1. $(2xy + y - \tan y)dx + (x^2 - x\tan^2 y + \sec^2 y)dy = 0$

$$\frac{\partial M}{\partial y} = 2x + 1 - \sec^2 y = 2x - \tan^2 y$$

$$\frac{\partial N}{\partial x} = 2x - \tan^2 y .$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\therefore \quad 2y \times \frac{x^2}{2} + yx - \underline{\underline{x\tan y}} + \tan y = c$$

Q2. $(y^2 e^{xy^2} + 4x^3)dx + (2xy\, e^{xy^2} - 3y^2)dy = 0$

$$\frac{\partial M}{\partial y} = y^2 e^{xy^2} \times 2xy + e^{xy^2} \times 2y$$

$$\frac{\partial N}{\partial x} = 2xy \times e^{xy^2}(y^2) + e^{xy^2} \times 2y = \frac{\partial M}{\partial y}$$

$$\frac{\cancel{y^2}\, e^{xy^2}}{y^2} + \frac{\cancel{4}x^4}{\cancel{4}} + \frac{-\cancel{3}y^3}{\cancel{3}} = c$$

$$\underline{\underline{e^{xy^2} + x^4 - y^3 = c}}$$

Q3. $(xy\cos xy + \sin xy)dx + x^2\cos(xy)dy = 0$

$$\frac{\partial M}{\partial y} = -xy\sin xy \times x + \cos xy \times x + \cos xy \times x$$

$$= -x^2 y \sin xy + 2x\cos xy$$

$$\frac{\partial N}{\partial x} = -x^2 \sin xy \times y + \cos y \times 2x = \frac{\partial M}{\partial y}$$

$$\therefore \quad \underline{\underline{x \times \left[\frac{\sin xy}{\cancel{x}}\right] = c}}$$

Q4. $(1 + e^{x/y})dx + e^{x/y}\left(1 - \frac{x}{y}\right)dy = 0$

$$\frac{\partial M}{\partial y} = e^{x/y} \times \frac{-x}{y^2}$$

$$\frac{\partial N}{\partial x} = e^{x/y}\left(-\frac{1}{y}\right) + \left(1 - \frac{x}{y}\right) e^{x/y} \times \frac{1}{y} = \frac{-x e^{x/y}}{y^2} = \frac{\partial M}{\partial y}$$

$$\therefore \quad \underline{\underline{x + \frac{e^{x/y}}{1/y} = c}}$$

MODEL V :- Equations reducible to exact eqn form.

If the given eqn

$$Mdx + Ndy = 0 \longrightarrow ①$$

is not exact equation.

ie $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

then it can be made exact equation by multiplying with a suitable IF. But to get the appropriate IF of ① we can follow certain rules.

Rule I

If the given eqn $Mdx + Ndy = 0 \longrightarrow ①$ is not an exact equation but it is a homogeneous eqn and

$$Mx + Ny \neq 0$$

then $\frac{1}{Mx+Ny} \longrightarrow ②$ is ~~not~~ an integrating factor of ①.

Q1. The solution of $(x^2y - 2xy^2)dx + (3x^2y - x^3)dy = 0$ —— ①

Note: If we observe that every term of both M & N of ① is with same degree, then we can recognize it as a homogeneous eqn.

$$Mx + Ny = \cancel{x^3y} - 2x^2y^2 + 3x^2y^2 - \cancel{x^3y}$$

$$= x^2y^2$$

$$IF = \frac{1}{Mx+Ny} = \frac{1}{x^2y^2} \longrightarrow ②$$

$$① \times ② \Longrightarrow \frac{x^2y - 2xy^2}{x^2y^2}dx + \frac{3x^2y - x^3}{x^2y^2}dy = 0$$

$$\frac{1}{y}x - 2\log x + 3\log y = c$$

Q2. $x^2ydx - (x^3+y^3)dy = 0$

$$Mx + Ny = \cancel{x^3y} - \cancel{x^3y} - y^4 = -y^4 \neq 0$$

$$IF = \frac{1}{Mx+Ny} = \frac{-1}{y^4}$$

$$\frac{-x^2y\,dx}{y^4} + \frac{x^3+y^3}{y^4}dy = 0 \qquad y^{-4} = -4y^{-3}$$

$$\frac{-1}{y^3}\times\frac{x^3}{3} + \underline{\underline{\frac{x^3\times -4}{y^3}}} + \log y = 0$$

Q3. $(3xy - 2ay^2)dx + (x^2 - 2axy)dy = 0$

$$Mx + Ny = 3x^2y - 2axy^2 + x^2y - 2axy^2 = 4x^2y - 4axy^2$$

$$\text{IF} = \frac{1}{4x^2y - 4axy^2}$$

$$\frac{3xy - 2ay^2}{4x^2y - 4axy^2}dx + \frac{x^2 - 2axy^2}{4x^2y - 4axy^2}dy = 0$$

$$\left[\frac{1}{4(x-ay)} + \frac{2y}{4xy}\right]dx + \left[\frac{x}{4xy} - \frac{a}{4(x-ay)}\right]dy = 0$$

$$\frac{1}{\cancel{4}}\log(x-ay) + \frac{\cancel{2}1}{\cancel{2}}\log x^2 + \frac{1}{\cancel{4}}\log y = \frac{1}{\cancel{4}}\log c$$

$$\log(x-ay) + \underline{\underline{\log x^2}} + \log y = \log c$$

Rule II

In the given eqn $Mdx + Ndy = 0 \longrightarrow ①$ is not exact equation but it is in the form

$$f(xy)y\,dx + g(xy)x\,dy = 0 \quad \text{and} \quad (Mx - Ny) \neq 0$$

then $\text{IF} = \frac{1}{Mx - Ny}$ ——②

Q1. $(x^2y^2 + xy + 1)y\,dx + (x^2y^2 - xy + 1)x\,dy = 0$ ——①

$$Mx - Ny = x^3y^3 + x^2y^2 + xy - (x^3y^3 + 2x^2y^2 + xy) = \underline{\underline{2x^2y^2}}$$

$$\text{IF} = \frac{1}{2x^2y^2} \longrightarrow ②$$

$$① \times ② \Longrightarrow \frac{x^2y^2 + xy + 1}{x^2y^2}y\,dx + \frac{x^2y^2 - xy + 1}{x^2y^2}x\,dy = 0 \times 2$$

$$yx + \log x + \underline{\underline{\frac{1}{y}(-1/x)}} - \log y = c$$

Q2. $y(xy + 2x^2y^2)dx + x(xy - x^2y^2)dy = 0 \longrightarrow$ ①

<u>Ans</u> $Mx - Ny = x^2y^2 + 2x^3y^3 - x^2y^2 + x^3y^3 = 3x^3y^3$

$IF = \dfrac{1}{3x^3y^3} \longrightarrow$ ②

①×② $\Longrightarrow$ $\dfrac{\cancel{y}(xy+2x^2y^2)}{x^3y^{\cancel{3}\,2}}dx + \dfrac{\cancel{x}(xy-x^2y^2)dy}{x^{\cancel{3}\,2}y^3} = 0$

$$\frac{1}{3y} \times \left(\frac{-1}{x}\right) + \frac{2}{3}\log x - \frac{1}{3}\log y = c$$

Q3. $[xy\cos(xy) + \sin(xy)]\,y\,dx + [xy\cos(xy) - \sin(xy)]\,x\,dy = 0 \longrightarrow$ ①

$Mx - Ny = 2xy\sin xy$

$IF = \dfrac{1}{2xy\sin xy} \longrightarrow$ ②

①×② $\Longrightarrow$ $\dfrac{xy\cos(xy) + \sin xy}{2xy\sin xy}dx + \dfrac{xy\cos xy - \sin xy}{2xy\sin xy}dy = 0$

$$\frac{\cancel{y}}{\cancel{2}}\left[\frac{\log\sin(xy)}{\cancel{y}}\right] + \frac{1}{\cancel{2}}\log x - \frac{1}{\cancel{2}}\log y = \frac{1}{\cancel{2}}\log c$$

$$\underline{\underline{x\sin xy = cy}}$$

<u>Note</u>: If the given eqn $Mdx + Ndy = 0$ is not exact eqn then rule I & II is not applicable. Then we can apply rule III or IV given below.

<u>RULE III</u>:

If the given eqn $Mdx + Ndy = 0 \longrightarrow$ ① is not exact eqn but

$\dfrac{1}{N}\left(\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}\right) = f(x)$ [only a fn of x]

then $e^{\int f(x)dx} \longrightarrow$ ② is an IF of ①.

<u>RULE IV</u>:

If the given eqn ① is not exact eqn but

$\dfrac{1}{M}\left(\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}\right) = g(y)$ (only a fn of y)

then $e^{\int g(y)dy}$ is an IF of ①.

Note : while applying Rule III or IV first we consider .

$$\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right) \longrightarrow \begin{cases} =0 & (\text{Exact equ}) \\ \neq 0 & (\text{Apply R - III/IV}) \end{cases}$$

Q1. $(x^2+y^2+x)dx + xy\,dy = 0 \longrightarrow ①$

$$\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right) = 2y-y = y$$

Rule III $\therefore \frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right) = \frac{1y}{xy} = 1/x = f(x)$

$$\text{IF} = e^{\int 1/x\,dx} = x \text{ —— } ②$$

$$① \times ② \Longrightarrow \underline{\underline{\frac{x^4}{4}+\frac{x^3}{3}}} + x^2\left(\frac{y^2}{2}\right) = c$$

Q2. $(xy^3+y)dx + 2(x^2y^2+x+y^4).dy = 0 \longrightarrow ①$

$$\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x} = x\times 3y^2+1-(4xy^2+1) = \underline{\underline{-(xy^2+1)}}$$

Rule IV : $\therefore \frac{1}{M}\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right) = \frac{xy^2+1}{xy^2+1}\times\frac{1}{y} = g(y)$

$$\text{IF} = e^{\int 1/y\,dy} = y \text{ —— } ②$$

$$① \times ② \Longrightarrow y^4\left(\frac{x^2}{2}\right) + \underline{\underline{y^2(x)}} + \cancel{2}\left(\frac{y^6}{\cancel{6}_3}\right) = c$$

Q1. $(x^3+xy^4)dx + (e^{x^3}-2y^3)dy = 0$ —— ①

$$\frac{\partial M}{\partial y} = 4y^3x$$

$$\frac{\partial N}{\partial x} = e^{x^3}\times 3x^2 - 0$$

$$\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x} = 4y^3x - 2xe^{x^2} = 2x(2y^3-e^{x^2})$$

$$\frac{1}{e^{x^2}-2y^3}\times 2x(2y^3-e^{x^2}) = -2x = \underline{\underline{f(x)}}$$

Q2. $(3x^2y^4 + 2xy)dx + (2x^3y^3 - x^2)dy = 0.$

IF = e^{-x^2}

Rule III

$$e^{-x^2}(x^3 + xy^4)dx + e^{-x^2}(e^{x^2} - 2y^3)dy = 0$$

$$\int x^2 e^{-x^2} \times x\,dx + y^4 \int x e^{-x^2} dx + \int dy = c$$

$x^2 = t$

$$\underline{\frac{-1}{2} e^{-x^2}(x^2+1) + \frac{y^4}{2}\left(\frac{e^{-x^2}}{-1}\right) + y = c}$$

Q2. $(3x^2y^4 + 2xy)dx + (2x^3y^3 - x^2)dy = 0.$

$$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 3x^2 \times 4y^3 + 2x - (2 \times 3x^2 \times y^3 - 2x)$$

$$= 6x^2y^3 + 4x = 2x(\underline{3xy^3} + 2)$$

Rule IV

$$\frac{-1}{M}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{-1}{3x^2y^4 + 2xy} \times 2x(3xy^3 + 2) = \frac{-1}{\cancel{x}y\cancel{(3xy^3+2)}} \times 2\cancel{x}\cancel{(3xy^3+2)}$$

$$= -2/y = \underline{g(y)}$$

$$\text{IF} = e^{\int g(y)dy} = e^{-2\times\log y} = \underline{1/y^2}$$

$$\therefore \quad \underline{3y^2 \times \frac{x^3}{3} + \frac{2}{y} \times \frac{x^2}{2} = c}$$

Q3. $(2xy+1)y\,dx + (1 + 2xy - x^3y^3)x\,dy$

$$Mx - Ny = \cancel{2x^2y^2} + \cancel{xy} - (\cancel{xy} + \cancel{2x^2y^2} + x^4y^4) = x^4y^4 \neq 0$$

$$\frac{(2xy+1)y}{x^4y^{4}}dx + \frac{(1 + 2xy - x^3y^3)x}{x^4y^4}dy = 0$$

$$\left(\frac{2}{x^3y^2} + \frac{1}{x^4y^3}\right)dx + \left(\frac{1}{x^3y^4} + \frac{2}{x^2y^3} + \frac{1}{y}\right)dy = 0$$

$$\frac{2}{y^2} \times \frac{-2}{x^2} + \underline{\frac{1}{y^3} \times \frac{-3}{x^3}} + \log y = c$$

Q4. $y(x+y+1)dx + x(x + 3y + 2)dy = 0 \longrightarrow ①$

$$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = x + 2y + 1 - (2x + 3y + 2) = -x - y - 1 = -(x+y+1)$$

Rule IV

$$\frac{1}{M} \times \left(-\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right) = +1/y = g(y)$$

$$IF = e^{\int 1/y \, dy} = \underline{\underline{y}}$$

$$IF \times ① \Longrightarrow y^2(x^2+y+1)dx + xy(x+3y+2)\,dy = 0$$

$$\int xy^2 dx + \cancel{y}^3 + \cancel{y}^2 = c$$

$$\underline{\underline{\frac{x^2y^2}{2} + y^3x + y^2x = c}}$$

Q5. $y^2dx + (x^2 - xy - y^2)\,dy = 0$ ——— ①

Homogeneous eqn.

$$Mx + Ny = y^2x + x^2y - xy^2 - y^3 = y(x^2-y^2)$$

$$\frac{1}{Mx+Ny} = \frac{1}{y(x^2-y^2)}$$

$$① \times IF = \frac{y^{2}dx}{\cancel{y}(x^2-y^2)} + \frac{x^2 - xy - y^2}{y(x^2-y^2)}\,dy = 0$$

$$\frac{y\,dx}{x^2-y^2} + \left(\frac{\cancel{x^2-y^2}}{y\cancel{(x^2-y^2)}} - \frac{xy}{\cancel{y}(x^2-y^2)}\right)dy = 0$$

$$\underline{\underline{\cancel{y}\,\frac{1}{2\cancel{y}}\,\cancel{\sin^{-1}}\,\log\left|\frac{x-y}{x+y}\right| + \log y = c}}$$

Q6. $y(axy + e^x)\,dx - e^x dy = 0$

$$axy^2\cancel{dx} + ye^x dx - e^x dy = 0$$

$\div$ by y^2

$$ax\,dx + \frac{ye^x dx - e^x dy}{y^2} = 0$$

$$\underline{\underline{ax\,dx + d\left(\frac{e^x}{y}\right) = c}}$$

$$\underline{\underline{\frac{ax^2}{2} + \frac{e^x}{y} = c}}$$

ALGORITHM OF SOLVING EQN OF FORM $Mdx + Ndy = 0$ —— ①

1) If it is possible rearranging the terms of ①, then apply Inspection method.
2) If it is not possible, if ① is a homogeneous eqn apply Rule ①.
3) If ① is in the form of $f(xy)y\,dx + g(xy)x\,dy = 0$ then apply Rule II.
4) If step 1, 2, 3 failed (not applicable), then consider

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \begin{cases} \to 0 & \text{exact eqn} \\ \to \neq 0 & (\text{Rule ③ or ④}) \end{cases}$$

Solution Of a Higher Order Linear Diff. Eqn with Constant Coeff. Form

The general form is given by.

$$\frac{d^n y}{dx^n} + K_1 \frac{d^{n-1}y}{dx^{n-1}} + K_2 \frac{d^{n-2}y}{dx^{n-2}} + \dots + K_{n-2}\frac{d^2y}{dx^2} + K_{n-1}\frac{dy}{dx} + K_n y = X \text{ —— ①}$$

where $K_1, K_2 - - - K_n$ are constants and.

X = fn of x.

Let $D = \frac{d}{dx}$

$$① \implies (D^n + K_1 D^{n-1} + K_2 D^{n-2} + \dots + K_{n-2}D^2 + K_{n-1}D + K_n)y = X.$$

ie $f(D)y = X \longrightarrow ②$

Suppose $X = 0$ in ②

we get $f(D)y = 0 \longrightarrow ③$

which is called Homogeneous Linear differential Eqn.

Otherwise $(X \neq 0)$ ② is called non homogeneous linear differential eqn.

Solution of $f(D)y=0$ form:

Given Eqn	General soln
1) $(D-a)y=0$	1) $y = ce^{ax}$
2) $(D+b)y=0$	2) $y = ce^{-bx}$
3) $Dy=0$	3) $y = c$
4) $(D-a)(D-b)y=0$	4) $y = c_1e^{ax}+c_2e^{bx}$
5) $(D-a)(D+b)(D-c)y=0$	5) $y = c_1e^{ax}+c_2e^{-bx}+c_3e^{cx}$

Note: we can observe that the solution of $f(D)y=0$ is dependent on the roots of the eqn $f(D)=0$. Hence it is called Auxillary Eqn (A.E).

6) $(D-a)(D-b)y=0$ where $a = \alpha+i\beta$ $b = \alpha-i\beta$	6) $y = e^{\alpha x}[K_1\cos\beta x + K_2\sin\beta x]$
7) $(D-a)(D-b)y=0$ where $a = \alpha+\sqrt{\beta}$ $b = \alpha-\sqrt{\beta}$	7) $y = e^{\alpha x}[K_1\cosh\sqrt{\beta}x + K_2\sinh\sqrt{\beta}x]$
8) $(D-a)^2y=0$	8) $y=(C_1+C_2x)e^{ax}$
9) $(D-a)^3y=0$	9) $y=(C_1+C_2x+C_3x^2)e^{ax}$
10) $(D-a)^4y=0$	10) $y=(C_1+C_2x+C_3x^3+C_4x^4)e^{ax}$.
11) $(D-a)^2(D+b)^3y=0$	11) $y=(C_1+C_2x)e^{ax}+(C_3+C_4x+C_5x^2)e^{-bx}$
12) $(D-a)^2(D-b)^2y=0$ $a=\alpha+i\beta$ $b=\alpha-i\beta$	12) $y=e^{\alpha x}[(K_1+K_2x)\cos\beta x+(K_3+K_4x)\sin\beta x]$
13) $(D-a)^2(D-b)^2y=0$ $a=\alpha+\sqrt{\beta}$	13) $y=e^{\alpha x}[(K_1+K_2x)\cosh\sqrt{\beta}x+(K_3+K_4x)\sinh\sqrt{\beta}x]$
14) $(D-a)^2(D-b)(D-c)(D+d)y=0$ $b=\alpha+i\beta$ & $c=\alpha-i\beta$	14) $y=(C_1+C_2x)e^{ax}+e^{\alpha x}(C_3\cos\beta x+C_4\sin\beta x)$ $+C_5e^{-dx}$

** In writing the general solution ~~of~~ or complete solution of the given differential eqn it must be observed that the no. of arbitrary constants in the solution is equal to the order of the differential eqn given.

