Probability & Statistics

#1 Sample space :

$1 \text{ coin} = 2^1$

$2 \text{ coin} = 2^2$

$\vdots$

$n \text{ coin} = 2^n$

~~Ex~~ Throwing a dice 10 times $= 6^{10}$

Throwing a pair of dice 10 times $= 36^{10} \approx 6^{20}$

2.

i) $0 \leq P(E) \leq 1$

ii) $P(E) = 1 - P(\bar{E})$

3.

$^nC_r = \frac{n!}{r!\,(n-r)!}$ $\qquad$ $^nP_r = \frac{n!}{(n-r)!}$

- $^nC_0 = 1$
- $^nC_n = 1$
- $^nC_r = {^nC_{n-r}}$

4. Types of event :

i) independent event : occurance of 1 event does not effect another event.

$P(E_1 \cap E_2) = P(E_1) \cdot P(E_2)$

ii) Mutually exclusive event : $E_1 \cap E_2 = \phi$

$P(E_1 \cap E_2) = 0$

iii) Exhaustive event :

- $E_1 \cup E_2 = S$
- $E_1 \cup E_2 \cup E_3 = P(S) = 1$

5. Alzebra of events

i) not A : $\bar{A}$

ii) Both A & B : $A \cap B$

iii) A or B
(atleast A or B) : $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

iv) Neither A nor B : $\overline{A \cup B}$ or $\bar{A} \cap \bar{B}$

$P(\overline{A \cup B}) = 1 - P(A \cup B)$

v) A but not B : $A \cap \bar{B}$

vi) Exactly one of A & B : $(A \cap \bar{B}) \cup (\bar{A} \cap B)$

vii) All of A, B, C occur : $A \cap B \cap C$

viii) Atleast one of A, B, C : $\cancel{P}(A \cup B \cup C)$

$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) - P(A \cap B \cap C)$

ix) Exactly two of A, B, C : $(A \cap B \cap \bar{C}) \cup (A \cap \bar{B} \cap C) \cup (\bar{A} \cap B \cap C)$

x) A or B for
(for mutually exclusive event) : $P(A \cup B) = P(A) + P(B)$

$(P(A \cap B) = 0)$

6. Problems on Cards & balls :

i) By default: items are drawn simultaneously (if not given)

ii) • One by One withdrawn :- case of 1^{st} ball, 2^{nd} balls etc.

• Simultaneously :- No case of 1^{st} ball or 2^{nd} ball.

iii) By default :- without replacement

iv) imp • Withdrawn one by one : Fraction method

• Simultaneously : Combination method.

Face cards :- K, Q, J

Honoured card : Ace, K, Q, J.

♣ club

7. ~~Conditional Probability~~

a = favourable , b = unfavourable.

Odds in favour = $\frac{a}{b}$

Odds against = $\frac{b}{a}$

* If a+b is sample space then,

$P(E) = \frac{\text{Odds in favour}}{\text{Odds in favour + Odds against}}$

#8. Conditional Probability : $P(B/A)$ = conditional probability of B given A has already occured.

$$\boxed{P\left(\frac{B}{A}\right) = \frac{P(A \cap B)}{P(A)}} \quad \& \quad \boxed{P\left(\frac{A}{B}\right) = \frac{P(A \cap B)}{P(B)}}$$

* If A & B are independent event : $P(A \cap B) = P(A) \cdot P(B)$.

$$\boxed{P\left(\frac{A}{B}\right) = P(A)} \qquad \boxed{P\left(\frac{B}{A}\right) = P(B)}.$$

#9. Total probability :- $E_1, E_2 \ldots$ are Ex & Exhau.

$$P(A) = P(E_1) \times P(A/E_1) + P(E_2) P(A/E_2) + \ldots\ldots$$

$$P(A) = P(E_1 \cap A) + P(E_2 \cap A) + \ldots\ldots\ldots P(E_n \cap A).$$

#10. Bayes Theorem :- $E_1, E_2 \ldots$ are Exclusive & Exhaustive

$$P\left(\frac{E_i}{A}\right) = \frac{P(E_i) \times P(A/E_i)}{P(E_1) \times P(A/E_1) + P(E_2) P(A/E_2) + \ldots\ldots}$$

#1. Random variables and distribution — discrete r.v / continuous r.v

Random variable — toss 2 coins, X = the no of times H comes up

X :	0	1	2
P(X)	$\frac{1}{4}$	$\frac{2}{4}$	$\frac{1}{4}$

Distribution

i) Discrete distribution or discrete probability distribution (also called propability mass func^n)

i) $p(x) = P[X = x_i]$

ii) $P(x_i) \geq 0 \quad \forall x_i$

iii) $\sum_i P(X = x_i) = 1$

iv)* $E(x) = \sum x P(X)$

x	x_1	x_2
$p(x)$		

ii) Continuous probability distribution (Probability density func^n)

i) $P(a < x < b) = \int_a^b F(x)\,dx$

ii) $F(x) \geq 0$

iii) $\int_{-\infty}^{\infty} F(x)\,dx = 1$ or over x

#2. Expectation, variance, standard deviation of random variable X

i) Discrete distribution : $\boxed{E(x) = \sum_{i=1}^{n} x_i P(x_i)}$

ii) Continuous distribution : $\boxed{E(x) = \int_{-\infty}^{\infty} x f(x)\,dx}$

Variance : $V(x) = E(x-\bar{x})^2 = E(x^2) - \{E(x)\}^2$

$= \int_{-\infty}^{\infty} t^2 f(t)\,dt$

Standard deviation : $\sigma_x = \sqrt{V(x)}$

* $\mu = E(t)$.

#

1. Binomial Distribution : (n) - no of trials is large. $n \to \infty$

$$\boxed{P(r) = {}^nC_r\, p^r q^{n-r}}$$

, n = no of trials.
$p = P(\text{success})$, $q = P(\text{failure})$
r = no of times success is achieved

Mean ($\bar{x}$) : $\boxed{\mu = np}$

Variance : $\boxed{V(x) = npq}$

2. Poission's distribution :
i) $n \to \infty$ (very large) 50, 100, 1000
ii) $p \to 0$ (very small

$$\boxed{P(r) = \frac{e^{-\lambda}\lambda^r}{r!} \quad \text{where } \lambda = np = \mu}$$

n = no of trials

** $\boxed{\left.\begin{array}{l}\mu = np \\ V(x) = np\end{array}\right\} = \lambda}$ **

np = finite.

3. Normal distribution
i) n is very large
ii) p is not very small

$$\boxed{P(X=x) = \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-\frac{(x-\mu)^2}{2\sigma^2}}}$$

μ, σ

* $\phi(x) = \frac{1}{\sqrt{2\pi x^2}}\, e^{-x^2/2}$ (z)

$\mu = 0$, $\sigma = 1$

standard normal distribution

μ = mean

σ = standard deviation. $= \sqrt{V(x)} = \sqrt{npq}$

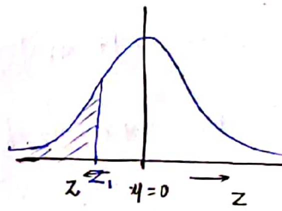
$$\boxed{\begin{array}{l}\mu = np \\ V(x) = npq\end{array}}$$

Standard normal distribution (z)

Mean: $\mu = 0$
Variance $V(x) = 1$

$$\boxed{Z = \frac{x-\mu}{\sigma}}$$

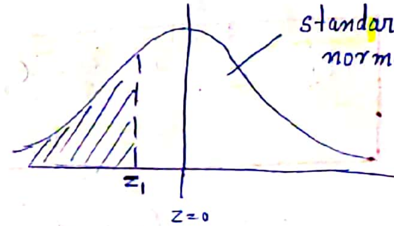
σ = S.D



Total Area = 1

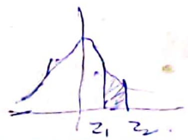
i) $P(X < x)$

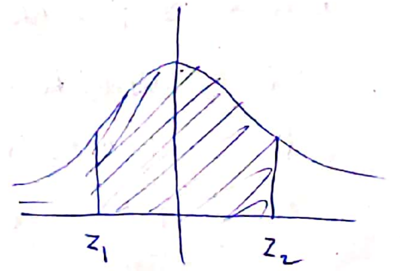
$z_1 = \frac{x-\mu}{\sigma}$



ii) $P(a < X < b)$

$z_1 = \frac{a-\mu}{\sigma}$, $z_2 = \frac{b-\mu}{\sigma}$





Statistics

1. Mean.

i) ungrouped data : $\bar{x}_{UGD} = \frac{\sum_{i=1}^{n} x_i}{n}$

ii) grouped data : $\bar{x}_{GD} = \frac{\sum f_i x_i}{\sum f_i}$

2. Median : First arrange data in increasing or decreasing order.

i) ungrouped data :

a) odd no of obs : Median $= \left(\frac{n+1}{2}\right)^{th}$ value.

b) even no of obs : Median $= \frac{\frac{n}{2}^{th} \text{ value} + \left(\frac{n}{2}+1\right)^{th} \text{ value}}{2}$

ii) grouped data :

$$\boxed{M_d = l + \frac{\left(\frac{N}{2} - C_f\right) \times h}{F}}$$

l = lower limit of median class

N = $\sum f_i$ = total no of obs

C_f = cummulative freq. of class preceeding the median class

h = class interval or height

F = Freq of median class.

3. Mode - Arrange in asc. or desc order.

i) ungrouped data :- Most frequently repeated obs.

ii) grouped data :-

$$\boxed{M_o = l + \left(\frac{f_1 - f_0}{2f_1 - f_0 - f_2}\right) \times h}$$

l = lower limit of modal class

f_1 = freq of modal class

f_0 = freq of class preceeding the modal class

f_2 = freq of class succeeding the median class

h = high of modal class.