Solve

Q1. $\frac{d^3y}{dx^3} - \frac{3dy}{dx} + 2y = 0$

Auxillary equation is $(D^3 - 3D + 2)y = 0$

$(D^3 - D - 2D + 2)y = 0$

$[D(D^2 - 1) - 2(D-1)]y = 0$

$(D-1)[D(D+1) - 2]y = 0$

$(D-1)[D^2 + D - 2]y = 0$

$(D-1)[(D+2)(D-1)]y = 0$

$(D-1)^2(D+2)y = 0$

$D = 1, 1, -2.$

$\therefore \quad y = (C_1 + C_2 x)e^x + C_2 e^{-2x}$

Q2. $(D^4 + 8D^2 + 16)y = 0$

$(D^4 + 4D^2 + 4D^2 + 16)y = 0$

$[D^2(D^2+4) + 4(D^2+4)]y = 0$

$(D^2+4)^2 = 0$

$D^2 = -4, -4$

$D = \pm 2i, \pm 2i$

$\therefore \quad y_1 = (C_1 + C_2 x)\cos 2x + (C_3 + C_4 x)\sin 2x$

Q3. $(D^4 - 2D^3 - 3D^2 + 4D + 4)y = 0$

$(D+1)(D^3 - 3D^2 + 4)y = 0$

$(D+1)(D+1)(D^2 - 4D + 4) = 0$

$(D+1)^2(D-2)^2 = 0$

$D = -1, -1, 2, 2$

$\therefore \quad y = (C_1 + C_2 x)e^{-x} + (C_3 + C_4 x)e^{2x}$

$$\begin{array}{r|rrrrr} 1 & 1 & -2 & -3 & 4 & 4 \\ & 0 & 1 & -1 & -4 & 0 \\ \hline & 1 & -1 & -4 & 0 & 4 \end{array}$$

$$\begin{array}{r|rrrrr} -1 & 1 & -2 & -3 & 4 & 4 \\ & 0 & -1 & 3 & 0 & -4 \\ \hline & 1 & -3 & 0 & 4 & 0 \end{array}$$

$$\begin{array}{r|rrrr} -1 & 1 & -3 & 0 & 4 \\ & 0 & -1 & 4 & -4 \\ \hline & 1 & -4 & 4 & 0 \end{array}$$

Q4. The solution of $y'' - 2y' + 10y = 0$ satisfying $y(0) = 4$ & $y'(0) = 1$

Ans $(D^2 - 2D + 10)y = 0$

$D^2 - 2D + 10 = 0$

$$D = \frac{2 \pm \sqrt{4 - 40}}{2} = \frac{2 \pm 6i}{2} = 1 \pm 3i$$

$y = e^x(c_1 \cos 3x + c_2 \sin 3x)$

$y(0) = 4$

$\therefore c_1 = \underline{\underline{4}}$

$y'(0) = 1$

$$\frac{dy}{dx} = e^x(-4 \times 3\sin 3x + 3c_2 \cos 3x) + (4\cos 3x + c_2 \sin 3x)e^x$$

$y'(0) = 1$

$1 = 3c_2 + 4$

$\therefore c_2 = 1$

$$\therefore \; y = \underline{\underline{e^x(4\cos 3x - \sin 3x)}}$$

Q5. The solution of $\frac{d^2y}{dx^2} + \frac{2dy}{dx} + 17y = 0$ satisfying the condition $y(0) = 1, y'(\pi/4) = 0$

Ans $(D^2 + 2D + 17)y = 0$

$D^2 + 2D + 17 = 0$

ie $$D = \frac{-2 \pm \sqrt{4 - 68}}{2} = \frac{-2 \pm 8i}{2} = -1 \pm 4i$$

$y = e^{-x}(c_1 \cos 4x + c_2 \sin 4x)$

$y(0) = 1$

$c_1 = 1$

$$\frac{dy}{dx} = e^{-x}(-4c_1 \sin 4x + 4c_2 \cos 4x) + -e^{-x}(c_1 \cos 4x + c_2 \sin 4x)$$

$$= e^{-x}(-4\sin 4x + 4c_2 \cos 4x) + -e^{-x}(\cos 4x + c_2 \sin 4x)$$

$\frac{dy}{dx}(\pi/4) = 0$

ie $c_2 = 1/4$

$$y = \underline{\underline{e^{-x}\left(\cos 4x + \frac{1}{4}\sin 4x\right)}}$$

Q6. $\frac{d^2y}{dx^4} + 4y = 0$

$[(D^2)^2 + (2)^2]y = [(D^2+2)^2 - 2D^2 \times 2]y = 0$

$(D^2+2)^2 - (2D)^2 = (D^2+2D+2)(D^2-2D+2)$

$D = \frac{2 \pm \sqrt{4-8}}{2}$ $\quad D = \frac{-2 \pm \sqrt{4-8}}{2}$

$D = 1 \pm i$ $\quad D = -1 \pm i$

$y = e^{-x}(c_1 \cos x + c_2 \sin x) \underline{+ e}^{-x}(c_3 \cos x + c_4 \sin x)$

Q7. $\frac{d^3y}{dx^3} + \frac{d^2y}{dx^2} + 100\frac{dy}{dx} + 100y = 0$.

$(D^3 + D^2 + 1000D + 100)y = 0$

$D^2(D+1) + 100(D+1) = 0$

$(D^2+100)(D+1) = 0$

$D = -1, \; D = \pm 10i$

$y = c_1 e^{-x} + \underline{c_2 \cos 10x} + c_3 \sin 10x$

Q8. i) The solution of $\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = 0$ is given as.

$y = c_1 e^{-x} + c_2 e^{-3x}$ then $(a, b) = (\_\_, \_\_)$

a) $(-4, 3)$ c) $(-4, -3)$

b) $(4, -3)$ d) $(4, 3)$

ii) Which of the following is a solution of

$\frac{d^2y}{dx^2} + a\frac{dy}{dx} + (b+1)y = 0$.

a) xe^{-4x} b) xe^{-3x} c) xe^{-2x} d) x^2e^{-2x}

Ans i) $(D^2 + aD + b)y = 0$

$D = \frac{-a \pm \sqrt{a^2 - 4b}}{2}$

$y = c_1 e^{-x} + c_2 e^{-3x}$

$(D+1)(D+3) = 0 \implies D^2 + 4D + 3 = 0$

Given $D^2 + aD + b = 0$

$\therefore \; a = 4, \; \underline{b = 3}$

ii) $[D^2 + Da + (b+1)]y = 0$

$a = 4, \; b = 4$

$(D^2+4D+4)y=0$

$(D+2)^2y=0$

$D = -2, -2$

$\therefore y = c_1e^{-2x} + c_2xe^{-2x} = \underline{(c_1+c_2x)}e^{-2x}$

Solution of $f(D)y = X$ form

$f(D)y = X$ ——①

From ① we can observe that $y = \frac{X}{f(D)}$ is also a solution to it which is called particular integral or particular solution of ① and is denoted by y_p.

i) $\frac{X}{D-a} = e^{ax}\int Xe^{-ax}dx$

2) $\frac{X}{D+a} = e^{-ax}\int Xe^{ax}dx$

3) $\frac{X}{D} = \int X dx$

Note : Now by selecting $X = e^{ax+b}$ or $\sin(ax+b)$ or $\cos(ax+b)$ or x^m. . . . or the RHS of ①, we can analyze various DI. in various cases

Case 1 : Consider the equation $f(D)y = e^{ax+b}$ then

$$y_p = \frac{e^{ax+b}}{f(D)} = \frac{e^{ax+b}}{f(a)}$$

$$D(e^{ax+b}) = a(e^{ax+b})$$

$$D^2(e^{ax+b}) = a^2(e^{ax+b})$$

$$\vdots \qquad \vdots$$

$$D^n(e^{ax+b}) = a^n(e^{ax+b})$$

$$f(D)e^{ax+b} = f(a)e^{ax+b}$$

ie $$\frac{e^{ax+b}}{f(a)} = \frac{e^{ax+b}}{f(D)}$$

Note :

i) $\frac{e^{2-3x}}{(D+1)(D+2)} = \frac{e^{2-3x}}{(-3+1)(-3+2)} = \frac{e^{2-3x}}{2}$

ii) $$\frac{e^{ax+b}}{(D-a)} = e^{ax}\int e^{ax+b} \times e^{-ax}\,dx = \underline{\underline{x e^{ax+b}}}$$

iii) $$\frac{e^{ax+b}}{(D-a)^2} = \frac{1}{(D-a)}\, x e^{ax+b} = e^{ax}\int x e^{ax+b} \times e^{-ax}\,dx = \underline{\underline{\frac{x^2}{2}}}\, e^{ax+b}$$

iv) $$\frac{e^{ax+b}}{(D-a)^3} = \frac{1}{(D-a)} \times \frac{x^2}{2} e^{ax+b} = e^{ax}\int \frac{x^2}{2} e^{ax+b} \times e^{-ax} = \underline{\underline{\frac{x^3}{3!}}}\, e^{ax+b}$$

v) $$\frac{e^{ax+b}}{(D-a)^K} = \frac{x^K}{K!}\, \underline{\underline{e^{ax+b}}}$$

vi) $$\frac{e^{ax+b}}{(D-a)^m(D-n)(D+p)^r} = \frac{x^m}{m!}\, \underline{\underline{\frac{e^{ax+b}}{(a-n)(a+p)^r}}}$$

Case II

$f(D)y = \sin(ax+b)$ or $\cos(ax+b)$

$$y_p = \frac{\sin(ax+b)}{f(D)} = ?$$

$D(\sin(ax+b)) = a\cos(ax+b)$

$D^2(\sin(ax+b)) = -a^2\sin(ax+b)$

$\underline{\underline{D^2 = -a^2}}$

$$y_p = \frac{\sin(ax+b)}{\phi(D^2)} = \frac{\sin(ax+b)}{\phi(-a^2)} \quad \text{Given } \phi(-a^2) \neq 0$$

IIIly $$y_p = \frac{\cos(ax+b)}{\phi(D^2)} = \frac{\cos(ax+b)}{\phi(-a^2)} \quad [\phi(-a^2) \neq 0]$$

Note:

i) $$\frac{\cos(3x+4)}{(D^2+4)} = \frac{\cos(3x+4)}{(-9+4)} = \frac{\cos(3x+4)}{\underline{-5}}$$

ii) $$\frac{\sin(2-x)}{D^3+1} = \frac{\sin(2-x)}{D \times D^2+1} = \frac{\sin(2-x)}{D\times -1+1} = (1+D)\frac{\sin(2-x)}{1-D^2}$$

$$= \frac{(1+D)\sin(2-x)}{1+1} = \underline{\underline{\frac{1}{2}\left[\sin(2-x) - \cos(2-x)\right]}}$$

iii) $$\frac{\cos(ax+b)}{D^2+a^2} = \frac{x}{2a}\sin(ax+b)$$

iv) $$\frac{\sin(ax+b)}{D^2+a^2} = \frac{-x}{2a}\cos(ax+b)$$

$$\frac{e^{i(ax+b)}}{D^2+a^2} = \frac{e^{iax+ib}}{(D+ia)(D-ia)}$$

ie $$\frac{\cos(ax+b)+i\sin(ax+b)}{(D+ia)(D-ia)} = \frac{e^{iax+ib}}{(D+ia)(D-ia)}$$

$$\frac{\cos(ax+b)}{D^2+a^2} + \frac{i\sin(ax+b)}{D^2+a^2} = \frac{ix}{2a}\left[\cos(ax+b)+i\sin(ax+b)\right]$$

Case I using Note (ii)

Comparing, we get the results

v) $$\frac{\sin(3x+4)}{D^2+9} = \frac{-x}{2\times 3}\cos(3x+4) = \underline{\underline{\frac{-x}{6}\cos(8x+4)}}$$

vi) $$\frac{\cos(2x-3)}{D^2+4} = \frac{x}{2}\,\frac{\sin(2x-3)}{2} = \underline{\underline{\frac{x}{4}\sin(2x-3)}}$$

Remarks: By solving the given eqn.

$$f(D)y = X \text{ —— ①}$$

we can observe that the solution of $f(D)y=0$ —— ② as a part of solution of ① which is called complimentary function (CF) and is denoted by y_c.

we know that $y_p = \frac{X}{f(D)}$ —— ③ as a particular solution of ①.

$\therefore$ $y = y_c + y_p$ gives the complete solution of ①.

Q1. $(D^3-5D^2+8D-4)y = e^{2x}$

Aux. eqn $(D^3-5D^2+8D-4)=0$

$$\begin{array}{r|rrrr} 1 & 1 & -5 & 8 & -4 \\ & 0 & 1 & -4 & 4 \\ \hline & 1 & -4 & 4 & 0 \end{array}$$

$(D-1)(D^2-4D+4)=0$

$(D-1)(D-2)^2=0$

$D = 1, 2, 2$

$$y_c = c_1 e^{x} + (c_2 + c_3 x)e^{2x}$$

$$y_p = \frac{X}{f(D)} = \frac{e^{2x}}{(D-1)(D-2)^2} = \underline{\underline{x^2\frac{e^{2x}}{2!}}}$$

Q₂. $y'' - 8y' + 16y = 3e^{4x}$ satisfying $y = 0$ at $x = 0$ & $x = 2$.

Ans $(D^2 - 8D + 16)y = 0$

$(D-4)^2 = 0$

$D = 4, 4.$

$y_c = (c_1 + c_2 x)e^{4x}$

$$y_p = \frac{3e^{4x}}{(D-4)^2} = \frac{x^2}{2!}\,3e^{4x}$$

$\therefore\ y = y_c + y_p = (c_1 + c_2 x)e^{4x} + \frac{3x^2}{2}e^{4x}$

$0 = c_1 + 0$

$c_1 = 0$

$0 = 2c_2 e^8 + 6e^8$

$c_2 = -3$

$$\therefore\ y = -3xe^{4x} + \frac{3x^2}{2}e^{4x}$$

Q₃. $\frac{d^4y}{dx^4} - y = 15\cos 2x$

$(D^4 - 1)y = 0$

$(D^2)^2 - 1^2 = 0$

$(D^2+1)(D^2-1)^2 = 0$

$D = \pm 1,\ \pm i$

$y_c = c_1 e^{x} + c_2 e^{-x} + e^{0}(c_3\cos x + c_4 \sin x)$

$$y_p = \frac{15\cos 2x}{(D^2+1)(D^2-1)} = \frac{15\cos 2x}{-3 \times -5} = \cos 2x$$

Q₄. $(D^2 - 4)y = \cosh(2x-1) + 3^x$

Ans $(D^2 - 4) = 0$

$(D+2)(D-2) = 0$

$D = \pm 2$

$y_c = c_1 e^{2x} + c_2 e^{-2x}$

$$\cosh(2x-1) + 3^x = \frac{e^{2x-1} + e^{-2x+1}}{\underset{2}{\cancel{(D+2)(D-2)}}} + e^{x\log 3}$$

$$y_p = \frac{\frac{1}{2}\left[e^{2x-1} + e^{-2x+1}\right] + e^{x\log 3}}{(D+2)(D-2)} = \frac{1}{2}\left[\frac{x}{4}e^{2x-1} + \frac{xe^{-2x+1}}{-4}\right] + \frac{3^x}{(2+\log 3)(-2+\log 3)}$$

$$= \frac{x}{4}\sinh(2x-1) + \frac{3^x}{(\log 3)^2 - 4}$$

$$= 1 + \frac{(8D+2)\,2\cos 2x}{-40} = 1 - \frac{1}{40}\left[4\cos 2x + 6x - 2\sin 2x\right]$$

$$= 1 - \frac{1}{10}\left[\cos 2x - 3\sin 2x\right]$$

Q8. $y'' + y = \sin x \sin 2x$

$D^2 + 1 = 0$

$D = \pm i$

$\therefore y_c = C_1 \cos x + C_2 \sin x$

$\sin x \sin 2x = \frac{1}{2}(2\sin x \sin 2x)$

$= \frac{1}{2}[\cos x - \cos 3x]$

$$y_p = \frac{\sin x \sin 2x}{D^2+1} = \frac{1}{2}\left[\frac{\cos x}{D^2+1} - \frac{\cos 3x}{D^2+1}\right] = \frac{1}{2}\left[\frac{x}{2}\sin x + \frac{\cos 3x}{8}\right]$$

Q9. The solution of $K^2 \frac{d^2y}{dx^2} = y - y_2$ satisfying the conditions

i) $y = y_1$ at $x = 0$

ii) $y = y_2$ at $x = \infty$ $\quad K, y_1, y_2$ are constants.

Ans $K^2 D^2 y = y - y_2$

$(K^2 D^2 - 1)y = -y_2$

$D = \pm \frac{1}{K}$

$y = c_1 e^{x/K} + c_2 e^{-x/K}$

$$y_p = \frac{-y_2 \times e^{0x}}{K^2D^2 - 1} = \frac{+y_2 \times e^{0x}}{+1} = y_2$$

$\therefore y = c_1 e^{x/K} + c_2 e^{-x/K} + y_2$

$y_1 = c_1 e^{0/K} + c_2 e^{-0/K} + y_2$

$y_1 = c_1 + c_2 + y_2$

$\cancel{y_2} = c_1 \infty + c_2 0 + \cancel{y_2}$

$c_1 \infty = 0 \implies c_1 = 0$

$c_2 = y_1 - y_2$

$\therefore y = (y_1 - y_2)e^{x/K} + y_2$

Case (iii)

Consider $f(D)y = x^m \quad (m \in Z^+)$

then $y_p = \frac{x^m}{f(D)} = f(D)^{-1} x^m$

Expand $[f(D)]^{-1}$ as ascending powers of D.

By using Binomial expansion and then apply on x^m.

$$(1-x)^{-1} = 1+x+x^2+x^3+\ldots$$

$$(1+x)^{-1} = 1-x+x^2-x^3+\ldots$$

$$(1 \pm x)^{-2} = 1 \pm 2x+3x^2 \pm 4x^3+\ldots$$

$$(1 \pm x)^{-3} = 1 \pm 3x + 6x^2 \pm 10x^2 + \ldots$$

Q1. $y'' - 4y' + 4y = x^3$

$$(D^2 - 4D + 4)y = x^3$$

$$(D-2)^2 = 0$$

$$D = 2, 2.$$

$$y_c = (c_1 + c_2 x)e^{2x}$$

$$y_p = \frac{x^3}{(D-2)^2} = \frac{x^3}{4(1-\frac{D}{2})^2} = \frac{1}{4}\left(1-\frac{D}{2}\right)^{-2} x^3$$

$$= \frac{1}{4}\left[1 + 2\times\frac{D}{2} + \frac{3D^2}{4} + \frac{4D^3}{8}\right]x^3 = \frac{1}{4}\left[x^2 + 3x^2 + \frac{3}{4}\times 6x + \frac{1}{2}\times 6\right]$$

$$= \underline{\underline{\frac{1}{4}\left[x^3 + 3x^2 + \frac{9}{2}x + 3\right]}}$$

Q2. $\frac{d^2y}{dx^2} + \frac{dy}{dx} = x^2 + 2x + 4.$

$$(D^2 + D) = 0$$

$$D(D+1) = 0$$

$$D = 0, -1$$

$$y_c = c_1 + c_2 e^{-x}$$

$$y_p = \frac{x^2+2x+4}{D(D+1)} = \frac{1}{D}(D+1)^{-1}(x^2+2x+4) = \frac{1}{D}(1-D+D^2-D^3)(x^2+2x+4)$$

$$= \left(\frac{1}{D} - 1 + D - D^2\right)(x^2+2x+4) = \frac{x^3}{3} + \frac{\cancel{2x^2}}{\cancel{2}} + 4x - \cancel{x^2} - \cancel{2x} - 4$$

$$+ \cancel{2x} + \cancel{2} - \cancel{2}$$

$$y_p = \underline{\underline{\frac{x^3}{3} + 4x - 4}}$$

Q3. $\frac{d^2y}{dx^2} - \frac{3dy}{dx} + 2y = x^2+x$

$(D^2-3D+2) = 0$

$(D-1)(D-2) = 0$

$D = 1, 2$

$y_c = c_1e^x + c_2e^{2x}$

$y_p = \frac{x^2+x}{(D^2-3D+2)} = \frac{x^2+x}{2(1-\frac{3}{2}D+\frac{D^2}{2})} =$ [illegible] $\left(\frac{3}{2}D - \frac{D^2}{2}\right)\Big]^{-1}(x^2+x)$

$= \frac{1}{2}\left[1+\left(\frac{3}{2}D-\frac{D^2}{2}\right)+\left(\frac{3}{2}D-\frac{D^2}{2}\right)^2\right](x^2+x)$

$= \frac{1}{2}\left[1+\frac{3}{2}D-\frac{D^2}{2}+\frac{9}{4}D^2\right](x^2+x) = \frac{1}{2}\left[x^2+x+\frac{3}{2}(2x+1)-\frac{1}{2}\times 2+\frac{9}{\cancel{4}_2}\times\cancel{2}\right]$

$= \frac{1}{2}\left[x^2+4x+5\right]$

Q4. $\frac{d^2y}{dx^2} = 3x-2$ satisfying $y(0) = 2$ and $y'(1) = -3$

Ans $\frac{dy}{dx} = \frac{3x^2}{2} - 2x + C_1$

$y = \frac{3x^3}{2\times 3} - \frac{\cancel{2}x^2}{\cancel{2}} + C_1x + C$

$y'(1) = -3$

$-3 = \frac{3}{2} - 2 + C_1$

$C_1 = 5/2$

$y(0) = 2$

ie $2 = \frac{0}{2} + 0 + C$

$C = 2.$

$\therefore\ y = \frac{x^3}{2} - x^2 + \frac{5}{2}x + 2$

Case (iv)

Consider the equations $f(D)y = e^{ax}.V$ where V is also a function of x. It may be $\sin(bx+c)$ or $\cos(ax+b)$ or x^m etc. . .

$f(D)y = e^{ax}.V$

$y_p = \frac{e^{ax}.V}{f(D)} = e^{ax}\left[\frac{V}{f(D+a)}\right]$

$$D[e^{ax}.V] = e^{ax}.D(V) + aVe^{ax} = e^{ax}[D+a]V$$

$$D^2[e^{ax}.V] = e^{ax}[D^2V+aDV] + (D+a)Ve^{ax} = e^{ax}[D^2+2aD+a^2]V$$

$$= e^{ax}\,\underline{\underline{(D+a)^2V}}$$

Q1. $(D^2-5D+6)y = e^{2x}.x^3$

$(D-2)(D-3)=0$

$D = 2,3$

$\therefore y_2 = c_1e^{2x} + c_2e^{3x}$

$$y_p = \frac{e^{2x}.x^3}{(D-2)(D-3)} = e^{2x}\left[\frac{x^3}{(D+2-2)(D+2-3)}\right] = -e^{2x}\left[\frac{1}{D}\;\frac{x^3}{\cancel{D}(1-D)}\right]$$

$$= -e^{2x}\left[\frac{1}{D}\;(1-D)^{-1}\;x^3\right] = -e^{2x}\left[\frac{1}{D}\left[1+D+D^2+D^3+D^4\right]x^3\right]$$

$$= e^{-2x}\left[\left(\frac{1}{D}+1+D+D^2+D^3\right)x^3\right] = e^{-2x}\left[\frac{x^4}{4}+\underline{\underline{x^3+3x^2+6x}}+6\right]$$

Q2. $y''+4y = 2e^x\sin^2x.$

$(D^2+4)y = e^x(1-\cos 2x) = e^x - e^x\cos 2x$

$D = \pm 2i$

$y_c = C_1\cos 2x + C_2\sin 2x$

$$y_{p_1} = \frac{e^x}{D^2+4} = \frac{e^x}{5}$$

$$y_{p_2} = \frac{e^x\cos 2x}{D^2+4} = e^x \times \frac{\cos 2x}{(D+1)^2+4} = e^x\,\frac{\cos 2x}{D^2+2D+5} = e^x\,\frac{\cos 2x}{\cancel{-4}+2D+\cancel{4}+1}$$

$$= \frac{e^x(2D-1)\cos 2x}{4D^2-1} = \frac{e^x(2D-1)\cos 2x}{-17} = \frac{-e^x}{17}\left[2x - \sin 2x \times 2 - \cos 2x\right]$$

$$= \frac{e^x}{17}\left[\cos 2x + 4\sin 2x\right]$$

$y_p = y_{p_1} \mp y_{p_2}$

$$y = y_c+y_p = C_1\cos 2x + C_2\sin 2x + \frac{e^x}{5} - \frac{e^x}{17}\left[\cos 2x + 4\sin 2x\right]$$

Q3. $(D^2-2D+4)y = e^x \cos x.$

$$D = \frac{2 \pm \sqrt{4-4\times 4}}{2} = 1 \pm 3\sqrt{2}\, i$$

$$y_c = e^x\left[C_1 \cos 3\sqrt{2}\,x + C_2 \sin 3\sqrt{2}\,x\right]$$

$$y_p = \frac{e^x \cos x}{D^2-2D+4} = e^x \frac{\cos x}{(D+1)^2-2D+4-2} = e^x \frac{\cos x}{D^2 \cancel{+2D} \cancel{+2} \cancel{-2D} +4 \cancel{-2}}$$

$$= e^x \frac{\cos x}{D^2+4} = e^x \frac{\cos x}{-1+4} = \underline{\underline{\frac{e^x \cos x}{3}}}$$

Q4. $y'' - 7y' + 6y = e^{2x}(1+x)$

$$(D^2-7D+6) = 0$$

$$(D-6)(D-1) = 0$$

$$D = 6, 1$$

$$y_c = C_1 e^{6x} + C_2 e^x.$$

$$y_p = \frac{e^{2x}(1+x)}{D^2-7D+6} = \frac{e^{2x}(1+x)}{(D-6)(D-1)} = e^{2x} \frac{1+x}{(D-4)(D+1)}$$

$$= e^{2x} \frac{(1+x)}{D^2-3D-4} = -4e^{2x}\left[(D+1)^{-1} \times \left(1-\frac{D}{4}\right)^{-1}\right](1+x)$$

$$= -4e^{2x}\left[(1-D)\left(1+\frac{D}{4}\right)\right](1+x)$$

$$= -4e^{2x}\left[1 - D + \frac{D}{4} - \frac{D^2}{4}\right](1+x) = -4e^{2x}\left[1+x-1\right] = \underline{\underline{-4xe^{2x}}}$$

Q5. $\dfrac{d^2y}{dx^2} - \dfrac{5\,dy}{dx} + 6y = x^3 e^{2x}$

$$(D^2 - 5D + 6)y = 0$$

$$(D-2)(D-3) = 0$$

$$D = \underline{\underline{2, 3}}$$

$$\therefore y_c = C_1 e^{2x} + C_2 e^{3x}$$

$$y_p = \frac{e^{2x} \times x^3}{(D-2)(D-3)} = e^{2x} \frac{x^3}{D\cdot(+D-1)} = -e^{2x} \times \frac{x^3}{D(1-D)}$$

$$= -e^{2x}\left[\frac{1}{D}(1-D)^{-1} x^3\right] = -e^{2x}\left[\frac{1}{D}(1+D+D^2+D^3+D^4)x^3\right]$$

$$= -e^{2x}\left[\left(\frac{1}{D}+1+D+D^2+D^3\right)x^3\right]$$

$$= -e^{2x}\left[\frac{x^4}{4} + x^3 + \underline{\underline{3x^2}} + 6x + 6\right]$$

Q5. $\frac{d^2y}{dx^2} - \frac{2dy}{dx} + y = xe^x\sin x$

$(D^2-2D+1)y = e^x(x\sin x)$

$(D-1)^2y = e^x\ x\sin x$

$D = 1, 1$

$y_c = (c_1+c_2x)e^x$

$y_p = \frac{e^x(x\sin x)}{(D-1)^2} = \frac{e^x\ x\sin x}{(D+1-1)^2} = e^x\left[\frac{1}{D^2}\ x\sin x\right]$

$= e^x\left[\frac{1}{D}\left(x\times -\cos x - \int 1\times -\cos x\right)\right] = e^x\left[\frac{1}{D}(-x\cos x + \sin x)\right]$

$= e^x\left[-(x\sin x - \int\sin x) - \cos x\right] = e^x\left[-x\sin x - \cos x - \cos x\right]$

$= -e^x\left[x\sin x + 2\cos x\right]$

Q6. $(D^4-1)y = \sinh x\cos x$

$(D^2-1)(D^2+1) = \frac{1}{2}\left[e^x\cos x - e^{-x}\cos x\right]$

$D = \pm 1, \pm i$

$\therefore y_c = c_1e^x + c_2e^{-x} + c_3\cos x + c_4\sin x$

$y_{p_1} = \frac{e^x\cos x}{(D^2-1)(D^2+1)} = \frac{e^x\cos x}{[(D+1)^2-1][(D+1)^2+1]} = e^x\frac{\cos x}{(\underset{-1}{\cancel{D^2}}+2D)(\underset{-1}{\cancel{D^2}}+2D+2)}$

$= e^x\frac{\cos x}{(2D-1)(2D+1)} = \frac{e^x\cos x}{(4D^2-1)} = \frac{e^x\cos x}{-5}$

$y_{p_2} = \frac{e^{-x}\cos x}{(D^2-1)(D^2+1)} = e^{-x}\frac{\cos x}{[(D-1)^2-1][(D-1)^2+1]}$

$= \frac{e^{-x}\cos x}{[D^2-2D]\cdot[D^2-2D+2]} = e^{-x}\frac{\cos x}{-(1+2D)(1-2D)}$

$= e^{-x}\frac{\cos x}{4D^2-1} = e^{-x}\frac{\cos x}{-5}$

Method Of Variation Of Parameters

Q1 $(D^2+a^2)y = \tan ax$

Q2. $y''+4y = \sec 2x$

Q3. $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = \frac{e^{3x}}{x^2}$

Q4. $(D^2-1)y = e^{-2x}\sin(e^{-x})$

Q5. $\frac{d^2y}{dx^2} - y = \frac{2}{1+e^x}$

General Description

Consider the eqn $\frac{d^2y}{dx^2} + K_1\frac{dy}{dx} + K_2y = X \longrightarrow ①$

where K_1, K_2 are constants and X is a function of **x**. Now we can have the CF of ① in the form.

$$y_c = c_1y_1 + c_2y_2 \longrightarrow ②$$

where c_1 & c_2 are arbitrary constants on parameters and y_1, y_2 are functions of X.

By the method of variation of parameters it is possible to write the PI of ① as in the form of ② given by $y_p = Ay_1+By_2 \longrightarrow ③$ where

$$A = -\int \frac{Xy_2}{w}\,dx \longrightarrow ④$$

$$B = \int \frac{Xy_1}{w}\,dx \longrightarrow ⑤$$

Comm Imp.

$$w = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \longrightarrow ⑥$$

[Called wronskian's of y_1, y_2]

Therefore the complete solution of ① is given by

$$y = y_c + y_p$$

$$= (c_1+A)y_1 + (c_2+B)y_2$$

Q1. $(D^2+a^2)y = \tan x$

$D = \pm ai$

$$y_c = \underbrace{c_1\cos ax}_{y_1} + \underbrace{c_2\sin ax}_{y_2}$$

$$\omega = \begin{vmatrix} \cos ax & \sin ax \\ -a\sin ax & a\cos ax \end{vmatrix} = a\times 1 = \underline{\underline{a}}$$

$$A = -\int \frac{X\, y_2}{\omega}\, dx = -\int \frac{\tan ax \times \sin ax}{a}\, dx$$

$$= \frac{-1}{a}\int \frac{\sin^2 ax}{\cos ax} = \frac{-1}{a}\int \frac{1-\cos^2 ax}{\cos ax}\, dx$$

$$= \frac{-1}{a}\left[\left(\frac{\log(\sec ax + \tan ax)}{a} \right) - \frac{\sin ax}{a} \right]$$

$$= \frac{-1}{a^2}\underline{\underline{\left[\log(\sec ax + \tan ax) - \sin ax \right]}}$$

$$B = \frac{1}{a}\int \overset{\sin ax}{\cancel{\tan ax}} \times \cancel{\cos ax}\, dx = \frac{1}{a}\int \sin ax\, dx$$

$$= \frac{1}{a} \times \frac{-\cos ax}{a} = \underline{\underline{\frac{-1}{a^2}\cos ax}}$$

$$y_p = \underline{\underline{Ay_1 + By_2}}$$

Q3. $(D^2-6D+9)y = \dfrac{e^{3x}}{x^2}$

$$(D-3)^2 y = \frac{e^{3x}}{x^2}$$

$D = 3, 3$

$$y = (c_1 + c_2 x)e^{3x} = \underbrace{c_1 e^{3x}}_{y_1} + \underbrace{c_2 x e^{3t}}_{y_2}$$

$$W = \begin{vmatrix} e^{3x} & xe^{3x} \\ 3e^{3x} & 3xe^{3x} + e^{3x} \end{vmatrix} = e^{6x}$$

$$A = -\int \frac{e^{3x}}{x^2} \times \frac{xe^{3x}}{e^{6x}}\, dx = -\log x$$

$$B = \int \frac{e^{3x}}{x^2} \times \frac{e^{3x}}{e^{6x}}\, dx = \underline{\underline{\frac{-1\,\cancel{\log x}}{x}}}$$

$$y_p = \underline{\underline{Ay_1 + By_2}}$$

Q4. $(D^2-1)y = e^{-2x}\sin(e^{-x})$

$D^2 - 1 = 0$

$D = \pm 1$

$$y_c = \underset{y_1}{c_1 e^{x}} + \underset{y_2}{c_2 e^{-x}}$$

$$W = \begin{vmatrix} e^{x} & e^{-x} \\ e^{x} & -e^{-x} \end{vmatrix} = -1-1 = \underline{\underline{-2}}$$

$$A = +\int \frac{e^{-2x}\sin(e^{-x})\,e^{-x}}{+2}\,dx$$

$e^{-x} = z$

$dz = -e^{-x}dx.$

$$= -\frac{1}{2}\int z^{2}\sin z\,dz = \frac{-1}{2}\left[z^{2}\times -\cos z + \int 2z\cos z\,dz\right]$$

$$= \frac{-1}{2}\left[z^{2}\times -\cos z + 2\times(z\sin z + \cos z)\right] = \frac{+1}{2}\left[e^{-2x}\cos e^{-x} - 2e^{-x}\sin e^{-x} + 2\cos e^{-x}\right]$$

$$B = \frac{-1}{2}\int e^{-2x}\sin e^{-x}\times e^{x}\,dx = \frac{1}{2}\int \sin t\,dt = \underline{\underline{\frac{-1}{2}\cos(e^{-x})}}$$

Equation Reducible To Linear Eqn with Constant Coeff. Form

Q1. $x^{2}\frac{d^{2}y}{dx^{2}} + x\frac{dy}{dx} - 4y = 0$ satisfying $y(0) = 0$ or $y(1) = 1$

Q2. $x^{2}\frac{d^{2}y}{dx^{2}} - 2y = x^{2} + \frac{1}{x}$

Q3. $(x+1)^{2}\frac{d^{2}y}{dx^{2}} + (x+1)\frac{dy}{dx} + y = 2\sin(\log(x+1))$

Q4. $(2x+3)^{2}\frac{d^{2}y}{dx^{2}} + 6(2x+3)\frac{dy}{dx} + 6y = \log(2x+3)$

Q5. $x^{2}\frac{d^{2}y}{dx^{2}} - 2x\frac{dy}{dx} + 2y = x\log x$

Q6. $x^{2}\frac{d^{2}y}{dx^{2}} + 4x\frac{dy}{dx} + 2y = e^{x}$

Euler Cauchy's Form (1,2,5,6)

Legendre's Form (3,4)

Solution

Q1. The given eqn is Euler's Cauchy's Form. To reduce this equation into constant coeff form we can substitute the following.

$$x = e^{z} \Longrightarrow z = \log z \quad \& \quad D = \frac{d}{dz}$$

then $x\frac{dy}{dx} = Dy$

$$x^{2}\frac{d^{2}y}{dx^{2}} = D(D-1)y$$

$$x^{3}\frac{d^{3}y}{dx^{3}} = D(D-1)(D-2)y$$

$D(D-1)y + Dy - 4y = 0$

$(D^2 - D + D - 4)y = 0$

$(D+2)(D-2)y = 0$

$D = \pm 2$

$y = c_1 e^{-2z} + c_2 e^{+2z}$

$= \frac{c_1}{x^2} + c_2 \times x^2$

$0 = c_1 x + c_2 \times 0 \implies c_1 = 0$

$1 = c_2 \times 1 \quad \therefore c_2 = 1.$

$\therefore \underline{y = x^2}$

Q2. $x^2 \frac{d^2y}{dx^2} - 2y = x^2 + \frac{1}{x}$

$D(D-1)y - 2y = x^2 + 1/x$

$(D^2 - D - 2)y = x^2 + 1/x = e^{2z} + e^{-z}$

$(D-2)(D+1)y = 0$

$D = 2, -1$

$y_c = c_1 e^{2z} + c_2 e^{-z}$

$y_p = \frac{e^{2z} + e^{-z}}{(D-2)(D+1)} = \underline{\frac{z}{3} e^{2z} - \frac{ze^{-z}}{3}}$

$y = y_c + y_p = \left(c_1 x^2 + \frac{c_2}{x}\right) + \underline{\frac{\log x}{3}\left(x^2 - \frac{1}{x}\right)}$

Q3. The given eqn is in Legender eqn form. To reduce this eqn to constant coeff. form we can substitute the following.

Let $(ax+b) = e^z$

$z = \log(ax+b)$ and $D = d/dz$

$(ax+b)\frac{dy}{dx} = aDy$

$(ax+b)^2 \frac{d^2y}{dx^2} = a^2 D(D-1)y$

$(ax+b)^3 \frac{d^3y}{dx^2} = a^3 D(D-1)(D-2)y$

$$(x+1)^2 \frac{d^2y}{dx^2} + (x+1)\frac{dy}{dx} + y = 2\sin(\log(x+1))$$

$$D(D-1)y + Dy + y = 2\sin z$$

$$(D^2 - D + D + 1)\,y = 2\sin z$$

$$(D^2+1)\,y = 2\sin z$$

$$D = \pm i$$

$$y_c = C_1\cos z + C_2\sin z$$

$$= C_1\cos(\log(x+1)) + C_2\sin(\log(x+1))$$

$$y_p = \frac{2\sin z}{D^2+1} = -\frac{2z\cos z}{2} = \underline{\underline{-\log(x+1)\cos(\log(x+1))}}$$

Q4. $2x+3 = e^z \Longrightarrow z = \log(2x+3)$

$$4D(D-1)y + 6\times 2Dy + 6y = z$$

$$\cancel{2}(2D^2 - \cancel{2D} + \overset{4D}{\cancel{6D}} + 3)\,y = \frac{z}{2}$$

$$(2D^2 + 4D + 3)\,y = \frac{z}{2}$$

Q5. $D(D-1)y - Dy + 2y = e^z \times z$

$$(D^2 - D - D + 2)\,y = e^z \times z$$

$$(D^2 - 2D + 2)\,y = e^z \times z.$$

Q6. $D(D-1)y + 4Dy + 2y = e^{e^z}$

$$(D^2 - D + 4D + 2)\,y = e^{e^z}$$

$$(D^2 + 3D + 2)\,y = e^{e^z}$$

NUMERICAL METHODS

Topics

1) Solutions to Algebraic and Transcandantal Eqns.
   - Bisection
   - Regula False
   - Secent.
   - ***** – Newton Raphson.
2) Solution to Integration Of Fn.
   - Trapezoidal Rule
   - Simpson's 1/3 rule.
   - Simpson's 3/8 rule.
3) Solution to S/m of Linear Eqns.
   - Gauss Elimination
   - LU Decomposition
4) Solutions to Differential Eqn.
   - Eulers Method.
   - Runge Kutta Method.

Gate 2005

Q. Which of the following method is useful for solving Algebraic eqn.

a) Coloumb's Method
b) Eulers Method.
c) Simpson's Rule
[✓ d) Newton Raphson.]

Q. Match the following

① Simpson Rule (B)
② Eulers Method. (C)
③ Newton Raphson method (D)
④ Gauss Elimination (F)

Ⓐ Eigen values
Ⓑ Integration
Ⓒ Differential Eqn.
Ⓓ Algebraic Eqn
Ⓔ Interpolation
Ⓕ S/m of eqns.

Mathematical Methods are of Two types.

① Analytical method

② Numerical method.

Analytical Method

eg. ① Find roots of $x^2 - 5x + 6 = 0$

Analytical solution is $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \underline{\underline{3, 2}}$

② Find $\int_1^2 x\,dx$. $\left.\frac{x^2}{2}\right]_1^2 = \frac{4}{2} - \frac{1}{2} = \underline{\underline{3/2}}$

③ Solve $\frac{dy}{dx} = x$

$dy = x\,dx$

$\int ~~1~~dy = \int x\,dx$

$y = \underline{\underline{\frac{x^2}{2} + C}}$

Drawbacks of Analytic Method.

- Not applicable for higher degree non linear equation.
- Not gives stepwise analysis

Inorder to overcome this we use help of numerical methods.

Solutions to Algebraic & Transcandal Eqns

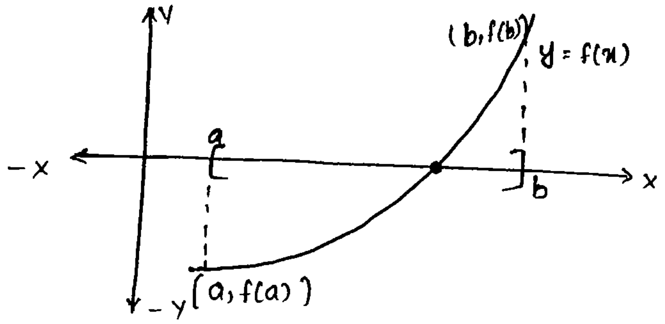
An eqn which involves trignometric fns is called Transcandal eqns

eg: $f(x) = x - \cos x = 0$

Intermediate Mean Value Theorem (I.M.T)

Let $f(x)$ is continuous fn defined in the interval $[a, b]$.

$f(a)$ and $f(b)$ having opposite signs (say $f(a) < 0$, $f(b) > 0$), then there exists atleast one root $f(x)=0$ in the interval $[a,b]$.



eg : $f(x) = x^3 - 4x - 9 = 0 \quad [2,3]$.

$f(2) = 2^3 - 4\times2 - 9 = -9 < 0$

$f(3) = 3^3 - 4(3) - 9 = 6 > 0$

Since $f(2)<0$ and $f(3)>0$. So atleast one root of $x^3-4x-9=0$ is in $[2,3]$

BISECTION METHOD

1) Let $f(x)$ is continuous fn defined in $[a,b]$.

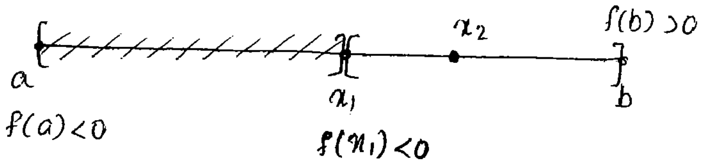
2) Let $f(a)<0$ and $f(b)>0$.

Using IMT there exists atleast one root of $f(x)=0$ in $[a,b]$.

3) Let x_1 is first approximation root of $f(x)=0$. and $x_1 = \frac{a+b}{2}$

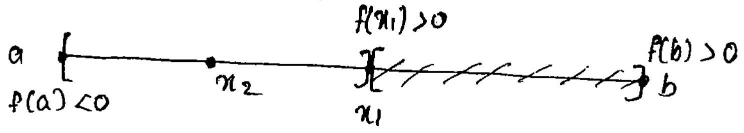
Case 1 : If $f(x_1)=0$ then x_1 is root, stop the process.

Case 2 : If $f(x_1)<0$ then compute x_2 using $x_2 = \frac{x_1+b}{2}$



Case 3 : If $f(x_1)>0$ then compute x_2 using

$$x_2 = \frac{a+x_1}{2}$$



Continue above process until desired accuracy of root is found.

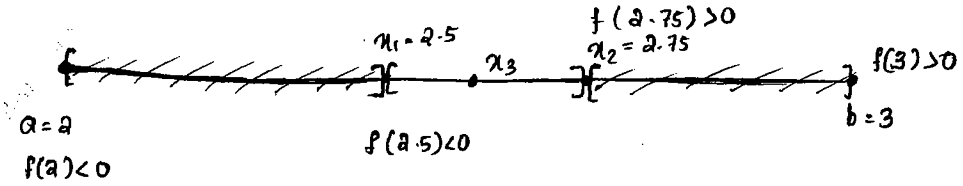
Q, Find x_2 and x_3 using Bisection, where $f(x) = x^3 - 4x - 9 = 0$ in $[2,3]$.