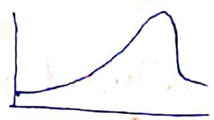
Properties of mean, median & mode.

$$\boxed{\text{Mode} = 3\,\text{Median} - 2\,\text{Mean}}$$

i) +vely skewed distribution : $M_o \leq M_d \leq M_e$

ii) Symmetric distribution $M_o = M_d = M_e$

iii) -vely skewed distribution $M_o \geq M_d \geq M_e$



- Skewness = opposite of symmetry.
- Symmetric curve ∵ skewness = 0
- Standard normal curve : skewness = 0.
- +vely or -vely skewed curve : skewness ≠ 0

Variance : sum of squares of the deviation from mean.
($V(x) = \sigma_x^2$)

i) ungrouped data :-

$$\sigma_x^2 = \frac{\sum (x_i - \bar{x})^2}{n} = \frac{1}{n}\sum x_i^2 - \bar{x}^2$$

$\bar{x}$ = mean of obs

n = total obs.

ii) For grouped data :

$$\sigma_x^2 = \frac{1}{N}\sum f_i x_i^2 - \left(\frac{1}{N}\sum f_i x_i\right)^2 , \quad N = \sum f_i$$

Note :-

i) $V(x) = \sigma_x^2$ never -ve

ii) $V(x)$ of const = 0

iii) $\sum_{i=1}^{n} (x_i - \bar{x}) = 0$

Standard deviation :- (σ)

$$\boxed{S.D = \sqrt{V(x)}}$$

Coefficient of variation (CV) - relative measure of dispersion

$$\boxed{CV = \frac{\sigma}{\mu}}$$

S.D

Mean

Regression line ~~...~~ establish cause and effect relation b/w dependent & independent variable.

Regression line of y on x-axis :-

$$Y = a + bX$$

$$a = \frac{\sum y + b\sum x}{n} \qquad b = \frac{n\sum xy - \sum x \sum y}{n(\sum x^2) - (\sum x)^2}$$

n = no of obs.

Complex Number

1. **Properties**

1. Sum of consecutive 4 powers of iota (i) is 0

$$i^n + i^{n+1} + i^{n+2} + i^{n+3} = 0$$

2. $|z| = |a+ib| = \sqrt{a^2+b^2}$

3. $\arg(z) = \tan^{-1}(b/a)$ $\qquad \angle a+ib = \arg(z)$

2. Polar form of complex no:-

$$z = a+ib = r(\cos\theta + i\sin\theta) = re^{i\theta}$$

i) $a = r\cos\theta$

$b = r\sin\theta$

ii) $r = \sqrt{a^2+b^2}$ = modulus

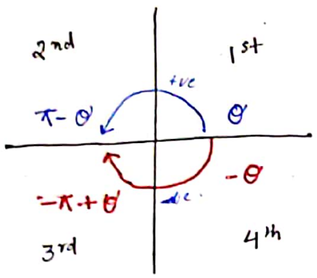
$\theta = \tan^{-1}(b/a)$ = argument

\#

$\frac{\pi}{2}+\theta$ or $\pi-\theta$	$\frac{\pi}{2}-\theta$ or θ
$\pi+\theta$ or $\frac{3\pi}{2}-\theta$	$2\pi-\theta$ or $\frac{3\pi}{2}+\theta$

for normal case

3. # Steps to determine the argument of a complex no.



for complex use this

3.

1. Determine value of the angle that lies in the 1^{st} quadrant according to magnitude (do not consider sign eg $-(1), -(\sqrt{3})$)

2. Identify the quadrant in which ~~angle is~~ $\tan\theta$ is +ve or -ve & use above sign conversion for complex no.

3: ~~The final ans~~

3. The final ans of the argument depends upon the location of complex no.

$z = a+ib$,

$a>0,\ b>0$	-	1^{st} quad.
$a<0 .\ b>0$	-	2^{nd} "
$a<0\ \ b<0$		3^{rd} "
$a>0\ \ b<0$		4^{th} "

4. Euler's form:

$$z = a+ib = r(\cos\theta + i\sin\theta) = re^{i\theta}$$

i) * $\cos\theta = \dfrac{e^{i\theta}+e^{-i\theta}}{2}$, $\sin\theta = \dfrac{e^{i\theta}-e^{-i\theta}}{2i}$

* hyperbolic func:-

$\cosh x = \dfrac{e^{x}+e^{-x}}{2}$, $\sinh x = \dfrac{e^{x}-e^{-x}}{2}$.

ii) De Moivre's theorem for complex no

$$\boxed{(\cos\theta + i\sin\theta)^n = (e^{i\theta})^n = e^{in\theta} = \cos n\theta + i\sin n\theta}$$

5. Properties of complex no

. Let $z_1 = a_1+ib_1$ & $z_2 = a_2+ib_2$

i) $z_1 = z_2 \Rightarrow a_1 = a_2$ & $b_1 = b_2$, ($r_1 = r_2$ & $\theta_1 = \theta_2$ in polar form)

ii) Inequality sign i.e $z_1 > z_2$ or $z_1 < z_2$ has no meaning.
But $|z_1| > |z_2|$ or $|z_1| < |z_2|$ has meaning.

iii) $z = x+iy$, $\bar{z} = x-iy$

iv) a) $z+\bar{z} = 2\,Re(z)$

b) $z-\bar{z} = 2\,Img(z)$

c) $z.\bar{z} = x^2+y^2$

v) 1. $\overline{z_1+z_2} = \bar{z}_1 + \bar{z}_2$

2. $\overline{z_1 z_2} = \bar{z}_1 . \bar{z}_2$

3. $\overline{z^n} = (\bar{z})^n$

vi)

1: $\cancel{|z|=|z_1 z_2|} = \cancel{|z_1||z_2|}$, $\cancel{\arg(z)} = 0$

If $z = z_1 z_2$., then

$|z| = |z_1|.|z_2|$ & $\arg(z) = \arg(z_1) + \arg(z_2)$

2 If $z = \frac{z_1}{z_2}$ .., then.

$|z| = \frac{|z_1|}{|z_2|}$, $\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$.

3. If $z = z_1^n$.

$|z^n| = |z|^n$, $\arg(z^n) = n.\arg(z)$.

4 imp $\ln(-a) = \ln(-1 \times a) = \ln(-1) + \ln(a)$

$= \ln(e^{i\pi}) + \ln(a)$

$= i\pi + \ln(a)$

$\boxed{\ln(-1) = \ln(e^{i\pi}) = i\pi}$ $-1 = e^{i\pi}$

vii)

1. $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2\,Re(z_1 \bar{z}_2)$

2. $|z_1 - z_2|^2 = |z_1|^2 + |z_2|^2 - 2\,Re(z_1 . \bar{z}_2)$

#6. Triangular inequalities in complex no.

1. $|z_1 + z_2| \leq |z_1| + |z_2|$

2. $|z_1 + z_2| \geq \big||z_1| - |z_2|\big|$

#7. Cube root of unity $z^3 = 1$, $z = 1^{1/3}$, $(z-1)(z^2+z+1) = 0$.

$z = 1, \frac{-1+i\sqrt{3}}{2}, \frac{-1-i\sqrt{3}}{2}$

$1, \omega, \omega^2$

$e^{i\frac{2\pi}{3}}$, $e^{i\frac{4\pi}{3}} = e^{(i\frac{2\pi}{3})^2}$

i) $|\omega| = |\omega^2| = 1$

ii) $\omega^{3n} = 1$

iii) $1 + \omega^n + \omega^{2n} = \begin{cases} 3 & , n = 3k , k \in Z \\ 0 & . \text{ else} \end{cases}$

iv) $1 + \omega + \omega^2 = 0$

Numerical Method

I Solution of transidental eq^n ($f(x)=0$)

1. Bisection method (Method of averages)

$$\underset{\displaystyle x_0}{\underset{\downarrow}{f(x)}} = 0$$

i) at $x=a$, $f(a) \lesssim 0$ & at $x=b$, $f(b) > 0$

$$x_1 = \frac{\overset{-ve}{a} + \overset{+ve}{b}}{2},$$

ii) $f(x_1) \lesssim 0$, $x_2 = \frac{x_1+b}{2}$ ....

2. Regular Falsi method (Method of chord), Interpolation, false position

$$\boxed{x_1 = a - \left(\frac{b-a}{\underset{+ve}{f(b)} - \underset{-ve}{f(a)}}\right) f(a)}$$

$$x_1 = \frac{a f(b) - b f(a)}{f(b) - f(a)} *$$

$$\left\{\begin{array}{l} a = \text{the root at which } f(x) = -ve \\ b = \text{the root at which } f(x) = +ve \end{array}\right\} *$$

3. Secant method

It is same as Regular Falsi method but in this we replace only one root throughout.

4. Newton Rapson method (Method of tangent)

$$\boxed{x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}}$$

* if initial value a & b given.
then, $x_0 = \frac{a+b}{2}$

* can solve real roots & complex roots

Newton Rapson iterative formula or recurrence formula.

1. Iterative formula to find $\frac{1}{N}$ is $\quad x_{n+1} = x_n(2 - Nx_n)$

2. Iterative formula to find $\sqrt{N}$ is $\quad x_{n+1} = \frac{1}{2}\left(x_n + \frac{N}{x_n}\right)$

3. Iterative formula to find $\frac{1}{\sqrt{N}}$ is $\quad x_{n+1} = \frac{1}{2}\left[x_n + \frac{1}{N x_n}\right]$

4. Iterative formula to find $\sqrt[k]{N}$ is $\quad x_{n+1} = \frac{1}{k}\left[(k-1)x_n + \frac{N}{x_n^{k-1}}\right]$

Method: i) $f(x) - N = 0$ form.

ii) $F(x) = f(x) - N$

$F'(x) = $:

iii) $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ — v.i.e Newton Rapson formula

If recurrance formula given, find the problem.