Ans $f(2) = -9 < 0$

$f(3) = 6 > 0$

Let $x_1 = \frac{a+b}{2} = \frac{2+3}{2} = 2.5$

$f(x_1) = f(2.5) = (2.5)^3 - 4 \times 2.5 - 9 = \underline{\underline{-3.375 < 0}}$



Now $x_2 = \frac{x_1 + b}{2} = \frac{2.5+3}{2} = \underline{\underline{2.75}}$

$f(x_2) = f(2.75) = (2.75)^3 - 4(2.75) - 9$

$= 0.7969 > 0$

$\therefore x_3 = \frac{x_1 + x_2}{2} = \frac{2.5 + 2.75}{2}$

$x_3 = \underline{\underline{2.62}}$

$\underline{\underline{x_2 = 2.75 \ \& \ x_3 = 2.62}}$

Newton Raphson

Let $f(x)$ is continuous fn. Newton Raphson Iteration Formula for finding root of the eqn $f(x) = 0$ is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Note: The difference b/w x_n & x_{n+1} is very small quantity which is negligible.

Gate '95

Q. Find N-R Iteration formula for square root of c where $c > 0$.

Ans Let $x = \sqrt{c}$

Squaring Both sides

$$x^2 = c$$

$$f(x) = x^2 - c = 0$$

$$f'(x) = 2x$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^2 - c}{2x_n}$$

$$x_{n+1} = \frac{x_n^2 + c}{2x_n}$$

N-R method is Quadratic convergence. So $f''(x)$ must exist.

In N-R method keep constant term free from the calculator.

Gate '09

Q Find N-R iteration formula for $f(x) = x^2 - 117 = 0$.

$$f(x) = x^2 - 117 = 0$$

$$f'(x) = 2x$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^2 - 117}{2x_n}$$

$$x_{n+1} = \frac{x_n^2 + 117}{2x_n}$$

Gate 2010

Q. $f(x) = x^2 - 2 = 0$ $x_0 = 3.5$.

Find x_1 using N-R method.

Ans

$$x_{n+1} = \frac{x_n^2 + 2}{2x_n}$$

$$x_1 = \frac{x_0^2 + 2}{2x_0} = \frac{3.5^2 + 2}{2 \times 3.5} = 2.035$$

Gate'96

Q. Find N-R iteration formula for $\sqrt[3]{c}$ where $c > 0$

Ans Let $x = \sqrt[3]{c}$

$$x^3 = c$$

$$f(x) = x^3 - c = 0$$

$$f'(x) = 3x^2$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \frac{x_n^3 - c}{3x_n^2} = \underline{\underline{\frac{2x_n^3 + c}{3x_n^2}}}$$

$$\boxed{\begin{array}{lll} \sqrt{c} \Longrightarrow & x_{n+1} = & \dfrac{1x_n^2 + c}{2x_n} \\ \sqrt[3]{c} \Longrightarrow & x_{n+1} = & \dfrac{2x_n^3 + c}{3x_n^2} \\ \sqrt[k]{c} \Longrightarrow & x_{n+1} = & \dfrac{(k-1)x_n^k + c}{k\, x_n^{k-1}} \end{array}}$$

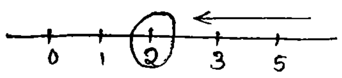
Gate'97

Q. $f(x) = x^2 - 2 = 0$, $x_0 = -1$ then N-R iteration formula will be

a) converges to -1 [✓ b) Conv. to $-\sqrt{2}$]

c) converges to $\sqrt{2}$ d) Not convergent.

Ans Assume $x = 2$ is exact root of $f(x) = 0$.



Iteration	Method 1	Method 2	Method 3
1	3	6	6
2	5	4	5
3	7	2.01	4
4	10		3
5	12		2.01
6			

$$x_{n+1} = \frac{x_n^2 + 2}{2x_n}, \quad x_0 = -1$$

$$x_1 = \frac{x_0^2 + 2}{2x_n} = \frac{(-1)^2 + 2}{2 \times (-1)} = \underline{\underline{-1.5}}$$

$$x_2 = \frac{x_1^2 + 2}{2x_1} = \frac{(-1.5)^2 + 2}{2 \times -1.5} = -1.416$$

$$x_3 = \frac{x_2^2 + 2}{2x_2} = -1.414$$

$$= \underline{\underline{-\sqrt{2}}}$$

** If $x_0 = 1$ then it converges to $\sqrt{2}$

If x_0 is not provided then it is converging to $\pm\sqrt{2}$.

Rate Of Convergence

The fastness of convergence to the root is called rate of convergence. In the above table rate of convergence of method 2 is higher than method 3. In method 1 when iterations are increasing it is moving away from the root. So method 1 is said to be not converging to the root.

Gate '08

Q. N-R iteration formula $x_{n+1} = \frac{1}{2}\left(x_n + \frac{R}{x_n}\right)$ evaluates.

a) log R b) $1/R$ c) R^2 d) $\sqrt{R}$

Ans $x_{n+1} = \frac{x_n^2+R}{2x_n} \xrightarrow{\text{By comparing}} \sqrt{R}$ (already done)

OR

Let $f(x) = 0 \rightarrow$ Root

$$x_{n+1} = x_n = x$$

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{R}{x_n}\right)$$

$$x = \frac{1}{2}\left(x + \frac{R}{x}\right)$$

$$x - \frac{x}{2} = \frac{R}{2x}$$

$$\frac{x}{\cancel{2}} = \frac{R}{\cancel{2}x} \implies x^2 = R \implies x = \pm\sqrt{R}$$

Gate 2008

Q. If $x_{n+1} = \frac{x_n}{2} + \frac{9}{8x_n}$ and $x_0 = 0.5$ then N-R iteration formula will be

A) -1.5 B) 1.5 c) 3 d) 3.5.

Ans $x_{n+1} = \frac{x_n}{2} + \frac{9}{8x_n}$

$$x_{n+1} = x_n = x$$

$$x = \frac{x}{2} + \frac{9}{8x}$$

$$\frac{x}{\cancel{2}} = \frac{9}{\underset{4}{\cancel{8}}x}$$

$x^2 = 9/4$ $x = \pm 3/2$ $\therefore x = +1.5,\ x = -1.5$

Since $x_0 = .5$ $x = 1.5$

Gate 2005

Q. Find N-R iteration formula for reciprocal of 'a' where $a > 0$.

Ans Let $x = \frac{1}{a}$

$$\Longrightarrow \frac{1}{x} = a$$

$$f(x) = \frac{1}{x} - a = 0$$

$$f'(x) = -1/x^2$$

$$x_{n+1} = x_n - \frac{\left(\frac{1}{x_n} - a\right)}{\left(\frac{-1}{x_n^2}\right)} = x_n + x_n^2\left(\frac{1}{x_n} - a\right) = \underline{\underline{2x_n - a x_n^2}}$$

$$\boxed{x_{n+1} = 2x_n - a x_n^2}$$

Gate '05

Q. Find x_1 & x_2 using N-R iteration formula for reciprocal of a. where $a = 7$, $x_0 = 0.2$.

Ans $x_{n+1} = 2x_n - a x_n^2$

Put $x_n = 0$

$n=0$ $x_1 = 2x_0 - a x_0^2 = 2 \times 0.2 - 7(0.2)^2 = \underline{\underline{0.12}}$

$n=1$ $x_2 = 2x_1 - a x_1^2 = 2(0.12) - 7(0.12)^2 = \underline{\underline{0.1392}}$

Gate '99

N-R iteration formula for finding roots of $f(x) = 0$ converging to the root.

Ⓐ If $f(x)$ is polynomial. Ⓑ If $f'(x_0) < 0$

Ⓒ Converges always to the root [✓ Ⓓ None of these.]

- Theorem :- Let $\phi(x_n)$ is iteration formula for finding roots of $f(x) = 0$ in the interval I.

$\phi(x_n)$ is said to be converging to the root

if $|\phi'(x_n)| < 1$ for all $x_n \in I$

Ans $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

$$\text{Let } \phi(x_n) = x_n - \frac{f(x_n)}{f'(x_n)} \quad \sim\sim\sim \circledast$$

$$\phi'(x_n) < 1$$

$$\Longrightarrow \left| 1 - \frac{f'(x_n)^2 - f(x_n) f''(x_n)}{(f'(x_n))^2} \right| < 1$$

$$\Longrightarrow \boxed{|f(x_n) f''(x_n)| < |(f'(x_n))^2|}$$

Practice Questions

G 1995
$f(x) = x - \cos x$, $x_{n+1} = x_n - \underline{\underline{\frac{x_n - \cos x_n}{1 + \sin x_n}}}$

G 2005
$f(x) = x^3 + 3x - 7 = 0$, $x_0 = 1$ then $x_1 = \underline{\underline{1.5}}$

G 2005
$f(x) = xe^x - 2 = 0$, $x_0 = 0.8679$ then $x_1 = \underline{\underline{0.853}}$

G 2007
$f(x) = x^3 + 4x - 9 = 0$, $x_{n+1} = \underline{\underline{\frac{2x_n^3 + 9}{3x_n^2 + 4}}}$

G 2007
$f(x) = x^3 - x^2 + 4x - 4 = 0$, $x_0 = 2$, then $x_1 = \underline{\underline{4/3}}$

G 2008
$f(x) = x - e^{-x} = 0$ then $x_{n+1} = \underline{\underline{\frac{e^{-x_n}(1 + x_n)}{(1 + e^{-x_n})}}}$

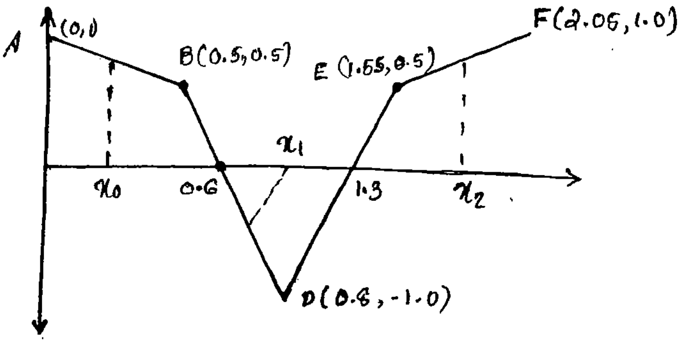
G 2011
$f(x) = x + \sqrt{x} - 3 = 0$ then $x_1 = \underline{\underline{1.69}}$ and $\underline{\underline{x_0 = 2}}$

G 2006
$x_{n+1} = \frac{2}{3} x_n + \frac{5}{3} x_n^{-2}$ then N-R converges to.

[✓ Ⓐ] 1.7099 Ⓑ 2.2361 Ⓒ 3.1251.

Gate '03
If we use N-R method to find roots of the eqn $f(x) = 0$ using x_0, x_1 & x_2 resp. as initial guess values the roots obtained will be

Ⓐ 1.3, 0.6, 0.6 resp.
Ⓑ 0.6, 0.6, 1.3 "
Ⓒ 1.3, 1.3, 0.6 "
[✓ Ⓓ] 1.3, 0.6, 1.3 "



Eqn of $\overline{AB}$, $y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$

$$y - 1 = \frac{0.5 - 1}{0.5 - 0}(x - 0)$$

$$y - 1 = -1(x)$$

$$x + y = 1$$

Put $y = 0$ we get $x = 1$

$\therefore x_0 = 1$ which is nearer to 1.3

$$\therefore \underline{\underline{x_0 = 1.3}}$$

Eqn for $\overline{EF}$, $y - 0.5 = \frac{1 - 0.5}{2.05 - 1.55}(x - 1.55)$

$$y - 0.5 = \frac{\cancel{0.5}}{\cancel{0.5}}(x - 1.55)$$

$$x - y = 1.05$$

Put $y = 0$, $x = 1.05$

Since $x = 1.05$ nearer to 1.3

$$\therefore \underline{\underline{x_2 = 1.3}}$$

REGULA FALSE METHOD

i) Let $f(x)$ is continuous fn. x_0, x_1 are initial guess values such that $f(x_0)$ and $f(x_1)$ having opposite signs ii) (say $f(x_0) < 0$ and $f(x_1) > 0$)

iii) Regula False iteration formula for finding roots of $f(x) = 0$ is

$$x_{n+1} = \frac{f_n x_{n-1} - f_{n-1} x_n}{f_n - f_{n-1}}$$

In particular for $n = 1$

$$x_2 = \frac{f_1 x_0 - f_0 x_1}{f_1 - f_0} \quad \text{------} (*)$$

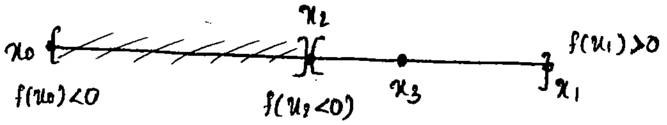
Case (i)

i) If $f(x_2)=0$ then $x_2 \longrightarrow$ root, stop the process.

Case(ii)

If $f(x_2)<0$ then compute x_3 by replacing x_0 by x_2 and f_0 by f_2 in ⊛

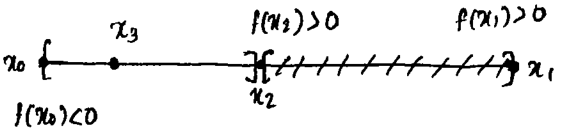
$$ⓧ \implies x_3 = \frac{f_1 x_2 - f_2 x_1}{f_1 - f_2}$$



Case(iii)

If $f(x_1)>0$ compute x_3 by replacing x_1 by x_2 and f_1 by f_2 in ⊛.

$$\text{Then ⊛} \implies x_3 = \frac{f_2 x_0 - f_0 x_2}{f_2 - f_0}$$



Continue above process until desired accuracy of root is found.

Q. Find x_2 & x_3 using Regula False. $f(x) = x^3 + x - 1 = 0$, $x_0 = 0.5$, $x_1 = 1$

Ans $f_0 = f(0.5) = -0.375 < 0$

$f_1 = f(1) = 1 > 0$

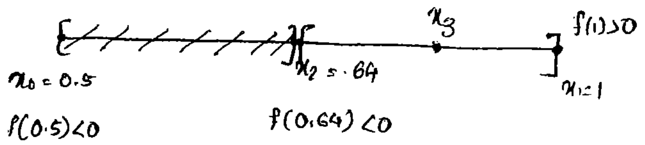
$$x_2 = \frac{f_1 x_0 - f_0 x_1}{f_1 - f_0} \rightsquigarrow ⊛$$

$$= \frac{1 \times (0.5) - (-0.375)(1)}{1 - (-0.375)}$$

$x_2 = 0.64$

$f_2 = f(x_2) = f(0.64) = -0.0979 < 0$

$$⊛ \implies x_3 = \frac{f_1 x_2 - f_2 x_1}{f_1 - f_2}$$



$$= \frac{1 \times 0.64 - (.0979)(1)}{1 - (-0.0979)}$$

$$x_3 = \underline{\underline{0.672}}$$

SECENT METHOD

Secent Method is similar to Regula - Falsi except that in Secent method initial values x_0, x_1 neednot satisfy the condition $f(x_0)f(x_1)<0$ ie these two neednot have opposite signs.

Secent method doesnot provide guarantee for existancy of root in the interval $[x_0, x_1]$. So it is unreliable.

Secent method iteration formula for finding roots of the eqn of $f(x) = 0$, is

$$x_n = \frac{f_n x_{n-1} - f_{n-1} x_n}{f_n - f_{n-1}}$$

In particular for $n=1$ $x_2 = \dfrac{f_1 x_0 - f_0 x_1}{f_1 - f_0} \rightsquigarrow \circledast$

In Secent Method to compute x_3 replace x_0 by x_1 and x_1 by x_2 in $\circledast$. Therefore

$$\circledast \Longrightarrow x_3 = \frac{f_2 x_1 - f_1 x_2}{f_2 - f_1}$$

Q. Find x_2 & x_3 using Secent method. $f(x) = x^3 - 2x - 5 = 0$, $x_0 = 2$, $x_1 = 3$.

Ans $f_0 = f(2) = -1$

$f_1 = f(x_1) = f(3) = 16$.

$$x_2 = \frac{f_1 x_0 - f_0 x_1}{f_1 - f_0} = \frac{16 \times 2 + 1 \times 3}{16 + 1} = 2.058$$

$f_2 = f(x_2) = f(2.058) = -0.3907$

$$x_3 = \frac{f_2 x_1 - f_1 x_2}{f_2 - f_1} = \frac{-0.3907 \times 3 - 16 \times 2.058}{-0.3907 - 16} = 2.0812$$

Method	Order Of Convergence		
① Bisection	$\mathcal{E}_{n+1} = K\mathcal{E}_n^1$	linear ,	O(1)
② Regula Falsi	$\mathcal{E}_{n+1} = K\mathcal{E}_n^1$	linear ,	O(1)
③ Secant	$\mathcal{E}_{n+1} = K\mathcal{E}_n^{1.62}$	Superlinear	O,(1.62)
④ Newton Raphson	$\mathcal{E}_{n+1} = K \cdot \mathcal{E}_n^2$	Quadratic	O,(2)

Order Of Convergence

A method is said to be convergence of order 'p' if $\underline{\mathcal{E}_{n+1} = K\mathcal{E}_n^p}$ where K is const.

Eg : f(x) = 0 and x = 2 is exact root of f(x) = 0.

Method 1

$$\left\{\begin{array}{ll} x_n = 2.004 & ; \mathcal{E}_n = -0.004 \\ x_{n+1} = 2.002 & ; \mathcal{E}_{n+1} = -0.002 \end{array}\right\} \mathcal{E}_{n+1} = \frac{1}{2}\mathcal{E}_n^{①} \rightarrow \text{linear}$$

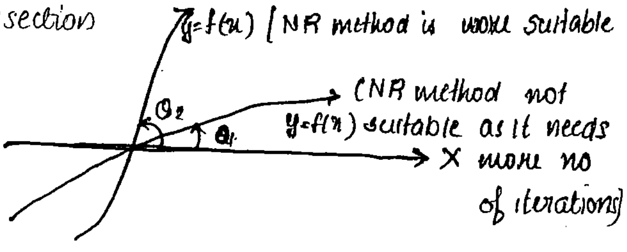
Method 2

$$\left\{\begin{array}{ll} x_n = 2.004 & ; \mathcal{E}_n = -0.004 \\ x_{n+1} = 2.000016 & ; \mathcal{E}_{n+1} = -0.000016 \end{array}\right\} \mathcal{E}_{n+1} = K\mathcal{E}_n^{②} \rightarrow \text{Quadratic}$$

where K = 1

Note

- N-R method is better than remaining methods but it is applicable only for the curves which are having large slope values i.e where the graph is crossing x axis if it is nearly vertical N-R method require few iterations to reach the required root. Otherwise apply any one of the remaining 3 methods.



- Secant method is better than Regula Falsi & Bisection But in secant method, there is a possibility that iteration formula will be invalid.

Eg : $f(x) = x^2$

$x_0 = -1 \rightarrow f_0 = f(-1) = (-1)^2 = 1$

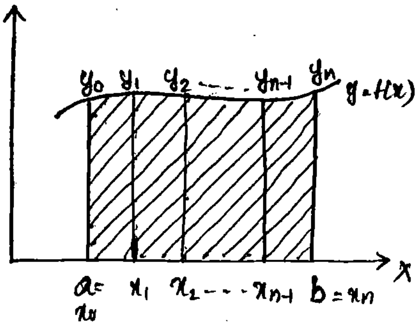
$x_0 = 1 \rightarrow f_1 = f(1) = 1^2 = 1$

$x_2 = \frac{f_1 x_0 - f_0 x_1}{f_1 - f_0}$ ∵ $f_1 = f_0$ so above formula is invalid. So Secant method is unreliable.

II. Solutions To Integration Of a Function

The area bounded by f(x) and x axis b/w the limits a & b is denoted by

$$\int_a^b f(x)\,dx \quad \text{———} \circledast$$



Divide the interval (a, b) into n equal subintervals where length of each interval is h (step size).

ie $[a, b] = \{a = x_0, x_1, x_2, ----, x_n = b\}$

where $a = x_0$

$x_1 = x_0 + h$

$x_2 = x_1 + h = x_0 + 2h$

$\vdots$

$x_n = x_0 + nh$

ie $b = a + nh$

$$\boxed{n = \frac{b-a}{h}}$$

(or) $$\boxed{h = \frac{b-a}{n}}$$

Eqn ⊛ can be evaluated by using.

① Trapezoidal Rule

$$\int_a^b f(x)\,dx = \frac{h}{2}\left[(y_0 + y_n) + 2(y_1 + y_2 + \cdots + y_{n-1})\right]$$

② Simpson's 1/3rd Rule

$$\int_a^b f(x)\,dx = \frac{h}{3}\left[(y_0 + y_n) + 2(y_2 + y_4 + \cdots + y_{n-2}) + 4(y_1 + y_3 + y_5 + \cdots + y_{n-1})\right]$$

③ Simpson's 3/8 Rule

$$\int_a^b f(x)\,dx = \frac{3h}{8}\left[(y_0 + y_n) + 2(y_3 + y_6 + \cdots + y_{n-3}) + 3(y_1 + y_2 + y_4 + y_5 \cdots + y_{n-1})\right]$$

Model No: I

Gate 2010

Q. Evaluate $\int_0^1 f(x)dx$ using Simpson's Rule.

	x_0	x_1	x_2	x_3	x_4
X	0	0.25	0.5	0.75	1
f(x)	1	.9412	0.8	0.64	0.5
	y_0	y_1	y_2	y_3	y_4

Ans By default Simpson's 1/3rd rule is called Simpson's rule.

$h = .25$

$$S\ 1/3 \text{ Rule} = \frac{h}{3}\left[(y_0+y_4) + 2y_2 + 4(y_1+y_3)\right]$$

$$= \frac{0.25}{3}\left[(1+0.5) + 2(0.8) + 4(0.9412+0.64)\right]$$

$$= \underline{0.7854}$$

Model II

GATE 2011

Q. The integral $\int_1^3 \frac{1}{x}dx$. Evaluate using Simpsons Rule on two equal intervals with length of each interval is 1.

Ans

x	y_x	
$x_0 = 1$	1	y_0
$x_1 = 2$	1/2	y_1
$x_3 = 3$	1/3	y_2

$$S\ 1/3 \text{ Rule} = \frac{1}{3}\left[(y_0+y_2) + 4y_1\right]$$

$$= \frac{1}{3}\left[1+\frac{1}{3} + 4\times\frac{1}{2}\right] = \underline{10/9}$$

Model III

Gate 2007

Q. The integral $\int_0^{2\pi} \sin x\,dx$ is evaluated using T. Rule on 8 equal intervals with '5' significant digits.

Ans

$$h = \frac{b-a}{n} \quad \frac{2\pi-0}{8} = \pi/4$$

x	$\sin x$
$x_0 = 0$	$y_0 = 0$
$x_1 = \pi/4$	$y_1 = .70710$
$x_2 = \pi/2$	$y_2 = 1$
$x_3 = 8\pi/4$	$y_3 = .70710$
$x_4 = \pi$	$y_4 = 0$
$x_5 = 5\pi/4$	$y_5 = -.70710$
$x_6 = 6\pi/4$	$y_6 = -1$
$x_7 = 7\pi/4$	$y_7 = -0.70710$
$x_8 = 2\pi$	$y_8 = 0$

$$T.\text{Rule} = \frac{h}{2}\left[(y_0+y_8) + 2(y_1+y_2+y_3+y_4+y_5+y_6+y_7)\right]$$

$$= \frac{\frac{\pi}{4}}{2}\left[0+0+2(.7071+1+.7071+0+-.7071 + -1 - 0.7071+0\right]$$

5 significant digits are req. $= \underline{\underline{0.00000}}$

Gate 2010

Q.

	x_0	x_1	x_2	x_3	x_4	x_5	x_6
x	0	60	120	180	240	300	360
$f(x)$	0	1068	−323	0	323	−355	0

. Evaluate $\int_0^{2\pi} f(x)dx$ using simpson's Rule.

Ans $h = 60 = \pi/3$ $n = 6.$

$$\int_0^{2\pi} f(x)dx = \frac{\overset{\pi/3}{\cancel{60}}}{3}\left[(0+0)+2(-323+323)+4(1068-355)\right]$$

$$= \frac{\pi\,\cancel{30}}{3\times 3}\left[0+0+4(1068-355)\right] = \underline{\underline{995}}$$

Since $f(x)$ is evaluated in the interval $(0, 2\pi)$ take $h = \pi/3$ instead of 60

Model No IV

Gate 2006

Q. The second degree polynomial $f(x)$ takes following values. If $\int_0^2 f(x)dx$ is evaluated using T Rule then what is the error estimation?

x	0	1	2
$f(x)$	1	4	15

Ⓐ 4/3 Ⓑ −2 Ⓒ −4/3 Ⓓ 0.

Ans Error = Exact value − Approx. value.

T Rule $\longrightarrow$ Approx value $= \frac{h}{2}\left[(y_0+y_2)+2y_1\right]$

$$= \frac{1}{2}\left[(1+15)+2\times 4\right] = \underline{\underline{12}}$$

Exact solution $= \int_0^2 f(x)dx$

Let $f(x) = a_0 + a_1 x + a_2 x^2$ A

$f(0) = 1 \Longrightarrow a_0 = 1$

$f(1) = 4 \Longrightarrow a_0 + a_1 + a_2 = 4$

$a_1 + a_2 = 3$

$f(2) = 15 \Longrightarrow a_0 + 2a_1 + 4a_2 = 15$

$2a_1 + 4a_2 = 14$

$a_1 + 2a_2 = 7$

$a_1 = -1 \quad , a_2 = 4$

$$\int_0^2 f(x)dx = \int_0^2 1 - x + 4x^2 \, dx = x - \frac{x^2}{2} + \frac{4x^3}{3} \Big]_0^2 = \underline{\underline{32/3}}$$

$$\text{Error} = \frac{32}{3} - 12 = \underline{\underline{-4/3}}$$

① Error = Exact − Approximation value.

② Absolute Error = | Error |

③ Relative Error = $\dfrac{|\text{ Error }|}{|\text{ Exact value }|}$

Suppose error $= 10^{-8}$ and actual value $= 10^{-9}$. Then relative error

$$= \frac{|\text{ Error }|}{|\text{ Exact value }|} = 10$$ which is not negligible.

ie before neglecting any error consider the actual magnitude and if the relati error is small, neglect the error.

Truncation Error

Let $f(x)$ is a fn defined in the interval x_0, x where $(x - x_0) = h$

Expand $f(x)$ about x_0 using Taylor series expansion.

$$\text{ie } f(x) = f(x_0) + (x - x_0) f'(x_0) + (x-x_0)^2 f''(x_0) + \dots + \frac{(x-x_0)^n}{n!} f^n(x_0) + R$$

$$\text{where } R = \frac{(x - x_0)^{n+1}}{(n+1)!} f^{n+1}(\phi)$$

$$x_0 \leq \phi \leq x$$

R is called Truncation Error.

Truncation error bound denoted by $|R|$ and $|R| \leq \left| \max \left(\frac{(x-x_0)^{n+1}}{(n+1)!} f^{n+1}(\phi) \right) \right|_{[x_0, x]}$

Q. If we approximate $e^x = 1 + x + \frac{x^2}{2!}$ then find (i) T·E (ii) |T.E| in [2,3]

Ans
$$e^x = \underset{\uparrow}{1} + \underset{\nwarrow}{x} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$$f(x) = f(x_0) + (x - x_0) f'(x_0) + (x - x_0)^2 f''(x_0) + \cdots$$

Comparing

$f(x) = e^x$

$x_0 = 0$

$x = x$

$f(x_0) = e^0 = 1$

$f'(x_0) = e^0 = 1$

$$R = \frac{(x - x_0)^{n+1}}{(n+1)!} f^{n+1}(\phi) \qquad x_0 \le \phi \le x$$

$$R = \frac{(x - 0)^3}{3!} f'''(\phi)$$

$$|TE| = R = \frac{x^3}{3!} e^\phi \qquad x_0 \le \phi \le x$$

ii) $|TE| \le \left|\max\left(\frac{x^3}{3!} e^\phi\right)\right|_{[2,3]} \le \frac{3^3}{3!} e^3$

$$|TE| \le \underline{\underline{\frac{27}{6}}} e^3$$

Note: Truncation Error in

ⓐ Trapezoidal Rule $= \frac{-(b-a)}{12} h^2 f''(x)$

ⓑ Simpson's 1/3 Rule $= \frac{-(b-a)}{180} h^4 \times f^{IV}(x)$

ⓒ Simpson's 3/8 Rule $= \frac{-3}{80n}(b-a) h^4 \times f^{IV}(x)$ $\quad$ $h^4 \longrightarrow$ 4th order derivative

Q. The T Rule evaluates the polynomial with exact results if the polynomials are having degree Ⓐ 0 Ⓑ only 1 ⓒ 0 (or) 1 ⓓ 2.

Ans Trapezoidal rule evaluates the polynomial with exact results if they are having upto degree 1 (0 or 1)

In Simpson's rule we truncated 4th order derivative. It evaluates the polynomials with exact results if they are having degree upto 3 (0, 1, 2 or 3).

- T Rules is applicable on any number of intervals.
- Simpson's 1/3 Rule is applicable only if no of intervals are multiples of 2 ($n = 2$ (or) 4 (or) 6 . . .)
- Simpson's 3/8 Rule is applicable if the no. of intervals are multiples of 3 ($n = 3$ (or) 6 (or) 9 . . .)

Error order in Trapezoidal Rule is order of h^2

" " " Simpson 1/3 " " " " h^4

" " " " 3/8 " " " " h^5

(since Truncation error $= \frac{-3(b-a)}{80\,n} h^4 f''''(x)$

$= \frac{-3}{80} \times h \times h^4 \; f^{IV}(x)$

$= \frac{-3}{80} \times h^5 \times f^{IV}(x)$

$= \underline{O(h^5)}$

Gate 2008

Q. Minimum no. of equivalent subintervals needed to approximate $\int_1^2 x e^x dx$ to an accuracy atleast $1/3 \times 10^{-6}$ using Trapezoidal rule

Ⓐ 100e Ⓑ 1000e Ⓒ 1000 Ⓓ 100.

Ans $n = ?$

$f(x) = e^x$

$a = 1$, $b = 2$

$f'(x) = e^x + xe^x$

$f''(x) = \underset{\uparrow ing}{2e^x} + \underset{\uparrow ing}{xe^x} = \uparrow ing$

$\text{Max} \left| f''(x) \right|_{[1,2]} = 4e^2$ $(x = 2)$

accuracy $\geq \frac{1}{3} \times 10^{-6}$

$|\text{Trunc. error}| \leq \frac{1}{3} \times 10^{-6}$

T. Rule

$$\left| \frac{b-a}{12} \times h^2 \times \max f''(x) \right| \leq \frac{1}{3} \times 10^{-6}$$

$$\frac{2-1}{12} \times h^2 \times 4e^2 \leq \frac{1}{3} \times 10^{-6}$$

$$h^2 \times e^2 \leq 10^{-6}$$

$$\frac{1}{n^2} \times e^2 \leq 10^{-6}$$

$$e^2 \times 10^6 \leq n^2$$

$$\underline{\underline{n \geq 1000e}}$$

$$n = \frac{b-a}{h}$$

$$= \frac{2-1}{n} = \underline{\underline{1/n}}$$

Q. The min no of subintervals needed to approx. $\int_0^2 e^{2x} dx$ to an accuracy atleast $\frac{8}{45} \times 10^{-8}$ using Simpson's Rule is

Ⓐ 200e Ⓑ 2000e Ⓒ 2000 Ⓓ 200

Ans $n = ?$

$f(x) = e^{2x}$

$a = 0$, $b = 2$.

$f'(x) = 2e^{2x}$

$f''(x) = 4e^{2x}$

$f'''(x) = 8e^{2x}$

$f^{iv}(x) = 16e^{2x}$

$\max_{[0,2]} f^{iv}(x) = 16e^4$

accuracy $\geq \frac{8}{45} \times 10^{-8}$

$|\text{Trunc error}| \leq \frac{8}{45} \times 10^{-8}$

S 1/3 R.

$$\left| + \frac{(b-a)}{180} h^4 \times f^{iv}(x) \right| \leq \frac{8}{45} \times 10^{-8}$$

$$\Rightarrow \frac{\cancel{2}}{\underset{\underset{\cancel{2}}{\cancel{90}}}{\cancel{180}}} \times h^4 \times \overset{\cancel{8}}{\cancel{16}} e^4 \leq \frac{\cancel{8}}{\cancel{45}} \times 10^{-8}$$

$$h^4 e^4 \leq 10^{-8} \qquad n = \frac{b-a}{h} = \frac{2}{h}$$

$$\left(\frac{2}{n}\right)^4 e^4 \leq 10^{-8}$$

$$\frac{2^4}{n^4} e^4 \leq 10^{-8}$$

$$n^4 \geq 2^4 \times e^4 \times 10^8$$

$$n \geq 2 \times e \times 10^2$$

$$\underline{\underline{n \geq 200e}}$$

Note : Let $f(x)=0$ is nth degree polynomial.

a) No of +ve real roots of $f(x)=0$ $\leq$ no. of sign changes in $f(x)=0$

b) No of -ve real roots of $f(x)=0$ $\leq$ no of sign changes in $f(-x)=0$

c) No. of imaginary roots $\leq n - \{(\text{no. of +ve roots}) + (\text{no. of -ve roots})\}$

Gate 2005

Q. $f(x) = x^5 + x + 2$ has.

ⓐ All complex roots ⓑ All real roots [✓ ⓒ 1 real & 4 complex roots.]

ⓓ 2 real roots & 3 complex roots.

Ans $f(x) = \underbrace{x^5 + x + 2}_{0 \text{ sign changes}} = 0$

$f(-x) = \underbrace{-x^5 - x + 2}_{1 \text{ sign change}} = 0$

Solutions To Differential Eqn

Consider $\frac{dy}{dx} = f(x,y)$ with $y(x_0) = y_0$ ——— ⊛

Eqn ⊛ can be solved by using.

① Euler's method.

a) Forward Euler's Method (by default)

b) Backward " " (Implicit Eulers Method)

② Runge Kutta Method.

- 1st order. (Forward Euler's)
- 2nd order (Modified Euler's Method)
- 3rd order
- 4th order.

Euler's Method

Consider $\frac{dy}{dx} = f(x,y)$, $y(x_0) = y_0$ ——— ⊛

Euler Formula for solving eqn ⊛ is

$$y_{i+1} = y_i + hf(x_i, y_i)$$

In particular for $i = 0$, $y_1 = y_0 + hf(x_0, y_0)$

Q1. $\frac{dy}{dx} = x+y$, $y(0)=1$, $h = 0.1$ find $y(0.2)$ using Euler's method

Ans $\frac{dy}{dx} = x+y$, $y(0) = 1$ ∴, $x_0 = 0$, $y_0 = 1$

x	y	Comment
$x_0 = 0$	$y_0 = 1$	Initial condn
$x_1 = 0.1$	$y_1 = 1.1$	$y_1 = y_0 + hf(x_0, y_0) = 1 + 0.1(x_0 + y_0) = 1 + 0.1(0+1) = \underline{1.1}$
$x_2 = 0.2$	$y_2 = 1.22$	$y_2 = y_1 + hf(x_1, y_1) = 1.1 + 0.1(x_1 + y_1) = 1.1 + 0.1(0.1 + 1.1) = \underline{1.22}$

Gate 2010

Q2. $\frac{dy}{dx} - y = x$, $y(0)=0$, $h = 0.1$. Find $y(0.3)$ using Eulers method

Ans $\frac{dy}{dx} = f(x,y) = x+y$

$y_0 = 0$, $x_0 = 0$. $h = 0.1$

$$y_1 = y_0 + hf(x_0, y_0) = 0 + 0.1(x_0 + y_0) = 0$$

$$y_2 = y_1 + hf(x_1, y_1) = 0 + 0.1(0.1+0) = 0.01$$

$$y_3 = y_2 + hf(x_2, y_2) = 0.01 + 0.1(0.2+0.01) = \underline{\underline{0.03}}$$

Backward Euler's Method (Implicit Eulers Method)

$$y_{i+1} = y_i + hf(x_{i+1}, y_{i+1})$$

$\because$ y_{i+1} appears on LHS & RHS.

Q1. $\frac{dy}{dx} = x+y$, $y(x_0) = y_0$, $y(0) = 1$, $h = 0.1$ find $y(0.2)$ using Implicit Eulers.

Ans $\frac{dy}{dx} = f(x,y) = x+y$, $x_0 = 0$, $y_0 = 1$

$$y_{i+1} = y_i + hf(x_{i+1}, y_{i+1})$$

$$y_{i+1} = y_i + h(x_{i+1} + y_{i+1})$$

$$(1-h)\, y_{i+1} = y_i + h\, x_{i+1}$$

$$y_{i+1} = \frac{y_i + h x_{i+1}}{(1-h)}$$

x	y's	Comment
$x_0=0$	$y_0=1$	Initial condn
$x_1=0.1$	$y_1=1.12$	$y_1 = \frac{y_0 + hx_1}{(1-h)} = \frac{1+0.1(0.1)}{1-0.1} = \underline{1.12}$
$x_2=0.2$	$y_2=1.26$	$y_2 = \frac{y_1 + hx_2}{(1-h)} = \frac{1.12+0.1(0.2)}{1-0.1} = \underline{1.26}$

Q2. $\frac{dy}{dx} = 0.25y^2$, $y(0) = 1$, $h = 1$ Find $y(1)$ using implicit Eulerz.

Ⓐ 1 Ⓑ 2 Ⓒ 3 Ⓓ 4

Ans $y_{i+1} = \cancel{\frac{y_i + h x_{i+1}}{(1-h)}}$ $y_i + hf(x_{i+1}, y_{i+1}) = y_i + h($

$$i=0 \Rightarrow y_1 = y_0 + hf(0.25)y_1^2 \Longrightarrow 0.25y_1^2 - y_1 + 1 = 0$$

x	y	Comment
$x_0 = 0$	$y_0' = 1$	
$x_1 = 1$	$y_1 = 2$	$\cancel{y_1} = \cancel{\frac{y_0 + hx_1}{(1-h)}} = \cancel{\frac{1+1\times 1}{1}}$

$$\Longrightarrow y_1 = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(0.25)(1)}}{2(0.25)}$$

$$y_1 = 1/0.5 = \underline{\underline{2}}$$

Gate 1993

Q3. $\frac{dy}{dx} = x - y$, $y(0) = 0$, $h = 0.2$. Find $y(0.2)$ using Second Order Runge Kutta?

Ans $\frac{dy}{dx} = f(x,y) = x - y$ $\quad x_0 = 0, y_0 = 0, h = 0.2$

Second Order Formula is $y_0 = y_1 + \frac{1}{2}(K_1 + K_2)$ ——— ⊛

where $\underline{K_1 = hf(x_0, y_0)} = 0.2(x_0 - y_0) = \underline{\underline{0}}$

$$\underline{K_2 = hf(x_0+h, y_0+K_1)} = 0.2((x_0+h) - (y_0+k_1))$$

$$= 0.2((0+0.2) - (0+0))$$

$$= \underline{\underline{0.04}}$$

⊛ $\Longrightarrow y_1 = 0 + 1/2(0+0.04) = \underline{\underline{0.02}}$

Note: ① 3rd order $y_1 = y_0 + \frac{1}{6}(K_1 + 4K_2 + K_3)$ where $K_1 = hf(x_0, y_0)$

$K_2 = hf(x_0 + h/2, y_0 + K_1/2)$

$K_3 = hf(x_0 + h, y_0 + K')$

where $K' = hf(x_0 + h, y_0 + K_1)$

② 4th order.

$y_1 = y_0 + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$ where $K_1 = hf(x_0, y_0)$

$K_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}\right)$

$K_3 = hf(x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2})$

$K_4 = hf(x_0 + h, y_0 + K_3)$

Q. $\frac{dy}{dx} = x+y$, $y(0)=1$, $h=0.2$ find $y(0.2)$ using ⓐ Fourth order RK
ⓑ Third order RK.

Ans Fourth Order

$K_1 = 0.2$

$K_2 = 0.24$

$K_3 = 0.244$

$K_4 = 0.288$

$\therefore y_1 = \underline{\underline{1.2426}}$

Third Order

$K_1 = 0.2$

$K_2 = 0.24$

$K_3 = 0.296$

$y_1 = \underline{\underline{1.2428}}$

Stability Analysis In Euler's Method

Consider Euler's eqn $y_{i+1} = y_i + hf(x_i, y_i)$ ——— ⊛

Convert eqn ⊛ into the form $y_{i+1} = Ey_i + hk$

where k is constant or it contains 'x_i' terms.

Eqn ⊛ said to be stable if $\boxed{|E| \leq 1}$

G 2003

Q. $\frac{dx}{dt} = \frac{1-x}{T}$ with step size $\Delta T > 0$. is evaluated using Euler's method. what is the maximum permissible value of ΔT to ensure the stability in the solⁿ.

Ans Ⓐ T Ⓑ T/2 Ⓒ 2T Ⓓ 0

$\frac{dx}{dt} = \frac{1-x}{T}$

$\frac{dy}{dx} = f(x,y) = \frac{1-y}{T}$

$y_{i+1} = y_i + hf(x_i, y_i)$

$$y_{i+1} = y_i + h\left(\frac{1-y_i}{\tau}\right)$$

$$y_{i+1} = \left(1-\frac{h}{\tau}\right)y_i + \frac{h}{\tau} \quad \text{———} \; ⊛$$

$$y_{i+1} = Ey_i + hk$$

where $E = 1 - \frac{h}{\tau}$, $k = 1/\tau$

⊛ is stable if $|E| \leq 1$

$$\left|1-\frac{h}{\tau}\right| \leq 1$$

$$\left|1 - \frac{\Delta T}{\tau}\right| \leq 1$$

$$-1 \leq 1 - \frac{\Delta T}{\tau} \leq 1$$

$$-2 \leq -\frac{\Delta T}{\tau} \leq 0$$

$$2\tau \geq \Delta T \geq 0$$

$$\therefore 0 \leq \Delta T \leq 2T$$

Method	Error Order
① Eulers (first order RK)	$O(h^2)$
② 2nd order R·K	$O(h^3)$
③ 3rd " "	$O(h^4)$
④ 4th " "	$O(h^5)$

$$y_0' = dy/dx\Big|_{(x_0,y_0)} = f(x_0,y_0)$$

$$y_1 = y_0 + hf(x_0,y_0)$$

$$\begin{tikzcd} y_1 & = & y_0 & + & hy_0' \\ f(x) \arrow[u] & = & f(x_0) \arrow[u] & + & hf'(x_0) \arrow[u] & + \underbrace{\frac{h^2}{2!}f''(x_0) + \frac{h^3}{3!}}_{\text{principal term.}} f'''(x_0) + \end{tikzcd}$$

- IMT
- O(1) Bisection : $x = \frac{a+b}{2}$
- O(2) N-R
  - $x = \sqrt{c} \Longrightarrow x_{n+1} = ?$
  - $x_{n+1} = (\ ) \Longrightarrow x = ?$
  - $x = 1/a \Longrightarrow x_{n+1} = ?$
- O(1) RF — $f(x_0) \times f(x_1) < 0$
- O(1.62) Secent — $f(x_0) \times f(x_1) \begin{matrix} < 0 \\ > 0 \end{matrix}$

Ⅰ

- Table
- h → Table
- $\frac{b-a}{n}$ → h → Table
- a_0, a_1, a_2 → Error
- ~~Error~~ Accuracy atleast
- Error order
  - T $O(h^2)$
  - S $O(h^4)$
  - S(3/8) $O(h^5)$

Ⅲ

- Eulers $O(h^2)$
  - Forward → $y_{i+1} = y_i + hf(x_i, y_i)$
  - Backward (Bring all 'y_{i+1}' to left side)
  - Stability Stability ($|E| \le 1$)
- R.K
  - F $O(h^2)$
  - S $O(h^3)$
  - Third $O(h^4)$
  - 4th $O(h^5)$

COMPLEX VARIABLES

① Analyticity
② Complex Integration
③ Complex power series.
④ Zeros and type of singular points.
⑤ Residues.