$\boxed{x_{n+1} = x_n}$ two iteration will be same.

Note:

	Method	Order
1.	Bisection	1
2.	Regula Falsi	1.
3.	Secant Method	1.62
4.	Newton Rapson	2.

II Solution of Simultaneous linear eq^n

Analytical method

1. Gauss Elimination method (Direct method or Analytical method)
(Back substitution method)

* use this method

$$a_1x + b_1y + c_1z = d_1 \quad \text{—(1)}$$
$$a_2x + b_2y + c_2z = d_2 \quad \text{—(2)}$$
$$a_3x + b_3y + c_3z = d_3 \quad \text{—(3)}$$

a_1 largest, 1st pivot, 2nd pivot

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ 0 & b_2' & c_2' \\ 0 & b_3' & c_3'' \end{bmatrix}$$

Step 1. Check out of a_1, a_2, a_3 which one has the largest magnitude.

Let a_1 is largest.

Step 2: With the help of eq (1) eliminate x from eq (2) & (3)

$$a_1x + b_1y + c_1z = d_1 \quad \text{—(1)'}$$
$$b_2'y + c_2'z = d_2' \quad \text{—(2)'}$$
$$b_3'y + c_3'z = d_3' \quad \text{(3)'}$$

Step 3: Out of b_2' & b_3' check which one is largest

Let, b_2' = largest

Step 4: With the help of eq 2 eliminate y from 3 eq (3)'

$$a_1x + b_1y + c_1z = d_1 \quad \text{—(1)}$$
$$b_2'y \quad c_2'z = d_2' \quad \text{—(2)'}$$
$$c_3'z = d_3'' \quad \text{—(3)''}$$

then solve, x, y & z by back substitution

* **Partial pivoting:**

In step 2 - eq (1) - 1st pivotal eq^n & a_1 = 1st pivot

In step 4 - eq (2) - 2nd pivotal eq^n & b_2' = 2nd pivot

2. Gauss Jordan (Direct substitution method) Analytical method

$AX = B$

$$a_1 x + b_1 y + c_1 z = d_1$$
$$a_2 x + b_2 y + c_2 z = d_2$$
$$a_3 x + b_3 y + c_3 z = d_3$$

* use this method

$$\begin{bmatrix} a_1 & 0 & 0 \\ 0 & b' & 0 \\ 0 & 0 & c_3'' \end{bmatrix} = \begin{bmatrix} d_1'' \\ d_2'' \\ d_3'' \end{bmatrix}$$

Step 1 : Let a_1 = largest. Check (a_1, a_2, a_3) = largest.

Step 2 : With eq ① eliminate x from eq ② & ③

$$a_1 x + b_1 y + c_1 z = d_1$$
$$b_2' y + c_2' z = d_2'$$
$$b_3' y + c_3' z = d_3'$$

Step 3 :- Check $(b_2' ; b_3')$ = largest. Let b_2' = largest

Step 4 :- With eq ② eliminate y from eq ① & ③

$$a_1 x + c_1' z = d_1'$$
$$b_2' y + c_2' z = d_2'$$
$$c_3' z = d_3'$$

Step 5 :- With eq ③ eliminate z from eq ① & ②

$$a_1' x = d_1''$$
$$b_2' y = d_2''$$
$$c_3'' z = d_3''$$

In case of Gauss Jordan method, when ever we eliminate, we eliminate from all off the eq^n. but

In Gauss elimination whenever we elimate we eliminate from rest of the eq^n

3. Factorisation method or Doolittle's method. (Direct method).

$AX = B$

i) Let $A = LU$, where

lower triangular $L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}$ & $U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$ upper triangular

ii) It becomes $LUX = B$

iii) Write $UX = V \rightarrow \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$

iv) eq ② becomes $LV = B$

To compute L & U, we write eq ① as.

$$LU = A.$$

$$\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

** Crout's :

$$LU = A$$

$$\begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} 0 & u_{12} & u_{13} \\ 0 & 0 & u_{23} \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Numerical Method – soln of simultaneous eqn.

1 Jacobi's method :

$a_1x + b_1y + c_1z = d_1$ $\quad a_1$ - gre
$a_2x + b_2y + c_2z = d_2$ $\quad b_2$ - gre
$a_3x + b_3y + c_3z = d_3$. $\quad c_3$ - greatest

Step 1 :- Choose the eqn that contains the largest ~~value~~ coefficient x, y, z & calculate the values of x, y & z from the respective eqn.

$$x = \frac{d_1 - b_1y - c_1z}{a_1}$$

$$y = \frac{1}{b_2}(d_2 - a_2x - c_2z)$$

$$z = \frac{1}{c_3}(d_3 - a_3x - b_3y)$$

1st iteration : $x_0 = y_0 = z_0 = 0$.

x_1 y_1 z_1

2nd iteration : x_1, y_1, z_1 use.

* In this method whenever we use, we use the old values i.e values from last iteration

2. Gauss Seidel method :

Whenever we use, we use newly available value.

$x_1 \rightarrow y_0 = z_0 = 0$, $y_1 = x_1, z_0 = 0$, $z_1 = x_1, y_1, z = 0$

III. Solution of differential equation. $\frac{dy}{dx} = f(x,y)$

1. Euler's method. (Explicit Euler method)

$$\boxed{y_{i+1} = y_i + h f(x_i, y_i)}$$

x	y	$f(x,y)$	y_{i+1}
x_i	y_i	$f(x_i)$	

$$y_{new} = y_{old} + h f(x_{old}, y_{old})$$

step size

2. Modified Euler's method : (Predictor corrector method) (Implicit Euler method)

(n) modification no in y_{i+1}

$$\boxed{y^{n}_{i+1} = y_i + \frac{h}{2}\left[f(x_i, y_i) + f(x_{i+1}, y^{(n-1)}_{i+1})\right]}$$

new new

3. Rangi Kutta method:

$$\boxed{y_{i+1} = y_i + k.}$$

where,

$$k = \frac{k_1 + 2k_2 + 2k_3 + k_4}{6}$$

&

$k_1 = hf(x_0, y_0)$

$k_2 = hf(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2})$

$k_3 = hf(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2})$

$k_4 = hf(x_0 + h, y_0 + k_3)$

IV Solution of numerical integration :

1. Trapezoidal rule : 1^{st} order   Linear   true value   +ve error

$$\boxed{\int_a^b f(x)\,dx = \frac{h}{2}\left[(y_0+y_n) + 2(y_1+y_2+y_3+\dots\dots y_{n-1})\right]}$$

$\boxed{h = \frac{b-a}{n}}$ n = no of intervals h = step size .   2 ↓ all

2. Simpson $\frac{1}{3}$ rd rule. 2^{nd} order   Parabolic

$$\boxed{\int_a^b f(x)\,dx = \frac{h}{3}\left[(y_0+y_n) + 2(\underbrace{y_2+y_4+y_6\dots y_{n-2}}_{even}) + 4(\underbrace{y_1+y_3+y_5\dots}_{odd})\right]}$$

$h = \frac{b-a}{n}$ $n \rightarrow$ multiple of 2

2 ↓ ev   4 ↓ rest

3. Simpson $3/8$ th rule 3^{rd} order

$$\boxed{\int_a^b f(x)\,dx = \frac{3h}{8}\left[(y_0+y_n) + 3(y_1+y_2+y_4+y_5\dots y_{n-1}) + 2(\underbrace{y_3+y_6+y_9\dots y_{n-3}}_{mult})\right]}$$

$h = \frac{b-a}{n}$ $n \rightarrow$ multiple of 3

2 ↓ 6 mul   3 ↓ rest

Differential Equation < Ordinary / Partial .

#1. Order and Degree of ODE.   – free from radicals & fractions. (dependent variable)

#2 Formation of an ordinary differential eq^n

Let . $F(x, y, a, b) = 0$.

i) Differentiate the func^n such that
Order of differential eq^n = no of arbitrary constant.

ii) Eliminate the constants (a & b). such that

$$\phi\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}\right) = 0 .$$

3. Solution of differential eq^n < General sol^n (no value c) / Particular sol.

* Linear dependency :

The sol^{n}s y_1 & y_2 are said to the linearly independent if the value of the following determinant $\neq 0$. This is called Wronskian matrix

If, $\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \neq 0$ two sol^n are linearly independent

If $\begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} \neq 0$ two sol^n are linearly independent.

#4. Solution of differential eq^n of 1st order and 1st degree.

1. Variable separable
2. Equation reducible to variable separable
3. Homogeneous D.E.
4. Non-homogeneous D.E* (reducible to either variable separable or Homogeneous)
5. Linear D.E
6. Bernouli's D.E (reducible to linear D.E)
7. Exact D.E*
8. Reducible to exact D.E**

1. Variable separable :

2. ~~Note~~ Reducible to variable separable.

If D.E of form: $\frac{dy}{dx} = f(ax+by+c)$.

Put $t = ax + by + c$ to reduce to variable separable

3. Homogeneous differentiable eq^n

$\frac{dy}{dx} = F(x,y)$ or $Mdx + Ndy \Rightarrow$ Homogeneous D.E.

when M & N have same degree.