Analyticity

$$N \in W \in I \in R \in \underset{\substack{\downarrow \\ x+iy}}{C} \in \text{Extended complex set } (C \cup \{-\infty, \infty\})$$

$$\underset{\substack{\downarrow \\ x \in (-\infty, \infty)}}{R}$$

Complex Number : If x, y are two real numbers and i is an imaginary unit such that $i^2 = -1$ or $i = \sqrt{-1}$ then the number of the form $z = x+iy$ is called complex number.

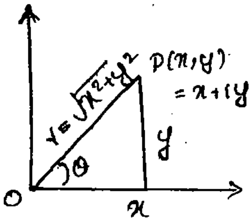
Therefore $z = x+iy$

where $x = Re(z)$ & $y = Im(z)$

Complex Variable : If x & y are two real variables then the variable of the form $z = x+iy$ is called complex variable.

- If $z = x+iy$, then $\bar{z} = x - iy$
- If $z = x+iy$ is a complex number then $|z| = |x+iy| = \sqrt{x^2+y^2}$
- $e^{i\theta} = \cos\theta + i\sin\theta$.
- $z = x+iy = re^{i\theta} = r[\cos\theta + i\sin\theta]$ where $r = \sqrt{x^2+y^2}$

  $\theta = \tan^{-1}(y/x)$



θ = amplitude of argument of $z = x+iy$.

- If $z = x+iy$ & $z_0 = x_0 + iy_0$ are two complex no's then the distance b/w z & z_0 is given by $|z - z_0|$ or $|z_0 - z|$

$$\therefore |z - z_0| = |x + iy - (x_0 + iy_0)| = |(x - x_0) + i(y - y_0)| = \underline{\underline{\sqrt{(x-x_0)^2 + (y-y_0)^2}}}$$

- $x^2 + y^2 = r^2$

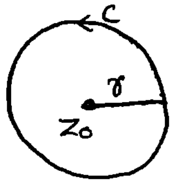
$$\Longrightarrow r = \sqrt{x^2+y^2}$$

$|z| = r$ is an equation of circle with centre at origin and radius r.

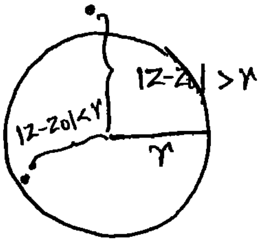
$(x - x_0)^2 + (y - y_0)^2 = r^2$

$\sqrt{(x - x_0)^2}$

$|z - z_0| = r$ is the eqn of circle with centre at z_0 and radius r.



- $|z - z_0| < r$ represents a set of all points lying within the circle $|z - z_0| = r$.
- $|z - z_0| > r$ represents set of all points lying outside the $\odot^{le}$ $|z - z_0| = r$



Complex Function

If A and B are two sets of complex numbers and every element of the form $z = x+iy$ in a set of A is associated with the unique element of the form $\omega = u+iv$ in a set B the $\omega = u+iv$ is called complex function of a complex variable $z = x+iy$ and it is denoted by $\omega = f(z)$ where $z = x+iy$ and $z = u+iv$.

Therefore
- $\omega = f(z) = f(x+iy) = u(x,y) + i\,v(x,y)$
- $\omega = f(z) = f(re^{i\theta}) = u(r,\theta) + iv(r,\theta)$

Neighbourhood of a pt z_0

The set of all points within the circle having centre at z_0 but not on the circle is called neighbourhood of a pt z_0 and it is also called open circular disc (region)

Therefore $N_d(z_0) = N(d, z_0) = \{z : |z - z_0| < d\}$

Analytic Function

If a complex function $f(z)$ is differentiable at a point z_0 and also differentiable at every point in some neighbourhood of a point z_0, then the function $f(z)$ is called Analytic function at a point z_0 and the point z_0 is called Analytic point of $f(z)$.

Singular Point

If a function $f(z)$ is not defined or not differentiable or not analytic, at a point z_0, then the point z_0 is called singular point of $f(z)$.

Eg: $f(z) = \dfrac{z+4}{z-2}$

$\therefore$ $z = 2$ is a singular point of $f(z)$.

② $f(z) = \sqrt{z-4}$

$\longrightarrow$ $z = 5$, $f(z) = 1$

$z = 4$, $f(z) = 0$

$z = 3$, $f(z) = i$

defined for all values of z.

Consider derivative

$$f'(z) = \frac{1}{2\sqrt{z-4}}$$

$z = 5$, $f'(z) = \underline{1/2}$

$z = 4$, $f'(z)$ doesnot exist.

$\therefore$ $z = 4$ is a <u>singular</u> point of $f(z)$

- <u>Entire Function</u>

If a complex function $f(z)$ is differentiable or analytic at every point throughout a complex plane, then the fn $f(z)$ is called an Entire function and it is also called Integrable function.

<u>Theorem</u> : Necessary conditions for a fn $f(z)$ to be analytic.

If $f(z) = u(x,y) + i\,v(x,y)$ is analytic function at a point z_0, then u_x, u_y, v_x, v_y exists and satisfy the Cauchy Riemann equations.

$$u_x = v_y \;\;\&\;\; v_x = -u_y$$

<u>Theorem 2</u> : Sufficient condition for a fn $f(z)$ to be analytic at every point in some neighbourhood of a point z_0.

If (i) $f(z) = u(x,y) + i\,v(x,y)$ is defined at every point in some neighbourhood of z_0.

(ii) u & v satisfy the C-R eqns at every point in some neighbourhood of a pt z_0.

(iii) u, v, u_x, u_y, v_x, v_y are continuous at every point in some neighbourhood of a pt z_0.

Then the fn $f(z) = u + iv$ is analytic at z_0 and $f'(z) = u_x + i v_x$.

Note : i) e^x, $\sin x$, $\cos x$, $\sinh x$, $\cosh x$ and every polynomial of the form $a_0+a_1x+a_2x^2$ $+ \cdots a_nx^n$ ($a_n \neq 0$ & $n \in N$) are everywhere defined, continuous, differentiable and also integrable.

ii) If f, g are two continuous functions, then

a) $f \pm g$ is also continuous

b) $f \cdot g$ is also continuous.

c) f/g $(g \neq 0)$ is also continuous.

Model 1

Q1. Test the analyticity of the following functions

i) $f(z) = x + e^x . \cos y + iy + ie^x \sin y$

Procedure : i) Given f(z), find u, v

ii) Find u_x, u_y, v_x, v_y.

iii) Check for C-R eqns. $u_x = v_y$ & $v_x = -u_y$

iv) Check the continuity of u, v, u_x, u_y, v_x, v_y

$$f(z) = x + e^x \cos y + iy + ie^x \sin y$$

$$u + iv = f(z) = (x + e^x \cos y) + i(y + e^x \sin y)$$

$$u = x + e^x \cos y \qquad v = y + e^x \sin y$$

$$u_x = \frac{\partial u}{\partial x} = 1 + e^x \cos y \qquad v_x = \frac{\partial v}{\partial x} = e^x \sin y$$

$$u_y = \frac{\partial u}{\partial y} = -(e^x \sin y) \qquad v_y = \frac{\partial v}{\partial y} = 1 + e^x \cos y$$

Here $u_x = v_y$ & $v_x = -u_y$ at every point and u, v, u_x, u_y, v_x, v_y are continuous at every point

$\therefore$ f(z) is an analytic fn.

Hence f(z) is also an entire fn.

ii) $f(z) = \bar{z}$

$\Longrightarrow u + iv = f(z) = x - iy$

$u = x \qquad v = -y.$

$u_x = 1 \qquad u_y = 0$

$V_x = 0 \qquad V_y = -1$

$u_x \neq V_y$ & $V_x = -u_y$

∴ $f(z)$ is not analytic function

iii) $f(z) = |z|^2$

$u + iV = f(z) = x^2 + y^2 + i \times 0$

$u = x^2 + y^2 \qquad V = 0$

$u_x = 2x \qquad u_y = 2y$

$V_x = 0 \qquad V_y = 0$

$u_x \neq V_y$ & $V_x \neq -u_y$

∴ $f(z)$ is not analytic

Note : ① $f(z) = |z|^2$ is differentiable only at the origin but not analytic at any point.

② $f(z) = \bar{z}$ is not differentiable and not analytic at any point.

③ $\log(-N) = i\pi + \log N$.

Eg : $\log(-4) = i\pi + \log(4)$

④ $\omega = z^c \iff \omega = e^{c\log(z)}$

eg : $y = x^x$

$\log y = x \log x$

$y = e^{x \log x}$

Q1. If $x = \sqrt{-1}$ then $x^x = ?$

a) $e^{\pi/2}$ b) $e^{-\pi/2}$ c) x d) 1

Ans $x^x = e^{x \log x}$

$= e^{x \log \sqrt{-1}} = e^{x(i\pi + \log 1)}$

Let $i = \sqrt{-1} = x$

then $\omega = i^i = x^x \qquad (e^{i\pi} = \cos\pi + i\sin\pi = -1)$

$= (e^{i\pi/2})^i = \underline{e^{-\pi/2}} \qquad (e^{i\pi/2} = \cos\pi/2 + i\sin\pi/2 = 1)$

⑤ C-R eqns in polar form are given by.

$$\boxed{U_r = \frac{1}{r}V_\theta \quad \& \quad V_x = \frac{-1}{r}V_\theta}$$

⑥ The derivative formula in polar form is given by.

$$\boxed{f'(z) = (U_r + iV_r)e^{-i\theta}}$$

⑦ $\sin(i\theta) = i\sinh\theta$

$\sinh(i\theta) = i\sin\theta$

$\cos(i\theta) = \cosh\theta$

$\cosh(i\theta) = \cos\theta$

$\tan(i\theta) = i\tanh(\theta)$

$\tan h(i\theta) = i\tan\theta$

⑧ $f(x) = x + e^x\cos y + iy + ie^x\sin y$.

$f(z) = (x + iy) + e^x(\cos y + i\sin y)$

$f(z) = z + e^x e^{iy}$

$= z + e^{(x+iy)} = z + e^z$

(OR)

$f(z) = x + e^x\cos y + iy + ie^x\sin y$ is a function

For a fn replace 'x' by 'z' & 'y' by '0'.

$f(z) = z + e^z$

⑨ $e^z, \sin z, \cos z, \sinh z, \cosh z$ & every polynomial of form $a_0 + a_1z + a_2z^2 + \dots + a_nz^n$ $(a_n \neq 0)$ & $(n \in N)$ are everywhere defined, continuous, differentiable, integrable functions.

MODEL No-2

Method :

1) Given V (or U). Find V_x, V_y (or U_x, U_y)

2) Consider $f'(z) = U_x + iV_x = U_y + iV_y$

3) Replace x by z and y by '0'. ie $f'(z) = g(z)$

4) $f(x) = \int g(x)\,dx + c$ where $c = c_1 + ic_2$.

Q1. If $V(r,\theta) = 3r^2 \sin 2\theta + 2r\sin\theta + 7$ then find analytic function $f(z) = u + iV$ where V is an imaginary part of analytic fn $f(z)$.

Ans $V_r = 6r\sin 2\theta + 2\sin\theta$ & $V_\theta = 6r^2\cos 2\theta + 2r\cos\theta$

Consider $f'(z) = (u_r + iV_r)e^{-i\theta} = \left(\frac{1}{r}V_\theta + iV_r\right)e^{-i\theta}$

$\Longrightarrow f'(z) = (6r\cos 2\theta + 2\cos\theta) + i[6r\sin(2\theta) + 2\sin\theta]e^{i\theta}$

Replace 'r' by 'z' & 'θ' by '0'.

$f'(z) = 6z + 2$.

$\therefore f(z) = 3z^2 + 2z + c$ where $c = c_1 + ic_2$

or $f(z) = 3z^2 + 2z + (c_1 + ic_2)$

$= 3z^2 + 2z + c_1 + i7$

(OR)

$f(z) = 3z^2 + 2z + c_1 + ic_2 = 3(re^{i\theta})^2 + 2(re^{i\theta}) + (c_1 + ic_2)$

$= 3r^2[\cos 2\theta + i\sin 2\theta] + 2r[\cos\theta + i\sin\theta] + c_1 + ic_2$

$= (3r^2\cos 2\theta + 2r\cos\theta + c_1) + i(3r^2\sin 2\theta + 2r\sin\theta + c_2)$

MODEL 3

Construction of Harmonic conjugate function.

Harmonic fn: If u_x, u_y, u_{xx} & u_{yy} are continuous functions.

& $u_{xx} + u_{yy} = 0$ or $\nabla^2 u = 0$ then $u(x,y)$ is called Harmonic function

Note

① If $f(x) = u + iV$ is analytic function then u & V satisfy Laplace eqns $\nabla^2 u = 0$ & $\nabla^2 V = 0$ ie (u & V are harmonic functions)

② If u & V are harmonic functions then $u + iV$ may or maynot be analytic fn.

<u>Harmonic Conjugate Fn</u> : If u & v are harmonic fn an u+iv is also analytic function, then v is called Harmonic conjugate fn of 'u'.

Note : ① If $u \xrightarrow{H.C.F} v$

then $v \xrightarrow{HCF} -u$

Method :

Step 1 : If $v(x,y)$ is given to find $u(x,y)$ then consider

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$$

Step 2 : $du = u_x\,dx + u_y\,dy = (v_y)\,dx + (-v_x)\,dy$

$(\because u_x = v_y \;\&\; v_x = -u_y)$

Step 3 : $u = \underbrace{\int (v_y)\,dx}_{\text{Treating } y \text{ as constant}} + \int\left[\text{terms not containing } x \text{ in } (-v_x)dy\right] + K$

$K \longrightarrow$ real integ~~al~~ ~~function~~ real constant.

Q1. If $v(r,\theta) = 3r^4 \sin(4\theta) + 4$, then find its harmonic conjugate function.

Ans Given $v(r,\theta) = 3r^4 \sin(4\theta) + 4$

$$\Longrightarrow v_r = 12r^3 \sin(4\theta) \;\&\; v_\theta = 12r^4 \cos(4\theta)$$

Consider $du = (u_r)\,dr + (u_\theta)\,d\theta = \left(\frac{1}{r}v_\theta\right)dr + (-r v_r)\,d\theta$

$\left(\because u_r = \frac{1}{r}v_\theta \;\&\; v_r = \frac{-1}{r}u_\theta\right)$

$$\Longrightarrow du = \left[12r^3\cos(4\theta)\right]dr + \left[-12r^4\sin(4\theta)\right]d\theta$$

$$\Longrightarrow u = \int 12r^3\cos(4\theta)\,d\theta + K.$$

$$\therefore u(r,\theta) = \underline{\underline{3r^4\cos(4\theta) + K}}$$

Note : 1) $F(z) = u(x,y) + iv(x,y)$ is analytic fn.

$$\begin{tikzcd}
F(z) \arrow[d] & u(x,y) \arrow[d] & iv(x,y) \arrow[d] \\
\text{Complex potential fn} & \text{Velocity potential fn} & \text{Stream fn} \\
f(z) \arrow[u] & \phi(x,y) \arrow[u] & i\Psi(x,y) \arrow[u]
\end{tikzcd}$$

Q2. If $f(z) = x^3 - 3xy^2 + i\varphi(x,y)$ where $i = \sqrt{-1}$ & $f(x+iy)$ is analytic function, then find the stream fn φ.

a) $x^3 + 3x^2y$ b) $3xy^2 - x^3$ c) $3x^2y - y^3$ d) $y^3 - 3x^2y$.

Ans Given $u = x^3 - 3xy^2$ $V = \varphi = ?$.

$u_x = 3x^2 - 3y^2$ $u_y = -6xy$.

Consider $dV = V_x dx + V_y dy$

$dV = -u_y dx + u_x dy$

$dV = 6xy dx + (3x^2 - 3y^2) dy$

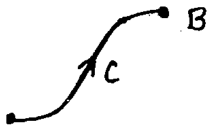
$V = \int (6y) x dx + \int (-3y^2) dy + K$.

$V = \underline{3x^2y - y^3 + K}$

COMPLEX INTEGRATION

1) Complex line integral

If a complex fn $f(z)$ is defined at every point on the curve C from a pt A to B, then the evaluation of integral of complex fn $f(z)$ is called line Integral of a complex fn $f(z)$ and is denoted by $\int_C f(z)dz$ where C is called path of integration.



2) The Relation b/w Real line Integral & Complex line Integral

If $f(z)$ is given by $f(z) = u + iV$ and $dz = dx + idy$ where $z = x + iy$

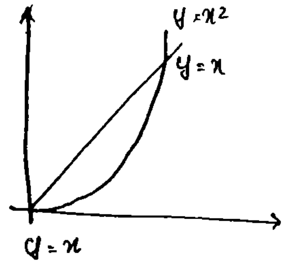
then $$\int_C f(z)dz = \int_C (u+iV)(dx+idy)$$

$$= \int_C (udx - Vdy) + i\int_C (Vdx + udy)$$

Q1. Evaluate $\int_0^{1+i} z\,dz$ along a curve c where C is the

i) $y = x$

ii) $y = x^2$



$y = x$
$dy = dx$ $x \to 0$ to 1
$y = t \Rightarrow x = t$ $dy = dx = dt$
$t \to 0$ to 1

Ans i) $$\int_0^{1+i} z\,dz = \int_{(x,y)=(0,0)}^{(1,1)} (x+iy)(dx+idy) = \int_0^1 (t+it)(dt+idt)$$

$$= (1+i)^2 \int_0^1 t\,dt = (1+i)^2 \left(\frac{t^2}{2}\right)_0^1 = (1+i)^2 \times \frac{1}{2} = \frac{2i}{2} = \underline{\underline{i}}$$

Q₂. Evaluate $\int_0^{1+i} (x^2 - iy)\,dz$ along the curve C where C is

i) $y = x$ ii) $y = x^2$.

Ans i) $$\int_{(0,0)}^{(1,1)} (x^2 - iy)(dx + idy) = \int_0^1 (t^2 - it)(dt + idt)$$

$$= \int_0^1 t^2dt + it^2dt - itdt + tdt = \left[\frac{t^2}{3} + i\frac{t^3}{3} - i\frac{t^2}{2} + \frac{t^2}{2}\right]_0^1$$

$$= \frac{1}{3} + \frac{i}{3} - \frac{i}{2} + \frac{1}{2} = \frac{1}{3}(1+i) + \frac{1}{2}(1-i) = \frac{5}{6} - \frac{i}{6} = \underline{\underline{\frac{5-i}{6}}}$$

ii) $$\int_{(0,0)}^{(1,1)} (x^2 - iy)(dx + idy) = \int_0^1 (t^2 - it^2)(dt + i2tdt)$$

$x = t$
$y = t^2$
$dx = dt$
$dy = 2tdt$

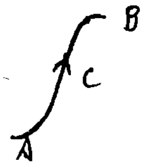
$$= \int_0^1 (1-i)t^2 \times dt(1 + i2t)$$

$$= (1-i)\int_0^1 t^2dt + i\,2t^3dt = (1-i)\left[\frac{t^3}{3} + i\frac{2t^4}{4}\right]_0^1$$

$$\overset{nt's}{=} (1-i)\left[1/3 + i/2\right] = \underline{\frac{5+i}{6}}$$

Note 3 If the integrant function is analytic, then the value of the integral depends on end points of the paths not on the path.

ie $\int_C f(z)dz = \int_A^B f(z)dz$



Eg: $$I = \int_{z=0}^{1+i} z\,dz = \left.\frac{z^2}{2}\right]_0^{1+i} = \frac{(1+i)^2}{2} = \frac{2i}{2} = \underline{\underline{i}}$$

Note 3 If the integrant function $f(z)$ is not analytic fn, then the value of integral depends on the path not on the end points of the path.

$$x^2 + y^2 = r^2 \Longrightarrow |z| = r = \sqrt{(x-x_0)^2 + (y-y_0)^2}$$

$x = r\cos\theta$ $x = x_0 + r\cos\theta$

$y = r\sin\theta$ $y = y_0 + r\cos\theta$

$r = |z - z_0|$ $\underline{\underline{z = z_0 + re^{i\theta}}}$

Note: The paramatric eqn of a circle $|z - z_0| = r$ is

$z = z_0 + re^{i\theta}$ where $\theta \longrightarrow 0$ to 2π for total θ^n path.

Q3. Evaluate $\int_C \frac{2z+3}{z}\,dz$ along a curve c, where c is $|z| = 3$.