Sol: Put $y = vx$, such that

$$\frac{dy}{dx} = v + x\frac{dv}{dx}.$$

4. Non-homogeneous differential eq^n (reducible to homogeneous form)

$$\frac{dy}{dx} = \frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2}$$ is non homogeneous eq^n

Case I

i) if $\frac{a_1}{a_2} = \frac{b_1}{b_2}$, then their exist a substitution which reduces above eq^n to a homogeneous eq^n

Case II

ii) But if $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$, then substitute $x = X + h$ & $y = Y + k$ such that $dx = dX$, $dy = dY$

$$\frac{dY}{dX} = \frac{a_1(X+h) + b_1(Y+k) + c_1}{a_2(X+h) + b_2(Y+k) + c_2}$$

$$\frac{dY}{dX} = \frac{a_1X + b_1Y + (a_1h + b_1k + c_1)}{a_2X + b_2Y + (a_2h + b_2k + c_2)}$$

Choose h & k such that $a_1h + b_1k + c_1 = 0$ & $a_2h + b_2k + c_2 = 0$

$$\frac{dY}{dX} = \frac{a_1X + b_1Y}{a_2X + b_2Y}$$

5. Linear differentiable eq^n

Linear D.E $\Rightarrow$ dependend variable & its derivative occur only in 1st degree & are not multiplied together.

i) Linear D.E in y

$$\frac{dy}{dx} + Py = Q \quad ; \quad P, Q \text{ are constants or } f(x)$$

$$I.f = e^{\int P\,dx}$$

$$y(I.F) = \int Q\,(I.f)\,dx + c.$$

ii) Linear D.E in x.

$$\frac{dx}{dy} + P'x = Q' \quad , \quad P' \,\&\, Q' \text{ are const. or } f(y)$$

$$I.f = e^{\int P'dy}$$

$$x\,(I.f) = \int Q'(I.f)\,dy + c.$$

6. Bernouli's differential eq^n (non-linear D.E).

$$\frac{dy}{dx} + Py = Qy^n.$$

to make it linear: divide by y^n (1)

$$y^{-n}\frac{dy}{dx} + P\,y^{1-n} = Q$$

Put $y^{1-n} = t$ (2) such that $(1-n)\,y^{-n}\dfrac{dy}{dx} = \dfrac{dt}{dx}$

or

$$y^{-n}\frac{dy}{dx} = \frac{1}{1-n}\frac{dt}{dx}. \quad (3)$$

dl

7. Exact differential eq^n.

$M\,dx + N\,dy = 0$ is exact d.e if $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$.

Sol^n of exact d.e

$$\underset{\substack{\text{keeping}\\ y \text{ const}}}{\int M\,dx} + \int \left(\text{terms of N which donot contain } x.\right) dy = c$$

$x = 0$

8. Reducible to exact differential eq^n

$M\,dx + N\,dy = 0$, but $\dfrac{\partial M}{\partial y} \neq \dfrac{\partial N}{\partial x} \Rightarrow$ not exact

To make it exact – the entire d.e is multiplied by an integrating factor (I.f)

i) If all the terms of M & N are of same degree then

$$I.f = \frac{1}{Mx + Ny}$$

ii) If d.e is of form : $y\,f(x,y)\,dx + x\,g(x,y)\,dy$

$$I.f = \frac{1}{Mx - Ny}$$

iii) If $\dfrac{\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}}{N} = f(x)$ only. then $I.f = e^{\int f(x)\,dx}$

iv) If $\dfrac{\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}}{M} = f(y)$ only the $I.f = e^{\int f(y)\,dy}$

Linear differential equations of higher order.

$$\frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + P_2 \frac{d^{n-2} y}{dx^{n-2}} + \dots\dots P_n y = X \text{ or } F(x).$$

nth order, linear differential eq^n. $P_1, P_2 \dots = \text{const}$

For above d.e, ~~has~~ the general sol^n consists of 2 parts.

$$\boxed{G.S = U + V}$$

U → complementary func

V → Particular integral.

$y = U + V$ $\quad X \neq 0$

** If $X = 0$, then it is called homogeneous n^{th} order linear d.e with const coefficient.

& In this case, general sol is given by y ~~y=u~~

$$\boxed{G.S = U}$$

U → complementary func^n.

$y = u$ $\quad X = 0$.

The eq^n $(D^n + P_1 D^{n-1} + P_2 D^{n-2} + \dots\dots P_n) y = X$

is written as $\{f(D)\} y = X$

The eq^n: $f(D) = 0 \rightarrow$ Auxillary eq^n

* Complementary func^n of a d.e. depends on the nature of roots of auxillary eq^n.

C.F

$f(D) = 0$, $f(m) = 0$.

Roots of A.E	C.F
1. Two distinct roots	$C_1 e^{m_1 x} + C_2 e^{m_2 x}$
2. Two repeated real roots $m_1, m_1 = m$ & a distinct real root m_2	$(C_1 + C_2 x) e^{mx} + C_3 e^{m_2 x}$
3. Three real repeated roots $m_1 = m_2 = m_3 = m$	$(C_1 + C_2 x + C_3 x^2) e^{mx}$
4. One pair of conjugate roots $\alpha \pm i\beta$	$(C_1 \cos \beta x + C_2 \sin \beta x) e^{\alpha x}$
5. Repeated conjugate pair of roots $\alpha \pm i\beta$, $\alpha \pm i\beta$	$e^{\alpha x}((C_1 + C_2 x) \cos \beta x + (C_3 + C_4 x) \sin \beta x)$
6. Conjugate pair of irrational roots $\alpha \pm \sqrt{\beta}$	$(C_1 \cosh(\beta x) + C_2 \sinh \beta x) e^{\alpha x}$

P.I

Particular integral.

If $X \neq 0$ then find both C.F & P.I.

$y = C.F + P.I.$

$$\boxed{P.I = \frac{1}{f(D)} \{X\}}$$

, $f(D)$ = Auxillary polynomial.

**
i) D = differential $(\frac{d}{dx})$.

ii) $\frac{1}{D} = \int$ integral.

$$P.I = \frac{1}{f(D)}\{X\}$$

Rule 1: If $X = e^{ax}$ or e^{ax+b} : $P.I = \frac{1}{f(D)} \cdot e^{ax+b}$

Put : $D = a$, if $f(a) \neq 0$, $P.I = \frac{1}{f(a)} e^{ax+b}$.

if $f(a) = 0$, $P.I = x \cdot \frac{1}{f'(D)} \cdot e^{ax+b}$

Again substitute $D = a$,

if $f'(a) \neq 0$, $P.I = x \cdot \frac{1}{f'(a)} e^{ax+b}$

if $f'(a) = 0$, $P.I = x^2 \frac{1}{f''(D)} \cdot e^{ax+b}$

Again repeat, till $f^n(a) \neq 0$,

Rule 2 :- If $X = \sin(ax+b)$ or $X = \cos(ax+b)$:

$PI = \frac{1}{f(D^2)} \cdot \sin(ax+b)$ or $\cos(ax+b)$

Substitute : $D^2 = -(a^2)$, ~~provi~~

if $f(-a^2) \neq 0$ $P.I = \frac{1}{f(-(a)^2)} \sin(ax+b)$

if $f(-a^2) = 0$ $P.I = \frac{x \cdot \sin(ax+b)}{f'(-(a^2))}$

Again repeat : till $f^n(-(a^2)) \neq 0$.

Rule 3: If $X = x^m$: $P.I = \frac{1}{f(D)} \cdot x^m$.

We write $\frac{1}{f(D)} = [f(D)]^{-1}$ &

& then expand by using binomial expansion.

$$\boxed{\begin{aligned}(1-x)^{-1} &= 1 + x + x^2 + x^3 + x^4 + \ldots\ldots \\ (1+x)^{-1} &= 1 - x + x^2 - x^3 + x^4 - x^5 + \ldots\ldots \\ (1-x)^{-2} &= 1 + 2x + 3x^2 + 4x^3 + \ldots\ldots \\ (1+x)^{-2} &= 1 - 2x + 3x^2 - 4x^3 + \ldots\ldots\end{aligned}}$$

Rule 4 : (i) If $X = e^{ax} \cdot V$, $V = \sin(bx+c)$ or $\cancel{V = x^m}$

$PI = \frac{1}{f(D)} \cdot e^{ax} \cdot V$ $\Rightarrow$ $P.I = e^{ax} \cdot \frac{1}{f(D+a)} \cdot V$

~~Rule~~ ~~(ii)~~ If $X = e^{ax} \cdot V$, $V = x^m$

Resolve $\frac{1}{f(D)}$ into partial fraction. & operate each partial fraction on X, remembering that

$$\frac{1}{D-a} \cdot X = e^{ax} \int X e^{-ax}\, dx$$

Two Other method of finding P.I.

1. Cauchy Euler differential eq^n.

$$x^n \frac{d^n y}{dx^n} + P_1 x^{n-1} \frac{d^{n-1}y}{dx^{n-1}} + P_2 x^{n-2} \frac{d^{n-2}y}{dx^{n-1}} + \dots P_n y = X \text{ or } F(x)$$

is called Cauchy. Euler. d.e., P_1, P_2, P_3 = const.
$X = f(x)$.

The above is reduced to n^{th} order liner d.e with const coefficient by substituting.

$$\boxed{z = \log x \quad \text{or} \quad x = e^z}, \quad \theta = \frac{d}{dz}$$

$$\left.\begin{aligned} x \frac{dy}{dx} &= \theta y \quad \text{or} \quad \frac{dy}{dz} \\ x^2 \frac{d^2y}{dx^2} &= \theta(\theta-1)y \\ x^3 \frac{d^3y}{dx^3} &= \theta(\theta-1)(\theta-2)y. \end{aligned}\right\}$$

Substitute all these values in d.e it results in linear. eq^n Which can be solved as before.

A last in sol back substitute $x = e^z$

2. Method of variation of parameters : use to find difficult P.I

imp

This method is quite general and applies to eq^n of the form.

$y'' + py' + qy = X$ where, p, q & $X = f(x)$.

It gives.

$$P.I = -y_1 \int \frac{y_2 X}{W} dx + y_2 \int \frac{y_1 X}{W} dx$$

where y_1 & y_2 are the solutions of $y'' + py' + qy = 0$ *

find. C.F $y = C_1 y_1 + C_2 y_2$

& $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$ is called Wronskian of y_1, y_2

Second Order Linear Partial Differential eq^n

1. Solution of one dimentional heat eq^n.

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad \text{where } c^2 = \text{thermal diffusivity } (\alpha)$$

$$c^2 = \alpha' = \frac{k}{\rho C_p}$$

The initial Boundary condition:

i) $u(0,t) = u(l,t) = 0$

ii) $u(x,0) = f(x)$.

Cases

case 1: $k = 0$, $u(x,t) = C_1(C_2 x + C_3)$

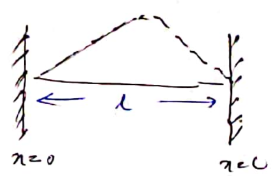
case 2: $k = p^2$ (+ve) $u(x,t) = (C_1 e^{px} + C_2 e^{-px}) C' e^{p^2 c^2 t}$

case 3: $k = -p^2$. $u(x,t) = C' e^{-p^2 c^2 t}(C_1 \cos px + C_2 \sin px)$

realistic situation.

$$u(x,t) = \sum_{n=1}^{\infty} A_n e^{-\frac{n^2\pi^2 c^2 t}{l^2}} \sin \frac{n\pi x}{l}$$

2. Sol^n of one dimentional wave eq^n.



$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

u = displacement

i) $u(0,t) = u(l,t) = 0$

ii) $u(x,0) = f(x)$

iii) $\left.\frac{\partial u}{\partial t}\right|_{t=0} = 0$

Laplace Transform

$f(t)$, $t > 0$

$$F(s) = L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt .$$

#1. Laplace transform of some basic func :

1. $L\{1\} = \frac{1}{s}$

* $L\{\delta(t)\} = 1$ — unit impulse funcⁿ

2. $L\{e^{at}\} = \frac{1}{s-a}$

3. $L\{\sin at\} = \frac{a}{s^2+a^2}$

4. $L\{\cos at\} = \frac{s}{s^2+a^2}$

5. $L\{\sinh at\} = \frac{a}{s^2-a^2}$

6. $L\{\cosh at\} = \frac{s}{s^2-a^2}$

7. $L\{t^n\} = \frac{n!}{s^{n+1}}$, where $n = 0, 1, 2, 3 ..$ $(Z+)$

$= \frac{\Gamma(n+1)}{s^{n+1}}$ if n is fraction but not a $-$ve integer

Note :

i) $\Gamma n = (n-1)!$ if n is a +ve integer

ii) $\Gamma n = (n-1)\,\Gamma(n-1)$

iii) $\Gamma \frac{1}{2} = \sqrt{\pi}$

iv) $\Gamma 1 = 1$

Properties of Laplace transform.

1. $L\{af(t) \pm b\,g(t)\} = aL\{f(t)\} \pm bL\{g(t)\}$ — L is linear operator

~~shifting property~~ :

If $L\{f(t)\} = F(s)$, then

1. 1^{st} shifting property : $L\{e^{at} f(t)\} = F(s-a)$.

or

$L\{e^{-at} f(t)\} = F(s+a)$

2. 2^{nd} shifting property :- $L\{f(t-a)\} = e^{-as}.F(s)$ * or

$L\{f(t-a)\,u(t-a)\} = e^{-as}.L(f(t-a))$ * imp

3. Multiplication by t^n : $L\{t^n f(t)\} = (-1)^n \dfrac{d^n}{ds^n}F(s)$

- $L\{t\,f(t)\} = -\dfrac{d}{ds}F(s)$
- $L\{t^2 F(t)\} = \dfrac{d^2}{ds^2}F(s)$

4. Division by t : $L\left\{\dfrac{f(t)}{t}\right\} = \int_s^\infty F(s)\,ds$

M ⇔ Diff

Div ⇔ Int

5. Laplace of derivatives :-

$L\left\{\dfrac{d^n}{dt^n}f(t)\right\} = s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0) - s^{n-3}f''(0) + \ldots\ldots s^0 f^{n-1}(0)$

- $L\left\{\dfrac{d}{dt}f(t)\right\} = sF(s) - s^0 f(0)$
- $L\left\{\dfrac{d^2}{dt^2}f(t)\right\} = s^2F(s) - s^1 f(0) - s^0 f'(0)$

6. Laplace of integrals : $L\left\{\int_0^t f(t)\,dt\right\} = \dfrac{F(s)}{s}$

7. Change of scale property : $L\{f(at)\} = \dfrac{1}{a}.F(s/a)$

8. Initial value theorem :

if $L\{f(t)\} = F(s)$

$\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s)$

9. Final value theorem.

if $L\{f(t)\} = F(s)$

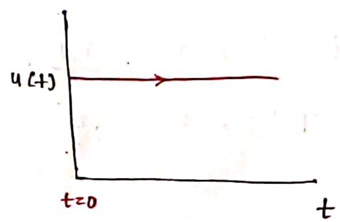
$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$

1. Laplace of periodic func :-

$L\left\{\begin{matrix}\text{periodic}\\ \text{func}\end{matrix}\right\} = \displaystyle\int_0^T \frac{e^{-st} f(t)\,dt}{1-e^{-sT}}$, T = time period of repeat

2. Unit Step Function ($u(t)$)

$u(t) = \begin{cases} 0, & t<0 \\ 1, & t \geq 0 \end{cases}$



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M ⇔ Diff
Div ⇔ Int

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$L\left\{\frac{d^n}{dt^n} f(t)\right\} = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - s^{n-3} f''(0) + \dots\dots s^0 f^{n-1}(0)$

- $L\left\{\frac{d}{dt} f(t)\right\} = s F(s) - s^0 f(0)$
- $L\left\{\frac{d^2}{dt^2} f(t)\right\} = s^2 F(s) - s^1 f(0) - s^0 f'(0)$

if $L\{f(t)\} = F(s)$

6. Laplace of integrals : $L\left\{\int_0^t f(t)\,dt\right\} = \frac{F(s)}{s}$

7. Change of scale property : $L\{f(at)\} = \frac{1}{a} \cdot F(s/a)$

#

8. Initial value theorem :

if $L\{f(t)\} = F(s)$

$\lim_{t \to 0} f(t) = \lim_{s \to \infty} s F(s)$

9. Final value theorem.

if $L\{f(t)\} = F(s)$,

$\lim_{t \to \infty} f(t) = \lim_{s \to 0} s F(s)$

#

1. Laplace of periodic func :-

$L\{\text{periodic func}\} = \int_0^T \frac{e^{-st} f(t)\, dt}{1 - e^{-sT}}$, T = time period of repeat

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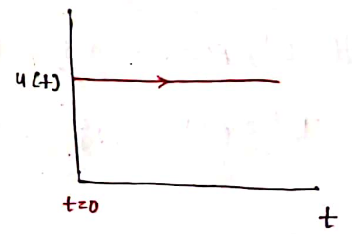
$u(t) = \begin{cases} 0 & , t < 0 \\ 1 & , t \geq 0 \end{cases}$



$L\{u(t)\} = \frac{1}{s}$

Inverse Laplace transform :

Some Formulas :

1. $L^{-1}\left[\frac{1}{(s-a)^2+b^2}\right] = \frac{1}{b} e^{at} \sin bt$

2. $L^{-1}\left[\frac{s-a}{(s-a)^2+b^2}\right] = e^{at} \cos bt$

3. $L^{-1}\left(\frac{s}{(s^2+a^2)^2}\right) = \frac{1}{2a} t \sin at$

4. $L^{-1}\left[\frac{1}{(s^2+a^2)^2}\right] = \frac{1}{2a^3}(\sin at - at\cos at)$

Shifting property of invere Laplace trasform

1. 1^{st} shifting therorm of inverse Laplace transform :

If $L^{-1}\{F(s)\} = f(t)$, then

$L^{-1}\{F(s-a)\} = e^{at} L^{-1}\{F(s)\}$

2. 2^{nd} shifting theorem in inverse laplace transform :

If $L^{-1}\{F(s)\} = f(t)$, then

$L^{-1}\{e^{-as} F(s)\} = f(t-a)$

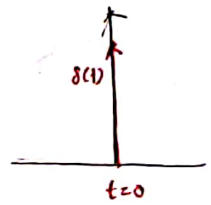
or

$L^{-1}\{e^{-as} F(s)\} = f(t-a)\, u(t-a)$

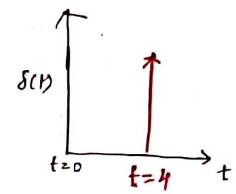
$L^{-1}\{\frac{d}{ds} F(s)\} = -t f(t).$

Impulse function.

$L^{-1}\{1\} = \delta(t)$
↓
unit impulse func



$\delta(t-4)$



Convolution theorem :

If $L^{-1}\{F(s)\} = f(t)$ & $L^{-1}\{G(s)\} = g(t)$, then

$$L^{-1}\{F(s) * G(s)\} = f(t) * g(t) = \int_0^t f(u)\, g(t-u)\, du$$

convolution sign

Vector Calculus

position vector & unit vector :-

$$\vec{OP} = a\hat{i} + b\hat{j} + c\hat{k}$$

$$\hat{OP} = \frac{a\hat{i} + b\hat{j} + c\hat{k}}{\sqrt{a^2 + b^2 + c^2}} = \frac{\vec{OP}}{|\vec{OP}|}$$

· Vector multiplication :

1. Dot product (Sclar product)

- $\vec{a} \cdot \vec{b} = |\bar{a}||\bar{b}| \cos\theta$.
- $\bar{a} \cdot \bar{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$

Projection of $\vec{a}$ in direction of vector $\vec{b}$:

$$\frac{\vec{a} \cdot \vec{b}}{|\bar{b}|} = |\bar{a}| \cos\theta$$

2. Cross product / Vetor product

$$\bar{a} \times \vec{b} = |\bar{a}||\bar{b}| \sin\theta \cdot \hat{n}$$, $\hat{n}$ is unit vector ~~to~~ $\perp$ to $\bar{a}$ & $\bar{b}$ both.

$$\bar{a} \times \bar{b} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

#

- Area of triangle $= \frac{1}{2} |\bar{A} \times \bar{B}|$
- Area of ||gm $= |\bar{A} \times \bar{B}|$

Vector differentiation

$$\nabla = \frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} \hat{k}$$

del operator

Vector differentiation

1. gradient
2. divergence
3. Curl.