Ans $C = |z| = 3$.

$$\Longrightarrow z = 3e^{i\theta}$$

$$dz = 3i\,e^{i\theta}d\theta$$

Here $\theta = 0$ to 2π

$$I = \int_c \frac{2z+3}{z}dz = \int_{\theta=0}^{2\pi} \frac{2\times 3e^{i\theta}+3}{3e^{i\theta}} \times i\,3e^{i\theta}d\theta$$

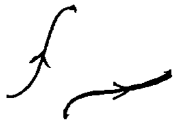
$$= \int_0^{2\pi}(6ie^{i\theta}+3i)d\theta = \left[\frac{6ie^{i\theta}}{i} + 3i\theta\right]_0^{2\pi}$$

$$= (6e^{i2\pi}+6i\pi) - (6e^0+0)$$

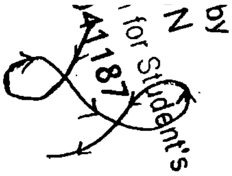
$$= \underline{6i\pi}$$

Different Type Of Curves

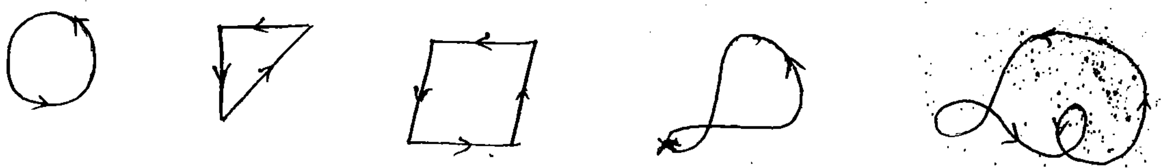
1) Simple curve



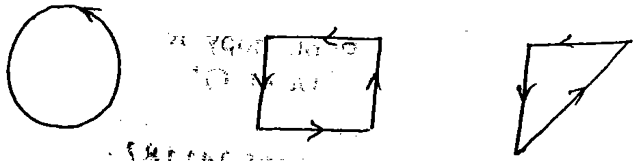
2) Multiple curve



3) Closed curve

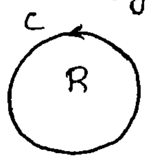


4) Simple closed curve.

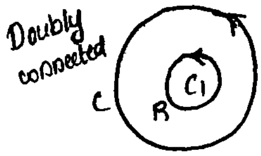


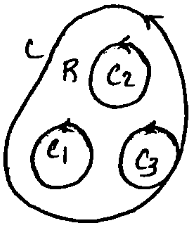
Type Of Regions

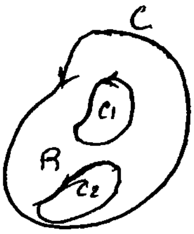
① Simply connected region

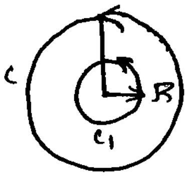


② Multiple connected region.







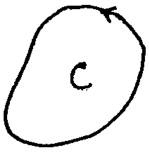


Ring shaped region.

Cauchy's Integral Theorem

If a fn f(z) is analytic at every point within and on a simple closed curve c, then integral over c.

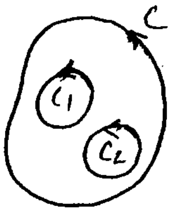
$$\oint_C f(z)dz = 0$$



CIT for a Multiply Connected Region

If a fn f(z) is analytic everywhere within and on a triply connected region R bounded by 3 simply closed curve C_1, C_2, C_3 but not analytic within C_1, C_2 then

$$\oint_C f(z)dz = \oint_{C_1} f(z)dz + \oint_{C_2} f(z)dz$$

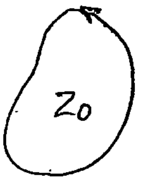


Cauchy's Integral Formula (CIF)

If f(z) is analytic at every point within and on a simple closed curve c and z_0 is any point within c then.

i) $$\oint_C \frac{f(z)}{z-z_0}dz = 2\pi i\, f(z_0)$$

ii) $$\oint_C \frac{f(z)}{(z-z_0)^{n+1}}dz = \frac{2\pi i}{n!} f^{(n)}(z_0)$$



Method

$\int_C (\ )dz,$

1) Let $\phi(z) = (\ )$

2) Find the singular pts of $\phi(z)$

3) c 3

A) No singular points : By CIT $\int ( )\, dz = 0$

Singular pts : CIF.

Q1. Evaluate $\int_C \frac{2z + \sin z + e^z}{(z-4)^{10}(z-6)^{100}} dz$ where C is $|z| = 3/2$

Ans Let $\phi(z) = \frac{2z + \sin z + e^z}{(z-4)^{10}(z-6)^{100}}$

Singular points : $z = 4, 6$.

$C : |z| = 3/2$

$|z - z_0| = r$

The singular points lie outside the curve.

$\therefore \int_C \phi(z)\, dz = \underline{\underline{0}}$

Q2. Evaluate $\oint \frac{2z+3}{z} dz$ ~~where~~ along ~~curve C~~ where $C : |z| = 3$.

Ans $\oint \frac{2z+3}{z} dz$

$\phi(z) = \frac{2z+3}{z}$

S. Pt : $z = 0$

$|z| = 3$

$|0| < 3$

$\therefore$ Singular point is inside.

$\therefore$ By CIF $\oint \frac{2z+3}{z-0} dz = \frac{2\pi i}{1} \times f(0) = 2\pi i \times (2 \times 0 + 3) = \underline{\underline{6\pi i}}$

Q3. Evaluate $\oint_C \frac{z}{(z-1)(z-2)^2} dz$ along curve C where $C : |z-2| = 1/2$

Ans $\phi(z) = \frac{z}{(z-1)(z-2)^2}$

Singular point : $z = 1, 2$.

$C : |z-2| = 1/2$

$z = 1$

$|-1| = 1 > 1/2$

$z = 2$

$|0| = 0 < 1/2$

$$\frac{\frac{z}{(z-1)}}{(z-2)^2} = \frac{f(z)}{(z-z_0)^{n+1}} \qquad \text{where } f(z) = \frac{z}{z-1}$$

$\therefore$ By CIF, we have $\oint_c \phi(z)dz = \frac{2\pi i}{1!} f'(z)$.

$$= 2\pi i \left[\frac{(z-1)-z}{(z-1)^2}\right]_{z=2}$$

$$= 2\pi i \, \frac{-1}{(2-1)^2} = \underline{\underline{-2\pi i}}$$

Q4. $\int_c \frac{e^z+\cos z}{(z-3)(z-2)} dz$, c : $|z| = 5$

$$\phi(z) = \frac{e^z+\cos z}{(z-3)(z-2)}$$

$$\boxed{\frac{1}{(z-a)(z-b)} = \frac{1}{(a-b)(z-a)} - \frac{1}{(a-b)(z-b)}}$$

S.pt : $z = 2, 3$

c: $|z| < 5$ $\qquad \phi(z) = \frac{e^z+\cos z}{z-3} - \frac{e^z+\cos z}{z-2}$

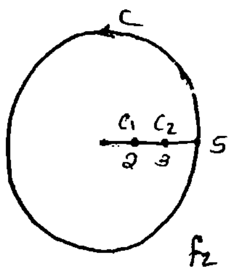
$$\therefore \oint_c \phi(z)dz = \int_c \frac{e^z+\cos z}{z-3}dz - \int_c \frac{e^z+\cos z}{z-2}$$

$$= 2\pi i \, f(3) - 2\pi i \, f(2) = 2\pi i\left[f(3) - f(2)\right]$$

$$= 2\pi i \left[(e^3+\cos 3) - (e^2+\cos 2)\right]$$

Method II

$$\phi(z) = \frac{e^z+\cos z}{(z-3)(z-2)}$$



$$\oint_c \phi(z)dz = \int_{c_1} \phi(z)dz + \int_{c_2} \phi(z)dz = \int_{c_1} \overset{f_1}{\boxed{\frac{e^z+\cos z}{(z-3)}}}(z-2) + \int_{c_2} \frac{\overset{f_2}{\boxed{e^z+\cos z}}}{\boxed{(z-3)}(z-2)}$$

$$= 2\pi i \, f(2) + 2\pi i \, f(3)$$

$$= 2\pi i\left[f(2) + f(3)\right] = 2\pi i\left[-(e^2+\cos 2) + (e^3+\cos 3)\right]$$

Q5. Evaluate $\int_{c} \frac{\bar{z}}{z} dz$ along a unit circle.

Ans Let $\phi(z) = \frac{\bar{z}}{z}$

$C : |z| = 1$

S.Pt : $z = 0$

$$\phi(z) = \frac{\bar{z}}{z-0} \cdot \frac{f(z)}{z - z_0}$$

$\bar{z}$ is not analytic anywhere.

But $\bar{z}$ can be made analytic.

Method I

$z = e^{i\theta}$

$dz = i e^{i\theta} d\theta$

For every θ : 0 to 2π

$$I = \int_{c} \frac{\bar{z}}{z} dz = \int_{0}^{2\pi} \frac{e^{-i\theta}}{e^{i\theta}} (i e^{i\theta}) d\theta$$

$$= i \int_{\theta=0}^{2\pi} e^{i\theta} d\theta = i \left(\frac{e^{-i\theta}}{-i}\right)_0^{2\pi} = e^{0} - e^{-i2\pi} = 1 - 1 = \underline{\underline{0}}$$

Method II

$z . \bar{z} = |z|^2$

$$\bar{z} = \frac{|z|^2}{z} \implies \frac{\bar{z}}{z} = \frac{|z|^2}{z^2} = \frac{1}{z^2} \quad \because |z| = 1$$

$$= \frac{1}{(z-0)^2} = \frac{f(z)}{(z - z_0)^{n+1}}$$

$\therefore$ By CIF we have $\int_{c} \frac{\bar{z}}{z} dz = \frac{2\pi i}{1!} f'(0)$

$$= 2\pi i \left[\frac{d}{dz} f(z)\right]_{z=0} = \underline{\underline{0}}$$

Q6. Evaluate $\int_{c} \frac{z}{z-2} dz$ along a circle $|z| = 2$.

Ans Let $\phi(z) = \frac{z}{z-2} = \frac{f(z)}{z - z_0}$

S.Pt = 2.

$C : |z| = 2$.

$z = 2e^{i\theta}$

$dz = 2ie^{i\theta} d\theta$

$\theta = 0$ to 2π

$$I = \int_C \frac{z}{z-2} dz = \int_0^{2\pi} \frac{\cancel{2}e^{i\theta}}{\cancel{2}e^{i\theta} - \cancel{2}} 2ie^{i\theta} d\theta = 2i \int_0^{2\pi} \frac{e^{2i\theta}}{e^{i\theta}-1} d\theta$$

The S.Pt lies on the curve C. so the ~~the~~ In cannot be evaluated.

Complex Power Series

An infinite series of the form

$$a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \dots + a_n(z-z_0)^n + \dots$$

or

$$\sum_{n=0}^{\infty} a_n (z-z_0)^n$$ is called Complex Power Series.

In powers of $(z-z_0)$ or about a point $(z-z_0)$. In the above power series a_n is a real or complex constant which is called coeff. of power series, z is a complex variable & z_0 is a fixed complex constant which is called centre of the power series.

For $a_n = 1$, $\sum_{n=0}^{\infty} (z-z_0)^n$

Region of Convergence (ROC): The set of all values of z for which the power series converges is called Region of convergence.

eg: $1 + z + z^2 + \dots = (1-z)^{-1}$; $|z|<1$

Here $|z|<1$ is an ROC, $|z|=1$ is a circle of convergence (COC) & radius $r = 1$ is a radius of convergence of the power series.

② $1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots = e^z$, $\forall z \in C$.

Here an entire complex plane is an ROC of power series.

Note: If $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ then

i) the radius of convergence of the above power series is given by

$$r = \frac{1}{\lim_{n\to\infty} |a_n|^{1/n}}$$ For $a_n = (\,)^n$ (or) $$r = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|$$

ii) The circle of convergence of above power series is given by $|z-z_0| = r$

iii) The region of convergence ROC of above power series is given by $|z-z_0| < r$.

Q1. Find the radius of convergence, COC & ROC of the given ~~by $|z-z_0|=r$~~ power series.

1) $\sum_{n=0}^{\infty} n!\, z^n$

Radius of convergence

Compare the given series with $\sum_{n=0}^{\infty} a_n (z-z_0)^n$

$$n!\, z^n = n!\, (z-z_0)^n$$

Here $a_n = n!$ & $z_0 = 0$

$$\therefore r = \lim_{n\to\infty} \left| \frac{n!}{(n+1)!} \right| = \lim_{n\to\infty} \left| \frac{1}{n+1} \right| = \underline{\underline{0}}$$

Here the above power series converges only at its centre. $z_0 = 0$ and $z_0 = 0$ is called centre of convergence of a power series.

ii) a) $\sum_{n=0}^{\infty} (3+4i)^n (z+2i)^n$

$a_n = (3+4i)^n$, $z_0 = -2i$

$$r = \frac{1}{\lim_{n\to\infty} |a_n|^{1/n}} = \lim_{n\to\infty} \frac{1}{|(3+4i)^n|^{1/n}} = \lim_{n\to\infty} \frac{1}{|(3+4i)|} = \frac{1}{\sqrt{9+16}}$$

$$= \underline{\underline{1/5}}$$

b) C.O.C

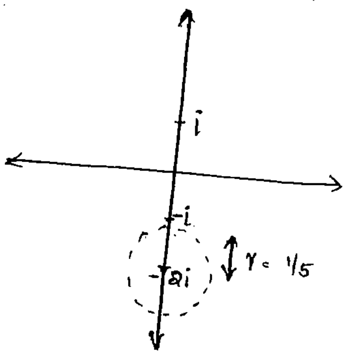
$|z-z_0| = r$

$|z-(-2i)| = \underline{\underline{1/5}}$

c) R.O.C

$|z-z_0| < r$

$\Longrightarrow |z-(-2i)| < \underline{\underline{1/5}}$



iii) $\sum_{n=0}^{\infty} \frac{2^n}{n!} (z-3)^n$ with $\sum_{n=0}^{\infty} a_n (z-z_0)^n$

Here $a_n = \frac{2^n}{n!}$ & $z_0 = 3$

a) $$r = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty}\left|\frac{2^n}{n!}\cdot\frac{(n+1)!}{2^{n+1}}\right| = \lim_{n\to\infty}\left|\frac{\cancel{2^n}}{\cancel{n!}}\frac{(n+1)\cancel{n!}}{\cancel{2^n}\times 2}\right|$$

$$= \lim\left|\frac{n+1}{2}\right| = \underline{\underline{\infty}}$$

b) C.O.C : $|z-z_0| = r \implies |z-3| = \infty$

c) R.O.C : $|z-z_0| < r \implies |z-3| < \infty$

Note 1 : i) If $f(z)$ is an analytic fn at z_0 (not a S.pt) $\implies$ Taylor series.

ii) If $f(z)$ is not an analytic fn at z_0 (S.pt) $\implies$ Laurentz series.

TAYLOR'S THEOREM

If a fn $f(z)$ is analytic at every point within a circle C, then for every pt z within the circle c, the fn $f(z)$ can be expressed as a power series in +ve powers of $(z-z_0)$ or about $z = z_0$

ie $$f(z) = f(z_0) + (z-z_0)f'(z) + \frac{(z-z_0)^2}{2!}f''(z_0) + \cdots + \frac{(z-z_0)^n}{n!}f^n(z_0) + \cdots$$

$$\sum_{n=0}^{\infty}\frac{(z-z_0)^n}{n!}f^n(z_0)$$

$$= \sum_{n=0}^{\infty} a_n(z-z_0)^n \quad \text{where } a_n = \frac{f^n(z_0)}{n!}$$

The RHS of above is called Taylor's series about $z = z_0$

The ROC of Taylor series is given by

$$|z-z_0| = r$$

where the radius of convergence r is a distance from a centre of the power series z_0 to its nearest singular point of the same fn $f(z)$.

Q. Find the Taylor series expansion of $f(z) = \frac{1}{z-2}$ about a point $z=1$. Hence find radius of convergence, C.O.C & ROC.

Ans Given $f(z) = \frac{1}{z-2}$, $z = 1$

Here $z_0 = 1$ & S.p.t $= 2$.

i) $r = |\, spt - z_0 | = |2-1| = 1$

ii) ROC : $|z - z_0| = r$

ie $|z-1| = 1$

iii) $|z - z_0| < r$

$|z-1| < 1$

Expansion :

$$f(z) = \frac{1}{z-2} \quad , \quad z = 1$$

Let $z - 1 = t$ then $z = 1+t$

$$f(z) = \frac{1}{t-1} = -(1-t)^{-1} \qquad |t|<1$$

$$= -[1 + t + t^2 + t^3 - - - - - + t^n + -$$

$$= (-1)[1 + (z-1) + (z-1)^2 + (z-1)^3 + \dots] \quad |z-1|<1$$

Laurent's Theorem

If a fn $f(z)$ is analytic at every point within a ring shaped region R bounded by two concentric circles C_1, C_2 having centre at z_0, with radii r_1, r_2 such that $r_2 < r_1$. then for every point z within R, the function $f(z)$ can be represented by a power series in both +ve & -ve powers of $z - z_0$ or about $z = z_0$.

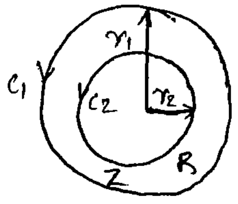
ie $$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} b_n (z - z_0)^{-n}$$

where $$a_n = \frac{1}{2\pi i} \oint_{C_1} \frac{f(z)}{(z - z_0)^{n+1}} dz$$

and $$b_n = \frac{1}{2\pi i} \oint_{C_2} \frac{f(z)}{(z - z_0)^{-n+1}} dz$$

The RHS of the above is called Laurent's series about $z = z_0$ and the ROC of a Laurents series is given by

$$r_2 < |z - z_0| < r_1$$



Property

- Expand $f(z) = \frac{e^{2z}}{(z-1)^2}$ as an infinite series about $z=1$ and also find ROC.

Ans Let $z-1 = t$

then $z = 1+t$

$$f(z) = \frac{e^{2(1+t)}}{t^2} = e^2 \times \frac{e^{2t}}{t^2}$$

$$= \frac{e^2}{t^2}\left[1 + 2t + \frac{(2t)^2}{2!} + \frac{(2t)^3}{3!} + \dots\right]$$

$$= e^2\left[\frac{1}{t^2} + \frac{2}{t} + 2 + \frac{4t}{3} + \dots\right]$$

$$= e^2\left[\frac{1}{(z-1)^2} + \frac{2}{(z-1)} + \frac{2^2}{2!} + \frac{2^3}{3!}(z-1) + \dots\right]$$

Therefore the above series is a Laurent's series about $z=1$ and entire complex plane is an ROC except $z=1$.

Q, Expand $f(z) = (z-3)\sin\frac{1}{(z+2)}$ as an infinite series about $z=-2$ and also find ROC.

Ans Let $z-(-2) = t$

then $z = t-2$

$f(z) = (t-5)\sin 1/t \qquad \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots$

$$f(z) = (t-5)\left[\frac{1}{t} - \frac{1}{t^3} * \frac{1}{3!} + \left(\frac{1}{t}\right)^5 \frac{1}{5!} + \dots\right]$$

$$= 1 - \frac{5}{t} - \frac{1}{3!\,t^2} + \frac{5}{3!\,t^3} + \dots$$

$$= 1 - \frac{5}{z+2} - \frac{1}{3!(z+2)^2} + \frac{5}{3!(z+2)^3} + \dots$$

Therefore the above power series is Laurent's series

Q2 Expand $f(z) = \frac{z}{(z+1)(z+2)}$ as an infinite series about $z=-2$.

Ans Let $z-(-2) = t \implies z = t-2$.

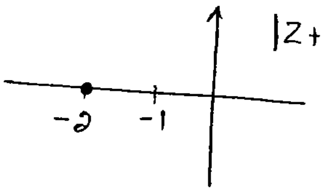
$$f(z) = \frac{t-2}{(t-1)t} = \frac{1-2}{t-1} - \frac{0-2}{t-0}$$

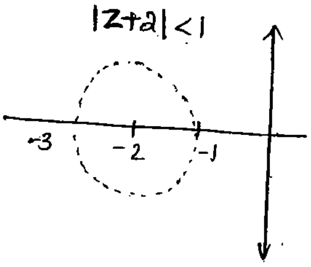
$$\frac{pz+q}{(z-a)(z-b)} = \frac{pa+q}{(a-b)(z-a)} - \frac{pb+q}{(a-b)(z-b)}$$

$$= \frac{2}{t} + \frac{1}{1-t} = \frac{2}{t} + (1-t)^{-1}$$

$$= \frac{2}{t} + 1+t+t^2+t^3+\dots \quad [|t|<1]$$

ie $f(z) = \frac{2}{z+2} + [1 + z+2 + (z+2)^2 + (z+2)^3 + \dots]$





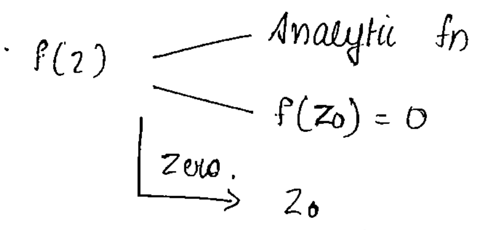
$\implies 0<|z+2|<1$

Therefore the ROC of above power series is $0<|z+2|<1$

Zeros & Types of Singular Points

1) Zero of an analytic function

If $f(z)$ is analytic at a point z_0 & $f(z_0)=0$ then the point z_0 is called zero of the fn $f(z)$.



ex: $f(z) = (z-3)^4$

Here the fn is analytic at $z=3$ and $f(3)=0$.

$z=3$ is a zero of $f(z)$

a) Order of Zero of Analytic fn

If $f(z)$ is an analytic fn at z_0 and $f(z_0)=0$, $f'(z_0)=0$, $f''(z_0)=0$ and so on... $f^{(m-1)}(z_0)=0$. but $f^{m}(z_0)\neq 0$ then the point z_0 is called zero of order m.

eg : ① $f(z) = (z-2)^3$

Here $z=2$ is a zero of $f(z)$.