1. Gradient of a scalar field : Scalar → Vector

$$\boxed{\text{grad} f = \vec{\nabla}.f = \frac{\partial f}{\partial x} i + \frac{\partial F}{\partial y} j + \frac{\partial F}{\partial z} \hat{k}}$$

Note :-

i) grad f i.e $\vec{\nabla} f$ is always $\perp$ to the surface

or

direction of grad f gives normal surface vector.

ii) It gives the direction of max increase in f. $\nabla.f = |\nabla||f| \cos\theta$

$\theta = 0$ for max

Directional derivative:

It gives the rate of change of func^n in the dir^n of $\vec{a}$.

D.D → gives the component of grad f in direction of $\vec{a}$

$$\boxed{D.D = \text{grad} f . \hat{a}}$$

2. Divergence of a vector field : Vector → Scalar.

$$\boxed{\text{div } \vec{F} = \vec{\nabla}.\vec{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}}$$

Physical meaning :- divergence gives the rate at which the field lines go out of a unit volume.

Note :-

If $\boxed{\text{div } \vec{F} = \vec{\nabla}.\vec{F} = 0}$, then $\vec{F}$ is known as solenoidal or conservative field

3. Curl of a vector field Vector → vector.

$$\text{Curl } \vec{F} = \vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}$$

Physical meaning :-

Curl denotes the rotational characteristic of a vector field.

Note :

If for the field $\vec{F}$, $\boxed{\text{Curl } \vec{F} = \vec{\nabla} \times \vec{F} = 0}$, then it is known as irrotational field

#

1. Line integral :

$$\int \bar{F}.\vec{dr} = \int_c F_x dx + \int_c F_y dy + \int_c F_z dz$$

Note:

$\boxed{\oint \bar{F}.dr = 0 . \Rightarrow \bar{F} \text{ is conservative field.}}$

i.e no work is done and energy is conserved.

2. Surface integral :

$$\iint_S \bar{F}.\hat{n}.ds$$

3. Volume integral :

$$\iiint_V \bar{F}.dv$$

#

1. Green's theorem : (2 D)

$$\oint_{\substack{S \\ \text{closed}}} \bar{F}.d\vec{r} = \iint_{\substack{S \\ \text{open}}} \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) dx\, dy .$$

for 2D

2. Stoke's theorem (3D) (line → surface)

$\hat{n}$ = normal unit vector on surface element.

$$\oint_{\substack{C \\ \text{closed}}} \bar{F}.d\bar{r} = \iint_{\substack{S \\ \text{open}}} (\vec{\nabla} \times \vec{F})\, \hat{n}.dS$$

3. Gauss divergence theorem. closed surface integral $\xrightarrow{\text{to}}$ open volume integral.

$$\oiint_{\substack{\text{closed} \\ \text{loop}}} \vec{F}.(\hat{n}.ds) = \iiint_{\text{open}} \underbrace{\bar{\nabla}.\vec{F}}_{\text{div } F}.dV$$

div F.dV

* Div Curl $\vec{V} = 0$

\# Vector triple product : $A \times (B \times C) = (A.C)B - (A.B)C$.

* $|\bar{a} \times \bar{b}|^2 = a^2 b^2 - (a.b)^2$

\# Sclar triple product : $a.(b \times c) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

$[a,b,c] = [b,c,a] = [c,a,b]$

$a.(b \times c)$ = Volume of parallelopipe.

Calculus

~~Limits~~

* $\lim_{x \to \infty} \frac{\sin x}{x} = 0$

Limits

7 - indeterminant forms

i) $\frac{0}{0}$ ii) $\frac{\infty}{\infty}$ $\longrightarrow$ Type 1.

iii) $\infty - \infty$ iv) $0 \times \infty$ $\longrightarrow$ Type 2

v) 1^∞ . vi) 0^0 , vii) ∞^0 $\longrightarrow$ Type 3.

Type 1 : i) $\frac{0}{0}$, ii) $\frac{\infty}{\infty}$

i) factorisation

ii) L' Hospital Rule.

Type 2 : iii) $\infty - \infty$, iv) $0 \times \infty$

Use

- rationalisation
- factorisation
- or any other alzebraic operations to convert Type 2 to Type 1

Type 1 : $\frac{0}{0}$, $\frac{\infty}{\infty}$ $\Rightarrow$ L.H.R.

Type 3:- 1^∞, 0^0, ∞^0 $\quad f(x)^{g(x)}$.

$$\lim_{x\to a} f(x)^{g(x)} = \lim_{x\to a} e^{g(x)\ln f(x)}.$$

$$\boxed{\lim_{x\to a} f(x)^{g(x)} = e^{\lim_{x\to a} g(x)\ln f(x)}}$$

Continuity

If a func^n is continuous at $x=a$, then,

$$\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = f(a).$$

Differentiability

$y=f(x)$ is said to be differentiable at x_0 when derivative has some finite value.

Mathematically,

$$f'(x) = \lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h} = \text{finite} \quad \text{or} \quad \lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0}$$

* L.H.D = R.H.D $\Rightarrow$ func^n is differentiable

Note:

i) sharp point $\Rightarrow$ not differentiable

ii) continuous $\Rightarrow$ it may or may not be differentiable

iii) differentiable $\Rightarrow$ continuous $\Rightarrow$ Limit exist

Continuity ~~in an open interval~~.

i) Continuity in an open interval.

A func^n f is said to be continuous in an open interval (a,b), if it it continuous at each point of open interval.

ii) Continuity in a closed interval. $[a,b]$

1. Continuous from the right a, $\rightarrow$ a^+
2. Continuous from the left of b, $\leftarrow$
3. Continuous on the open interval (a,b)

Differentiability.

i) Differentiability in an open interval.

Differentiable at each point of the open interval (a,b); for diff.

ii) Differentiabilility in closed inerval $[a,b]$

i) differentiable from right of a;

ii) differentiable from left of b,

iii) differentiable in the open interval (a,b)

Mean Value Theorem

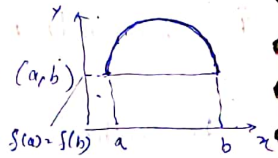
1 Rolle's Mean value theorem :

If $f(x)$ is a func^n defined in the interval $[a,b]$ such that

i) $f(x)$ is continuous in $[a,b]$

ii) $f'(x)$ exist in (a,b) → differentiable in (a,b)

iii) $f(a) = f(b)$



Then, there exist atleast one value of x (say c), $c \in (a,b)$ such that $f'(c) = 0$.

Geometrically : There is atleast one point in (a,b), such that tangent at which is ∥ to x-axis.

* $f'(c)=0$, conti, diff. $\not\Rightarrow$ $f(a)=f(b)$

2. Lagrange's mean value theorem :

If $f(x)$ is a func^n defined in $[a,b]$, such that

i) $f(x)$ is continuous in $[a,b]$

ii) $f(x)$ is differentiable in (a,b) or $f'(x)$ exist in (a,b)

Then, there exist at least one value of x (say c), $c \in (a,b)$

such that $f'(c) = \dfrac{f(b)-f(a)}{b-a}$ $f(a)=f(b) \to$ Rolle's

Geometrically : the theorem gives you value for which,

the avg rate of change of func = instantaneous rate of change of func^n

Cauchy's mean value theorem – same as Lagrange's – only parametric eq^n of curve is given. eg - circle.

If a curve is described by the parametric eq^n.

$y = f(t)$ and $x = g(t)$, where parameter t ranges from $[a,b]$ such that

i) $f(t)$ & $g(t)$ are continuous in $[a,b]$

ii) $f(t)$ & $g(t)$ are differentiable in (a,b)

then there exist some $c \in (a,b)$ such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$$

Taylor and Maclaurin's series

Taylor series : $y = f(x)$. at $x = x_0$ around/about

$$f(x) = f(x_0) + (x-x_0) f'(x_0) + \frac{(x-x_0)^2}{2!} f''(x_0) + \frac{(x-x_0)^3}{3!} f'''(x_0) + \ldots\ldots$$

$$\boxed{\text{General term}_{n\text{th}} = \frac{(x-x_0)^n}{n!} f^n(x_0)}$$

term used :- approximation

Maclurian's series :- Taylor's expansion at $x_0 = 0$.

$x = x_0 = 0$

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \ldots\ldots$$

$$\boxed{\text{General term}_{n\text{th}} = \frac{x^n}{n!} f^n(0)}$$

Expansion of Known series :-

1. $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$
2. $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$ odd ↓ diff.
3. $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}$ even
4. $\tan x = x + \frac{x^3}{3} + \frac{2}{15}x^5 + \ldots$
5. $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} +$
6. $a^x = 1 + x\log a + \frac{x^2}{2!}(\log a)^2 + \ldots$
7. $\sin^{-1} x = x + \frac{x^3}{6} + \frac{3}{40}x^5$
8. $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} -$

1 Partial derivative :

$z = f(x, y)$.

$\frac{\partial z}{\partial x}$: y const

$\frac{\partial z}{\partial y}$: x - const

$$\boxed{\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}}$$

2. Homogeneous func^n:

An expression in which every term has same total degree.

How to check homogenity

let $z = f(x, y)$.

If: $z = f(kx, ky) = k^n f(x, y)$, then. ($n$ → degree of homogeneous eq^n)

z is said to be homogeneous func^n of degree n

General form of homogeneous func^n

$$f(x,y) = a_0 x^n + a_1 x^{n-1} y + a_2 x^{n-2} y^2 + \ldots\ldots a_{n-1} x^1 y^{n-1} + a_n y^n.$$

where n = degree of homogeneous func.

Euler's theorem - Applicable for homogeneous func^n.

If $z = f(x, y)$ is a homogeneous func^n of degree n, then

$$\boxed{\begin{aligned} & x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz \\ & x^2 \frac{\partial^2 f}{\partial x^2} + 2xy\frac{\partial^2 f}{\partial x \partial y} + y^2 \frac{\partial^2 f}{\partial y^2} = n(n-1)z \end{aligned}}$$

2. Total derivative :

$$dz = \frac{\partial z}{\partial x} \cdot dx + \frac{\partial z}{\partial y} \cdot dy$$

total derivative coefficient .

$$\boxed{\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}}$$

Total derivative.

Jacobians

$$J\left(\frac{x,y}{r,\theta}\right) = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$$

$$J\left(\frac{u,v}{x,y}\right) = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

<u>Application of differentiation</u>

1. Increasing and decreasing func^n .

- increasing $\frac{dy}{dx} > 0$ $x \in (x_1, x_2)$
- decreasing $\frac{dy}{dx} < 0$ $x \in (x_1, x_2)$
- monotonically increasing $\frac{dy}{dx} > 0$ $\forall x \in R$
- monotonically decreasing $\frac{dy}{dx} < 0$ $\forall x \in R$

2. Maxima & Minima :

<u>Double derivative test</u> : Put $\frac{dy}{dx} = 0$, (x_1)

$\left.\frac{d^2y}{dx^2}\right|_{x_1} < 0$ Maxima

$\left.\frac{d^2y}{dx^2}\right|_{x_1} > 0$ Minima .

$\left.\frac{d^2y}{dx^2}\right|_{x_1} = 0$ Point of inflection

#* Maxima minima in [a, b] .

Also check at extreme points for maxa & minima .

#3. Application based problem of maxima & minima.

- Always try to draw figure related to question
- Find func^n ✓ which needs to be maximize / minimize
- variable
- contraint (relation)
  convert func^n into 1 variable.
- make differentiation easier by squaring func^n or do something else
  ↳ to remove $\sqrt{\ }$

7 Maxima minima for a func^n in 2 variable :

1. Find p, q, r, s & t

$$p = \frac{\partial z}{\partial x}, \quad q = \frac{\partial z}{\partial y}, \quad r = \frac{\partial^2 z}{\partial x^2}, \quad s = \frac{\partial^2 z}{\partial x \partial y}, \quad t = \frac{\partial^2 z}{\partial y^2}$$

2. Equate p & q $= 0$, for stationary points

$$\frac{\partial z}{\partial x} = 0, \quad \frac{\partial z}{\partial y} = 0$$

3. For each stationary point, check the value of $rt - s^2$

i) if $rt - s^2 > 0$ & $r > 0$ Minima

ii) if $rt - s^2 > 0$ & $r < 0$ Maxima

iii) $rt - s^2 < 0$, Point of inflection

Integral Calculus. * Definite $\int$

Application of integration :

1. Area under the curve.

a) $\int_a^b f(x)\,dx$ = Area bounded by $f(x)$, line $x=a$, $x=b$ & x-axis

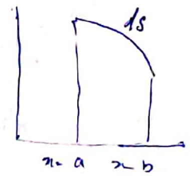
b) $\int_a^b f(y)\,dy$ = Area bounded by $f(y)$, line $y=a$, $y=b$ & y-axis

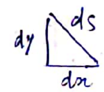
c) Area b/w two curves.

$$A = \int_a^b [f(x) - g(x)]\,dx$$

2. Length of curve

$$\boxed{L = \int_{x=a}^{x=b} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,.dx \qquad L = \int_{y=a}^{y=b} \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\,.dy}$$





$ds^2 = dx^2 + dy^2$

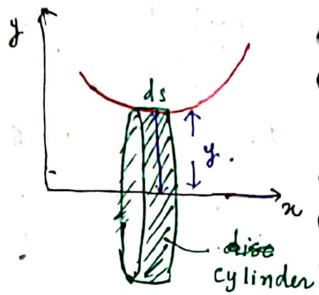
$ds = \sqrt{dx^2 + dy^2}$

$$L = \int_0^t \sqrt{\left(\frac{\partial x}{\partial t}\right)^2 + \left(\frac{\partial y}{\partial t}\right)^2 + \left(\frac{\partial z}{\partial t}\right)^2}$$ parametric func^n

3. Area of surface by revolution –

$$A = \int_a^b 2\pi y \, ds \quad \text{about } x\text{-axis}$$

$$A = \int_a^b 2\pi x \, ds \quad \text{about } y\text{-axis}$$



$A = \underline{2\pi r h}$

$V = \pi r^2 h$

4. Volume of curve by revolution :

$$V_x = \int_a^b \pi y^2 \, ds \quad \text{about } x\text{-axis}$$

$$V_y = \int_a^b \pi x^2 \, ds \quad \text{about } y\text{-axis}$$

Double integral.

$\underbrace{\iint_R z \, dx \, dy}_{\text{volume}}$ $\underbrace{\iint dx \, dy}_{\text{Area}}$

* i) Limit of integration cost : $\int_a^b \left(\int_c^b z \, dy \right) dx$ interior first

ii) Limits of integration is variable : $\int_a^b \left(\int_{x_1}^{x_2} z \, dy \right) dx$

imp

* If limits are func^n of x, then integration is to be done w.r.t y and vice versa.

Triple integral.

$$I = \int_{x_1}^{x_2} \left[\int_{y_1(x)}^{y_2(x)} \left[\int_{z_1(x,y)}^{z_2(x,y)} f(x,y,z) \, dz \right] dy \right] dx$$

Change of order of integration.

* Always make figure then change

$$\int_{y_1}^{y_2} \left(\int_{x_1}^{x} f(x,y) \, dx \right) dy = \int_{x_1'}^{x_2'} \left[\int_{y_1'}^{y_2'} f(x,y) \, dy \right] dx.$$

Linear Alzebra

I. Types of matrix :

1. Rectangular matrix : no of rows $\neq$ no of columns $(m \times n)$
2. Square matrix : no of rows = no of columns $(n \times n)$
3. Diagonal matrix :
   i) square matrix
   ii) Non-diagonal element = 0 i.e $a_{ij} = 0$ $i \neq j$
   iii) Diagonal element may or may not be 0. but atleast one of the diagonal element is non-zero.

   diag [ ]

4. Scalar matrix : (digonal matrix) $\left.\begin{array}{l} a_{ij} = 0 \; . \; i \neq j \\ a_{ii} = k , \; i = j \end{array}\right\}$
5. Identity matrix $a_{ij} = 0$, $i \neq j$
   $a_{ii} = 1$, $i = j$

* Identity matrix is use to represent any no. in matrix form

6. Null matrix $a_{ij} = 0$.
   [O]
   represented as capital zero.

* Null matrix may or may not be a square matrix.

II.

1. # Multiplication of matrix :

$A_{m \times n} \times B_{pq}$ is possible if $n = p$ (column 1 matrix = Row of 2nd matrix)

⇓

Resultant ~~real~~ matrix = $C_{m \times q}$

2. # Trace of matrix : sum of diagonal element.

$$\boxed{tr(A) = \sum a_{ii} = a_{11} + a_{22} + a_{33}}$$

3. Transpose of matrix :

i) $(A^T)^T = A$

ii) $(A+B)^T = A^T + B^T$

iii) $(AB)^T = B^T A^T$

Types of matrix on the basis of transpose → Real Matrix, Complex matrix

1. Real matrix :

i) symmetric matrix : $\boxed{A^T = A}$ i) $\boxed{a_{ij} = a_{ji}}$

ii) skew symmetric matrix : $\boxed{A^T = -A}$ i) $a_{ij} = -a_{ji}$ ii) $a_{diag} = 0$

- $A + A^T$ = symmetric version of matrix A
- $A - A^T$ = skew symmetric version of matrix A.

* A matrix can be represented as sum of symmetric & skew symmetric matrix

$$\boxed{A = \left(\frac{A+A^T}{2}\right) + \left(\frac{A-A^T}{2}\right)}$$

iii) Orthogonal matrix : $\boxed{A^T = A^{-1}}$ or $AA^T = I$

For orthogonal matrix : $|A| = 1$

A.B = 0 orthogonal

Conjugate of a matrix :- Change i to $-i$

i) $\overline{\overline{A}} = A$

ii) $\overline{A} = A$ ⇒ A is purely real matrix

iii) $\overline{A} = -A$ ⇒ A is purely imaginary matrix

→ Transpose of the conjugated matrix : $A^\theta = (\overline{A})^T$

2. Complex matrix :

i) Harmation matrix : $A^\theta = A$

ii) skew Harmation matrix : $A^\theta = -A$

iii) Unitry matrix : $A^\theta = A^{-1}$

III

1. Nil potent matrix of index x : (A^x) : if $A^x = 0$ but $A^{x-1} \neq 0$
2. Idempotent matrix : $A^2 = A$
3. Involutary matrix : $A^2 = I$

IV. # Determinant :

1. Minor : M_{ij}

2. Cofactor $A_{ij} = (-1)^{i+j} M_{ij}$

3. Determinant $|A| = a_{11} \times A_{11} + a_{12} \times A_{12}$ ← cofactor

Points :

i) If matrix $A_{n\times n}$ then Minor = $M_{(n-1)(n-1)}$

ii) For matrix of order $n \times n$ ∴ there will be n^2 minors of ~~n-1~~ order → $(n-1)$ order.