$f'(z) = 3(z-2)^2$

$f'(2) = 0$

$f''(z) = 6(z-2)$

$f''(2) = 0$

$f'''(z) = 6$

$f'''(2) \neq 0$

Therefore $z=2$ is a zero of order 3.

ii) $f(z) = \sin z$

$$\sin z = \frac{e^{iz}-e^{-iz}}{2i}$$

$$\cos z = \frac{e^{iz}+e^{-iz}}{2}$$

$$\sinh z = \frac{e^{z}-e^{-z}}{2}$$

$$\cosh z = \frac{e^{z}+e^{-z}}{2}$$

$f(z) = \sin z$, $z = n\pi$, $n \in I$

$f'(z) = \cos z \neq 0$ at $z = n\pi$, $n \in I$

$\therefore z = n\pi$, $n \in I$ are first order zeros.

iii) $f(z) = \cos z$, $z = (2n+1)\pi/2$, $n \in I$

$f'(z) = -\sin z \neq 0$ at $z = (2n+1)\pi/2$, $n \in I$

$z = (2n+1)\pi/2$, $n \in I$ are simple zeros of $f(z)$

iv) $f(z) = \sinh z$, $z = -n\pi i$, $n \in i$ (simple zeros)

v) $f(z) = \cosh z$; $z = (2n+1)\pi/2 \, i$ (simple zeros)

vi) $f(z) = e^z$ ($\because e^x \neq 0 \ \forall x \in R = (-\infty, \infty)$)

$e^z \neq 0 \ \forall z \in C$)

vii) $f(z) = e^z - 1$ $z = 2n\pi i$, $n \in I$ (simple zeros)

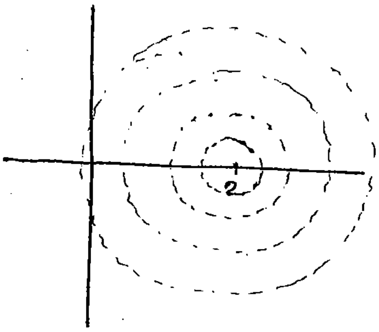
3) Singular Point

a) Isolated Singular point.

If z_0 is a singular point of $f(z)$ & $f(z)$ is analytic at every point except z_0 in atleast 1 neighbourhood of a point z_0, then the point z_0 is called Isolated singular point of $f(z)$.

eg : $f(z) = \dfrac{(z+4)^3}{z-2}$

S.pt is $z = 2 \longrightarrow$ Isolated S.pt



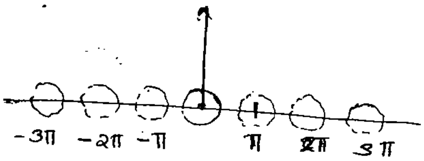
$f(z) = \dfrac{(z-2)^3(z-4)}{(z-5)^2(z-6)^3}$

S.pt is $z = 5, 6$

Atleast one region exists. So 5,6 are isolated S pts.

$f(z) = \dfrac{1}{\sin z}$

S pt : $z = n\pi$, $n \in I \implies \underbrace{z = 0, \pm\pi, \pm 2\pi - - - -}_{\text{Isolated pt.}}$



zero is isolated. π, 2π

we can't find any other singular point in this region. So isolated s.p.

Type of Singular Point

If z_0 is a singular point of $f(z)$ then

$$f(z) = \underbrace{\sum_{n=0}^{\infty} a_n (z-z_0)^n}_{\text{Analytic point (A.P)}} + \underbrace{\sum_{n=1}^{\infty} b_n (z-z_0)^n}_{\text{Principal point (P.P)}}$$

$$f(z) - \underset{\downarrow\; s\cdot pt}{Z_0} \longrightarrow LS = A\cdot P \longrightarrow ①$$

$$= P\cdot P \quad \Big\}\text{—②} \quad \text{Removable poles of order } m$$

$$= A\cdot P + P\cdot P \quad \Big\}\text{—③} \quad \text{Essential.}$$

① Removable s.pt

If the principal part of Laurentz series expansion of f(z) about $Z = Z_0$ doesnot exist then the singular point Z_0 is called Removable s.p of f(z)

eg : $f(z) = \dfrac{\sin z}{z}$

② Pole (of order m)

If the principal part of Laurentz expansion of f(z) about $Z = Z_0$ ~~exists~~ contains finite no. of -ve powers of $(Z - Z_0)$ then the s.pt 'Z_0' is called pole of order m ie say m terms.

eg : $f(z) = \dfrac{e^z}{z}$

$Z = 0$ is the singular point.

Now expand about this point.

$$= \frac{1}{Z}\left\{ 1 + Z + \frac{Z^2}{2!} + \frac{Z^3}{3!} + \cdots \right\}$$

$$= \frac{1}{Z-0} + 1 + \frac{Z-0}{2!} + \frac{(Z-0)^2}{3!} + \cdots$$

$$= 0 \times \frac{1}{(Z-0)^2} + 0 \times \frac{1}{(Z-0)} + \frac{1}{(Z-0)^3}$$

∴ $Z = 0$ is a pole of order 1 (simple pole).

③ Essential Singular Points (∞ no of -ve points)

If the principal part of Laurentz series expansion of f(z) about $Z = Z_0$ contains infinite number of -ve powers of $(Z - Z_0)$ then the singular point Z_0 is called essential singular point of f(z).

$$* \quad f(Z) = (Z-4)\, \sin\left(\frac{1}{Z-4}\right)$$

$$= (Z-4)\left\{ \frac{1}{Z-4} - \frac{1}{(Z-4)^3 3!} + \frac{1}{(Z-4)^5 5!} + \cdots \right\}$$

Q1. $f(z) = \dfrac{(z-4)^3(z-6)^2}{(z-5)^{10}(z-7)^5}$

i) $\dfrac{Nr}{Dr}$

ii) s.pt $\longrightarrow$ $5 \longrightarrow$ order 10

$7 \longrightarrow$ order 5

iii) $Nr \neq 0$ at s.pts.

$\therefore$ 5 & 7 are poles.

Q2. $f(z) = \dfrac{\sin z}{z - \pi/2}$

i) Nr/Dr.

ii) s pt $\longrightarrow$ $\pi/2$ order 1

iii) $Nr \neq 0$ at $z = \pi/2$ $\therefore$ $z = \pi/2$ pole of order 1 (simple pole)

Q3. $f(z) = \tan z$

$= \dfrac{\sin z}{\cos z}$

i) $\dfrac{Nr}{Dr}$

ii) s.pt $\longrightarrow$ $\pi/2\,(2n+1)$ $n \in I$

∞ no of singular points.

iii) $Nr \neq 0$ at s. pt.

$\therefore$ $(2n+1)\pi/2$ pole of order 1.

Q4. $f(z) = \dfrac{\cos z}{z - \pi/2} = \dfrac{\phi(z)}{(z-z_0)^m}$ $m = 1$

s. pt $= \pi/2$

$Nr = \cos z = 0 \longrightarrow$ pole or removable singular pt.

$m = 1$, $n = ?$

$\phi(z) = \cos z = 0$ at $z = \pi/2$

$\phi'(z) = -\sin z \neq 0$ at $z = \pi/2$

Therefore $z = \pi/2$ is a pole of order 1

$\therefore n = 1$.

$\therefore m = n = 1 \implies z = \pi/2$ is a removable s.pl.

Q5. $f(z) = \dfrac{1 - \cos z}{z}$

S.pt $\longrightarrow z = 0$

Nr $= 0$ at $z = 0$

$f(z) = \dfrac{1 - \cos z}{z} = \dfrac{\phi(z)}{(z - z_0)^m}$ $m = 1$.

$\phi(z) = 1 - \cos z = 0$ at $z = 0$

$\phi'(z) = \sin z = 0$ at $z = 0$

$\phi''(z) = \cos z \neq 0$ at $z = 0$.

$\therefore z = 0$ is a pole of order $2 = n$.

$m \neq n$, $n > m \implies z = 0$ is a removable s.pt.

Q6. The fn $f(z) = \dfrac{1 - e^{2z}}{z^4}$

Ans s.pt $\longrightarrow z = 0$ order $m = 4$

Nr $= 0$ at $z = 0$

$f(z) = \dfrac{\phi(z)}{(z - z_0)^m}$ $m = 4$.

$\phi(z) = 1 - e^{2z}$

$\phi'(z) = 1 - 2e^{2z} \neq 0$ at $z = 0$

$\therefore n = 1$

ie $m > n \implies z = 0$ is a pole of order 3

Q7. $f(z) = (z - 4) \sin\left(\dfrac{2}{z - 4}\right)$

$f(z) \neq \dfrac{\text{Nr}}{\text{Dr}}$

$$f(z) = (z-4)\left[\frac{2}{z-4} - \frac{2^3}{3!(z-4)^3} + \frac{2^5}{5!(z-4)^5} - \cdots \right]$$

$$= 2 - \frac{2^3}{3!(z-4)^2} + \frac{2^5}{(z-4)^4 4!} - \cdots$$

S.pt $\longrightarrow$ $z = 4$ $\Longrightarrow$ Essential singular point

Q. Find the residue of following functions.

i) $f(z) = \dfrac{z}{z^2+4}$

Ans $z^2+4 = 0$

$z = \pm 2i \longrightarrow$ S.pt, Simple poles

R_1 : Res $\{ f(z) : z = 2i \}$

$$= \lim_{z\to 2i} [(z-2i)f(z)] = \lim_{z\to 2i} \frac{(z-2i)\times z}{(z-2i)(z+2i)} = \frac{2i}{4i} = \underline{\underline{1/2}}$$

R_2 : Res $\{ f(z) : z = -2i \}$

$$= \lim_{z\to 2i} [(z+2i)f(z)]$$

$$= \frac{(z+2i)z}{(z+2i)(z-2i)} = \frac{-2i}{-2i-2i} = \underline{\underline{1/2}}$$ Ans

ii) $f(z) = \dfrac{\cos z}{z-\pi} = \dfrac{\phi(z)}{z-z_0}$

S.pt $z = \pi \longrightarrow$ Simple pole.

$\therefore$ Residue $\phi(\pi) = \underline{\underline{-1}}$

iii) $f(z) = \cot z = \dfrac{\cos z}{\sin z} = \dfrac{\phi(z)}{\psi(z)}$

S.pt $\longrightarrow$ $z = n\pi$ $\quad n \in I$

$$\text{Res}\{ f(z) : z = z_0 \} = \frac{\phi(n\pi)}{\psi'(n\pi)} = \frac{\cos n\pi}{\cos n\pi} = \underline{\underline{1}}$$

iv) $f(z) = \dfrac{e^z + \cos z}{(z-\pi)^2}$

$z = \pi \longrightarrow$ pole of order $2 = m$

$$R_1 = \frac{1}{(m-1)!} \lim_{z \to z_0} \left[\frac{d^{m-1}}{dz^{m-1}} \left((z-z_0)^m f(z) \right) \right]$$

$$= \frac{1}{(2-1)!} \lim_{z \to \pi} \left\{ \frac{d}{dz} (z-\pi)^2 \frac{e^z + \cos z}{(z-\pi)^2} \right\}$$

$$= \lim_{z \to \pi} \left\{ e^z - \sin z \right\} = \underline{\underline{e^\pi}}$$

v) $f(z) = \dfrac{\cos z}{z - \pi/2} = \dfrac{\phi(z)}{(z-z_0)^m}$ $\quad m = 1$

$z = \pi/2 \longrightarrow$ pole of order 1.

Nr $= 0$ at $z = \pi/2$

$\phi(z) = \cos z$

$\phi'(z) = -\sin z \neq 0 \quad \therefore n = 1$

$\because m = n = 1 \implies z = \pi/2 \longrightarrow$ Removable s.pt.

$\therefore R_1 = \text{Res}\left[f(z) : z = \pi/2 \right] = 0$

vi) $f(z) = (z-2)e^{3/z-2} \neq \dfrac{Nr}{Dr}$

$$= (z-2)\left[1 + \frac{3}{z-2} + \frac{3^2}{(z-2)^2} + \frac{3^3}{3!(z-2)^3} + \cdots \right]$$

$z = 2$ Essential s.pt (∞ no. of -ve powers of z)

$\therefore R_1 = \text{Res}\left\{ f(z) : z = 2 \right\} = b_1 = \dfrac{3^2}{2!} = \underline{\underline{9/2}}$

Q. Evaluate $\oint_C \dfrac{z}{(z-1)(z-2)^2} dz$, $C : |z-2| = 1/2$

Method :

1) Let $f(z)$ = Integrant fn.

2) Find singular pt of $f(z)$

3) C :

Choose region

4) ~~cohen~~ Check whether the s-pt is within the region or not.

No s-pt within the region $\implies$ CI ? $\oint f(z) dz = 0$

$$= 1 - \frac{1}{(z-4)^2 3!} + \frac{1}{(z-4)^4 5!} + \cdots$$

S.pt $\quad z = 4 \longrightarrow$ Essential singular point.

5) Residues

i) Residue of $f(z)$ at $z = z_0$

If z_0 is an isolated singular point of $f(z)$, then the coefficient of $\frac{1}{z-z_0}$ in Laurents series of $f(z)$ about $z = z_0$ is called residue of $f(z)$ and it is denoted by $Res[f(z) : z = z_0]$.

Therefore $Res[f(z) : z = z_0] =$ The coefficient of $\frac{1}{z-z_0}$ in Laurents series.

$$= b_1 = \frac{1}{2\pi i} \oint_c f(z) dz.$$

ii) Cauchy's Residue Theorem (C.R.T)

If $f(z)$ is analytic at every point within and on a simple closed curve c except at a finite number of isolated singular points $z_1, z_2, \ldots z_n$ within c, then

$$\oint_c f(z) dz = 2\pi i \left(\sum_{j=1}^{n} R_j \right) \qquad \text{where } R_j = Res(f(z) : z = z_j)$$

Methods To find Residues

① Removable Singular Point

If z_0 is a removable s.pt of $f(z)$ then $Res(f(z) : z = z_0) = b_1 = 0$

② Essential Singular Point

If the point z_0 is an essential spt of $f(z)$ then expand $f(z)$ as a Laurentz series, about $z = z_0$. and collect the coefficient of $\frac{1}{z-z_0}$ in the Laurentz series which gives the residue of $f(z)$

③ Pole

a) If $f(z) = \frac{P(z)}{Q(z)}$ has simple pole at z_0, then.

$$\left[Res[f(z) : z = z_0] = \lim_{z \to z_0} [(z - z_0) f(z)] \right]$$

b) If $f(z) = \frac{\phi(z)}{\psi(z)}$ has simple pole at z_0.

then $\left[\text{Res } (f(z) : z = z_0) = \frac{\phi(z_0)}{\psi'(z_0)} \right]$

where $\phi(z_0) \neq 0$ and $\psi'(z_0) \neq 0$.

c) $f(z) = \frac{\phi(z)}{z - z_0}$ has simple pole at z_0, then

$\left[\text{Res } [f(z) : z = z_0] = \phi(z_0) \right]$ where $\phi(z_0) \neq 0$

d) If $f(z)$ has pole at z_0 of order m, then

$$\text{Res } [f(z) : z = z_0] = \frac{1}{(m-1)!} \lim_{z \to z_0} \left[\frac{d^{m-1}}{dz^{m-1}} \left\{ (z - z_0)^m f(z) \right\} \right]$$

Note :

$f(z) = \frac{P(z)}{Q(z)}$ Nr & Dr are polynomials of z.

$= \frac{\phi(z)}{\psi(z)}$ Nr & Dr are not polynomials of z.

Problems

1) Classify the singular points of the following functions.

Method

1) $f(z) = \frac{Nr}{Dr}$

2) Find the singular points of $f(z)$ [ie zeros of Dr fn]

3) Nr (at singular pts) $\begin{cases} \neq 0 \implies \text{Poles} \\ = 0 \implies \text{pole or removable s.pt} \end{cases}$

If $m > n \implies$ pole order $= m - n$

$m \leq n \implies$ removable s:pt.

where $n \longrightarrow$ order of zero of Nr.

$m \longrightarrow$ " " " " Dr.

One or more s.pt within the region ~~$\Rightarrow$ CIT~~ Goto step ⑤.

5) Classify the singular points. ~~$\oint f(z)dz$.~~

6) Find the residues.

7) Substitute residue in CIT.

$$\oint_c f(z)dz = 2\pi i(R_1+R_2+\ldots+R_n)$$

Ans Let $f(z) = \dfrac{z}{(z-1)(z-2)^2}$

s.pt : $z = 1, 2$

$C: |z-2| = 1/2$

$z=1$ outside c $> 1/2$

$z=2$ inside c $< 1/2$

$z=2 \longrightarrow$ pole of order 2.

$Nr \neq 0$ at $z=2$.

$$R_1 = Res\{f(z) : z=2\} = \frac{1}{(2-1)!} \lim_{z\to 2}\left[\frac{d}{dz}(z-2)^2 \times \frac{z}{(z-1)(z-2)^2}\right\}$$

$$= \lim_{z\to 2} \frac{d}{dz}\left[\frac{z}{z-1}\right]$$

$$R_1 = \lim_{z\to 0}\left[\frac{-1}{(z-1)^2}\right] = \underline{\underline{-1}}$$

$\therefore$ By C.R.T, we have $\oint_c f(z)dz = 2\pi i R_1 = 2\pi i \times -1 = \underline{-2\pi i}$

Q. Evaluate $\oint_c z^2 e^{1/z} dz$ along a unit circle $|z|=1$.

Ans $f(z) = z^2 e^{1/z}$

$$= z^2\left[1+\frac{1}{z}+\frac{1}{2!z^2}+\frac{1}{3!z^3}+\ldots\right]$$

$$= z^2+z+\frac{1}{2}+\frac{1}{3!z}+\frac{1}{4!z^2}+\ldots\Big]$$

$$= (z-0)^2+(z-0)+\frac{1}{2}+\frac{1}{3!(z-0)}+\ldots$$

s.pt : $z=0 \longrightarrow$ Essential s.pt.

C : $|z| = 1$

$z = 0$ lies inside C.

Coeff. of $\frac{1}{z - z_0} = \frac{1}{3!}$

$\therefore R_1 = \text{Res}\{f(z) : z = 0\} = \frac{1}{3!} = \underline{1/6}$

$\therefore$ By C.R.T we have $\oint_C f(z)dz = 2\pi i \times R_1 = \underline{\frac{\pi i}{3}}$

<u>OR</u>

By CIF, $\int_C \frac{f(z)}{(z - z_0)^{n+1}} = \frac{2\pi i}{n!} f^n(z_0)$

$f(z) = \underline{ze^{1/z}}$

This will give no result. So CIF is not defined for all the integrals.

<u>Note</u> : So for evaluating $\int$als first find the singular point within the region and apply CIF (Easier)

If it is not applicable apply Residue Theorem.

<u>Isolated Essential Singular Point</u>

The limit points of zeros of f(z) is an isolated essential singular point.

Eg : 1) $f(z) = \sin\left(\frac{3}{z-4}\right)$

The zeros of f(z) are given by $\frac{3}{z-4} = n\pi$

$$\Longrightarrow z - 4 = \frac{3}{n\pi}$$

$$\Longrightarrow z = 4 + \frac{3}{n\pi} \qquad n \in I$$

∞ no of zeros.

Let $z_n = 4 + \frac{3}{n\pi}$, $n \in I$

$\therefore$ Isolated essential singular point of f(z)

$$= \lim_{n \to \infty} z_n$$

$$= \lim_{n \to \infty} 4 + \frac{3}{n\pi} = \underline{4}$$

$$= \sin\left(\frac{3}{z-4}\right) = \frac{3}{z-4} - \frac{3^3}{(z-4)^3 3!} + \frac{3^5}{(z-4)^5 5!} + \dots$$

S. pt $z = 4 \longrightarrow$ Essential singular point.

Single essential S. pt $\Longrightarrow$ Isolated S. pt.

Non Isolated Essential S. pt

The limit points of poles of $f(z)$ is a non isolated essential singular

Eg : 1) $f(z) = \dfrac{1}{\sin(\pi/z)}$

S. pt : $\pi/z = n\pi$, $n \in I$

$z = 1/n$ $n \in I \longrightarrow$ poles $\rightarrow \infty$ nos.

$\therefore$ There non isolated S. pt of

$f(z) = \lim_{n\to\infty} z_n = \lim_{n\to\infty} (1/n) = \underline{\underline{0}}$

Exercise Qns

Q20. $Im\{if'(z)\} = 12xy - 6x$, $f(0) = 3-2i$. $f(1) = 6-5i$ $f(1+i) = 9$

Ans $f'(z) = u_x + iv_x$

$F(z) = u + iv$

$F'(z) = u_x + iv_x$

$F'(z) = v_y + iv_y$

$F'(z) = 12x + i(12y - 6)$

$F'(z) = 12z + i(-6)$

$F(z) = 6z^2 - i6z + c$

$f'(z) = 6z^2 - i6z + c$

$f(z) = 2z^3 - i3z^2 + cz + K$

$f(0) = 3-2i$

$K = 3-2i$

$f(1) = 6-5i \Longrightarrow 6-5i = 2-3i + c + 3-2i$, $1 = c$

$\therefore f(z) = 2z^3 - i3z^2 + z + 3 - 2i \Longrightarrow f(1+i) = 2(1+i)^3 - i3(1+i)^2 + (1+i) + 3 - 2i$

$= 6+8i$