- Similarly for matrix of $(n-1)$ order, there will be $(n-1)^2$ minors of order $(n-2)$

Non-singular or invertible matrix : $|A| \neq 0$.

Singular matrix : $|A| = 0$.

Adjoint matrix : $\text{adj } A = [C_{ij}]^T$

Inverse of matrix :

$$A^{-1} = \frac{1}{|A|} \text{adj } A, \quad |A| \neq 0$$ non-singular matrix

Properties of Determinant :

i) All rows/ column interchange : $|A|$ remain same.

ii) Two rows/ column interchange : $|A| \Rightarrow -|A|$

iii) Two rows/ column same : $|A| = 0$

iv) Any row/ column is 0 : $|A| = 0$

v) If any row/column ~~repre~~ expressed as sum of 2 no. ⇒ Determinant can also expressed as sum of 2 determinant

* If 1 row ⇒ sum of p no's
2 row ⇒ sum of q no's
3 row ⇒ sum of r no's

then determinant can be expressed as a sum of $p \times q \times r$

vi) * $\boxed{|kA| = k^n |A|}$

vii) $R_1 \pm kR_2$ ⇒ $|A|$ remain same
elementary operation.

* If apply row than only row operation &
If apply column than apply only column operation.

* Shortcut for adj (A) 2×2 matrix

i) interchange digonal element

ii) reverse sign of non diagonal element.

Important Points :

i) $AA^{-1} = I = A^{-1}A \longrightarrow A\cdot\left(\frac{adj\,A}{|A|}\right) = I$

ii) $A\,adj\,A = |A|.I$ $\quad \frac{adj\,A}{|A|} = A^{-1}$

iii) Imp. $|A^{-1}| = \frac{1}{|A|}$

iv) $|A^T| = |A|$

~~|A| = 1 A orthogonal matrix~~

v) For orthogonal matrix : $|A| = 1$,
$(A^{-1} = A^T)$

vi) $|KA| = K^n|A|$

vii) Area of $\Delta = \frac{1}{2}\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}$

* $a^2+b^2+c - \cancel{2}ab - bc - ca = (a-b)^2 + (b-c)^2 + (c-a)^2$

#1 Linear dependency and independency of the vectors :

$$k_1E_1 + k_2E_2 + k_3E_3 + \ldots\ldots k_nE_n = 0 ,$$

where

$k_1, k_2, k_3 \ldots k_n$ are all sclars

$E_1, E_2, E_3 \ldots E_n$ are all vectors.

case 1 : Linearly dependent	case 2 : Linearly independent
* If $k_1 = k_2 = k_3 \ldots k_n \neq 0$ & $k_1E_1 + k_2E_2 \ldots k_nE_n = 0$ holds.	* If $k_1 = k_2 = k_3 = \ldots k_n = 0$ & $k_1E_2 + k_2E_2 \ldots + k_nE_n = 0$ holds.
* $\underline{\underline{|A| = 0}}$	* $\underline{\underline{|A| \neq 0}}$.

#2. Rank of matrix. [$\rho(A)$]

$\rho(A)$ = no of linearly independent rows or column.

dimention of null space of matrix = order of matrix − $\rho(A)$.

Finding the rank of the matrix

1. Determinant method. $A_{n\times n}$

i) Square matrix

case 1 - $|A| \neq 0 \Rightarrow \rho(A) = n$

case 2 $|A| = 0 \Rightarrow \rho(A) < n$

~~case~~

case 2.1 : If any $|\overset{\text{minor}}{M_{(n-1)}}| \neq 0 \Rightarrow \rho(A) = n-1$

case 2.2 If every $|M_{n-1}| = 0 \Rightarrow \rho(A) < n-1$

case 2.2.1 If any $|M_{n-2}| \neq 0 \Rightarrow \rho(A) = n-2$

case 2.2.2 If every $|M_{n-2}| = 0 \Rightarrow \rho(A) < n-2$.

(ii)** For rectangular matrix

We convert a non-square matrix into a square matix by adding a zero row or zero column.

Points :

i) For rectangular matrix of order $m \times n$.

$r(A_{m\times n}) \leq \min \text{ of } (m,n)$

ii) Rank $(AB) \leq [\min \text{ of } r(A) \text{ or } r(B)]$

eg $A_{2\times 5}$ $B_{5\times 3}$

$r(AB) \leq 2$

2. Normal form: $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ $r = \rho(A)$

3. Echelon form :

$\rho(A)$ = no of non-zero rows.

** Use only Row operation for normal & Echelon form.

Solution of Simultaneous Linear eqn:

$AX = B$
- → $AX = 0$ Homogeneous
- → $AX = B$, $B \neq 0$ Non-homogeneous

- → **consistant**: If sol exists then system is known as consistent system
- → **Inconsistant**: If sol. doesnot exists, then system is known as inconsistent system.

Cayley Hamilton theorem :

Every matrix satisfies its characteristic eqn

i) $|A - \lambda I| = 0$ $\qquad A \to$ matrix.

$f(\lambda) = a_0\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0$

ii) $f(A) = a_0A^2 + a_1A^2 + a_2A + a_3I = 0$.

Application:

i) It helps us to determine inverse of matrix ($\underline{\underline{A^{-1}}}$)

ii) Post / Pre-multiplying by A^{-1}

iii) It also helps us to determine higher power of A

$A^n = a_1A \pm b_1I$ (linear form)

Eigen system.

$AX = \lambda \cancel{X}$ ⇒ imp

$\boxed{(A-\lambda I)X = 0}$

eigen value

eigen vector

$BX = 0$.

Sol^n of system of eq^n (By Aumented matrix method) $(A|B)$ → augemented matrix

Homogeneous system is always consistant.

$AX = B$ Non-Homogeneous.

Unique sol	Infinite sol	No-solution.
$\rho(A\|B) = \rho(A) = n$ (no of variables)	$\rho(A\|B) = \rho(A) < \underline{\underline{n}}$	$\rho(A\|B) \neq \rho(A)$
$\left[\begin{array}{ccc\|c} 1 & - & - & - \\ 0 & 1 & - & - \\ 0 & 0 & - & - \end{array}\right]$	$\left[\begin{array}{ccc\|c} 1 & - & - & - \\ 0 & 1 & - & - \\ 0 & 0 & 0 & 0 \end{array}\right]$	$\left[\begin{array}{ccc\|c} 1 & - & - & - \\ 0 & 1 & - & - \\ 0 & 0 & 0 & k \end{array}\right]$
		$R_3 \Rightarrow 0x + 0y + 0z = k$
⇒ consistent.	⇒ consistent	⇒ In consistent
→ $\|A\| \neq 0$	⇒ $\|A\| = 0$	⇒ $\|A\| = 0$

$AX = 0$ Homogeneous

Unique sol.	Infinite sol^n
$\rho(A) = n$	$\rho(A) < n$
Sol^n = $(0,0,0)$ for $x\ y\ z$ ∴ is always a sol^n	
→ Trival sol / "Zero sol"	→ Non-trival sol^n
→ consistent	→ Consistent
→ $\|A\| \neq 0$	→ $\|A\| = 0$

i) Determining eigen ~~vectors~~ values :

$|A-\lambda I| = 0$

$\lambda^2 + a_0\lambda + a_1 = 0$ ⇒ characteristic eqn

$\lambda = \lambda_1, \lambda_2$ → characteristic roots (eigen values).

ii) Determining the eigen vectors :-

$AX = \lambda X$

$(A-\lambda I)X = 0$ where $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

Substituting the values of λ in the eqn & then solve for x.

imp **Properties of eigen values**

i) eigen values (A) = eigen values of A^T → ~~real matrix~~

complex matrix → eigen val ($A = A^{\theta}$)

ii) eigen values of diag(A) = eigen value (Upper triangular) = eigen (value) (Lower triangular) = diagonal elements.

iii) eigenvalue of symmetric matrix ($A = A^T$) = always real. (real matrix)

iv) eigen value of skew-symmetric matrix ($-A = A^T$) = purely imaginary or zero.

v) eigen values of orthogonal matrix = is of unit modulus. = $|\lambda| = 1$

vi) eigenvalue of orthogonal matrix is λ, then $\frac{1}{\lambda}$ will also be its eigen value

vii)** sum of eigen value = trace of matrix

viii)** product of eigen values = $|A|$ (|A|)

ix)

ix) for a given matrix, if there are n distinct eigen values then there will be n linearly independent eigen vector

x) If $\lambda_1, \lambda_2, \lambda_3$ are eigen values of matrix A, then

a) For matrix kA, the eigen values are $k\lambda_1, k\lambda_2, k\lambda_3$

b) For A^{-1}, the eigen values are $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \frac{1}{\lambda_3}$

*c) For A^k, the eigen values are $\lambda_1^k, \lambda_2^k, \lambda_3^k$. but eigen vectors remain same

d) For adj(A) ($A^{-1}.|A|$), the eigen values are $\frac{|A|}{\lambda_1}, \frac{|A|}{\lambda_2}, \frac{|A|}{\lambda_2}$

xi)* For singular matrix, one of the eigen values must be 0

[** But no effect on eigen vector due to these operations.]

* Similarity Transformation (Diagonalisation of matrix.)

$P^{-1}AP = D$ or $A = PDP^{-1}$

P: modal matrix of A — Matrix formed by using eigen vector of A

D: spectral matrix of A. — Diagonal matrix whose elements are eigen values of A

if repeated eigen vector then cannot be diagonalised.